One Remark About Brückner-Vostokov Explicit Reciprocity Law

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0. Introduction.

Let K be a local complete discrete valuation field of characteristic 0 with a finite residue field k of characteristic $p \geq 3$. Assume that K contains a primitive root of unity of degree p^M, $M \geq 1$, and denote by

$$(\ , \ ) : K^* \times K^* \longrightarrow \mu_{p^M}(K)$$

a pairing given by the Hilbert symbol. Fix a primitive root ζ of degree p^M in K. If $u, v \in K^*$, let $c(u,v) \in \mathbb{Z} \bmod p^M \mathbb{Z}$ be such that $(u,v) = \zeta^{c(u,v)}$. The explicit reciprocity law (ERL) gives a formula for the value of $c(u,v)$ in terms of expressions of u and v as power series in a uniformizer π of K. First results in this direction were obtained by Artin and Hasse, [A-H28], for the field $\mathbb{Q}_p(\zeta)$. A different approach to the problem to find general ERL belongs to I.R.Shafarevich, [Sh50]. His arguments were based on a deep analogy between algebraic numbers and functions on Riemann surfaces. The main idea of Shafarevich was to define an analog $< \ , \ >$ of the Hilbert symbol in a ring of formal power series in one variable X, to prove its independence of the choice of the variable X and to obtain the value of the Hilbert symbol by specialization $X \mapsto \pi$. Proceeding in the same way S.Vostokov, [Vo78], obtained explicit simple expression for $c(u,v)$ (c.f. also [Br64], [Br79]).

The higher r^{th} ($r \geq 1$) analogs of the Artin-Hasse ERL for the field $\mathbb{Q}_p(\zeta)$ were constructed by Coates and Wiles, [Wi78] (the classical ERL appears for $r = 1$). Bloch and Kato, [B-K90], found relation between higher Coates-Wiles ERL and cohomology of Fontaine-Messing. E.de Shalit,[dSh], gave a simple proof of Bloch-Kato ERL from Coates-Wiles ERL in the case $r = 1$. In a recent paper J.-M.Fontaine, [Fo94], deduced Bloch-Kato ERL from Witt ERL for local fields of characteristic p. His arguments use the Fontaine-Wintenberger functor "field of norms", which give a relation between local fields in characteristic p and characteristic 0. This functor depends on the choice of some infinite local field $\widetilde{K}$ with increasing ramification, and Fontaine uses p-cyclotomic extension of $\mathbb{Q}_p$. So, one can combine arguments of Fontaine and of E.de Shalit to deduce the classical Coates-Wiles ERL (in characteristic 0) from Witt ERL (in characteristic p) by the above choice of the field of norms functor. A different way to obtain this implication for extensions of degree p was given by E.de Shalit, [dSh92]. His arguments are based on the relation between local fields in characteristics p and 0, given by P.Deligne, [De84].

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To the best author's knowledge the opportunity to prove the general ERL in the form of Brückner-Vostokov from Witt ERL has not been yet systematically investigated (the simplest case of the problem: $M = 1, K = \mathbb{Q}_p(\zeta)$, was done in [DR89]). In this paper we give a proof of this implication. We use Fontaine's (resp., E.de Shalit's) way to relate Witt-Artin-Schreier (resp., Kummer) theory to Fontaine-Messing theory for $r = 1$. In fact, the only difference with Fontaine's approach is another choice of the field of norms functor. This functor corresponds to the extension $\widetilde{K} = K(\{\pi^{1/p^n} | n \geq 0\})$, where π is a fixed uniformizer of the field K. The answer is given by the natural specialisation of the symbol defined for a power series ring in one variable X_π. Though this symbol coincides with the symbol introduced in [Vo78] we can't explain its independence of the choice of a uniformizer in a power series ring (for different π these power series rings "live in different worlds"). We also restrict our attention to the study of values of the Hilbert symbol only on principal units of the field K. General formula can be easily derived from this study using explicit constructions of p^M-primary elements, [Vo78, n.4].

1. Preliminaries.

Let K be a local field from introduction with fixed uniformizer π.

1.1. *Functor "field of norms" $\mathcal{X}$.*

Consider an extension $\widetilde{K}$ of K given by $\widetilde{K} = \bigcup_{n \geq 0} K(n)$, where $K(0) = K$, $K(n) = K(n-1)(\pi_n)$, $\pi_n^p = \pi_{n-1}$ for $n \geq 1$ and $\pi_0 = \pi$.

$\widetilde{K}$ is an APF-extension in terminology of [Wtb] and, therefore, it defines the field of norms functor $\mathcal{X}$. This functor is defined on the category of algebraic extensions of the field $\widetilde{K}$, $\mathcal{X}(\widetilde{K}) = \mathcal{K}$ is a complete discrete valuation field of characteristic p with the same residue field k, and $\mathcal{X}$ induces an equivalence of the category of algebraic extensions of $\widetilde{K}$ and of the category of separable extensions of the field $\mathcal{K}$. In particular, if we choose an algebraic closure $\bar{K}$ of K, then one can use identifications $\mathcal{X}(\bar{K}) = \mathcal{K}_{\mathrm{sep}}$ and

$$\mathcal{G} = \mathrm{Gal}(\mathcal{K}_{\mathrm{sep}}/\mathcal{K}) = \mathrm{Gal}(\bar{K}/\widetilde{K}) \subset \Gamma = \mathrm{Gal}(\bar{K}/K).$$

If $\widetilde{E}$ is a finite extension of $\widetilde{K}$, then the multiplicative group $\mathcal{E}^*$ of nonzero elements of the field $\mathcal{E} = \mathcal{X}(\widetilde{E})$ is equal to a projective limit of groups E^*, where $K \subset E \subset \widetilde{E}$, $(E : K) < \infty$ and transition morphisms are induced by norm maps N_{E_1/E_2}, where $K \subset E_2 \subset E_1 \subset \widetilde{E}$ and $(E_1 : K) < \infty$. The sequence $\{\pi_n\}_{K(n)}$ gives the uniformizer of the field $\mathcal{K}$. The field of constants of $\mathcal{K}$ consists of sequences $\{\hat{\alpha}^{p^{-n}}\}_{K(n)}$ (where $\hat{\alpha} \in K$ is Teichmuller representative of $\alpha \in k$) and can be identified with k by the map $\alpha \mapsto \{\hat{\alpha}^{p^{-n}}\}$. So, if we denote the above uniformizer of $\mathcal{K}$ by $\widetilde{t_0}$, then $\mathcal{K}$ is identified with the quotient field $k((\widetilde{t_0}))$ of the power series ring $k[[\widetilde{t_0}]]$.

1.2. *Shafarevich's basis for K^*/K^{*p^M}.*

Let e be the absolute ramification index of K and let $\zeta \in K$ be a fixed primitive root of unity of degree p^M.

Let K^0 be the maximal subextension of K which is unramified over $\mathbb{Q}_p$. We have the natural identification of the valuation ring of this field with Witt vectors $W(k)$.

Denote by σ the absolute Frobenius automorphism of $W(k)$. For $\alpha \in W(k)$ and an indeterminate X set

$$E(\alpha, X) = \exp(\alpha X + \frac{(\sigma\alpha)X^p}{p} + \cdots + \frac{(\sigma^n\alpha)X^{p^n}}{p^n} + \dots).$$

One has (c.f. [Sh50])

$$E(\alpha, X) \in W(k)[[X]] \text{ for } \alpha \in W(k);$$

$$E(\alpha_1 + \alpha_2, X) = E(\alpha_1, X)E(\alpha_2, X) \text{ for } \alpha_1, \alpha_2 \in W(k);$$

$$E(a\alpha, X) = E(\alpha, X)^a \text{ for } a \in \mathbb{Z}_p.$$

Let $\{\alpha_i\}_{1 \leqslant i \leqslant N}$ be a $\mathbb{Z}_p$-basis of $W(k)$. Then Shafarevich's basis of K^*/K^{*p^M} consists of the uniformizer π, principal units $E(\alpha_i, \pi^a)$, where $1 \le i \le N, 1 \le a < ep/(p-1), (a,p) = 1$, and one additional so-called p^M-primary element. One can make a choice of a p^M-primary element by fixing $\alpha \in W(k)$ such that $\mathrm{Tr}_{K^0/\mathbb{Q}_p} \alpha \in \mathbb{Z}_p^*$. Then by definition

$$E(\alpha) = E(\beta, \xi_0)^{p^M},$$

where $\beta \in W(\bar{k})$ is such that $\sigma\beta - \beta = \alpha$ and $\xi_0 \in O_K$ is such that $E(1, \xi_0) = \zeta$.

One can show that $E(\alpha)$ can be expressed in terms of elements of the ground field K

$$E(\alpha) \equiv \prod_{n \geqslant 1} E(\sigma^n\alpha, \xi_0^{p^n})^{p^{M-n}} \bmod K^{*p^M}.$$

This expression was simplified by Vostokov, [Vo78, n.4].

The basic property of $E(\alpha)$ is that $E(\alpha)^{1/p^M}$ generates unramified extension of K of degree p^M. Therefore, all images of p^M-primary elements in K^*/K^{*p^M} create a cyclic subgroup of order p^M.

1.3. *Shafarevich's basis for $\mathcal{K}^*/\mathcal{K}^{*p^M}$.*

Here the elements $\widetilde{t_0}$ and $E(\alpha_i, \widetilde{t_0^a})$, where $\{\alpha_i\}_{1 \le i \le N}$ is a $\mathbb{Z}_p$-basis of $W(k)$ and $a \in \mathbb{N}$, $(a,p) = 1$, give a basis of $\mathcal{K}^*/\mathcal{K}^{*p^M}$.

Let

$$\mathcal{N} : \mathcal{K}^* = \varprojlim K(n)^* \longrightarrow K(0)^* = K^*$$

be a canonical projection.

Lemma.

a) $\mathcal{N}(\widetilde{t_0}) = \pi$;

b) *if* $\alpha \in W(k), a \in \mathbb{N}, (a,p) = 1$, *then* $\mathcal{N}(E(\alpha, \widetilde{t_0}^a)) = E(\alpha, \pi^a)$.

Proof.

a) Obviously, $\mathcal{N}(\widetilde{t_0}) = \mathcal{N}((\pi_n)_{K(n)}) = \pi$.

b) It is sufficient to check up that for any $n \geq 1$

$$\mathrm{N}_{K(n)/K(n-1)}(E(\sigma^{-n}\alpha, \pi_n^a)) = E(\sigma^{-(n-1)}\alpha, \pi_{n-1}^a).$$

This is implied by the following observation

$$\mathrm{Tr}_{K(n)/K(n-1)}(\pi_n^{ap^m}) = \begin{cases} 0, & \text{for } m = 0 \\ p\pi_{n-1}^{ap^{m-1}}, & \text{for } m \geq 1. \end{cases}$$

Corollary.

The group $\mathcal{N}(\mathcal{K}^*/\mathcal{K}^{*p^M})$ has an index p^M in K^*/K^{*p^M} and is generated by the images of the elements π and $E(\alpha, \pi^a)$, where $\alpha \in W(k)$, $1 \leq a < ep/(p-1)$, $(a,p) = 1$.

Proof.

By the above lemma $\pi \in \mathcal{N}(\mathcal{K}^*)$ and all $E(\alpha, \pi^a) \in \mathcal{N}(\mathcal{K}^*)$.

So, it is sufficient to show that the index

$$(K^*/K^{*p^M} : \mathcal{N}(\mathcal{K}^*/\mathcal{K}^{*p^M})) \geq p^M.$$

This follows from the inclusion

$$\mathcal{N}(\mathcal{K}^*)K^{*p^M} \subset N_M := \mathrm{N}_{K(M)/K}(K(M)^*)$$

and the equality $(K^* : N_M) = p^M$ given by local class field theory.

1.4. *Compatibility of class field theories.*

By the result of F. Laubie [La], we have a commutative diagramm

$$\begin{array}{ccc} \mathcal{K}^* & \xrightarrow{\mathcal{N}} & K^* \\ {\scriptstyle\psi_{\mathcal{K}}}\downarrow & & \downarrow{\scriptstyle\psi_K} \\ \mathcal{G}^{\mathrm{ab}} & \xrightarrow{i_{\mathcal{K}/K}} & \Gamma^{\mathrm{ab}} \end{array}$$

Here $\psi_{\mathcal{K}}$ and ψ_K are reciprocity maps for the fields $\mathcal{K}$ and K respectively, and homomorphism $i_{\mathcal{K}/K}$ appears from the identification $\mathrm{Gal}(\mathcal{K}_{\mathrm{sep}}/\mathcal{K}) = \mathrm{Gal}(\bar{K}/\widetilde{K}) \subset \Gamma$ given by the above functor $\mathcal{X}$.

Let $E(M)$ and $\mathcal{E}(M)$ be maximal abelian extensions of exponent p^M of the fields K and $\mathcal{K}$ in $\bar{K}$ and $\mathcal{K}_{\mathrm{sep}} = \mathcal{X}(\bar{K})$, respectively. Then we have

$$\mathcal{E}(M) \supset \mathcal{X}(E(M)\widetilde{K}) \supset \mathcal{K},$$

$$E(M) \cap \widetilde{K} = K(M),$$

$$\mathrm{Gal}(\mathcal{X}(E(M)\widetilde{K})/\mathcal{K}) = \mathrm{Gal}(E(M)/K(M)).$$

It follows now from Corollary of n.1 that

$$i_{\mathcal{K}/K}(\mathcal{G}^{\mathrm{ab}}/\mathcal{G}^{\mathrm{ab}\,p^M}) = \mathrm{Gal}(E(M)/K(M)).$$

1.5. *Explicit reciprocity law in characteristic p,* [Wtt].

The uniformizer $\widetilde{t_0}$ of the field $\mathcal{K}$ gives p-basis for any separable extension $\mathcal{E}$ over $\mathcal{K}$. Therefore (c.f. [Ab93]), one can define a compatible on $\mathcal{E} \subset \mathcal{K}_{\text{sep}}$ and $M \geq 1$ system of liftings $O_M(\mathcal{E})$ of the fields $\mathcal{E}$ modulo p^M ($O_M(\mathcal{E})$ is a flat $\mathbb{Z}/p^M\mathbb{Z}$-algebra such that $O_M(\mathcal{E})/pO_M(\mathcal{E}) = \mathcal{E}$).

Fix an element $\widetilde{t} \in O_M(\mathcal{K})$ such that $\widetilde{t} \bmod p = \widetilde{t_0}$ and define (the unique) lifting

$$\sigma : O_M(\mathcal{K}) \longrightarrow O_M(\mathcal{K})$$

of the absolute Frobenius of $\mathcal{K}$ by the condition $\sigma\widetilde{t} = \widetilde{t}^p$. Then for any $\mathcal{E} \subset \mathcal{K}_{\text{sep}}$ there exists only one lifting $\sigma_{\mathcal{E}}$ of the absolute Frobenius of the field $\mathcal{E}$ to $O_M(\mathcal{E})$ which is compatible with the above chosen lifting σ. We use the same symbol σ for all these liftings $\sigma_{\mathcal{E}}$.

Let $f \in O_M(\mathcal{K})$, $g \in \mathcal{K}^*$ and $(f, g) \in W_M(\mathbb{F}_p)$ be a pairing given (as usually) as follows

$$\tau T - T = (f, g),$$

where $T \in O_M(\mathcal{K}_{\text{sep}})$, $\sigma T - T = f$ and $\tau = \psi_{\mathcal{K}}(g) \in \mathcal{G}^{\text{ab}}$.

Witt explicit reciprocity law is given by the formula

$$(f, g) = \operatorname{Tr}(\operatorname{Res} f \frac{d\hat{g}}{\hat{g}}),$$

where $\operatorname{Tr} : W_M(k) \longrightarrow W_M(\mathbb{F}_p)$ is induced by the trace map of the extension $k/\mathbb{F}_p$ and $\hat{g} \in O_M(\mathcal{K})$ is the image of $g \in \mathcal{K}^*$ under the multiplicative morphism $\mathcal{K}^* \longrightarrow O_M(\mathcal{K})^*$ given by the formulae

$$\widetilde{t_0} \mapsto \widetilde{t}, E(\alpha, \widetilde{t_0}) \mapsto E(\alpha, \widetilde{t}),$$

where $\alpha \in W(k)$.

1.6. *Fontaine's ring R, c.f.* [Fo82].

R is a complete valuation ring of characteristic p and consists of sequences $(x^{(n)})_{n\geqslant 0}$, where all $x^{(n)}$ are in the valuation ring $O_{\mathbb{C}_p}$ of the completion $\mathbb{C}_p$ of the field $\bar{K}$ and $x^{(n+1)p} = x^{(n)}$ for all $n \geq 0$. If $(x^{(n)})_{n\geqslant 0}, (y^{(n)})_{n\geqslant 0} \in R$, then

$$(x^{(n)})_{n\geqslant 0} + (y^{(n)})_{n\geqslant 0} = (z^{(n)})_{n\geqslant 0}$$

$$(x^{(n)})_{n\geqslant 0}(y^{(n)})_{n\geqslant 0} = (w^{(n)})_{n\geqslant 0},$$

where $(z^{(n)})_{n\geqslant 0} = \lim_{m\to\infty}(x^{(n+m)} + y^{(n+m)})^{p^m}$ and $w^{(n)} = x^{(n)}y^{(n)}$.

Valuation v_R on R is given by $v_R((x^{(n)})_{n\geqslant 0}) = v(x^{(0)})$, where v is the valuation on $O_{\mathbb{C}_p}$ normalized by $v(p) = 1$. Residue fields of $O_{\mathbb{C}_p}$ and R are identified by the correspondence

$$\alpha \mapsto (\hat{\alpha}^{p^{-n}})_{n\geqslant 0},$$

where $\hat{\alpha}$ is Teichmuller representative of $\alpha \in \bar{k}$.

Let R_0 be the quotient field of the ring R. Clearly, R and R_0 are Γ-modules.

There exists a natural injection $j : \mathcal{K}_{\rm sep} \longrightarrow R_0$, c.f. [Wtb83]. Recall briefly its construction. Let $[\widetilde{E} : \widetilde{K}] < \infty$, $E_{\rm tr}$ be the maximal tamely ramified extension of K in $\widetilde{E}$, $\mathcal{F}_n$ be a family of fields E, such that $E_{\rm tr} \subset E \subset \widetilde{E}$, $[E : E_{\rm tr}]$ is finite and equivalent to 0 modulo p^n. If $a = (a_E)_{E\in\mathcal{F}_n} \in \varprojlim_{E\in\mathcal{F}_n} E^* = \mathcal{E}^*$, then $j(a) = (j(a)^{(n)})_{n\geq 0}$, where $j(a)^{(n)} = \lim_{E\in\mathcal{F}_n} a_E^{p^{-n}[E:E_{\rm tr}]}$.

One has $j(\widetilde{t_0}) = (\pi_n)_{K(n)}$ and j is compatible with given above identifications of the residue fields of $\mathcal{K}_{\rm sep} = \mathcal{X}(\bar{K})$ and of R with the residue field of K.

Let $\tau \in \Gamma = \mathrm{Gal}(\bar{K}/K)$. Then $\tau\widetilde{K}$ gives the field of norms functor which we denote by $\mathcal{X}_\tau$. Clearly, τ defines isomorphism of fields $\mathcal{X}(\bar{K}) = \mathcal{K}_{\rm sep}$ and $\mathcal{X}_\tau(\bar{K})$, which we denote by the same symbol τ. Let $j_\tau : \mathcal{X}_\tau(\bar{K}) \longrightarrow R_0$ be embedding analogous to the above embedding j. Then j and j_τ are compatible with the natural action of Γ on R_0, i.e. the following diagramm is commutative

$$\begin{array}{ccc} \mathcal{K}_{\rm sep} & \xrightarrow{\ j\ } & R_0 \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ \mathcal{X}_\tau(\bar{K}) & \xrightarrow{\ j_\tau\ } & R_0 \end{array}$$

1.7. *Ideal* $J \subset W(R)$.

1.7.1. Let

$$u = \sum_{n\geqslant 0} p^n[u_n] \in W(R),$$

where $[u_n]$ are Teichmuller representatives of $u_n \in R$ for all $n \geq 0$. Then the correspondence

$$u \mapsto \sum_{n\geqslant 0} p^n u_n^{(0)}$$

defines an epimorphism of rings $\gamma : W(R) \longrightarrow O_{\mathbb{C}_p}$ and $\mathrm{Ker}\,\gamma = J$ is a principal ideal, c.f. [Fo82].

Let

$$l(X) = X + \frac{X^p}{p} + \cdots + \frac{X^{p^n}}{p^n} + \cdots \in \mathbb{Q}_p[[X]]$$

be the Lubin-Tate logarithm. By the Hazewinkel functional equation lemma, [Ha78], the power series $\psi(X) = l^{-1}(\log(1+X))$ is in $\mathbb{Z}_p[[X]]$.

Fix an element $\varepsilon = (\varepsilon^{(n)})_{n\geq 0} \in R$ such that $\varepsilon^{(M)} = \zeta$.

We have, c.f. [Ab90, n.1.7]

a) $\sigma([\varepsilon]-1) = \psi([\varepsilon]^p - 1) = [p]_G(\psi)([\varepsilon]-1)$, where $[p]_G$ is the endomorphism of multiplication by p in the formal group G with logarithm $l(X)$.

b) There exists $s \in W(R)$ such that

$$\psi([\varepsilon]-1) = s\psi([\varepsilon]^{1/p} - 1)$$

and this element s generates the ideal J.

1.7.2. Set $j(w) = \sigma w/w^p$ for $w \in W(R)^*$.

Then $j(w) \in 1 + pW(R)$ and we have the following lemma, c.f. [dSh]

Lemma. *The sequence of (multiplicative) groups*

$$1 \longrightarrow [\varepsilon]^{\mathbb{Z}_p} \longrightarrow 1+J \xrightarrow{j} 1+pW(R) \longrightarrow 1$$

is exact.

Proof.

We repeat arguments of E. de Shalit. Nontrivial part of the proof is surjectivity of j. If $w_1 \in 1+pW(R)$, then there exists $w \in 1+W(R)$ such that $j(w)=w_1$ (it can be proved by successive approximations). Then one can take $u \in R$, such that $\gamma([u])=\gamma(w)$, and check up that $wu^{-1} \in 1+J$ and $j(wu^{-1})=w_1$.

1.8. *Fontaine's ring* B^+_{cris}.

1.8.1. B^+_{cris} is a p-adic completion of the divided power envelope of $W(R)$ (in $W(R)\otimes\mathbb{Q}_p$) with respect to the ideal $\operatorname{Ker}\gamma = sW(R)$. For an integral $i \geq 0$ denote by $\operatorname{Fil}^i B^+_{\text{cris}}$ a decreasing filtration of B^+_{cris} by its closed ideals, generated by all $s^n/n!$, where $n \geq i$. One has the natural structure of a continuous Γ-module on B^+_{cris}. The Frobenius σ on R induces Frobenius morphism on B^+_{cris}, which we denote also by σ.

Let $B^+_{\text{cris},M} = B^+_{\text{cris}}/p^M B^+_{\text{cris}}$ and $\operatorname{Fil}^1 B^+_{\text{cris},M}$ be the image of $\operatorname{Fil}^1 B^+_{\text{cris}}$ in $B^+_{\text{cris},M}$. One has

$$\operatorname{Fil}^1 B^+_{\text{cris}} \cap p^M B^+_{\text{cris}} = p^M \operatorname{Fil}^1 B^+_{\text{cris}}.$$

Indeed, $\operatorname{Fil}^1 B^+_{\text{cris}} + W(R) = B^+_{\text{cris}}$ and $\operatorname{Fil}^1 B^+_{\text{cris}} \cap W(R) = J$. Therefore,

$$\operatorname{Fil}^1 B^+_{\text{cris}} \cap p^M B^+_{\text{cris}} = p^M \operatorname{Fil}^1 B^+_{\text{cris}} + \operatorname{Fil}^1 B^+_{\text{cris}} \cap p^M W(R) \subset$$

$$\subset p^M \operatorname{Fil}^1 B^+_{\text{cris}} + J \cap p^M W(R) = p^M \operatorname{Fil}^1 B^+_{\text{cris}}.$$

It follows now that the exponential $\exp(x) = \sum_{n\geqslant 0} x^n/n!$ defines a bijection

$$\exp : \operatorname{Fil}^1 B^+_{\text{cris},M} \longrightarrow (1+\operatorname{Fil}^1 B^+_{\text{cris},M})^*$$

and the morphism $\sigma/p : \operatorname{Fil}^1 B^+_{\text{cris}} \longrightarrow B^+_{\text{cris}}$ induces morphism

$$\frac{1}{p}\sigma : \operatorname{Fil}^1 B^+_{\text{cris},M} \longrightarrow B^+_{\text{cris},M}.$$

1.8.2. Let $\psi([\varepsilon]-1), s \in W(R)$ be the elements from n.1.7. Then

a) $\psi([\varepsilon]-1)^{p-1}/p \in \operatorname{Fil}^1 B^+_{\text{cris}}$ and is a topologically nilpotent element of this ring, c.f. [Ab90, n.1.5].

b) In the ring B^+_{cris} we have $\sigma s = p\eta_1$, where

$$\eta_1 \in 1 + \frac{\psi([\varepsilon]-1)^{p-1}}{p} W(R)\left[\left[\frac{\psi([\varepsilon]-1)^{p-1}}{p}\right]\right] \subset B^+_{\text{cris}}$$

(and therefore η_1 is invertible in B^+_{cris}), c.f. [Ab90, n.1.8].

c) Let S_M be the ideal of $B^+_{\mathrm{cris},M}$ generated by $\psi([\varepsilon]-1)^{p-1}/p$. Then σ/p induces a nilpotent endomorphism of the $\mathbb{Z}_p$-module S_M.

Indeed,

$$\sigma\psi = [p]_G(\psi) = \psi^p + p\psi + \sum_{i\geqslant 2} c_i p\psi^i,$$

where $\psi = \psi([\varepsilon]-1)$. This expression gives

$$\frac{1}{p}\sigma\left(\frac{\psi^{p-1}}{p}\right) = \left(\frac{\psi^{p-1}}{p}\right)^2 w' + \left(\frac{\psi^{p-1}}{p}\right) w'',$$

where $w' \in W(R)$ and $w'' \in pW(R)$. This formula implies the above statement c).

1.8.3. Let $[\varepsilon] \in W(R)$ be the element from n.1.7. Then, c.f. [Fo82], $\log[\varepsilon] \in \mathrm{Fil}^1 B^+_{\mathrm{cris}}$ and

$$\{m \in \mathrm{Fil}^1 B^+_{\mathrm{cris}} \mid \sigma m = pm\} = \mathbb{Z}_p \log[\varepsilon].$$

From this it follows now that

a) $\{m \in \mathrm{Fil}^1 B^+_{\mathrm{cris},M} \mid \sigma m/p = m\} = W_M(\mathbb{F}_p)\log[\varepsilon] \subset B^+_{\mathrm{cris},M}$;

b) If $a \in \mathbb{Z}_p$, then $[\varepsilon]^a \in 1 + p^M B^+_{\mathrm{cris}} \Leftrightarrow a \equiv 0 \bmod p^M$.

2. General explicit reciprocity law.

We use all previous notation. In particular, K is a complete discrete valuation field of characteristic 0 with a finite residue field k of characteristic p, fixed uniformizer π and primitive root ζ of degree p^M, $M \geq 1$. If $u, v \in K$, then (u,v) is the Hilbert symbol given by the formula

$$(u,v) = \psi_K(v)(u_1)u_1^{-1},$$

where $\psi_K : K^* \longrightarrow \Gamma^{\mathrm{ab}}$ is the reciprocity map of local class field theory and $u_1 \in \bar{K}$, $u_1^{p^M} = u$.

2.1. Mostly essential part of description of the Hilbert symbol of the field K is related to its values on principal units of the form $E(\alpha, \pi^a)$, where $\alpha \in W(k)$, $a \in \mathbb{N}, (a,p) = 1$.

The explicit reciprocity law in the form of Vostokov, [Vo78], gives these values in the form

$$\left(E(\alpha,\pi^a), E(\beta,\pi^b)\right) = \zeta^{c(\alpha,\beta,a,b)},$$

where $\alpha, \beta \in W(k)$, $a, b \in \mathbb{N}$, $(ab,p) = 1$ and

$$(*) \qquad c(\alpha,\beta,a,b) = \frac{1}{2}\mathrm{Tr}_{K^0/\mathbb{Q}_p}\left\{\mathrm{Res}\left[\frac{\alpha\beta(b-a)X^{a+b-1}\,\mathrm{d}X}{H(X)^{p^M}-1}\right]\right\}$$

Here K^0 is the maximal subextension of K, which is unramified over $\mathbb{Q}_p$, Tr is the trace map, $H(X) \in W(k)[[X]]$ is such that $H(\pi) = \zeta$ and Res is the residue at $X = 0$.

Remarks.

a) Vostokov and Brückner use a different definition of the Hilbert symbol, where Kummer theory and reciprocity map are applied to the second and the first arguments, respectively. So, our expression for $c(\alpha, \beta, a, b)$ has the opposite sign compared with the expression given by the Brückner-Vostokov formula, because of the skew-symmetricity of the Hilbert symbol.

b) We use slightly modified expression, which can be obtained from Vostokov's or Brückner's formula by skew-symmetrization. Other simplifications are related to the assumption $(ab, p) = 1$ and to the special properties of the power series $H(X)^{p^M} - 1$ (c.f. n. 2.4 a,b below).

2.2 *Reformulation of* $(*)$.

Let $\mathcal{X}$ be the "field of norms" functor from n.1.4, $\mathcal{K} = \mathcal{X}(\widetilde{K})$, $\widetilde{t_0}$ be the uniformizer of $\mathcal{K}$ constructed in n.1.4, $O_M(\mathcal{K})$ be the lifting of $\mathcal{K}$ modulo p^M chosen in n.1.5, $\widetilde{t} \in O_M(\mathcal{K})$ be such that $\sigma\widetilde{t} = \widetilde{t}^p$ and $\widetilde{t} \bmod p = \widetilde{t_0}$. Clearly, $O_M(\mathcal{K})$ can be identified with the $W_M(k)$-algebra of Laurent series in the variable $\widetilde{t}$.

Denote by $\mathcal{U}_\mathcal{K}$ the group of principal units of the field $\mathcal{K}$. Any $u \in \mathcal{U}_\mathcal{K}$ can be uniquelly expressed in the form

$$u = \prod_{\substack{a \in \mathbb{N} \\ (a,p)=1}} E(\alpha_a, \widetilde{t}^a),$$

where $\alpha_a = \alpha_a(u) \in W(k)$, $a \in \mathbb{N}$, $(a, p) = 1$.

The correspondence

$$u \mapsto \sum_{\substack{a \in \mathbb{N} \\ (a,p)=1}} \alpha_a(u) \widetilde{t}^a \bmod p^M$$

defines the homomorphism $L : \mathcal{U}_\mathcal{K} \longrightarrow O_M(\mathcal{K})$. Clearly, $\operatorname{Ker} L = \mathcal{U}_\mathcal{K}^{p^M}$, $\operatorname{Im} L \subset W_M(k)[[\widetilde{t}]]$.

If $G \in O_M(\mathcal{K})^*$, define symbol $< \, , \, >_G$ on the group $\mathcal{U}_\mathcal{K}$ with values in $W_M(\mathbb{F}_p)$ by the following formula

$$< u, v >_G = \operatorname{Tr} \left\{ \operatorname{Res} \left(\frac{L(u)(\mathrm{d}\, L(v)) - (\mathrm{d}\, L(u))L(v)}{G} \right) \right\}.$$

Here $\operatorname{Tr} : W_M(k) \longrightarrow W_M(\mathbb{F}_p)$ induced by the trace of the extension $k/\mathbb{F}_p$ and Res is a residue at $\widetilde{t} = 0$.

Similarly to the n. 2.1 introduce $H \in W_M(k)[[\widetilde{t}]]$ such that $\hat{H}(\pi) \equiv \zeta \bmod p$, where $\hat{H} \in W(k)[[\hat{t}]]$, $\hat{H} \bmod p^M = H$, and set $H_M = H^{p^M} - 1$.

Consider the projection, c.f. n.1.3, $\mathcal{N} : \mathcal{U}_\mathcal{K} \longrightarrow U_K$, where U_K is the group of principal units of the field K. Now the formula $(*)$ of n.2.1 is equivalent to the following statement

Theorem. *If* $u, v \in \mathcal{U}_\mathcal{K}$, *then*

$$(\mathcal{N}(u), \mathcal{N}(v)) = \zeta^{\frac{1}{2} < u,v >_{H_M}}.$$

Proof of theorem.

2.3. Let $f \in L(\mathcal{U}_{\mathcal{K}}) \subset O_M(\mathcal{K})$. Consider Witt-Artin-Schreier extension $\mathcal{E}_f$ of $\mathcal{K}$ given by a solution T of the equation

$$(*_1) \qquad \sigma T - T = \frac{f}{H_M}$$

in $O_M(\mathcal{K}_{\rm sep})$. For any $\tau \in \mathcal{G}$ we have

$$\tau T - T = a_\tau \in W_M(\mathbb{F}_p).$$

Let $u \in \mathcal{U}_{\mathcal{K}}$ be such that $L(u) = f$. Then a solution Z of the equation

$$Z^{p^M} = \mathcal{N}(u)$$

defines Kummer extension of K. This extension does not depend on the choice of u and will be denoted by E_f. For any $\tau \in \Gamma$ we have

$$\frac{\tau Z}{Z} = \zeta^{b_\tau},$$

where all $b_\tau \in \mathbb{Z} \bmod p^M\mathbb{Z}$.

Proposition.
In the above notation for any $\tau \in \mathcal{G} \subset \Gamma$ we have

$$a_\tau = b_\tau \bmod p^M\mathbb{Z}.$$

Remark.
The embedding $\mathcal{G} \subset \Gamma$ is given by the construction of the functor $\mathcal{X}$, c.f. n.1.4.

2.4. Deduce theorem of n.2.2 from the above proposition.

By bilinearity it is sufficient to consider the case $u = E(\alpha, \tilde{t}_0^{\,a})$, $v = E(\beta, \tilde{t}_0^{\,b})$, where $\alpha, \beta \in W(k)$, $a, b \in \mathbb{N}, (ab, p) = 1$.

Consider the equation $(*_1)$ of n.2.3 for $f = L(u) = \alpha\tilde{t}^{\,a}$.

Let $\psi_{\mathcal{K}} : \mathcal{K}^* \longrightarrow \mathcal{G}^{\rm ab}$ be the reciprocity map and $\tau \mapsto \psi_{\mathcal{K}}(v)$ under canonical projection $\mathcal{G} \longrightarrow \mathcal{G}^{\rm ab}$.

Then the explicit reciprocity law in characteristic p case, c.f. n.1.5, gives (in notation of n.2.3) that

$$(*) \qquad a_\tau = \operatorname{Tr}\left\{ \operatorname{Res}\left[\frac{\alpha\tilde{t}^{\,a}}{H_M} \sum_{n\geqslant 0} b(\sigma^n\beta)\tilde{t}^{\,p^n b} \right] \frac{d\tilde{t}}{\tilde{t}} \right\}.$$

Remark, that

a) $H_M = H^{p^M} - 1 = G_0(\tilde{t}^{\,p})$ for some power series $G_0 \in W_M(k)[[\tilde{t}\,]]$, therefore, we can omit all terms of the above formula $(*)$ with $n > 0$.

b) $H_M = G_M(\tilde{t}^{p^M}) + pG_{M-1}(\tilde{t}^{p^{M-1}}) + \dots + p^{M-1}G_1(\tilde{t}^p)$ for some power series $G_1, \dots, G_M \in W_M(k)[[\tilde{t}]]$. This gives skew-symmetricity of the right-hand side of $(*)$ as a function of pairs (α, a) and (β, b). So, one can rewrite the formula $(*)$ in the form

$$a_\tau = \frac{1}{2} < u, v >_{H_M} .$$

The commutative diagramm of n.1.4 gives $i_{\mathcal{K}/K}(\psi_{\mathcal{K}}(v)) = \psi_K(\mathcal{N}(v))$. Therefore, (in notation of n.2.3)

$$(\mathcal{N}(u), \mathcal{N}(v)) = \zeta^{b_\tau}$$

and our theorem follows from the above proposition.

2.5. *Proof of proposition n.2.3.*

It is sufficient to treat the case $f = \alpha\tilde{t}^a$, where $\alpha \in W(k)$, $a \in \mathbb{N}, (a, p) = 1$. Let T be a solution of the equation

$$(*_1) \qquad \sigma T - T = \frac{\alpha\tilde{t}^a}{H_M}$$

in $O_M(\mathcal{K}_{\rm sep})$. Then for any $\tau \in \mathcal{G}$

$$\tau T - T = a_\tau = a_\tau(\alpha, a) \in W_M(\mathbb{F}_p).$$

2.5.1. *One equivalence.*

Consider the embedding $j : \mathcal{K}_{\rm sep} \longrightarrow R_0$ from no.1.6. This j can be prolonged uniquely to the imbedding

$$O_M(j) : O_M(\mathcal{K}_{\rm sep}) \longrightarrow W_M(R_0)$$

which transforms σ to Frobenius morphism of $W_M(R_0)$ (so, we can denote the Frobenius on $W_M(R_0)$ also by σ) and is compatible with the inclusion of Galois groups $\mathcal{G} \subset \Gamma$.

Let $H \in W(k)[[\tilde{t}]]$ be the power series from n.2.2 and (as earlier) $\varepsilon = (\varepsilon^{(n)})_{n \geq 0} \in R$ be such that $\varepsilon^{(M)} = \zeta$. Clearly, this means that $(\sigma^{-M}\varepsilon)^{(0)} = \zeta$. Therefore, we have the following equivalence in $W_M(R)$

$$\sigma^{-M}[\varepsilon] \equiv H \bmod(\tilde{t}^e W_M(R) + pW_M(R)).$$

It is easy to see that we have an equality of ideals

$$(\varepsilon - 1)R = \tilde{t}_0^{\frac{ep}{p-1}} R,$$

in the ring R (here e is the absolute ramification index of K). Therefore,

$$[\varepsilon] \equiv H^{p^M} \bmod(([\varepsilon] - 1)^{p^{M-1}(p-1)} W_M(R) + p([\varepsilon] - 1)^{p^{M-2}(p-1)} W_M(R) + \dots$$
$$\dots + p^{M-1}([\varepsilon] - 1)^{(p-1)} W_M(R)).$$

This gives for any $\alpha \in W(k)$ and $a \in \mathbb{N}$

$$\frac{\alpha \tilde{t}^a}{[\varepsilon]-1} \equiv \frac{\alpha \tilde{t}^a}{H_M} \bmod(([\varepsilon]-1)^{p^{M-1}(p-1)-2} W_M(m_R) + p([\varepsilon]-1)^{p^{M-2}(p-1)-2} W_M(m_R) + \dots$$

$$\dots + p^{M-1}([\varepsilon]-1)^{(p-1)-2} W_M(m_R)),$$

where m_R is the maximal ideal in R.

2.5.2. *Interpretation of a_τ in $W_M(R_0)$.*

Consider an analoug of the above equation $(*_1)$

$$(*_2) \qquad \sigma T - T = \frac{\alpha \tilde{t}^a}{[\varepsilon]-1}$$

in $W_M(R_0)$. The last equivalence of n.2.5.1 gives

$$O_M(j)(T_1) \equiv T_2 \bmod(([\varepsilon]-1)^{p^{M-1}(p-1)-2} W_M(m_R) + p([\varepsilon]-1)^{p^{M-2}(p-1)-2} W_M(m_R) + \dots$$

$$\dots + p^{M-1}([\varepsilon]-1)^{(p-1)-2} W_M(m_R)),$$

where T_2 is a solution of the equation $(*_2)$ in $W_M(R_0)$. Therefore, for any solution $T \in W_M(R_0)$ of the equation $(*_2)$ and any $\tau \in \mathcal{G}$ we have

$$\tau T - T = a_\tau,$$

because $W_M(R_0)^{\mathcal{G}} = W_M(\mathbb{F}_p)$.

2.5.3. *Interpretation of a_τ in $W_M(R)$.*

We use notation of n.1.7.

Remark, that

$$\psi(X) \equiv \widetilde{\log}(1+X) \bmod X^p \mathbb{Z}_p[[X]],$$

where $\widetilde{\log}(1+X) = X - X^2/2 + \dots - X_{p-1}/(p-1)$ is the truncated logarithm.

Let T be a solution of the equation

$$(*_3) \qquad \sigma T - T = \frac{\alpha \tilde{t}^a}{\psi([\varepsilon]-1)}$$

in $W_M(R_0)$.

By the above remark

$$\frac{\alpha \tilde{t}^a}{\psi([\varepsilon]-1)} \equiv \frac{\alpha \tilde{t}^a}{[\varepsilon]-1} \bmod W_M(m_R).$$

Therefore, T is equivalent modulo $W_M(m_R)$ to some solution of the equation $(*_2)$. This gives

$$\tau T - T = a_\tau$$

for any $\tau \in \mathcal{G}$.

Lemma.

$$T\psi([\varepsilon]^{1/p}-1)\in W_M(R).$$

Proof.

Clearly, $T_1=T\psi([\varepsilon]^{1/p}-1)$ satisfies in $W_M(R_0)$ the equation

$$\sigma T_1 - sT_1 = \alpha\tilde{t}^{a}.$$

R is integrally closed in R_0, therefore, $T_1 \bmod p\in R$. Now one can apply induction on M to prove $T_1\in W_M(R)$.

Let $X_0=T\psi([\varepsilon]-1)$. Then $X_0=sT_1\in J_M=J \bmod p^M W(R)$ and satisfies the equation

$$(*_4)\qquad \frac{\sigma X}{\sigma s}-X=\alpha\tilde{t}^{a}$$

in $W_M(R)$.

Clearly, multiplication by $\psi([\varepsilon]-1)$ defines a one-to-one correspondence between all solutions of the equation $(*_3)$ in $W_M(R_0)$ and all solutions of the equation $(*_4)$ in J_M. Therefore, if $X\in J_M$ satisfies $(*_4)$ and $\tau\in\mathcal{G}$, then

$$\tau X - X = a_\tau\psi([\varepsilon]-1).$$

2.5.4. *Interpretation of a_τ in $B_{\mathrm{cris},M}$.*

Consider a morphism

$$\delta_M : W_M(R)\longrightarrow B^+_{\mathrm{cris},M}$$

induced by the natural inclusion $W(R)\subset B^+_{\mathrm{cris}}$.

According to the property 1.8.2 b) δ_M transforms solutions of the equation $(*_4)$ to solutions $Y\in \mathrm{Fil}^1 B^+_{\mathrm{cris},M}$ of the equivalence

$$(*_5)\qquad \frac{\sigma\widetilde{m}}{p}-\widetilde{m}\equiv\alpha\tilde{t}^{a} \bmod S_M,$$

where S_M is the ideal of $B^+_{\mathrm{cris},M}$, generated by $\psi([\varepsilon]-1)^{p-1}/p$. From the property 1.8.2 c) it follows now that every solution $\widetilde{m}$ of the above equivalence $(*_5)$ gives rise to the unique solution $m\in \mathrm{Fil}^1 B^+_{\mathrm{cris},M}$ of the equation

$$(*_6)\qquad \frac{\sigma m}{p}-m=\alpha\tilde{t}^{a}$$

in $B^+_{\mathrm{cris},M}$, such that $\widetilde{m}=m \bmod S_M$. Now the property 1.8.3 a) gives

$$\tau m - m = a_\tau \log[\varepsilon],$$

for any $\tau\in\mathcal{G}$ and arbitrary solution $m\in\mathrm{Fil}^1 B^+_{\mathrm{cris},M}$ of the equation $(*_6)$.

Multiplying $(*_6)$ by p and taking an exponential we obtain that for every

$$Y \in (1 + \mathrm{Fil}^1 B^+_{\mathrm{cris}}) \bmod p^{M+1} B^+_{\mathrm{cris}},$$

satisfying the equality

$$(*_7) \qquad \sigma Y = Y^p \exp(p\alpha \widetilde{t}^{\,a})$$

in $B^+_{\mathrm{cris},M+1}$, one has

$$(\tau Y)Y^{-1} = [\varepsilon]^{a_\tau} \bmod p^M B^+_{\mathrm{cris}}.$$

2.5.5. *Interpretation of a_τ in $W_{M+1}(R)^*$.*

Consider the exact sequence from lemma of n.1.7.2. This sequence gives solvability in $(1+J)^* \bmod p^{M+1} W(R)$ of the equation

$$(*_8) \qquad \sigma Z = Z^p \exp(p\alpha \widetilde{t}^{\,a})$$

in $W_{M+1}(R)$ and the cocycle $\{c_\tau \in \mathbb{Z}_p \mid \tau \in \mathcal{G}\}$, such that $\tau Z = Z[\varepsilon]^{c_\tau}$.

Clearly, the imbedding $W(R) \subset B^+_{\mathrm{cris}}$ maps solutions $Z \in (1+J)^* \bmod p^{M+1} W(R)$ of $(*_8)$ to solutions $Y \in (1 + \mathrm{Fil}^1 B^+_{\mathrm{cris}}) \bmod p^{M+1} B^+_{\mathrm{cris}}$ of $(*_7)$, and, therefore,

$$a_\tau = c_\tau \bmod p^M \mathbb{Z}_p$$

for any $\tau \in \mathcal{G}$.

2.5.6. *Relation to Kummer theory.*

Let $Z \in (1+J)^* \bmod p^{M+1} W(R)$ be some solution of $(*_8)$ in $W_{M+1}(R)$. Then

$$\sigma^M Z = Z^{p^M} \exp(p\sigma^{M-1}(\alpha) \widetilde{t}^{\,ap^{M-1}} + \cdots + p^M \alpha \widetilde{t}^{\,a})$$

in $W_{M+1}(R)$.

The power series $E(\alpha, X)$ from n.1.2 satisfies the following identity

$$E(\alpha, X)^{p^M} = \exp(p^M \alpha X + p^{M-1} \sigma(\alpha) X^p + \cdots + p\sigma^{M-1}(\alpha) X^{p^{M-1}}) E(\sigma^M(\alpha), X^{p^M}).$$

Take $\widetilde{t}_1 \in W_{M+1}(R)$, such that $\widetilde{t} = \widetilde{t}_1 \bmod p^M, \sigma \widetilde{t}_1 = \widetilde{t}_1^{\,p}$ and $\tau \widetilde{t}_1 = \widetilde{t}_1$ for any $\tau \in \mathcal{G}$ (such element $\widetilde{t}_1$ can be constructed similarly to $\widetilde{t}$ in the lifting $O_{M+1}(\mathcal{K})$ and then one can take $\widetilde{t} = \widetilde{t}_1 \bmod p^M$).

Now the above identity gives an equality

$$\sigma^M \left(Z E(\alpha, \widetilde{t}_1^{\,a}) \right) = \left(Z E(\alpha, \widetilde{t}_1^{\,a}) \right)^{p^M}.$$

Therefore,

$$Z_1^{p^M} = Z E(\alpha, \widetilde{t}_1^{\,a})$$

for $Z_1 = (\sigma^{-M} Z) E(\sigma^{-M} \alpha, \sigma^{-M} \widetilde{t}_1^{\,a})$, because σ is bijective on $W_{M+1}(R)$.

For every $\tau \in \mathcal{G}$ we have

$$\tau Z_1 (Z_1)^{-1} = [\sigma^{-M}\varepsilon]^{c_\tau}.$$

Consider the homomorphism of rings $\gamma : W(R) \longrightarrow O_{\mathbb{C}_p}$ from n.1.7.1. This γ induces the homomorphism

$$\gamma_{M+1} : W_{M+1}(R) \longrightarrow O_{\bar{K}}/p^{M+1}O_{\bar{K}}$$

and, clearly, $\gamma_{M+1}(Z) = 1 \bmod p^{M+1}O_{\bar{K}}$, $\gamma_{M+1}(E(\alpha, \tilde{t_1}^a)) = \mathcal{N}(E(\alpha, \tilde{t_1}^a)) \bmod p^{M+1}O_{\bar{K}}$, $\gamma_{M+1}(\sigma^{-M}\tilde{t_1}^a) = \pi_M \bmod p^{M+1}O_{\bar{K}}$, c.f. n.1.3, and $\gamma_{M+1}([\sigma^{-M}\varepsilon]) = \zeta \bmod p^{M+1}O_{\bar{K}}$. Therefore, if $W \in O_{\bar{K}}$ be such that $\gamma_{M+1}(Z_1) = W \bmod p^{M+1}O_{\bar{K}}$, then

$$W^{p^M} \equiv E(\alpha, \pi^a) \bmod p^{M+1}O_{\bar{K}}$$

and for any $\tau \in \mathcal{G}$ one has

$$(\tau W)W^{-1} \equiv \zeta^{c_\tau} \bmod p^{M+1}O_{\bar{K}}.$$

Obviously, this implies that Kummer extension

$$Z^{p^M} = \mathcal{N}(u) = E(\alpha, \pi^a)$$

has a cocycle $(\tau Z)Z^{-1} = \zeta^{c_\tau}$. Proposition is proved, because $a_\tau = c_\tau \bmod p^M\mathbb{Z}_p$, c.f. n.2.5.5.

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