Stratifying k-points Algebraic Quotients

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§0. Introduction.

Let k be a field of characteristic zero. Let G be a reductive algebraic group and V an affine G-variety (both with points in $\overline{k}$, an algebraic closure of k). As usual, we define the *algebraic quotient* $V/\!\!/G$ to be the affine variety whose coordinate ring is $\mathcal{O}(V)^G$, the algebra of G-invariant regular functions on V.

In this paper, we introduce and explore basic properties of a stratification of $(V/\!\!/G)_k$, in the case that V, G, and the action of G on V are defined over k. The stratification comes from the k-structure on the Zariski-closed orbits of the action, and coincides with the usual isotropy-type stratification when k is algebraically closed. Behavior very much like the case $k = \mathbf{C}$ occurs when $k = \mathbf{R}$ or a p-adic field, and in these circumstances we also consider the space $V_k/\!\!/G_k$ of closed G_k-orbits in V_k, the map $V_k/\!\!/G_k \to (V/\!\!/G)_k$, and an interesting stratification of $V_k/\!\!/G_k$. The language of relative Galois cohomology, introduced by Springer [**Sp**], is essential for our purposes and is reviewed here (§2). The main technical tool we develop is a version of Luna's étale slice theorem for non-algebraically closed fields (§3); this allows us to give a k-stucture to the normal bundle of any Zariski-closed orbit in V which is defined over k. This was implicit in Luna's paper [**Lu2**], and some of his results in the case $k = \mathbf{R}$ required only minor modifications to become valid in more general situations.

Notation and Conventions: Unless otherwise indicated, G denotes a reductive algebraic group defined over k (with points in $\overline{k}$). All G-varieties are understood to be affine. The quotient map $V \to V/\!\!/G$ is denoted by $\pi_{V,G}$ or simply π. The notation T_vS means the tangent space at v to S. The words "closure," "closed," "open," "neighborhood," etc. refer to the topology on the k-points of an affine variety, which comes from a topology on the field k. All field topologies are assumed to be nontrivial and nondiscrete. We use the notation $G *^H N$ to denote twisted products (see [Sl]); points in a twisted product are writen as $[g, n]$. If H is a subgroup of G, then (H) denotes the set of G-conjugates of H. We write $V/\!\!/G = \bigcup_H (V/\!\!/G)_{(H)}$ for the isotropy-type stratification of $V/\!\!/G$. Finally, we use Gal as a shorthand for the Galois group $\mathrm{Gal}(\overline{k}/k)$.

§1. Algebraic quotients over k.

Let V be an affine G-variety over k, with k-structure $\mathcal{O}(V) = \overline{k} \otimes_k \mathcal{O}(V)_k$. It follows that $\mathcal{O}(V)^G = \overline{k} \otimes_k \mathcal{O}(V)_k^G$, since $\mathcal{O}(V)^G \subset \mathcal{O}(V)$ is stable under the action of Gal. Hence $V /\!\!/ G$ and $\pi : V \to V /\!\!/ G$ are defined over k. We set $Z = (V /\!\!/ G)_k$ and $X = \pi(V_k)$.

The following proposition describes $V /\!\!/ G$, Z, and X as spaces of certain Zariski-closed orbits:

PROPOSITION 1.1:

(1) π is surjective.
(2) Each fiber of π is a union of G-orbits and contains a unique Zariski-closed orbit, which is of minimum dimension among orbits in the fiber.
(3) If $y \in V /\!\!/ G$, then the following are equivalent:
  (a) $y \in Z$.
  (b) $\pi^{-1}(y)$ is defined over k.
  (c) The Zariski-closed orbit in $\pi^{-1}(y)$ is defined over k.
(4) If $y \in V /\!\!/ G$, then the following are equivalent:
  (a) $y \in X$.
  (b) The Zariski-closed orbit $G \cdot v \subset \pi^{-1}(y)$ contains a k-point of V.

PROOF: (1) and (2) can be found in [**Kr**]. In (3), $(a) \Leftrightarrow (b) \Leftarrow (c)$ is trivial since the action of Gal on V maps fibers to fibers. The implication $(b) \Rightarrow (c)$ follows from (2). In (4), $(b) \Rightarrow (a)$ is trivial. To prove $(a) \Rightarrow (b)$: there is a G-equivariant retraction $\pi^{-1}(y) \to G \cdot v$ which is defined over k (3.4), which must carry k-points to k-points. ∎

LEMMA 1.2: If Y is an affine G-variety over k, then there is a G-equivariant, Zariski-closed embedding over k of Y into a G-module V which is defined over k.

PROOF: Suppose that $\mathcal{O}(Y)$ is generated by $\{f_1, \ldots, f_n\}$. Then $f_1, \ldots, f_n$ lie in a finite-dimensional G-module W', and $W = \mathrm{span}_{\sigma \in \mathrm{Gal}}\{\sigma(W')\}$ is again finite-dimensional (any $f \in W'$ has a finite Gal-orbit, and W is the span of the Gal-orbits of a basis of W'). Finally, from the surjection of algebras $S^{\cdot}(W) \twoheadrightarrow \mathcal{O}(Y)$, we obtain an embedding of Y into the G-module $V = W^*$. ∎

§2. Compatible k-structures on homogeneous spaces.

In this section, G is an affine algebraic group, not necessarily reductive. For background on homogeneous spaces and k-structures on varieties, see [**Bo**].

We begin by recalling a fact about coset spaces. If H is a (Zariski-closed) subgroup of G, both defined over a field K (not necessarily algebraically closed), then G/H has the structure of a quasiprojective variety over K (with the action of $\mathrm{Gal}(\overline{K}/K)$ given by $gH \overset{\sigma}{\mapsto} \sigma(g)H$). The variety structure on G/H comes via an embedding into $\mathbf{P}(V)$, where V is a representation of G defined over K. If G and H are reductive, then G/H is affine.

We review relative Galois cohomology, which was introduced by Springer [**Sp**]. We return to our field k and group G defined over k, but only assume that H is defined over $\overline{k}$; we consider the k-structures on G/H such the left action of G on G/H is defined over k. Thus we call an action $(\sigma, gH) \mapsto \sigma(gH)$ of Gal on G/H a *compatible k-structure on G/H* if it comes from a k-structure on G/H, and if

$$(*) \qquad \sigma(g_1 g_2 H) = \sigma(g_1) \cdot \sigma(g_2 H)$$

for all $g_1, g_2 \in G$ and $\sigma \in$Gal. We describe these structures, using the language of Galois cohomology:

By $(*)$, we need only know $\sigma(eH)$ for each $\sigma \in$Gal. Suppose that $\sigma(eH) = s_\sigma H$, where $s_\sigma \in G$. The map (Gal $\to G$, $\sigma \mapsto s_\sigma$) has the following properties:

(1) $s_{\sigma_1\sigma_2} H = \sigma_1(s_{\sigma_2}) \cdot s_{\sigma_1} H$ for all $\sigma_1, \sigma_2 \in$Gal.
(2) $s_\sigma H s_\sigma^{-1} = \sigma(H)$ for all $\sigma \in$Gal.
(3) $\{\sigma \in \text{Gal} : s_\sigma \in H\} \supset \text{Gal}(\overline{k}/k')$ for some finite Galois extension $k' \supset k$.

(1) follows since $(\sigma_1\sigma_2)(eH) = \sigma_1(\sigma_2(eH))$, and (2) is true since $\sigma(eH) = \sigma(hH)$ for all $h \in H$. Finally, $eH \in (G/H)_{k'}$ for some finite Galois $k' \supset k$, hence $eH = \sigma(eH) = s_\sigma H$ for all $\sigma \in \text{Gal}(\overline{k}/k')$.

Conversely, suppose that $\sigma \mapsto s_\sigma$ is a map having these properties. By (1) and (2), it gives a well-defined action of $\text{Gal}(\overline{k}/k)$ on G/H which satisfies $(*)$. We show that the action comes from a k-structure. Let $g_0 H \in G/H$, and let K be a finite Galois extension of k' such that H is defined over K and $g_0 \in G_K$. Then G/H is defined over K and the action of $\text{Gal}(\overline{k}/K)$ is the same as the one coming from the inclusion $\text{Gal}(\overline{k}/K) \hookrightarrow \text{Gal}(\overline{k}/k)$. We may find an affine open neighborhood of $g_0 H$ in G/H of the form $U = \text{Spec}\, A$, where U is defined over K. Hence $A = \overline{k} \otimes_K A_0$, where $A_0 = \{f \in A : f(\sigma(g)H) = \sigma(f(gH)) \text{ for all } \sigma \in \text{Gal}(\overline{k}/K\}$. Moreover it is clear that $A_0 = K \otimes_k A_1$, where $A_1 = \{f \in A_0 : f(\sigma(g)s_\sigma H) = \sigma(f(gH)) \text{ for all } \sigma \in \text{Gal}\}$ since K/k is finite. Hence $A = \overline{k} \otimes_k A_1$ as was required.

Functions $s : \text{Gal} \to G$ with the above properties are called *cocycles* (relative to H). We let $\mathcal{Z}(k, G, H)$ denote the set of cocycles, and let $(G/H, s)$ denote G/H with the compatible k-structure induced by s. If H is defined over k, then the cocycle which is the constant function $\sigma \mapsto e$ will be denoted 1.

We wish to identify two compatible k-structures on G/H if they are related by a G-equivariant automorphism of G/H. Such automorphism are of the form $gH \mapsto gnH$ for some $n \in N_G H$. We obtain a corresponding equivalence relation on $\mathcal{Z}(k, G, H)$ as follows: for any $n \in N_G H$ and $\{h_\sigma\}_{\sigma \in \text{Gal}} \subset H$, we declare

$$(\sigma \mapsto s_\sigma) \equiv (\sigma \mapsto \sigma(n) \cdot s_\sigma \cdot n^{-1} \cdot h_\sigma).$$

Let $\mathcal{H}^1(k, G, H)$ denote the set of equivalence classes of cocycles relative to H. We have proven

PROPOSITION 2.1: $\mathcal{H}^1(k, G, H)$ parametrizes the equivalence classes of compatible k-structures on G/H. ∎

If $H = \{e\}$, we shall use the briefer notation $\mathcal{Z}(k, G)$ and $\mathcal{H}^1(k, G)$. In nonabelian Galois cohomology, the case $H = \{e\}$ has received the most attention. It is a special case of [**Se1**, Proposition 5, pg. III–6] that the sets $\mathcal{H}^1(k, G, H)$ may be viewed in the "absolute" framework:

Suppose that $\mathcal{H}^1(k, G, H)$ is nonempty. Fix $s \in \mathcal{Z}(k, G, H)$. We obtain a k-structure on the algebraic group $\mathcal{W} := N_G H/H$, by demanding that the action of $\mathcal{W}$ on $(G/H, s)$ be defined over k. Specifically, σ $\in$Gal sends $nH \in \mathcal{W}$ to $s_\sigma^{-1} \cdot \sigma(n) \cdot s_\sigma H$. Then

PROPOSITION 2.2: The map $\begin{pmatrix} \mathcal{Z}(k, G, H) & \longrightarrow & \mathcal{Z}(k, \mathcal{W}) \\ (\sigma \mapsto t_\sigma) & \longmapsto & (\sigma \mapsto s_\sigma^{-1} \cdot t_\sigma H) \end{pmatrix}$ induces an isomorphism $\mathcal{H}^1(k, G, H) \simeq \mathcal{H}^1(k, \mathcal{W})$. ∎

Of course if H is defined over k, then we may take $s_\sigma = 1$.

NOTATION 2.3: If $a \in G$, then there is an isomorphism

$$\mathcal{H}^1(k, G, H) \xrightarrow{\sim} \mathcal{H}^1(k, G, aHa^{-1})$$

induced by the map

$$(\sigma \mapsto s_\sigma) \longmapsto (\sigma \mapsto \sigma(a) \cdot s_\sigma \cdot a^{-1})$$

on cocycles. We use this to identify $\mathcal{H}^1(k, G, H)$ and $\mathcal{H}^1(k, G, H')$ if H and H' are conjugate in G, and we shall write $\mathcal{H}^1(k, G, (H))$ when we choose not to draw attention to a particular element in (H). If $s \in \mathcal{Z}(k, G, H)$, let $[s]$ denote its image in $\mathcal{H}^1(k, G, (H))$. Let $\mathcal{Z}(k, G, H)_0 = \{s \in \mathcal{Z}(k, G, H) : (G/H, s) \text{ has a } k\text{-point}\}$, and let $\mathcal{H}^1(k, G, (H))_0$ be the image of $\mathcal{Z}(k, G, H)_0$ in $\mathcal{H}^1(k, G, (H))$.

Let $\mathcal{C}(H) = \{H' \in (H) : H' \text{ is defined over } k\}$. If $s \in \mathcal{Z}(k, G, H)$, let $\mathcal{C}(H, s) = \{H' \in \mathcal{C}(H) : H' \text{ is the isotropy group of a } k\text{-point of } (G/H, s)\}$. Let $G'(H) = \{g \in G : \sigma(g^{-1}) \cdot g \in N_G H \text{ for all } \sigma \in \text{Gal}\}$. ∎

LEMMA 2.4: Suppose that H is defined over k.

(1) The map $(G'(H) \to \mathcal{C}(H), g \mapsto gHg^{-1})$ induces a bijection $G'(H)/N_G H \simeq \mathcal{C}(H)$. Likewise the map $(G'(H) \to G/N_G H, g \mapsto gN)$ induces a bijection between $G'(H)/N_G H$ and the set of k-points of $(G/N_G H, 1)$.

(2) If $g \in G'(H)$, then the map $(\text{Gal} \to G, \sigma \mapsto \sigma(g^{-1}) \cdot g)$ is an element of $\mathcal{Z}(k, G, H)$.

(3) If $g \in G'(H)$ and $n \in N_G H$, then the cocycles $(\sigma \mapsto \sigma((gn)^{-1}) \cdot (gn))$ and $(\sigma \mapsto \sigma(g^{-1}) \cdot g)$ are equivalent in $\mathcal{H}^1(k, G, (H))$.

(4) If $s, t \in \mathcal{Z}(k, G, H)$ are equivalent, then $\mathcal{C}(H, s) = \mathcal{C}(H, t)$.

PROOF: (1)–(3) are trivial. We prove (4). Suppose that $t_\sigma = \sigma(n) \cdot s_\sigma \cdot n^{-1} \cdot h_\sigma$, for some $n \in N$ and $h_\sigma \in H$. We compute easily that gH is a k-point of $(G/H, t)$ if and only if gnH is a k-point of $(G/H, s)$. However, gH and gnH have the same isotropy group. ∎

THEOREM 2.5.

(1) *The map* $\begin{pmatrix} G'(H) & \longrightarrow & \mathcal{Z}(k,G,H) \\ g & \longmapsto & (\sigma \mapsto \sigma(g^{-1})\cdot g) \end{pmatrix}$ *induces a map*

$$\Phi : \mathcal{C}(H) \simeq G'(H)/N_G H \longrightarrow \mathcal{H}^1(k,G,(H))$$

with image $\mathcal{H}^1(k,G,(H))_0$.

(2) *If* $s \in \mathcal{Z}(k,G,H)$, *then* $\Phi^{-1}([s]) = \mathcal{C}(H,s)$.

(3) *If* $\{s^i\}_{i\in I} \subset \mathcal{Z}(k,G,H)$ *and* $\mathcal{H}^1(k,G,(H))_0$ *is the disjoint union of* $\{[s^i]\}_{i\in I}$, *then* $\mathcal{C}(H) = \bigsqcup_{i\in I} \mathcal{C}(H,s^i)$.

PROOF: By (2.4(2)) and (2.4(3)), we obtain a map $G'(H)/N_GH \to \mathcal{H}^1(k,G,(H))$. The image is contained in $\mathcal{H}^1(k,G,(H))_0$ since if $g \in G'(H)$, then then gH is a k-point of $(G/H,s)$ (where $s_\sigma = \sigma(g^{-1})\cdot g$). Conversely, if $s \in \mathcal{Z}(k,G,H)$ is such that $[s] \in \mathcal{H}^1(k,G,(H))_0$, and gH is a k-point of $(G/H,s)$, then $s_\sigma \in \sigma(g^{-1})\cdot g \cdot H$. This proves (1).

To prove (2): by virtue of (1) and (2.4(4)), we may assume that $s = \sigma(g^{-1})\cdot g$ for some $g \in G'(H)$, and then (2) follows easily. Part (3) follows directly from (2). ∎

Otherwise said:

(1) If S is a G-homogeneous space, then every compatible k-structure on S, for which S has a k-point, arises as $S \simeq (G/H,1)$, where H is defined over k.

(2) If $H_1, H_2 \subset G$ are defined over k and are G-conjugate, then $(G/H_1,1) \simeq (G/H_2,1)$ if and only if H_2 is the isotropy group of a k-point of $(G/H_1,1)$.

REMARK 2.6: Using (2.4(1)), we obtain bijections

$$G_k\backslash G'(H)/N_GH \simeq \{G_k\text{-conjugacy classes in } \mathcal{C}(H)\} \simeq \{G_k\text{-orbits in } (G/N_GH)_k\}.$$

By [**Se1**], the map $(G'(H) \to \mathcal{H}^1(k,N_GH), g \mapsto (\sigma \mapsto \sigma(g^{-1})\cdot g))$ induces a bijection

$$G_k\backslash G'(H)/N_GH \simeq \text{Kernel}\left(\mathcal{H}^1(k,N_GH) \to \mathcal{H}^1(k,G)\right). \quad ∎$$

Suppose now that H is defined over k. Let $G''(H) \subset G'(H)$ denote $\{g \in G : \sigma(g^{-1})g \in H \text{ for all } \sigma \in \text{Gal}\}$.

REMARK 2.7: The map $(G''(H) \to G/H, g \mapsto G_k\cdot g\cdot H)$ induces a bijection

$$G_k\backslash G''(H)/H \simeq \{G_k\text{-orbits in } (G/H,1)_k\}$$

and as in (2.6), these sets are isomorphic to the kernel of $\mathcal{H}^1(k,H) \to \mathcal{H}^1(k,G)$. ∎

REMARK 2.8: We obtain a map $G_k\backslash G''(H)/H \to G_k\backslash G'(H)/N_GH$ with image $G_k\backslash\mathcal{C}(H,1)$. We give an example to show that this map is not injective in general. Let $G = SL(2,\mathbf{C})$, $G_{\mathbf{R}} = SL(2,\mathbf{R})$, and $H = SO(2,\mathbf{C})$. It is easily seen that the $\mathbf{R}$-points of $(G/H,1)$ consist of two $G_{\mathbf{R}}$-orbits, containing $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}H$ and $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}H$ respectively, but the two points have the same G-isotropy.

§3. The étale slice theorem over k.

We recall Luna's slice theorem ([**Lu**],[**Sch1**],[**Sl**], [**Kn**]). In the statement, we only assume that G and V are defined over $\overline{k}$.

THEOREM 3.1. *Let V be an affine G-variety. Suppose that $v \in V$ lies on a Zariski-closed G-orbit and has isotropy group H. Then there exists $S \subset V$ such that*

(1) *$v \in S$.*
(2) *S is affine, locally closed in the Zariski topology, and stable under H.*
(3) *The morphism $(G \times S \to V,\ (g,s) \mapsto g \cdot s)$ induces an étale morphism $G *^H S \twoheadrightarrow U \subset V$, where U is affine, Zariski-open, and G-saturated, and where $U/\!\!/G \hookrightarrow V/\!\!/G$ as a Zariski-open, affine subvariety.*
(4) *The induced morphism $S/\!\!/H \simeq (G *^H S)/\!\!/G \to V/\!\!/G$ is étale.*
(5) *The map*
$$\begin{array}{ccc} G *^H S & \longrightarrow & S/\!\!/H \underset{V/\!\!/G}{\times} V \\ [g,s] & \longmapsto & (\pi_{S,H}(s), g \cdot s) \end{array}$$
is an isomorphism of G-varieties. In particular, if $y \in U$, then the isotropy of y is G-conjugate to a subgroup of H.
(6) *If V is a G-module and N is an H-stable complement of $T_v(G \cdot v)$ in V, then we may choose S to be an affine, Zariski-open neighborhood of v in $v + N \subset V$.*
(7) *If V is smooth at v, we may assume that S is smooth, and that there is an H-equivariant map $\phi : S \to T_v(S)$ (with $\phi(v) = 0$) which is étale with affine image. Furthermore, $\phi/\!\!/H : S/\!\!/H \to (T_vS)/\!\!/H$ is étale, and we may assume that*
$$\begin{array}{ccc} G *^H S & \longrightarrow & S/\!\!/H \underset{(T_vS)/\!\!/H}{\times} (G *^H T_vS) \\ [g,s] & \longmapsto & (\pi_{S,H}(s), [g, \phi(s)]) \end{array}$$
is an isomorphism of G-varieties. ∎

We now return to the situation where G, V, and the action of G on V are defined over k (notation as in (3.1)).

THEOREM 3.2. *Suppose that $G \cdot v$ is Zariski-closed and $\pi_{V,G}(v) \in Z$. Then we may choose S such that the following also hold:*

(1) *U is defined over k.*
(2) *There are k-structures on $G *^H S$ and $S/\!\!/H$ such that the maps*
$$G *^H S \twoheadrightarrow U \hookrightarrow V$$
$$S/\!\!/H \twoheadrightarrow U/\!\!/G \hookrightarrow V/\!\!/G$$
$$G *^H S \twoheadrightarrow S/\!\!/H,$$

the projection $G *^H S \twoheadrightarrow G/H \simeq G \cdot v$,
and the action of G on $G *^H S$ are all defined over k.

(3) *If furthermore V is smooth at v, then S may be chosen such that there are k-structures on $G *^H (T_v S)$ and $(T_v S)/\!\!/H$, for which the maps*

$$S/\!\!/H \longrightarrow (T_v S)/\!\!/H$$
$$G *^H S \longrightarrow S/\!\!/H \underset{(T_v S)/\!\!/H}{\times} (G *^H (T_v S)),$$

the projection $G *^H (T_v S) \longrightarrow (T_v S)/\!\!/H$,
and the action of G on $G *^H (T_v S)$ are all defined over k.

PROOF: We indicate the points in the proof of (3.1) (we use the proof in [**Kn**]) where care must be taken when working over k.

Step 1. The main tool in the proof of (3.1) is Luna's "lemme fondamental", which describes the local behavior of a morphism $A \to B$ between G-varieties satisfying certain conditions. This lemma is applied to certain morphisms described below, and produces the variety U in the statement of (3.1). It is an easy consequence of the proof of the lemme fondamental [**Kn**] that under our hypotheses, *we may choose U to be defined over k.*

Step 2. We require the following lemma:

LEMMA 3.3: Let $s \in \mathcal{Z}(k, G, H)$. Let V be a G-module, defined over k, and let $W \subset V$ be an H-submodule such that $s_\sigma \cdot W = \sigma(W)$ for all $\sigma \in$ Gal. Then there is an H-stable splitting $V = W \oplus W'$, where $s_\sigma \cdot W' = \sigma(W')$ for all $\sigma \in$Gal.

PROOF: Since H is reductive, the restriction res : $\mathrm{Hom}_H(V, W) \to \mathrm{Hom}_H(W, W)$ is surjective. We can define k-structures on $\mathrm{Hom}_H(V, W)$ and $\mathrm{Hom}_H(W, W)$ as follows: if β is in either set, let ${}^\sigma\beta = s_\sigma^{-1} \circ \sigma \circ \beta \circ \sigma^{-1} \circ s_\sigma$. It is straightforward to check that this gives k-structures and that res is defined over k. Since these are just $\overline{k}$-vector spaces and res is linear, we may find a k-point $\theta \in \mathrm{Hom}_H(V, W)$ such that $\mathrm{res}(\theta) = \mathrm{Id} \in \mathrm{Hom}_H(W, W)_k$. Then let $W' = \ker(\theta)$. ∎

We continue with the proof of (3.2). Let $s \in \mathcal{Z}(k, G, H)$ satisfy $\sigma(v) = s_\sigma \cdot v$ for all $\sigma \in$Gal.

Step 3. Suppose that V is a G-module. By (3.3), we may choose an H-stable complement $N \subset V$ to $T_v(G \cdot v)$ such that $s_\sigma \cdot N = \sigma(N)$ for all $\sigma \in$Gal. Applying Luna's lemme fondamental to the morphism $(G *^H v + N \to V, [g, v+n] \mapsto g \cdot (v + n))$, we obtain $U \subset V$. Following [**Kn**], we let $S = (v + N) \cap U$.

We define a k-structure on $N/\!\!/H$ by letting $\sigma \in$Gal send a closed orbit $H \cdot n$ to the closed orbit $H \cdot s_\sigma^{-1} \cdot \sigma(n)$. We likewise define a k structure on $G *^H (v + N)$ via $[g, v+n] \overset{\sigma}{\mapsto} [\sigma(g)\, s_\sigma, v + s_\sigma^{-1} \cdot \sigma(n)]$. Since U is defined over k, it follows that $G *^H S$ (resp. $S/\!\!/H$) is stable under this action of Gal on $G *^H (v + N)$ (resp. $N/\!\!/H$). Thus we obtain k-structures on $G *^H S$ and $S/\!\!/H$. The verification that all the requisite maps in (3.2) are defined over k is straightforward.

Step 4. Let V be an arbitrary affine G-variety over k. The variety S is constructed as follows: we embed V (equivariantly, over k) in a G-module V' (1.2). Choose $N \subset V'$ as in Step 3, and let $S' = V \cap (v + N)$. Applying the lemme fondamental to the morphism $G *^H S' \to V$, we obtain our $U \subset V$; let $S = U \cap S'$. We then take the restrictions of the k-structures on $G *^H N$ and $N /\!\!/ H$ defined in Step 3 to obtain k-structures on $G *^H S$ and $S /\!\!/ H$.

Step 5. Suppose that V is smooth at v. We must still verify (3.2(3)). We do this, perhaps for a smaller S than the one constructed above.

First, we construct the map ϕ from (3.1(7)). Recall the notation $S' = V \cap (v + N)$ from Step 4. Let $\mathfrak{m}$ be the maximal ideal of $v \in \mathcal{O}(S')$. There is an exact sequence of (locally finite) H-modules

$$(*) \qquad 0 \longrightarrow \mathfrak{m}^2 \longrightarrow \mathfrak{m} \longrightarrow (T_v S)^* \longrightarrow 0.$$

For each $\sigma \in$Gal, we can define an automorphism α_σ of S' (or S) via $\alpha_\sigma = s_\sigma^{-1} \circ \sigma$. We obtain an automorphism of $\mathcal{O}(S')$ which leaves $\mathfrak{m}$ and $\mathfrak{m}^2$ fixed (since $\alpha_\sigma v = v$ for all σ). As in the proof of (3.3), one may find an H-stable splitting of $(*)$ such that that $(T_v S)^* \subset \mathfrak{m}$ is stable under each α_σ. The composite $(T_v S)^* \hookrightarrow \mathfrak{m} \hookrightarrow \mathcal{O}(S')$ induces a morphism $S \to T_v(S)$. If we then define k-structures on $(T_v S)/\!\!/ H$ and $G *^H (T_v S)$ via

$$H \cdot X \overset{\sigma}{\longmapsto} H \cdot \alpha_\sigma(X) \qquad \text{and} \qquad [g, X] \overset{\sigma}{\longmapsto} [\sigma(g) s_\sigma, \alpha_\sigma(X)]$$

respectively, then the maps in (3.2(3)) are defined over k.

Let B denote the points in S at which either S is not smooth or ϕ is not étale. Then B is Zariski-closed, H-stable, and stable under each α_σ. Let $f \in \mathcal{O}(S/\!\!/ H)_k$ vanish on B. If we replace S by $\{s \in S : f(s) \neq 0\}$, then the new S satisfies all the requirements of the theorem. This concludes the proof of (3.2). ∎

COROLLARY 3.4: Under the hypotheses of (3.2), let $F = \pi_{V,G}^{-1}(\pi_{V,G}(v))$. Then there is a G-equivariant retraction $F \to G \cdot v$ which is defined over k.

PROOF: Using the notation from (3.1), we have $F \simeq \pi_{S,H}(v) \underset{\pi_{V,G}(v)}{\times} F \subset S/\!\!/ H \underset{U/\!\!/ G}{\times} U$. By restricting the isomorphism $S/\!\!/ H \underset{U/\!\!/ G}{\times} U \simeq G *^H S$, we see that $F \simeq G *^H F'$, where $F' = \pi_{S,H}^{-1}(\pi_{S,H}(v))$. By (3.2), the (equivariant) retraction $(G *^H S' \to G/H \simeq G \cdot v$, $[g, s'] \mapsto [g, v])$ is defined over k. ∎

REMARK 3.5: If $G \cdot v$ contains a k point, we may assume that $v \in V_k$ and $s_\sigma = e$ for all σ. We then have k-structures on H and S via the inclusions of H into G and S into some G-module over k, and the k-structures on $G *^H S$ and $S/\!\!/ H$ described in the last theorem, come in the obvious way from the k-structures on G, H, and S.

If $G \cdot v$ contains no k-points, we can at least say the following: the map $n \overset{\sigma}{\mapsto} s_\sigma^{-1} \cdot \sigma(n)$ gives a k-structure on S^H. From it and the k-structure on $G/H \simeq G \cdot v$ we obtain k-structures on $G *^H (v + S^H) \simeq G/H \times S^H$ and $(S/\!\!/H)_{(H)} \simeq S^H$, and these k-structures coincide with the ones coming from the inclusions $G *^H (v + S^H) \hookrightarrow G *^H (v + S)$ and $(S/\!\!/H)_{(H)} \hookrightarrow S/\!\!/H$.

Suppose that $v \in V_k$ lies on a Zariski-closed orbit, and has isotropy H. We assume that S (as in (3.2)) lies in some G-module defined over k. We see that the set of k-points of $G *^H S$ is a disjoint union of the subsets $[G_k g_i, S^i]$, where $\{g_i\}$ is a set of representatives of $G_k \backslash G''(H)/H$ and where $S^i = \{n \in S : \sigma(n) = \sigma(g_i^{-1}) g_i \cdot n\} = g_i^{-1} \cdot (g_i \cdot S)_k$. The set $[G_k g_i, S^i]$ is isomorphic to $G_k *^{H^i_k} (g_i \cdot S)_k$ (where $H^i = g_i H g_i^{-1}$), and represents the k-points of $G *^H N$ which retract to the G_k-orbit $G_k \cdot g_i \cdot v$ under the map $G *^H N \to G/H$. ∎

§4. $\mathcal{H}^1$-strata in Z.

In this section, we use a partial order on the set of homogeneous spaces with compatible k-structures (due to Springer [**Sp**]) to stratify $Z = (V/\!\!/G)_k$.

Let $\mathcal{H}^1$ (resp. $\mathcal{H}^1_0$) denote the disjoint union of the sets $\mathcal{H}^1(k, G, (H))$ (resp. $\mathcal{H}^1(k, G, (H))_0$) over all conjugacy classes of reductive subgroups of G. We define a partial order on $\mathcal{H}^1$ as follows: if s is an element of $\mathcal{Z}(k, G, H)$ and t is an element of $\mathcal{Z}(k, G, K)$, we declare that $[s] \leq [t]$ if there exists a G-equivariant map $(G/H, s) \to (G/K, t)$ which is defined over k.

LEMMA 4.1: For H, K, s, t as above, if $[s] \leq [t]$, then there exists $K' \in (K)$ such that

(1) $H \subset K'$
(2) $s : \mathrm{Gal} \to G$ is an element of $\mathcal{Z}(k, G, K')$
(3) $s \in \mathcal{Z}(k, G, K')$ and $t \in \mathcal{Z}(k, G, K)$ give the same element of $\mathcal{H}^1(k, G, (K))$.

PROOF: Since the map $\phi : G/H \to G/K$ is G-equivariant, it must be of the form $gH \mapsto g \cdot g_0 K$, where $H \subset g_0 K g_0^{-1}$. If we follow ϕ by the G-isomorphism

$$\begin{array}{ccc} G/K & \xrightarrow{\theta} & G/g_0 K g_0^{-1} \\ gK & \longmapsto & g g_0^{-1} \cdot g_0 K g_0^{-1} \end{array}$$

(and give $G/g_0 K g_0^{-1}$ the unique k-structure such that θ is defined over k), we obtain the G-equivariant map

$$\begin{array}{ccc} G/H & \longrightarrow & G/g_0 K g_0^{-1} \\ gH & \longmapsto & g \cdot g_0 K g_0^{-1} \end{array}$$

defined over k. If we let $K' = g_0 K g_0^{-1}$, then the claims of the lemma are easily verified. ∎

We remark that if $[s] \in \mathcal{H}^1_0$, $[t] \in \mathcal{H}^1$, and $[s] \leq [t]$, then $[t] \in \mathcal{H}^1_0$.

Now let V be a G-variety defined over k, and let $\pi, V/\!\!/G, Z$ and X be as in §1. We define $\Psi : Z \to \mathcal{H}^1$, to be the function which gives, for each $z \in Z$, the compatible k-structure on the unique Zariski-closed orbit in $\pi^{-1}(z)$. If $[s] \in \mathcal{H}^1$, let

$Z_{[s]} = \Psi^{-1}([s])$ If $[s] \in \mathcal{H}^1_0$, we also write $X_{[s]}$ for $Z_{[s]}$ (this is justified by (1.1(4))). Thus

$$Z_{(H)} := (V/\!\!/G)_{(H)} \cap Z = \bigcup_{[s] \in \mathcal{H}^1(k,G,(H))} Z_{[s]}$$
$$X_{(H)} := (V/\!\!/G)_{(H)} \cap X = \bigcup_{[s] \in \mathcal{H}^1(k,G,(H))_0} X_{[s]}.$$

For $s \in \mathcal{Z}(k, G, H)$, let

$$\begin{aligned} V^H &= \{v \in V : h \cdot v = v \text{ for all } h \in H\} \\ V^{\langle H \rangle} &= \{v \in V^H : v \text{ has isotropy } H\} \\ V^H_s &= \{v \in V^H : \sigma(v) = s_\sigma \cdot v \text{ for all } \sigma \in \mathrm{Gal}\} \\ V^{\langle H \rangle}_s &= V^{\langle H \rangle} \cap V^H_s. \end{aligned}$$

The following proposition contains well-known consequences of (3.1):

PROPOSITION 4.2: (see [**Sch1**, pg. 56]) Let V be a G-variety, and let (H) be an isotropy class of V.

(1) $V^{\langle H \rangle}$ is Zariski-open in V^H.
(2) All orbits intersecting $V^{\langle H \rangle}$ are Zariski-closed.
(3) $\pi\left(V^{\langle H \rangle}\right) = (V/\!\!/G)_{(H)}$.
(4) $\pi(V^H) = \bigcup_{(H') \geq (H)} (V/\!\!/G)_{(H')} \supset$ Zar $\mathrm{Cl}(V/\!\!/G)_{(H)}$, with equality if V is a G-module. ▮

We obtain the following analogues over k:

PROPOSITION 4.3: Let V be a G-variety, defined over k, and let (H) be an isotropy class of V. Let $s \in \mathcal{Z}(k, G, H)$.

(1) $v \overset{\sigma}{\mapsto} s_\sigma^{-1} \cdot \sigma(v)$ gives a k-structure on V^H with k-points V^H_s.
(2) $\pi\left(V^{\langle H \rangle}_s\right) = Z_{[s]}$.
(3) $\pi\left(V^H_s\right) = \bigcup_{[s'] \geq [s]} Z_{[s']}$.
(4) If V is a G-module, then $Z_{[s]} \neq \emptyset$. Furthermore, for any field topology on k, $\mathrm{Cl}\left(Z_{[s]}\right) \supset \bigcup_{[s'] \geq [s]} Z_{[s']}$, and if (H) is the principal isotropy class of V, then $X_{(H)}$ is dense in X.

PROOF: (1) and (2) are trivial. We prove (3). By (4.1), if $[s'] \geq [s]$, we may assume that $s \in \mathcal{Z}(k, G, H)$, $s' \in \mathcal{Z}(k, G, H')$ where $H \subset H'$ and $s' = s$ as maps from Gal to G. The inclusion ($\supset$) follows from (2). Conversely, if $v' \in V^H_s$, let v denote the image of v' under the retraction described in (3.4). (Here $G \cdot v$ is the Zariski-closed orbit in the Zariski-closure of $G \cdot v'$.) Then $v \in V^H_s$, and the map

$((G/H, s) \to G\cdot v, gH \mapsto g\cdot v)$ is G-equivariant and defined over k. If $G\cdot v$ is of type $[s']$, this shows that $[s'] \geq [s]$. Since $\pi(v') = \pi(v) \in Z_{[s']}$, we obtain the inclusion $(\subset)$. This proves (3).

To prove (4), we need the following easy fact: ***Given any (nondiscrete) topology on an (infinite) field k, the complement of the zero set of a finite number of polynomials on k^n, is dense in k^n.*** To prove the first part of (4), we note that $V_s^H \simeq k^n$ for some n; also, $V^{\langle H\rangle} \subset V^H$ is stable under the action of Gal in (4.3(1)) (under $\sigma \in \mathrm{Gal}$, $V^{\langle H\rangle}$ is mapped to $V^{\langle s_\sigma^{-1}\cdot\sigma(H)\cdot s_\sigma\rangle} = V^{\langle H\rangle}$). Hence by (4.2(1)), $V_s^{\langle H\rangle}$ is the complement in k^n of the zero set of a finite number of polynomials with coefficients in k. In particular, it is not empty; hence $Z_{[\sigma]} = \pi(V_s^{\langle H\rangle})$ is nonempty. If k has a field topology, then by the above fact,

$$\bigcup_{[s']\geq[s]} Z_{[s']} = \pi(V_s^H) = \pi(\mathrm{Cl}\,(V_s^{\langle H\rangle})) \subset \mathrm{Cl}\,(\pi(V_s^{\langle H\rangle})) = \mathrm{Cl}\,(Z_{[s]}).$$

To prove the last part, we consider $V_k \cap \pi^{-1}\left((V/\!\!/G) \setminus (V/\!\!/G)_{(H)}\right)$ and again apply the above remark. ∎

§5. Normal types.

If V is a *smooth* G-variety, then as is well known, $V/\!\!/G$ may be given a stratification finer than the one by isotropy type. To a point $z \in V/\!\!/G$, one associates (the isomorphism class of) the normal bundle to the Zariski-closed orbit in $\pi^{-1}(z)$. If V is a G-module, then the two stratifications of $V/\!\!/G$ coincide. In this section, for smooth G-varieties defined over k, we discuss the stratification of Z by "normal type with k-structure."

For us, an *associated bundle* will mean a G-variety of the form $G *^H N$, where H is a reductive subgroup of G and N is an H-module. (It is the G-fibration associated to N, coming from the principal H-fibration $G \to G/H$.) If G is defined over k, then a *compatible k-structure on $G *^H N$* is a k-structure on (the affine variety) $G *^H N$ such that the action of G on $G *^H N$, the projection of $G *^H N$ onto the zero-section $\{[g, 0] : g \in G\} \simeq G/H$, and addition and scalar multiplication on sections, are all defined over k. A morphism $G *^{H_1} N_1 \to G *^{H_2} N_2$ is a *morphism of associated bundles* if it is a G-equivariant morphism of varieties; if it commutes with projection onto the zero-fibers; and if it is linear on fibers. Two compatible k-structures on $G *^H N$ are *equivalent* if they differ by an automorphism of $G *^H N$. Let $\mathcal{M}$ denote the set of (equivalence classes of) compatible k-structures on G-associated bundles.

REMARK 5.1: Let $G' = N_G H \times GL(N)$ and $H' = \{(h, h) \in G' : h \in H\}$. Then $\mathrm{Aut}\,(G *^H N) = N_{G'}H'/H'$. In particular, $\mathrm{Aut}\,(G *^H N)$ is reductive.

We arrive at the same situation as (2.1). From any compatible k-structure on $G *^H N$, we can obtain a k-structure on $\mathrm{Aut}\,(G *^H N)$, and then by [**Se1**, Prop. 5, pg. III–6]:

PROPOSITION 5.2: $\mathcal{H}^1(k, \mathrm{Aut}\,(G *^H N))$ parametrizes the equivalence classes of compatible k-structures on $G *^H N$. ▮

Specifically, given a compatible k-structure on $G *^H N$, $\sigma \in \mathrm{Gal}$ acts by the rule

$$[g, n] \overset{\sigma}{\mapsto} [\sigma(g)s_\sigma, m_{\sigma,n}],$$

where $(\sigma \mapsto s_\sigma) \in \mathcal{Z}(k, G, H)$, and where $\{m_{\sigma,n}\}$ satisfies

(1) $m_{\sigma,n} \in N$.

(2) $m_{\sigma,\lambda n_1+n_2} = \sigma(\lambda)m_{\sigma,n_1} + m_{\sigma,n_2}$ for all $\lambda \in \overline{k}$ and $n_1, n_2 \in N$.

(3) $m_{\sigma_1,m_{\sigma_2,n}} = h \cdot m_{\sigma_1\sigma_2,n}$ if $g_{\sigma_1\sigma_2} = \sigma_1(g_{\sigma_2}) \cdot g_{\sigma_1} \cdot h$.

REMARK 5.3: We have already encountered compatible k-structures on associated bundles $G *^H N$ in Step 3 of the proof of (3.2). We show that all k-structures arise in this way. Let $G *^H N \subset V$ be a G-equivariant embedding of $G *^H N$ in a G-module, defined over k (1.2). (Note that $N \hookrightarrow G *^H N \subset V$.) Using the notation from the last paragraph, we see that $\sigma(n) = \sigma[e, n] = [s_\sigma, m_{\sigma,n}] = s_\sigma \cdot m_{\sigma,n}$ for all $n \in N$ (in particular, $\sigma(N) = s_\sigma \cdot N$). Hence $m_{\sigma,n} = s_\sigma^{-1} \cdot \sigma(n)$, and finally,

$$\sigma([g, n]) = [\sigma(g)s_\sigma, m_{\sigma,n}] = [\sigma(g)s_\sigma, s_\sigma^{-1} \cdot \sigma(n)] \qquad \text{as in (3.2).}$$ ▮

Suppose that V is a smooth G-variety, defined over k. We have a function $\Lambda : Z \to \mathcal{M}$ which assigns to each $z \in Z$, the isomorphism class of the normal bundle (with k-structure) to the Zariski-closed orbit in $\pi^{-1}(z)$. We obtain a stratification $Z = \bigcup_{\lambda \in \mathcal{M}} Z_\lambda$.

PROPOSITION 5.4: Let V be a smooth G-variety, defined over k. The stratification of Z by $\mathcal{M}$ is a refinement of the stratification by $\mathcal{H}^1$. If V is a G-module, then the two stratifications coincide.

PROOF: The first part is trivial: if two Zariski-closed, Gal-stable orbits in V have k-isomorphic normal bundles, then the zero-sections are k-isomorphic.

We prove the second part. Suppose we have $z_1, z_2 \in Z$ which lie in the same $\mathcal{H}^1$-stratum. We must show that they have the same normal type. Suppose we have $v_1, v_2 \in V$, lying on Zariski-closed orbits and having isotropy H, such that $\pi(v_i) = z_i$ and $\sigma(v_i) = s_\sigma \cdot v_i$ for some $s \in \mathcal{H}^1(k, G, H)$. Let $T_i \subset V$ denote the tangent space to $G \cdot v_i$; we know that $\sigma(T_i) = s_\sigma \cdot T_i$ for all $\sigma \in \mathrm{Gal}$. By (3.3), we may pick H-stable complements N_i to T_i, with $\sigma(N_i) = s_\sigma \cdot N_i$ for all σ. As in the proof of (3.3), we may define a k-structure on $\mathrm{Hom}_H(N_1, N_2)$, and then clearly there is a k-point in $\mathrm{Hom}_H(N_1, N_2)$ which is a nonsingular linear transformation. In this way we obtain a map $\theta : N_1 \to N_2$ which is H-equivariant and commutes with each $s_\sigma^{-1} \circ \sigma$. Finally, the map $(G *^H N_1 \to G *^H N_2, [g, n] \mapsto [g, \theta(n)])$ is an isomorphism which is defined over k. ▮

§6. Complete fields.

In this section we consider, for the more part, only fields of characteristic zero which are complete under a (nontrivial) real absolute value. We use elementary facts about analytic manifolds and analytic groups over such fields (see [**Se2**]). We do not distinguish between equivalent absolute values on a field. We need the following facts:

PROPOSITION 6.1 (see [**Cas**]): Let k be complete under a nontrivial absolute value. If the absolute value is archimedean, then $k = \mathbf{R}$ or $\mathbf{C}$ (with the standard absolute value). If the absolute value is nonarchimedean, then the following are equivalent:

(1) k is locally compact.
(2) $\{\alpha \in k : |\alpha| \leq 1\}$ is compact.
(3) The value group of $|\ |$ on k^* is discrete, and the residue class field is finite.
(4) k is a finite extension of $\mathbf{Q}_p$ (p a prime). ∎

An example of a complete but not locally compact field is $k((T))$, the field of formal Laurent series over an (infinite) field k, where $|\sum_{i=n}^{\infty} \alpha_i T^i| = 1/2^n$ if $\alpha_n \neq 0$.

We also consider "fields of type (F)" (we still only consider characteristic zero). These are defined by the following equivalent statements [**Se1**]:

(1) k has only finitely many extensions of a given degree.
(2) $\mathcal{H}^1(k, G)$ is finite for all finite groups G.
(3) $\mathcal{H}^1(k, G)$ is finite for all (affine) algebraic groups G.

Examples include $\mathbf{R}$, p-adic fields, and $K((T))$, where K is algebraically closed. Type (F)-fields have the following properties:

(1) Any affine algebraic group G has only finitely many inequivalent k-forms.
(2) If G is an algebraic group defined over k, then the set of k-points of any homogeneous space defined over k, consists of finitely many G_k-orbits.

We begin by recalling a theorem of Kempf which is valid for all perfect fields k:

THEOREM 6.2 [**Ke**]. *Let V be a G-module, defined over k. Suppose that the G-orbit of $v \in V_k$ is not Zariski-closed. Then there is a homomorphism $\lambda : \overline{k}^* \to G$, defined over k, such that $\lim_{t \to 0} \lambda(t) \cdot v$ exists and lies on a Zariski-closed orbit.* ∎

From now on, we assume that k is complete under a real absolute value.

PROPOSITION 6.3: If $v \in V_k$, then $G \cdot v$ is Zariski-closed if and only if $G_k \cdot v$ is closed (in the k-topology). Each G_k-orbit in $(G \cdot v)_k$ is open and closed in $(G \cdot v)_k$.

PROOF: If $G_k \cdot v$ is closed, then $G \cdot v$ is Zariski-closed by (6.2). (For this, k need not be complete.)

Conversely, if $G \cdot v$ is Zariski-closed, then $(G \cdot v)_k$ is closed. Consequently $(G \cdot v)_k = \bigcup_{i \in I} G_k \cdot v_i$, a union of G_k-orbits. Since the map $(G_k \to (G \cdot v)_k, g \mapsto g \cdot v_i)$ has everywhere surjective differential, it follows that each $G_k \cdot v_i$ is open in $(G \cdot v)_k$. Hence each $G_k \cdot v_i$ is closed in $(G \cdot v)_k$ and therefore closed in V_k. ∎

PROPOSITION 6.4: If $v \in V_k$, then $\mathrm{Cl}\,(G_k \cdot v)$ contains a unique closed G_k-orbit.

PROOF: The existence follows from (6.2) and (6.3). By (3.4), (3.5), and (6.3), we see that $\pi^{-1}(\pi(v)) \cap V_k$ is a union of open subsets, each containing exactly one closed G_k-orbit; hence the uniqueness. ∎

Let $V_k/\!\!/G_k$ denote the set of closed G_k-orbits in V_k. By (6.4), there is a map $p : V_k \to V_k/\!\!/G_k$ which is constant on G_k-orbits. We give $V_k/\!\!/G_k$ the quotient topology. A set $F \subset V_k$ is *G_k-saturated* if $p^{-1}(p(F)) = F$, or equivalently, if F contains v whenever F contains a point in the unique closed G_k-orbit in the closure of $G_k \cdot v$.

COROLLARY 6.5: We obtain a (continuous) map $P : V_k/\!\!/G_k \to X$, which identifies closed G_k-orbits which lie on the same Zariski-closed G-orbit. ∎

REMARK 6.6: For any $U \subset X$, $\pi^{-1}(U) \cap V_k$ is G_k-saturated. ∎

THEOREM 6.7. *Let V be a G-variety, defined over k, and let (H) be an isotropy class of V such that $Z_{(H)} \neq \emptyset$. Let $\psi \in \mathcal{H}^1$ (and $\lambda \in \mathcal{M}$, if V is smooth) be such that the corresponding strata are nonempty subsets of $Z_{(H)}$.*

1. *Given v as in (3.2), there are neighborhoods U of $\pi_{S,H}(v)$ in $(S/\!\!/H)_k$ and U' of $\pi_{V,G}(v)$ in $(V/\!\!/G)_k = Z$ (in the k-topology), which are analytically isomorphic. Furthermore, the map $G *^H S \to V$ yields a G-equivariant bijection $\pi^{-1}_{G*^H S,G}(U) \simeq \pi^{-1}_{V,G}(U')$ commuting with the action of Gal. The map restricts to a G_k-equivariant analytic isomorphism $\pi^{-1}_{G*^H S,G}(U) \cap (G *^H S)_k \simeq \pi^{-1}_{V,G}(U') \cap V_k$, and these sets are G_k-saturated.*
2. *Ψ is locally constant on $Z_{(H)}$, and if V is smooth, then Λ is also locally constant on $Z_{(H)}$.*
3. *$Cl\,(Z_\psi) \subset \bigcup_{\psi' \geq \psi} Z_{\psi'}$, with equality if V is a G-module.*
4. *If V is smooth, then $Z_{(H)}$, Z_ψ, and Z_λ are analytic manifolds, of dimension equal to the dimension of $(V/\!\!/G)_{(H)}$ as a variety over $\overline{k}$.*
5. *X is closed in Z.*

PROOF: The first part of (1) is true since the map $(S/\!\!/H)_k \to (V/\!\!/G)_k$ is étale at $\pi_{S,H}(v)$. Note that by (3.1) and (3.2), the morphism $G *^H S \to V$ restricts to a G-equivariant bijection $G *^H \left(\pi^{-1}_{S,H}(U)\right) \to \pi^{-1}_{V,G}(U')$ which commutes with the action of Gal. With (6.6), this proves the rest of (1). Since there is a G-equivariant retraction of $G *^H S$ to the zero-section, the same is true for $\pi^{-1}_{V,G}(U') \simeq G *^H \left(\pi^{-1}_{S,H}(U)\right) \subset G *^H S$. It follows that Ψ must be constant on $U' \cap Z_{(H)}$, hence locally constant on $Z_{(H)}$. By (5.4), Λ must be locally constant on $Z_{(H)}$. Also, we have shown that every point in Z has a neighborhood on which Ψ can only increase; this proves the first part of (3). The second part follows from (4.3(4)). Next, we prove (4). We have seen that $U'_k \cap (V/\!\!/G)_{(H)} \simeq U_k \cap (S/\!\!/H)_{(H)}$, and near $\pi_{S,H}(v)$, the latter is analytically isomorphic to a neighborhood of $\pi_{T_vS,H}(0)$ in the k-points of $(T_vS/\!\!/H)_{(H)}$. However, $(T_vS/\!\!/H)_{(H)} \simeq (T_vS)^H \simeq \overline{k}^n$ for some

n. Hence $U'_k \cap (V/\!\!/G)_{(H)}$ is analytically isomorphic near $\pi_{V,G}(v)$ to k^n, and (4) follows. Finally, (5) follows from (1): if $z \in \mathrm{Cl}(X)$, then the neighborhood of U' of z described in (1) intersects X. Using (1), there is a retraction defined over k, to the closed orbit in $\pi^{-1}(z)$, of a set in V containing k points. Hence $\pi^{-1}(z)$ contains k-points, and $z \in X$. ∎

THEOREM 6.8. *Let $v \in V_k$ lie on a closed orbit, and let H and S be as in (3.2). Then there is an open, G_k-saturated neighborhood of $v \in V_k$ which is isomorphic to an open, G_k-saturated neighborhood of $[e, v]$ in $G_k *^{H_k} S_k$. If V is smooth, then the same is true if S_k is replaced by $(T_vS)_k$.*

PROOF: By (6.7(1)), v has an open, G_k-saturated neighborhood which is isomorphic to an open, G_k-saturated set $A \subset (G *^H S)_k$. By (3.5), $(G *^H S)_k$ is a union of subspaces $[G_k g_i, S^i]$ which retract to the different G_k-orbits in $(G \cdot v)_k$; by (6.3), these spaces are open in $(G *^H S)_k$, hence they must be G_k-saturated. The one containing $[e, v]$ is isomorphic to $G_k *^{H_k} S_k$. We take $A \cap G_k *^{H_k} S_k$ as the desired neighborhood of $[e, v]$ in $G_k *^{H_k} S_k$. Similar arguments and (3.2(3)) complete the proof for $(T_vS)_k$. ∎

REMARK 6.9: We have shown that if $v \in V_k$ is on a closed orbit, then $(G \cdot v)_k$ has a G_k-saturated neighborhood, equal to the union of open, G_k-saturated sets U_i, with each U_i containing exactly one G_k-orbit in $(G \cdot v)_k$, and having the form described in (6.8). ∎

COROLLARY 6.10: *$V_k/\!\!/G_k$ is Hausdorff.*

PROOF: Let $z_1 \neq z_2 \in V_k/\!\!/G_k$; we need disjoint open sets containing these points. If $P(z_1) \neq P(z_2)$, the result is clear. If $P(z_1) = P(z_2)$, the result follows from (6.9). ∎

We give a stratification of $V_k/\!\!/G_k$, using ideas from §2. If $H' \in \mathcal{C}(H)$ for some H, let $[H']$ denote its G_k-conjugacy class. Let $\mathcal{C} = \bigcup_H [H]$; that is, $\mathcal{C}$ consists of the disjoint union of all G_k-conjugacy classes of reductive subgroups of G which are defined over k. We define a partial order on $\mathcal{C}$ by declaring that $[H] \leq [H']$ if there exists $H'' \in [H']$ such that $H \subset H''$. Clearly we may use $\mathcal{C}$ to stratify $V_k/\!\!/G_k$; we denote a typical stratum by $(V_k/\!\!/G_k)_{[H]}$.

REMARK 6.11: This stratification is a refinement of the stratification of X by $\mathcal{H}^1_0$, pulled back to $V_k/\!\!/G_k$ via the map P. We give an example to show that it may indeed be finer. Let $G = SL(2, \mathbf{C})$, $G_{\mathbf{R}} = SL(2, \mathbf{R})$, and let H be the normalizer of the set of diagonal elements of G. Computation shows that $\mathcal{H}^1(\mathbf{R}, G)$ and $\mathcal{H}^1(\mathbf{R}, H)$ have one and two elements, respectively. By (2.6), $\mathcal{C}(H)$ has two conjugacy classes, even though $\mathcal{H}^1(\mathbf{R}, G, H)$ has only one element (this is clear from (2.2) since H is self-normalizing). Specifically, if $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$, then g_1H and g_2H are real points of $(G/H, 1)$ with G-isotropy groups which are not $G_{\mathbf{R}}$-conjugate (and therefore g_1H and g_2H lie on different $G_{\mathbf{R}}$-orbits). ∎

LEMMA 6.12: Let V be a G-variety, defined over k.

(1) $p(V_s^H) = \bigcup_{[H']\geq[H]} (V_k/\!\!/G_k)_{[H']}$.

(2) $p(V_s^{\langle H\rangle}) = (V_k/\!\!/G_k)_{[H]}$.

(3) $\mathrm{Cl}\left((V_k/\!\!/G_k)_{[H]}\right) \subset \bigcup_{[H']\geq[H]} (V_k/\!\!/G_k)_{[H']}$.

PROOF: Part (1) is proved easily using (3.4). Part (2) is immediate, and (3) follows from (6.8). ∎

THEOREM 6.13. *Let V be a G-module, defined over k. Let (H) be an isotropy class of V and suppose that we may choose H to be defined over k. Then $(V_k/\!\!/G_k)_{[H]} \neq \emptyset$, and $Cl\,(V_k/\!\!/G_k)_{[H]} = \bigcup_{[H']\geq[H]} (V_k/\!\!/G_k)_{[H']}$.*

PROOF: We need only show the reverse inclusion in (6.12(3)), and this follows from (6.12) as in the proof of (4.3(4)). ∎

PROPOSITION 6.14: Let k be a complete field of type (F). Let V be a G-module, defined over k, where G acts effectively on V. If $\{z_i\} \subset X$ converges to z, then for some subsequence of $\{z_i\}$, there are points $v_i \in V_k$ on closed orbits such that $\pi(v_i) = z_i$ and $\{v_i\}$ converges. Furthermore, each v_i has the same isotropy group, and if $v = \lim v_i$, then v lies on a closed orbit.

PROOF: Use induction on $\dim V$. Since k is of type (F), the $\mathcal{H}^1$-stratification of Z is finite, and we may assume that all z_i lie in a single stratum. Hence there is a (Gal-stable) subgroup $H \subset G$ such that $\{z_i\} \subset \pi(V_k^{\langle H\rangle})$. If $H = G$ then the proposition is trivial. We suppose that $H \neq G$, so that $V^H \neq V$. We consider the map $\alpha : V^H/\!\!/N_GH \to V/\!\!/G$, and see that for each z_i, we may pick a point $z_i' \in \left(V^H/\!\!/N_GH\right)_k$ such that $\alpha(z_i') = z_i$.

By a theorem of Luna [**Lu3**, §2], α is finite. We claim that by refining the sequence, we may assume that $\{z_i'\}$ converges to some $z' \in \alpha^{-1}(z)$. More generally, let $\alpha : X \to Y$ be a finite map of affine varieties over k, and suppose that $\{y_i\} \subset Y_k$ converges to y_0. Further suppose that there exists $\{x_i\} \subset X_k$ with $\alpha(x_i) = y_i$. If $g \in \mathcal{O}(X)_k$, then there exists a polynomial $f(T) = \sum_{j=1}^n a_j(y)T^j$ (where $a_j \in \mathcal{O}(Y)_k$ and $a_n = 1$) such that $f(g) = 0$ in $\mathcal{O}(X)$. We may write $f(T) = f_1(T) - f_2(T)$, where $f_1(T) = \sum_{j=1}^n a_j(y_0)T^j$, $f_2(T) = \sum_{j=1}^{n-1} b_j(y)T^j$ and $b_j(y) = a_j(y_0) - a_j(y)$. In some finite extension of k, we may factor $f_1(T)$ as $\prod_{j=1}^n (T-t_j)$. Since $f(g(x)) = 0$ for all $x \in X$, we conclude that for all i,

$$(*) \qquad \prod_{j=1}^{n}(g(x_i) - t_j) = \sum_{j=1}^{n-1} b_j(y_i)g^j(x_i).$$

From $(*)$, it is clear that $\{g(x_i)\}$ is bounded. But then the right side of $(*)$ approaches 0 as $i \to \infty$, and hence some infinite subsequence of $\{g(x_i)\}$ approaches one of the t_j's. We may then repeat this procedure to obtain a subsequence (still denoted $\{x_i\}$) such that $\{g(x_i)\}$ converges for all g is a finite set of generators of $\mathcal{O}(X)_k$. It follows that $\{x_i\}$ converges.

By induction, we obtain points $v_i \in V_k^{(H)}$ and $v \in V_k^H$ on closed N_GH-orbits, such that $\pi_{V^H, N_GH}(v_i) = z_i'$, and hence $\pi_{V,G}(v_i) = z_i$. By [**Lu3**, §3], since $N_GH \cdot v_i$ and $N_GH \cdot v$ are Zariski-closed, so are $G \cdot v_i$ and $G \cdot v$. ∎

THEOREM 6.15. *Let V be a G-variety defined over k, where k is complete and of type (F). If $S \subset V_k$ is closed and G_k-stable, then $\pi(S) \subset X$ is closed.*

PROOF: By embedding, we may assume that V is a G-module, and the result then follows from (6.14). ∎

THEOREM 6.16. *Under the same hypotheses as in (6.15), it follows that $p(S)$ is closed in $V_k /\!/ G_k$.*

PROOF: Let $\{z_i\} \to z$ in $V_k /\!/ G_k$, with $z_i \in p(S)$. We must show that $z \in p(S)$. By continuity, $\{P(z_i)\} \to P(z)$. Let $U \subset V_k /\!/ G_k$ be an open set containing all points of $P^{-1}(P(z))$ except for z, and avoiding a neighborhood of z (this is possible by (6.9)). Let $S' = S \setminus p^{-1}(U)$; it is closed and G_k-stable. By (6.15), $\pi(S')$ is closed in X, hence $P(z) \in \pi(S')$. By construction, $z \in p(S') \subset p(S)$. ∎

THEOREM 6.17. *With the same hypotheses on k, let $v \in V_k$ lie on a closed orbit. Suppose that $U \subset V_k$ is open, G_k-stable, and contains v. Then there exists $v \in U' \subset U$ such that U' is open and G_k-saturated.*

PROOF: Let $S = V_k \setminus U$. It is closed and G_k-stable. By (6.16), $p(S)$ is closed, hence $p^{-1}(p(S)) \cap V_k$ is closed; also it is G_k-saturated and does not contain v. Let U' be its complement in V_k. ∎

REMARK 6.18: (6.3), (6.4), (6.7(1)), and (6.8) were proved by Luna in [**Lu2**] for $k = \mathbb{R}$, and our proofs are essentially the same. In the same paper one will find a rather delicate proof of (6.15). Completely different proofs of several results of this section, including (6.15), using a result of Kempf and Ness, can be found in papers of Schwarz [**Sch2**] (for $k = \mathbb{C}$) and Richardson & Slodowy [**RS**] (for $k = \mathbb{R}$). ∎

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