Distance matrices, random metrics and Urysohn space.

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ABSTRACT

We introduce an universum of the Polish (=complete separable metric) spaces - the convex cone of distance matrices and study its geometry. It happened that the generic Polish spaces in this sense of this universum is so called Urysohn spaces defined by P.S.Urysohn in 20-th, and generic metric triple (= metric space with probability borel measure) is also Urysohn space with non-degenerated measure. We prove that the complete invariant of the metric triples up to measure preserving isometries is so called matrix distribution - a measure of the special type on the cone of distance matrices, it defined an important class of S_∞ random metrics on the set of naturals.

CONTENT

1 The cone of distance matrices - the universum of the Polish spaces

Denote by $\mathcal{R}$ a set of all infinite real matrices

$$\mathcal{R} = \{\{r_{i,j}\}_{i,j=1}^{\infty} : r_{i,i} = 0, r_{i,j} \geq 0, r_{i,j} = r_{j,i}, r_{i,k} + r_{k,j} \geq r_{i,j}, i, j, k = 1 \dots\}$$

We will call such matrices (finite or infinite) *the distance matrices.* Each such a matrix defines a semi-metric ρ on an ordered countable set - for definiteness - on the set of naturals $\mathbf{N}$: $\rho(i,j) = r_{i,j}$. We allow zeros out from principle diagonal, so it is only semi-metric in general. If matrix has no zeros out of principle diagonal we will call it *true distance matrix.*

The set of all distance matrices is a weakly closed convex cone in the real linear space $Mat_{\mathbf{N}}(\mathbf{R}) = \mathbf{R}^{\mathbf{N}^2}$ which equipped with ordinary weak topology. We denote this cone as $\mathcal{R}$ and will quote as the (topological) space of distance matrices. Subset of true distance matrix is every dense open strata in $\mathcal{R}$.

Suppose now that (X, ρ) is a complete separable metric (=Polish) space with metric ρ, and $\{x_i\}_{i=1}^{\infty}$ is an ordered dense countable set in it. Define the matrix $r = \{r_{i,j}\} \in \mathcal{R}$ where $r_{i,j} = \rho(x_j, x_j), i, j = 1 \dots$ is a distance matrix with positive elements out from diagonal; we interpret it as a *metric on the set of naturals.* Evidently this distance matrix includes all information about initial space (X, ρ), because (X, ρ) is canonical completion of the set of naturals with that metric.

Any invariant metric property of the space (like compactness, topological, homological properties etc,) could be expressed in terms of the distance matrix for any dense countable subsets of that space. Some of them is easy to rewrite (say, compactness) another more difficult (dimension).

General distance matrix (with possible zeros out of diagonal) defines on the set naturals the structure of *semi-metric space.* By completion of naturals in that case we mean the completion of corresponding quotient metric space of the classes of points with zero distances. Number of the classes we will call *geometric rank* of distance matrix or of the corresponding semi-metric space. For example zero matrix is a distance matrix of the naturals (or countable semi-metric space) with zero distance between each two points and "completion" of it is one point metric space. So finite metric spaces also could be considered in this setting.

The space of distance matrices $\mathcal{R}$ is an *universum of all separable complete metric spaces* with a fixed dense countable subsets: we can look at $\mathcal{R}$ as a "fiber bundle", the base of which is a set of all individual Polish spaces and the fiber over given space is the set of all countable ordered dense subsets in this metric space. Because of universality of Urysohn space $\mathcal{U}$(see below) the set of all closed subsets of $\mathcal{U}$ could be considered as a base of that bundle.

Let ξ is a partition of $\mathcal{R}$ on the classes of matrices which produce the isometric completions of the set of naturals with that metric as complete Polish spaces. The quotient space over partition ξ is the space of the *classes* of the Polish spaces up to isometry. As was conjectured in [4] and proved in the paper [1] that partition (or equivalence relation) is not "smooth" so, the quotient has no good borel or topological structure and the problem of the classification of the Polish spaces up-to isometry is wild (non smooth) problem. In the same time the restriction of this problem to the case of compact Polish spaces is smooth (see [2]) and a space of all classes up to isometry of compact metric spaces has a natural topology.

The space $\mathcal{R}$ plays role of "tautology fibration" over the space of classes of isometrcal Polish spaces similar to ordinary topological constructions. We will see that the problem of classification of the metric space with borel probability measure is "smooth" problem and the complete invariant is a measure on $\mathcal{R}$ with a special properties.

In this paper we study geometry of the space $\mathcal{R}$ itself and describe its generic properties Polish spaces in terms of that cone, and prove that Urysohn space is generic. Then we consider the metric triples (Gromov's mm-spaces), find its invariant and prove that probability measures on the space $\mathcal{R}$. which are concentrated on so called universal matrices are also generic, so typical metric space with measure is also Urysohn space with measure. Roughly speaking - random choice of the Polish space gives you that remarkable space. One of the analogs of this theorem in another categories - Erdos theorem about random graphs.

2 Geometry and topology of the cone $\mathcal{R}$

2.1 Convex structure

Let us denote the finite dimensional cone of distance matrices of order n as $\mathcal{R}_n = \mathcal{R} \cap Mat_n(\mathbf{R})$. Cone $\mathcal{R}_n$ is polyhedral cone inside the positive orthant in $Mat_n(\mathbf{R}) \equiv \mathbf{R}^{n^2}$. Denote as $\mathbf{M}_n^s(\mathbf{R}) \equiv \mathbf{M}_n^s$ the space *symmetric matrices with zeros on the principle diagonal*, the cone $\mathcal{R}_n$ is contained in this space: $\mathcal{R}_n \subset \mathbf{M}_n^s$ and the last space is evidently the linear hull of the cone: $span(\mathcal{R}_n) = \mathbf{M}_n^s$, because the interior of $\mathcal{R}_n$ in that space is not empty. It is not so evident that $span(\mathcal{R}) = \mathbf{M}_{\mathbf{N}}^s$, where $\mathbf{M}_{\mathbf{N}}^s$ is the space of all real infinite symmetric matrices with zero principle diagonal (we do not need in weak closure of linear hull as it usually happened in infinite dimensional cases), but we will not use this fact.

Each matrix $r \in \mathcal{R}_n$ defines the (semi)metric space X_r on the sets of n points

Define projection

$$p_{m,n} : \mathbf{M}_m^s \longrightarrow \mathbf{M}_n^s, m > n$$

which associates to the matrix r of order m its of NW-corner of order n. The cones $\mathcal{R}_n$ are preserved by $p_{n.m} : p_{m,n}(\mathcal{R}_m) = \mathcal{R}_n$ Projection $p_{n,m}$ are natural extends to the space of infinite symmetric matrices with zero diagonal - $p : M_{\mathbf{N}}^s \longrightarrow M_n^s(\mathbf{R})$ and p_n are also preserved the cones: $p_n(\mathcal{R}) = \mathcal{R}_n$. It is clear that $\mathcal{R}$ is inverse limit as topological space (in weak topology) of the finite dimensional cones of $\mathcal{R}_n$ under projections $\{p_n\}$. We will omit the first index and denote $p_{N,n} = p_n$.

Let us consider a geometrical structure of $\mathcal{R}_n$ and $\mathcal{R}$.

It is easy to describe the extremal rays (in the sense of convex geometry) of the convex polyhedral cone $\mathcal{R}_n, n = 2, \dots, \infty$. (for $n = 1$ the cone $\mathcal{R}_1$ consist with one point - zero).

Lemma 1 *Each extremal ray in $\mathcal{R}_n, n > 1$ is a ray of type $\{\lambda \cdot l\}_{\lambda \geq 0}$, where λ runs over all real nonnegative numbers, and l is a symmetric $0-1$- distance matrix which corresponds to semi-metric space metric quotient of which has just two points. The same is true for $\mathcal{R}$.*

In another words all nonzero matrix l which belongs to extremal ray is a distance matrix of the finite or countable semi-metric space which divides on two nonempty subset the distance between two points from the same subset is zero and between two points of the different subsets equal to one. The number of extremal rays of $\mathcal{R}_n, n > 1$ is equal to $2^{n-1} - 1$.

Proof. If the ray $\{\lambda \cdot l_{\lambda \geq 0}\}, l \in \mathcal{R}$ is extremal then the non zero matrix l must have at least one zero coordinates out from diagonal. If we facotrize the space modulus zero distance we obtain a new metric space which can not contain a non degenerated triangles because existence of such triangles contradicts to extremality of the ray. Consequently the corresponding metric space has two points only. ■

The cones $\mathcal{R}_n$ as the topological spaces could be described in the different ways. They have very interesting stratifications as semi-algebraic sets and various systems of coordinates. In order to clarify topological structure of the cones we will use the system of the coordinates which is well-organized as a sets of vectors of entries of matrices, so called admissible vectors, which is very convenient for the natural construction and studying of Urysohn space and measures on it (see next paragraph).

2.2 Admissible vectors and structure of the $\mathcal{R}$

Suppose $r = \{r_{i,j}\}_1^n$ is a distance matrix of order n, choose a vector $a \equiv \{a_i\}_{i=1}^n \in \mathbf{R}^n$ such that if we border a matrix r with vector a as the last column and the last row then the new matrix of order $n+1$ still will be also a distance matrix. We will call such a vector *admissible vector* for fixed distance matrix r and denote the set of of all admissible vectors for r as $A(r)$. For given $a \in A(r)$ denote as (r^a), a distance matrix of order $n+1$ which is obtains from matrix r with adding vector $a \in A(r)$ as the last row and column. It is clear that $p_n(r^a) = r$. The matrix r^a has the form:

$$\begin{pmatrix} 0 & r_{1,2} & \dots & r_{1,n} & a_1 \\ r_{1,2} & 0 & \dots & r_{2,n} & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{1,n} & r_{2,n} & \dots & 0 & a_n \\ a_1 & a_2 & \dots & a_n & 0 \end{pmatrix}$$

The (semi)metric space X_{r^a} corresponding to matrix r^a is extension of of X_r - we add one new point x_{n+1} and $a_i, i = 1 \dots n$ is a distance between x_{n+1} and x_i. The admissibility of a is equivalent to the following set of inequalities : vector $a = \{a_i\}_{i=1}^n$ must subtract to the series of triangle inequalities for all $i, j, k = 1 \dots n$; (matrix $\{r_{i,j}\}_{i,j=1}^n$ is fixed):

$$|a_i - a_j| \le r_{i,j} \le a_i + a_j \tag{1}$$

So, for given distance matrix r of order n the set of admissible vectors is $A(r) = \{\{a_i\}_{i=1}^n : |a_i - a_j| \le r_{i,j} \le a_i + a_j, i, j = 1 \dots n\}$. The set $A(r)$ is the intersection of cone $\mathcal{R}_{n+1}$ with affine subspace which consists with matrices of order $n+1$ with matrix r as a NW-corner of order n. It is clear from the linearity of inequalities that set $A(r)$ is an unbounded closed convex polytope in $\mathbf{R}^n$. If $r_{i,j} \equiv 0, i, j = 1 \dots n \ge 1$, then $A(r)$ is diagonal: $A(r) = \Delta_n \equiv \{\lambda, \dots (n) \dots \lambda\}_{\lambda \ge 0} \subset \mathbf{R}_+^n$. Let us describe its structure more carefully.

Lemma 2 *For each true distance matrix r of order n the set of admissible vectors $A(r)$ is a closed convex polyhedron in orthant $\mathbf{R}_+^n$, namely this is a Minkowski sum:*

$$A(r) = M_r + \Delta_n,$$

where $\Delta_n \equiv \{\lambda, \dots (n) \dots \lambda\}_{\lambda \geq 0}$ *is a half-line – positive diagonal in the space* $\mathbf{R}^n_+$*, and* $M_r =$ *conv(ext* $A(r)$*) is a compact convex polytope of the dimension* n*. This polytope is a convex hull of extremal points of the set* $A(r)$*.*

The set $A(r)$ *is homeomorphic to the product of the simplex and half-line, or, equivalently to the half-space* $R^n_+ = \{(b_1, \dots b_n) : b_n \geq 0\}$*) (this homeomorphism could be chosen as piecewise linear but not canonically). If* r *is not true distance matrix of order* n *and has geometric rank* $m < n$ *then* $A(r)$ *is homeomorphic to the product of* $(m-1)$*-dimensional simplex and half-line.*

Proof. The set $A(r) \subset \mathbf{R}^n$ is intersection of finitely many of the closed subspaces, evidently it does not contain straight lines, so, because of general theorem of convex geometry $A(r)$ is a sum of the convex closed polytope and some cone with the vertex at origin. That convex polytope is the convex hull of the extremal points of convex set $A(r)$. But this cone must be one dimensional, namely - diagonal in $\mathbf{R}^n$ because if it contains any half-line differ from diagonal then the triangle inequality (left side) will violate. Dimension of $A(r)$ is a number which depends on matrix r and could be less n for degenerated r, evidently dimension of M_r is equal to dimension of $\dim A(r)$ or to $\dim A(r) - 1$. The assertion about topological structure of $A(r)$ follows from what was claimed above. ■

The convex structure of polytopes $M_r, A(r)$ is very interesting and it seems never had been studied. For dimensions more than 3 combinatorial type of the polytope M_r hardly depends on r. For our purpose here it is important only to establish *topological* isomorphisms of the special type with half-spaces for true distance matrices of the given order.

In dimension three the *combinatorial type* of polytopes M_r, and consequently combinatorial structure of the sets $A(r)$ is the same for all true distances matrices r.

Example For $n = 3$ the description of the set $A(r)$ and its of extremal points is the following. Let r^a is the matrix:

$$\begin{pmatrix} 0 & r_{1,2} & r_{1,3} & a_1 \\ r_{1,2} & 0 & r_{2,3} & a_2 \\ r_{1,3} & r_{2,3} & 0 & a_3 \\ a_1 & a_2 & a_3 & 0 \end{pmatrix}$$

There are seven extremal points (a_1, a_2, a_3) of $A(r)$: the first one is a vertex which is the closest to origin: $(a_1, a_2, a_3) = (\frac{1}{2}(r_{1,2} + r_{1,3} - r_{2,3}, \frac{1}{2}(r_{1,2} - r_{1,3} + r_{2,3}), \frac{1}{2}(-r_{1,2} + r_{1,3} + r_{2,3}))$;

three non degenerated extremal points: $(\frac{1}{2}(r_{1,2} + r_{1,3} + r_{2,3}), \frac{1}{2}(-r_{1,2} + r_{1,3} + r_{2,3}), \frac{1}{2}(r_{2,3} + r_{1,3} - r_{2,3}))$ and the rest two are cyclic permutations of that in the natural sense.

and three degenerated extremal points $(r_{1,2}, 0, r_{2,3}), (0, r_{1,2}, r_{2,3}), (r_{1,3}, r_{2,3}, 0)$ which defined the metric spaces with two distinguish points and with the third point which coincide with one of those.

If $r_{1,2} = r_{1,3} = r_{2,3} = 1$ then those seven points are as follow
$(1/2, 1/2, 1/2), (1, 0, 1), (0, 1, 1), (1, 1, 0), (3/2, 1/2, 1/2), (1/2, 3/2, 1/2), (1/2, 1/2, 3/2)$.

It makes sense to mention that all non-degenerated extremal points defines the finite metric spaces which can not be isometrically embedded to Euclidean space. The cone $\mathcal{R}_3$ is simplicial cone of dimension 3, and for all matrices $r \in \mathcal{R}_3$ which does not belong to the boundary the set of admissible vectors $A(r)$ (extremal points of which we had defined above) have the same convex combinatorial type although they are no affine isomorphism between them.

2.3 Projections and isomorphisms

Suppose we have distance matrix r of order N and its NW-corner of order $n < N$ - $p_n(r)$. Then we can define neglecting projection χ_n^r of $A(r)$ to $A(p_n(r)$: $\chi_n^r : (b_1, \dots b_n, b_{n+1}, \dots b_N) \mapsto (b_1 \dots b_n)$. (We omit index N in the denotation of χ_n^r) The next simple lemmas play very important for our construction.

Lemma 3 *Let $r \in \mathcal{R}_n$ is a distance matrix of order n and two vectors $a = (a_1, \dots a_n) \in A(r), b = (b_1, \dots b_n) \in A(r)$ there exist real nonnegative number $h \in \mathbf{R}$ such that vector $\bar{b} = (b_1, \dots b_n, h) \in A(r^a)$.*

The claim of this lemma is equivalent to the assertion that for each r the projection χ_n^r defined above is epimorphism of $A(r^a$ to $A(r)$:

Corollary 1 *For each $r \in \mathcal{R}_n$ and $a \in A(r)$ the map: $\chi_{n+1,n}^r : (b_1, \dots b_n, b_{n+1}) \mapsto (b_1 \dots b_n)$ of $A(r^a) \to A(r)$ is epimorphism of $A(r^a)$ on $A(r)$.(by definition $p_{n+1,n}(r^a) = r$)*

Proof. The assertion of lemma as we will see, is a simple geometrical observation: suppose we have two finite metric space $X = \{x_1, \dots x_{n-1}, x_n\}$ with metric ρ_1 and $Y = \{y_1, \dots y_{n-1}, y_n\}$ with metric ρ_2. Suppose the subspaces of the first $n-1$ points $\{x_1, \dots x_{n-1}\}$ and $\{y_1, \dots y_{n-1}\}$ are isometric – $\rho_1(x_i, x_j) = \rho_2(y_i, y_j), i, j = 1, \dots n-1$,

then there exists the third space $Z = \{z_1, \dots z_{n-1}, z_n, z_{n+1}\}$ with metric ρ and two isometries I_1, I_2 of both spaces X and Y to the space $Z, I_1(x_i) = z_i, I_2(y_i) = z_i, i = 1, \dots n-1, I_1(x_n) = z_n, I_2(y_n) = z_{n+1}$. In order to prove existence of Z we need to prove that it is possible to define only nonnegative number h which will be the distance $\rho(z_n, z_{n+1}) = h$ between z_n and z_{n+1} (images of x_n and y_n in Z correspondingly) such that all triangle inequalities took place in the space Z. The existence of h follows from the inequalities:

$$\rho_1(x_i, x_n) - \rho_2(y_i, y_n) \leq \rho_1(x_i, x_n) + \rho_1(x_i, x_j) - \rho_2(y_i, y_n) =$$

$$= \rho_1(x_i, x_n) + \rho_2(y_i, y_j) - \rho_2(y_i, y_n) \leq \rho_1(x_i, x_n) + \rho_2(y_i, y_n)$$

for all $i, j = 1, \dots n-1$.

Consequently

$$\max_i |\rho_1(x_i, x_n) - \rho_2(y_i, y_n)| \equiv m \leq M \equiv \min_j(\rho_1(x_i, x_n) + \rho_2(y_i, y_n)).$$

So, a number h could be chosen as an arbitrary number from the nonempty closed interval $[M, m]$ and we define $\rho(z_n, z_{n+1}) \equiv h$; it follows from the definitions that all triangle inequalities are satisfied. Now suppose we have a distance matrix r of order $n-1$ and admissible vector $a \in A(r)$, so we have metric space $\{x_1, \dots x_{n-1}, x_n\}$ (first $n-1$ points corresponds to matrix r and all space - to extension matrix r^a. Now suppose we choose another admissible vector $b \in A(r)$, distance matrix r^b defined space $\{y_1, \dots y_{n-1}, y_n\}$ where subset of first $n-1$ points is isometric (the same) as space $\{x_1, \dots x_{n-1}\}$. As we proved we can define space Z whose distance metric $\bar{r}$ of order $n+1$ gives has needed property. ∎

Now we can formulate the general assertion about projections χ^r.

Lemma 4 *For each naturals N, n and $r \in \mathcal{R}_N$ the map χ_n^r is epimorphism of $A(r))$ onto $A(p_n(r))$. In another words for each $a = (a_1, \dots a_n) \in A(p_n(r))$ there exist vector $(b_{n+1}, \dots b_N)$ such that $b = (a_1, \dots a_n, b_{n+1}, \dots b_N) \in A(r)$.*

Proof. Previous proof shows how to define the first number b_{n+1}. But the projection χ_n^r as a map from $A(r), r \in \mathcal{R}_N$ to $A(p_n(r)$ is the product of projections $\chi_n^r \cdots \chi_{N-1}^r$ Previous lemma shows that all those factors are epimorphisms. ∎

It is useful to consider each infinite distance matrix $r \equiv \{r_{i,j}\} \in \mathcal{R}$ as as a sequences of the admissible vectors of the increasing lengths

$$r(1) = \{r_{1,2}\}, r(2) = \{r_{1,3}, r_{2,3}\}, \dots r(k) = \{r_{1,k+1}, r_{2,k+1}, \dots r_{k,k+1}\} \dots, k = 1, 2 \dots, \quad (2)$$

with conditions $r(k) \in A(p_k(r))$, (remember that $p_k(r)$ is NW-projection of matrix r on the space $\mathbf{M}_k^s$ which was defined above) - each vector $r(k)$ is admissible for the *previous distance matrix*. This is a realization of the infinite distance matrix using the projections of the cones. We can consider the following sequences of the cones and maps:

$$0 = \mathcal{R}_1 \xleftarrow{p_2} \mathcal{R}_2 = \mathbf{R}_+ \xleftarrow{p_3} \mathcal{R}_3 \longleftarrow \dots \longleftarrow \mathcal{R}_{n-1} \xleftarrow{p_n} \mathcal{R}_n \longleftarrow \dots \quad (3)$$

where projections p_n now is restriction of the projection p_n (defined above) onto the cone $\mathcal{R}_n$. A preimage of the point $r \in \mathcal{R}_{n-1}$ (fiber over r) is the set $A(r)$ which structure had been described in the lemma 2. The fibers are not even homeomorphic to each other for the various r (even dimensions could be different), so the sequences (3) are not the sequences of fiber bundles in usual sense. But it defines a very interesting stratifications of all cones, and the corresponding complex structure on $\mathcal{R}$. More careful studying of the stratification of the cones $\mathcal{R}_n$ and $\mathcal{R}$ must take in account the combinatorial and semi-algebraic structures of its. We postpone the discussion on this interesting subject.

For our goals in this paper it is enough to consider the open cell (main strata) of each cones which is simply *the set of distance matrices for which all triangle inequalities are strict inequalities*. Denote this open part (interior) of the cone $\mathcal{R}_n$ as $\mathcal{R}_n^0$. The cone $\mathcal{R}^0$ is a set of all distance matrices for which any triangle inequalities are strict, it is every dense subset of $\mathcal{R}$ Let $r \in \mathcal{R}_n^0$, denote as $\dot{A}(r)$ the interior of $A(r)$ (of those admissible vectors for which again all triangle inequalities are strict). So the cones $\mathcal{R}^0$ equipped with the structure of inverse limit. For each n projection $\mathcal{R}_{n-1}^0 \overset{p_n}{\longleftarrow} \mathcal{R}_n^0$ this is a trivial fibration because now all fibers are homeomorphic to each other and to open half-space, and the base is homeomorphic to affine space, because this is an open nonempty convex set. We can to refine this picture with the trivialization of this fibration because now all fibers for given n are open cells of the same dimension.We obtain

Lemma 5 *1.The structure of the cone $\mathcal{R}^0$ is described with the sequence of the maps:*

$$0 = \mathcal{R}_2^0 = \mathbf{R}_+ \overset{p_3}{\longleftarrow} \mathcal{R}_3^0 \longleftarrow \ldots \longleftarrow \mathcal{R}_{n-1}^0 \overset{p_n}{\longleftarrow} \mathcal{R}_n^0 \longleftarrow \ldots \longleftarrow \mathcal{R}^0 \tag{4}$$

For each n the map $\mathcal{R}_{n-1}^0 \overset{p_n}{\longleftarrow} \mathcal{R}_n^0$ defines a trivial fiber bundle and natural decomposition: $\mathcal{R}_n^0 \simeq \mathcal{R}_{n-1}^0 \times \dot{\mathbf{R}}_+^{n-1}$ where $\dot{\mathbf{R}}_+^{n-1}$ is open half-space, π_n is the projection on the first summand in that decomposition; and the second factor is homeomorphic to $\dot{A}(r)$ for all r. So we have a realization of open strata:

$$\mathcal{R}^0 \simeq \prod_{k=1}^{\infty} \dot{\mathbf{R}}_+.$$

Remark 1 *The topology which is induced on $\mathcal{R}^0$ from $\mathcal{R}$ does not coincide with topology of inverse limit in (2), but we will not use it. The borel structure defined by the family of cylindric sets on that direct product coincides with borel structure which is induced on $\mathcal{R}^0$ from $\mathcal{R}$. There is no canonical isomorphism in the theorem: there are many ways to identify the fibers with affine open half-spaces (or simply with open discs) according to lemma 2.*

3 Universality and Urysohn space

3.1 Universal distance matrices

The following definition plays a crucial role.

Definition 1 *The infinite true distance matrix $r = \{r_{i,j}\}_{i,j=1}^{\infty} \in \mathcal{R}$ called as universal matrix if the following condition is satisfied:*

for each $\epsilon > 0$, each $n \in \mathbf{N}$ and for each vector $a = \{a_i\}_{i=1}^{n} \in A(p_n(r))$ there exists such $m \in \mathbf{N}$ that $\max_{i=1\dots n} |r_{i,m} - a_i| < \epsilon$.

In another words: for each $n \in \mathbf{N}$ the set of vectors $\{\{r_{i,j}\}_{i=1}^{n}\}_{j=n+1}^{\infty}$ is every dense in the set of admissible vectors $A(p_n(r))$.

Let us denote the set of universal distance matrices as $\mathcal{M}$

Theorem 1 *The set of universal matrix in $\mathcal{R}$ is nonempty and moreover - is an every dense G_δ-set in weak topology of the cone $\mathcal{R}$.*

Proof. We will use representation of the lemmas of previous section for the construction of at least one universal true distance matrix in the cone $\mathcal{R}_0$.

Let us fixed sequence $\{m_n\}_{n=1}^{\infty}$ of naturals in which each natural number occurs infinitely many times and with property: for each $n, m_n \leq n, m_1 = 1$. For each finite distance matrix $r \in \mathcal{R}_n$ let us choose an ordered countable dense subset $\Gamma_r = \{\gamma_k^r\}_{k=1}^{\infty} \subset \dot{A}(r) \subset \mathbf{R}^n$ and choose any metric in $\dot{A}(r)$, say Euclidean norm from $\mathbf{R^n}$.

The first step consists with the choice of positive real number $\gamma_1^1 \in \Gamma_1 \subset \dot{A}(0) = \dot{\mathbf{R}}_+^1$ so we define a distance matrix r of order 2 with element $r_{1,2} = \gamma_1^1$.

Our method of the construction of the universal matrix r used its representation as sequence of the admissible vectors $\{r(1), r(2), \dots\}$ of increasing lengths (see (2)), - the index in the brackets is a dimension of the vector, -the conditions on the vectors are as follow: $r(k) \in A(p_k(r_{k+1})$. The sequence of the corresponding matrices $r_n, n = 1\dots$ are stabilized to the infinite matrix r. Suppose after $n-1$-th step we obtain finite matrix r_{n-1}, now we choose a new admissible vector $r(n) \in \dot{A}(r_{n-1})$. The choice of this vector (denote it as a) is defined by condition: the distance in $A(r_{m_n})$ (in norm) between projection $\chi_{m_n}^r(a)$ of the vector a onto subspace of admissible vectors $A(r_{m_n})$ and point $\gamma_s^{m_n} \in \Gamma_{r_{m_n}} \subset A(r_{m_n}$ must be less than 2^{-n}, here $s = |i : m_i = m_n, 1 \leq i \leq n| + 1$:

$$\|\chi_{m_n}^r(a) - \gamma_s^{r_{m_n}}\| < 2^{-n}.$$

Remain that projection $\chi_{m_n}^r$ is epimorphism from $A(r)$ to $A(p_{m_n}(r))$, (lemma 4), so a vector $a \in A(r_n)$ with this properties does exist. Number s is nothing more than the numbers of the points of $\Gamma_{r_{m_n}}$ which occur on the previous steps of the construction. After countable many of steps we obtain the infinite distance matrix r.

Universality of r is evident, because for each n projection χ_n^r of vectors $r(k), k = n+1\dots$ is a dense set in $A(r_n)$ by construction. This proves the existence of universal matrix.

From the previous proof it is easy to extract by induction over n the following very important property of any universal matrix r. Suppose $r = \{r_{i,j}\} \in \mathcal{R}$ is an arbitrary universal distance matrix, and q is a finite distance matrix of order $n \geq 2$ and $\epsilon > 0$ then there exists a submatrix $\{r_{i_s,i_t}\}_{s,t=1}^n$ of matrix r, such that $\|q - \bar{q}\| < \epsilon$ (here $\|.\|$ -is an arbitrary norm in the space of matrices of order n). Indeed, because $\{r_{1,n}\}_n^\infty$ is dense in $\mathbf{R}_+$ we can choose m_1 such that $|r_{1,m_1} - q_{1,2}| < \epsilon$, then because of density of the columns of length 2 we can choose m_2 such that $\|(r_{1,m_2}, r_{2,m_2}) - (q_{1,3}, q_{2,3})\| < \epsilon$ etc.

We can say that for each n the closure of the set of all submatrices $q = \{r_{i_s,i_t}\}_{s,t=1}^n$ of order n over all choice of n-plies $i_1 < i_2 < \dots < i_n$ of universal distance matrix $r = \{r_{i,j}\}$ coincides with the set $\mathcal{R}_n$ of all distance matrices of order n. Let us call this property a *weak universality.*

Now remark that the universality of the matrix is preserved under the action of any finite permutations simultaneously permute the rows and columns, and under the shift which cancels the first row and first column of the matrices. Also the set of universal matrices $\mathcal{M}$ is invariant under the changing of the finite part of the matrix. Consequently, $\mathcal{M}$ contains with the given matrix all its permutations and shifts. But because of the weak universality, the orbit of r under group of permutations $S_{\mathbf{N}}$ is every dense in $\mathcal{R}$ in weak topology.

Finally, the formula which follows directly from the definition of universality shows us immediately that the set of all universal matrices $\mathcal{M}$ is a G_δ-set:

$$\mathcal{M} = \cap_{k\in\mathbf{N}} \cap_{n\in\mathbf{N}} \cap_{a\in A(r^n)} \cup_{m\in\mathbf{N},m>n} \{r \in \mathcal{R} : \max_{i=1,\dots n} |r_{i,m} - a_i| < \frac{1}{k}\}.$$

∎

Remark 2 *The property of matrix to be universal is much more stronger than the property of weak universality - it is easy to give an example of non universal but weak universal matrix.*

3.2 Urysohn space

Now we introduce a remarkable Urysohn space. In one of his last papers [3] Urysohn gave a concrete construction of the universal Polish space which we will call "Urysohn space" and denote as $\mathcal{U}$. There are no the notion of universality in [3] because Urysohn did not consider infinite matrices at all but he actually have proved several theorems which we summarize as following theorem:

Theorem 2 *(Urysohn [3]) There exist the Polish space with the properties:*

1)(Universality) For each Polish space X there exists the isometric embedding to $\mathcal{U}$;

2)(Homogeneity) For each two isometric finite subsets $X = (x_1 \dots x_m)$ and $Y = (y_1 \dots y_m)$ of $\mathcal{U}$ there exists isometry I of whole space $\mathcal{U}$ which brings X to Y;

3)(Uniqueness) Polish spaces with the previous properties is unique up to isometry

In our terminology Urysohn actually have proved

Theorem 3 *Let r is an arbitrary infinite universal distance matrix, consider the set of naturals $\mathbf{N}$ with metric r. Then the completion of the space $(\mathbf{N}, r)$ with respect to metric r is Urysohn space*

The proof of all three items of the theorem follows directly and easily from the universality of the matrix r and does not use even the existence of universal matrices. That was a content of three main theorems in [3]. We do not repeated here very natural Urysohn's arguments and restrict ourself only with the hints. Universality of the matrix r allows to construct a family of fundamental sequences such their limits (in completion) gives the finite set of points with the given a prori distance matrix. This gives universality (item 1). If we have two given finite isometric sets of the points of completion $(x_1, \dots x_n)$ and $(y_1, \dots y_n)$ we can include each of them to two countable dense subsets which are still isometric - this gives homogeneity (item 2). Uniqueness follows from the fact that in each space with the properties 1),2) it is possible to find countable dense subset with given universal matrix.

Remark 3 *1.The inverse assertion is trivially true: the distance matrix of any countable ordered dense set in Urysohn space is an universal matrix. In another words, universality of the true distance matrix r is necessary and sufficient condition for the metric on the naturals $(\mathbf{N}, r)$ to have as a completion Urysohn space.*

2.Remark also, that if we use in our construction admissible vectors whose coordinates less or equal to 1 then we obtain universal metric space of diameter 1 with the same property of universality (for Polish space with diameter ≤ 1 etc. this universal space was also mentioned in [3, 2].

Urysohn's paper begun with a very concrete example of the countable space with rational distance matrix (indeed that was an universal space over rationales - $\mathbf{Q}$). More simple iterative method of the construction of the universal space was suggested in 80-th by Katetov, the similar idea was used later by Gromov in his book [2]. Our method of construction of the universal matrix differs from all those, - it used geometry of the cone $\mathcal{R}$ and as we will see in the next section allows interpret all the construction and Urysohn space in probabilistic. It also allows to study genericity of various property ties of Polish spaces.

As a result of the theorem 1 we obtain the remarkable fact that "typical" (generic) distance matrix is universal matrix, and *typical (generic) in our unversum $\mathcal{R}$ Polish metric space is Urysohn space $\mathcal{U}$*! In the next section we will prove the analog of this fact for metric spaces with measure.

4 Metric triples and its invariant - matrix distribution.

Suppose (X, ρ, μ) Polish space with metric ρ and with borel probability measure μ. We will call *metric triple* (In [2] the author used term "mm-space"). Two triples (X_1, ρ_1, μ_1) and (X_2, ρ_2, μ_2) are *equivalent* if there exists isometry V which preserve the measure

$$\rho_2(Vx, Vy) = \rho_2(x, y), V\mu_1 = \mu_2$$

. We have mentioned before that the classification of the Polish space is non smooth problem. But the classification of metric triple has a good answer.

For triple $T = (X, \rho, \mu)$ define the infinite product and Bernoulli measure $(X^{\mathbf{N}}, \mu^{\mathbf{N}}$ and define the map $F : X^{\mathbf{N}} \to \mathcal{R}$ as follow:

$$F_T(\{x_i\}_{i=1}^{\infty}) = \{\rho(x_i, x_j)\}_{i,j=1}^{\infty} \in \mathcal{R}$$

The F_T-image of the measure $\mu^{\mathbf{N}}$ which we denoted as D_T will called *matrix distribution of the triple* T :$F_T\mu^{\infty} \equiv D_T$.

Lemma 6 *Measure D_T is borel probability measure on $\mathcal{R}$ which is invariant and ergodic with respect to group of simultaneous permutations of rows and columns and to simultaneous shift in vertical and horizontal directions (NW-shift).*

Proof. All facts follow from the same properties of measure μ^{∞} and because map F_T commutes with action of the shifts and permutations. ■

Let us call a measure on the metric space non-degenerated if there are no nonempty open set of zero measure.

Theorem 4 *Two metric triples $T_1 = (X_1, \rho_1, \mu_1)$ and (X_2, ρ_2, μ_2) with non-degenerated measures are equivalent iff its matrix distributions are coincided: $D_{T_1} = D_{T_2}$ as the measures on the cone $\mathcal{R}$.*

Proof. The necessity of the coincidence of the matrix distribution is evident: if there exists the isometry V between T_1 and T_2 which preserves measures then the infinite power V^{∞} preserve the measures: $V^{\infty}(\mu_1^{\infty}) = \mu_2^{\infty}$ and because of equality $F_{T_1}X_1^{\infty} = F_{T_2}(V^{\infty}X_1^{\infty}$ the image of these measure are the same $D_{T_2} = D_{T_1}$. Suppose now that $D_{T_2} = D_{T_1} = D$. Then D-almost all distance matrices r are the images under the maps F_{T_1} and F_{T_2} say $r_{i,j} = \rho_1(x_i, x_j) = \rho_2(y_i, y_j)$ but this means that the identification of $x_i \in X_1$ and $y_i \in X_2$ for all i is isometry V between these countable sets $V(x_i) = y_i$. Now the main argument: by ergodic (with respect to NW-shift) theorem μ_1-almost all sequences of $\{x_i\}$ and μ_2-almost all sequences $\{y_i\}$ are uniformly distributed on X_1 and X_2 correspondingly. This means that the μ_1 measure of each ball $B^l(x_i) \equiv \{z \in X_1 : \rho_1(x_i, z) < l\}$ equal to

$$\mu_1(B^l(x_i)) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{\rho_1(x_k, x_i) \geq l}$$

which is equal because of isometry V ($r_{i,j} = \rho_1(x_i, x_j) = \rho_2(y_i, y_j)$ - see above) to the μ_2 measure of the ball: $B^l(y_i) \equiv \{u \in X_2 : \rho_2(y_i, u) < l\}$

$$\mu_2(B^l(y_i)) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{\rho_2(y_k, y_i) \geq l} = \mu_1(B^l(x_i))$$

But both measures are non-degenerated, consequently each sequences $\{x_i\}$ and $\{y_i\}$ are every dense in its own space and because both measures are borel it is enough to conclude their coincidence if we established that the measures of the all such balls are the same. ■

Corollary 2 *Matrix distribution is complete invariant of the equivalence of metric triples with non-degenerated measures.*

Firstly this theorem as “Reconstruction theorem” in another formulation had been proved in the book [2] pp.117-123 by Gromov. He formulated its in the terms of finite dimensional distributions of what we called matrix distribution and proved using analytical method. He asked me in 1997 about this theorem and I suggested the proof which is written here (see also in [4]) and which he had quoted (pp.122-123) in the book. Gromov has invited the readers to compare two proofs, one of which is rather analytical (Weierstrass approximations) and another (above) in fact uses only the reference on the ergodic theorem. One of the explanations of this difference is the same as in all applications of ergodic theorem - it helps us to substitute the methods of space approximation onto operations with infinite (limit) orbits. In our case the consideration of infinite matrices and cone $\mathcal{R}$ with invariant measures gives a possibility to reduce the problem to the investigation of ergodic action of infinite groups.

5 Properties of matrix distributions and random metrics on the naturals

A matrix distribution of the metric triple $T = (X, \rho, \mu)$ is by definition a measure D_T on the cone $\mathcal{R}$. It means that we can consider it as random (semi)metric on the naturals. Namely, we choose independently the points $\{x_i\}$ of metric space X with measure μ as a distribution and and look at their distance matrix $\{r_{i,j}\} = \{\rho(x_i, x_j)\}$.

Each distance matrix define the metric on $\mathbf{N}$ and consequently the Polish space which is completion of naturals with that metric. But if we have some probability measure on $\mathcal{R}$ then we need to put some condition on this measure in order it corresponds to some metric triple, or in another words - in order almost with respect to that measure distance matrices define the same (isometric) completion of naturals.

As we mentioned all measure D_T must be invariant and ergodic with respect to action of infinite symmetric group and to NW-shift. But this is not enough and not all restriction to which it satisfies. We mention necessary and sufficient conditions (see [4]) for that.

Theorem 5 *Let D - a probability measure on the cone $\mathcal{R}$,which is invariant and ergodic with respect to action of infinite symmetric group (=group of all finite permutations of the naturals). The following condition is necessary and sufficient in order D is matrix distribution for some metric triple $T = (X, \rho, \mu)$ or $D = D_T$:*

for each $\epsilon > 0$ there exists integer N such that

$$D\{r = \{r_{i,j}\} \in \mathcal{R} : \lim_{n\to\infty} \frac{|\{j : 1 \leq \ j \leq n, \min_{1\leq i\leq N} r_{i,j} < \epsilon\}|}{n} > 1 - \epsilon\} > 1 - \epsilon$$

The following more stronger condition is equivalent to the compactness of metric space: if for each $\epsilon > 0$ there exists integer N such that

$$D\{r = \{r_{i,j}\} \in \mathcal{R} : \textit{for all} j > N, \min_{1\leq i\leq N} r_{i,j} < \epsilon\} > 1 - \epsilon,$$

then $D = D_T$ where $T = (X, \rho, \mu)$ and (X, ρ) is compact.

The second condition is simply retelling of the existence of ϵ-net for each positive ϵ in the metric compactum; the first one expresses well-known property of metric space with measure: the set of full measure is sigma-compact, or for each ϵ there exist a compact of measure $> 1 - \epsilon$. The subset in the outer brackets in the first condition is rather complicate borel set.

Now let us consider the arbitrary measures on the cone $\mathcal{R}$, or - an arbitrary random metrics on the naturals and choose some denotations.

Denotations. Denote $\mathcal{V}$ the set of all probability borel measures [1] on the cone $\mathcal{R}$ and equip it with weak topology,- this is a topology of inverse limit of the sets of probability measures on the finite dimensional cones $\mathcal{R}_n$ with its usual weak topology. The convergence in this topology is convergence on the cylindric sets with open bases. Let $\mathcal{D}$ be the subset in $\mathcal{V}$ of the measures each of which is a matrix distributions for some metric triple and equip it with induced topology from $\mathcal{V}$. This set was described above in the theorem 5 in a constructive way. The set $\mathcal{P}$ is a subset of $\mathcal{V}$ of measures each of which concentrated on the set of universal distance matrices: $\nu \in \mathcal{P}$ iff $\nu(\mathcal{M}) = 1$, and finally $\mathcal{Q} \subset \mathcal{D}$ consists with the measures which corresponds to the metric triples $T = (X, \rho, \mu)$ in which (X, ρ) is Urysohn space. We have imbeddings

$$\mathcal{P} \subset \mathcal{V} \supset \mathcal{D} \supset \mathcal{Q} \subset \mathcal{P}; \mathcal{Q} = \mathcal{P} \cap \mathcal{D}$$

Remark that the set of non-degenerated measures is of course every dense G_δ set in $\mathcal{V}$

Theorem 6 *1.The subset $\mathcal{P} \subset \mathcal{V}$ is every dense G_δ in $\mathcal{V}$. This means that for generic measures ν on $\mathcal{R}$ ν-almost all distance matrices $r \in \mathcal{R}$ for $\nu \in \mathcal{P}$ are universal and consequently ν-almost all distance matrix r defines the metric on the naturals such that completion of naturals with respect to this metric is the Urysohn space.*

2.The subset $\mathcal{Q} \subset \mathcal{D}$ is every dense G_δ-set in $\mathcal{D}$. This means that generic metric triple $T = (X, \rho, \mu)$ has Urysohn space as the space (X, ρ).

Proof. The first claim follows from the theorem 1 which claimed in particular that the set of distance matrices which are universal is a set of type G_δ in $\mathcal{R}$, and from a general fact that the set all measures on separable metrizable space such that the given every dense G_δ-set (in our case - $\mathcal{M}$) has measure 1, is in its turn every dense G_δ-set in the space of all measures in weak topology. The second claim follows from the fact that intersection of G_δ-set with any subspace in Polish space is also G_δ-set in induced topology. ∎

Now we can give a probabilistic interpretation of our construction of universal matrix and Urysohn space. Namely, we will give a family of measures from the set $\mathcal{P}$.

Example.

Let γ is a continuous measure on open half-line $\dot{\mathbf{R}}^1_+$ with full support - for example Gaussian measure in the half-line. We will define inductively the measure ν on cone of distance matrices $\mathcal{R}$ (indeed on $\mathcal{R}^0$) by construction of its finite dimensional projection on the cones $\mathcal{R}^0_n$ as follow. The distribution of the entry $r_{1,2}$ is distribution γ. So we have defined the measure on $\mathcal{R}^0_2$, denote it as ν_2. Suppose we already define the joint distribution of the entries $\{r_{i,j}\}^n_{i,j=1}$ which means that we define a measure ν_n on $\mathcal{R}^0_n$. By lemma 5 the

[1] we use later on term “measure” in the meaning - “borel probability measure”

open cone $\mathcal{R}^0_{n+1}$ is homeomorphic to direct product of $\mathcal{R}^0_n$ and open half-space $\mathbf{R}^n_+$, but we use only the structure of fibratoin: projection $\mathcal{R}^0_n \stackrel{p_{n+1}}{\longleftarrow} \mathcal{R}^0_{n+1}$ - gives us this structure with the fibers - interior of $A(r)$. So in order to define a masure on $\mathcal{R}^0_{n+1}$ with the given measure ν_n on the base $\mathcal{R}^0_n$ it is enough to define the conditional measures on the fibers. Choose each $r \in \mathcal{R}_n$ and represent the set of admissible vectors $A(r)$ as Minkowski sum:

$$A(r) = M_r + \Delta_n,$$

(see 2.2) or as projection of the direct product $\pi : M_r \times \Delta_n \to M_r + \Delta_n = A(r)$. Consider product measure on $M_r \times \Delta_n : \gamma_r = m_r \times \gamma$ where m_r is normalized Lebesgue measure on the compact polytope M_r and measure γ was define before, and its projection $\pi\gamma_r$ on $A(r)$. We define *conditional measure* on $A(r)$ as $\pi\gamma_r$.

So we can define a measure on $\mathcal{R}_{n+1}$ denote it ν_{n+1} with given projection ν_n on $\mathcal{R}^0_n$ and with the given set of conditional measures $\pi\gamma_r$ on the fibers $A(r)$. By definition $\{\nu_n\}_{n=2}^{\infty}$ is a concordant systems of cylindric measures on $\mathcal{R}$ with respect to projections. $\mathcal{R}^0_n \stackrel{p_{n+1}}{\longleftarrow} \mathcal{R}^0_{n+1}$. Consequently applying Kolmogoroff's theorem we obtain the true measure ν on $\mathcal{R}$. The proof that for ν-almost matrices $r \in \mathcal{R}$ are universal repeats the arguments of the proof of theorem 1, so $\nu \in \mathcal{P}$. This construction indeed is pure combinatorial - we choose randomly a new points of metric space and if we do this sufficiently "random" we obtain in completion the Urysohn space. That is similar to the mentioned Erdos theorem about random graphs. The detales will be done elsewhere.

We can varied the parameters of this construction, but of course it does not give in general a measure from the set $\mathcal{D}$ (matrix distribution) because it could be not $S_{\mathbf{N}}$-invariant, and consequently not from the set $\mathcal{Q}$. It means that these measures does not define a measure on Urysohn space. What we did with this construction is *a random countable every dense subset in Urysohn space.* A very interesting problem is: how to obtain non degenerated measure on Urysohn space?

The natural but indirect method is to try to construct a measure on $\mathcal{R}$ which subtract to all conditions of the theorem 5. We know a few things about geometry, topology and other structures of Urysohn space $\mathcal{U}$ and about group of its isometries in order to make a direct constructions. Here is a concrete question about measures on it: is it possible to define a measure on $\mathcal{U}$ such that distance matrix of randomly and independently chosen n points has a given (for example Gaussian in the cone $\mathcal{R}_n$) distribution? Even the case $n = 4$ is interesting. Another important problem - calculation of the matrix distributions for concrete metric triples. Even finite dimensional distributions for compact Lie groups with Haar measure or for some simple manifolds, as I know, never was found.

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