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Welschinger invariants of real del Pezzo surfaces of degree ≥ 2

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Abstract

We compute the purely real Welschinger invariants, both original and modified, for all real del Pezzo surfaces of degree ≥ 2. We show that under some compatibility conditions, for any such surface X with a non-empty real part $\mathbb{R}X$ and a real nef and big divisor class $D \in \mathrm{Pic}(X)$, through any generic collection of $-DK_X - 1$ real points lying on a connected component of $\mathbb{R}X$ one can trace a real rational curve $C \in |D|$. This is derived from the positivity of appropriate Welschinger invariants. We furthermore show that these invariants are asymptotically equivalent, in the logarithmic scale, to the corresponding genus zero Gromov-Witten invariants. Our approach consists in a conversion of the Shoval-Shustin recursive formulas counting complex curves on the plane blown up at seven points and of Vakil's extension of the Abramovich-Bertram formula for Gromov-Witten invariants of almost Fano surfaces into formulas computing real enumerative invariants.

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Keywords: real rational curves, enumerative geometry, Welschinger invariants, Caporaso-Harris formula, Abramovich-Bertram-Vakil formula.

Contents

The cashier gave us a sad smile, took a small hammer out of her mouth, and moving her nose slightly back and forth, she said:

- In my opinion, a seven comes after an eight, only if an eight comes after a seven.

(Daniil Kharms "A sonnet")

1 Introduction

Welschinger invariants can be regarded as real analogues of genus zero Gromov-Witten invariants. They were introduced in [24], [25] and count, with appropriate signs, the real rational pseudo-holomorphic curves which pass through given real collections of points in a given real rational symplectic four-fold. In the case of real del Pezzo surfaces, the Welschinger count is equivalent to enumeration of real rational algebraic curves. In the present paper, we continue the study of purely real Welschinger invariants (that is, Welschinger invariants in the situation when all the point constraints are real) of del Pezzo surfaces. These invariants, as well as their modifications introduced in [15], can be used to prove the existence of interpolating real rational curves.

As we proved in [10, 11, 13, 14, 15], if X is either the plane blown up at a real points and b pairs of complex conjugate points, where $a+2b \le 6$, $b \le 1$, or a minimal two-component real conic bundle over $\mathbb{P}^1$, or a two-component real cubic surface, then the (modified) Welschinger invariants of X are positive and are asymptotically equivalent in the logarithmic scale to the corresponding Gromov-Witten invariants. These results not only prove the existence of interpolating real rational curves, but also show their abundance.

In the present paper, we extend these results to del Pezzo surfaces of degree ≥ 2 (see Theorem 6) and, in particular, cover all the missing cases in degree ≥ 3. The main novelty is the use of nodal del Pezzo surfaces in a way which is similar to Vakil's approach to computation of Gromov-Witten invariants of the plane blown up at six points [22]. We derive new real Caporaso-Harris type formulas (see Theorems 2 and 3) and real analogues of Abramovich-Bertram-Vakil formula [1, 22] (see Theorems 4 and 5). These formulas combined together allow one to compute the purely real Welschinger invariants of all real del Pezzo surfaces of degree ≥ 2 from finitely many explicitly determined initial values (see Propositions 9 and 14).

As a technical tool, we introduce certain numbers (called ordinary w-numbers and sided w-numbers) that count with signs some specifically constrained real rational curves on real nodal del Pezzo surfaces, and exhibit a case when sided w-numbers are independent of the choice of point constraints (see Corollary 25).

A new phenomenon for del Pezzo surfaces of degree 2 is the absence of real rational curves in some cases, namely, for several divisor classes on the surfaces whose real part is homeomorphic to the sphere S^2 (see Theorem 7(iv)). In this regard, note that in the case of multicomponent del Pezzo surfaces, the original Welschinger invariants often happen to vanish. For example, E. Brugallé and N. Puignau [4, Proposition 3.3] showed that, for real del Pezzo surfaces X of degree ≥ 3 with a disconnected real point set one has $W(X, D, F, 0) = 0$ whatever is the divisor class $D \in \text{Pic}^{\mathbb{R}}(X)$ with $-K_X D \ge 3$. We extend this result in Theorem 8. However, by (47) in Theorem 6, in most cases such a vanishing is not related to the non-existence of real rational curves, but only states that the real rational curves under consideration cancel each other when supplied with the original Welschinger signs.

A few results related to that of the present paper should be mentioned here. Similar real versions of Abramovich-Bertram-Vakil formula were obtained at the same time by E. Brugallé and N. Puignau, and then extended by them to the symplectic setting and arbitrary real rational symplectic 4-manifolds, see [4]. They mainly focus on the behavior of Welschinger invariants under Morse transformations and derive some vanishing results.

J. Solomon [21, 9] suggested a completely different and very powerful recursive tool for computing Welschinger invariants of real blown ups of the projective plane. His recursion is based on analogues of Kontsevich-Manin axioms and WDVV equation, and involves the Gromov-Witten invariants and a finite number of initial values. However, the presence of plenty of terms of opposite signs (contrary to our formulas which contain only non-negative terms) makes not evident the use of these recursive formulas for getting general statements on positivity and asymptotic behavior.

In an unpublished joint work with R. Rasdeaconu, J. Solomon has considered a kind of w-numbers which count curves subject to point constraints and odd tangency conditions to a fixed divisor, and showed that some combinations of such numbers are independent of point constraints. Let us underline that our sided w-numbers are defined via even tangency conditions and, in some cases, are individually invariant with respect to point constraints.

The paper is organized as follows. Section 2 contains a reminder on (modified) Welschinger invariants. In Section 3, we define ordinary and sided w-numbers and prove Caporaso-Harris type recursive formulas for these numbers in the case of real rational surfaces Y with a given real smooth rational curve E such that the classes $-K_Y$ and $-K_Y - E$ are nef (we call (Y, E) a monic log-del Pezzo pair). In Section 4, we consider nodal degenerations of del Pezzo surfaces and derive Abramovich-Bertram-Vakil type formulas relating the ordinary and sided w-numbers of the degeneration with Welschinger invariants. The further sections are devoted to applications of the results of Sections 3 and 4: positivity and asymptotics of Welschinger invariants are studied in Section 5, their monotonicity in Section 6, and Mikhalkin type congruences in Section 7.

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2 Purely real (modified) Welschinger invariants of real del Pezzo surfaces

Let X be a real del Pezzo surface (*i.e.*, a real smooth rational surface having an ample anticanonical class $-K_X$) with a non-empty real point set $\mathbb{R}X$. Let $D \in \text{Pic}(X)$ be a real effective divisor class with $D^2 \geq -1$, assumed to be primitive in $\text{Pic}(X)$ if $D^2 = 0$.

The set $R(X, D)$ of reduced irreducible rational curves in $|D|$ is a non-empty quasi-projective variety of pure dimension $-DK_X - 1$ with nodal curves as generic elements (see, for instance, [16, Lemma 4]). Denote by $\mathbb{R}R(X, D)$ the set of real rational curves in $R(X, D)$.

We intend to count curves in $\mathbb{R}R(X, D)$ that match a suitable number of real point constraints. If $-DK_X > 1$, we pick a generic collection $\boldsymbol{w}$ of $-DK_X - 1$ points in $\mathbb{R}X$. Since a curve in $\mathbb{R}R(X, D)$ passing through $\boldsymbol{w}$ must contain all these points in its (unique) real one-dimensional component, we have to suppose that $\boldsymbol{w}$ lies in one connected component of $\mathbb{R}X$. Notice also that if $-DK_X = 1$, each curve in the (finite) set $\mathbb{R}R(X, D)$ has a one-dimensional real branch. Indeed, a real curve with a finite real part must have an even self-intersection, whereas $D^2 \equiv -DK_X \mod 2$ by the adjunction formula.

To introduce (*modified*) *purely real Welschinger numbers*, let us fix a connected component F of the real part $\mathbb{R}X$ of X and, in addition, a conjugation invariant class $\varphi \in H_2(X \setminus F, \mathbb{Z}/2)$. If $-DK_X = 1$, we set $\mathbb{R}R(X, D, F) = \{C \in \mathbb{R}R(X, D) : |C \cap F| = \infty\}$ and put

$$W(X, D, F, \varphi) = \sum_{C \in \mathbb{R}R(X,D,F)} (-1)^{s(C) + C_{1/2} \circ \varphi} ,$$

where $C_{1/2}$ is the image of one of the halves of $\mathbb{P}^1 \setminus \mathbb{R}P^1$ by the normalization map $\mathbb{P}^1 \to C$, and $s(C)$ is the number of real solitary nodes of C. If $-DK_X > 1$, we pick a generic collection $\boldsymbol{w}$ of $-DK_X - 1$ points of F, set $\mathbb{R}R(X, D, \boldsymbol{w}) = \{C \in \mathbb{R}R(X, D) : C \supset \boldsymbol{w}\}$, and put

$$W(X, D, F, \varphi, \boldsymbol{w}) = \sum_{C \in \mathbb{R}R(X,D,\boldsymbol{w})} (-1)^{s(C) + C_{1/2} \circ \varphi} . \quad (1)$$

The following statement is a version of the Welschinger theorem [24] (*cf.* also [16, Theorem 2]).

Theorem 1 *(1) If $-DK_X > 1$, the number $W(X,D,F,\varphi,\boldsymbol{w})$ does not depend on the choice of a generic collection $\boldsymbol{w}$ of $-DK_X - 1$ points in F.*

(2) With the given data X, D, F, φ as above, let X_t, $t \in [0,1]$, $X_0 = X$, be a smooth family of smooth real rational surfaces with non-empty real part such that for all but finitely many $t \in [0,1]$, X_t is a real del Pezzo surface. Let $\theta_t : X_0 \to X_t$, $t \in [0,1]$, $\theta_0 = \mathrm{Id}$, be a smooth family of conjugation invariant C^∞-diffeomorphisms that trivializes our family of surfaces. Then

$$W(X,D,F,\varphi,\boldsymbol{w}) = W(X_1, (\theta_1)_*(D), \theta_1(F), (\theta_1)_*(\varphi), \theta_1(\boldsymbol{w})) \ .$$

In the sequel we write $W(X,D,F,\varphi)$ omitting the notation of point constraints.

3 Recursive formulas for w-numbers of real monic log-del Pezzo pairs

3.1 Surfaces under consideration

Let Y be a smooth rational surface which is a blow-up of $\mathbb{P}^2$, and let $E \subset Y$ be a smooth rational curve. Suppose that $-K_Y$ is positive on all curves different from E and $K_Y E \ge 0$, and that the log-anticanonical class $-(K_Y+E)$ is nef, effective, and satisfies $(K_Y+E)^2 = 0$. We call such a pair (Y,E) a *monic log-del Pezzo pair.* Throughout Section 3, we assume that (Y,E) is a monic log-del Pezzo pair.

Observe that $-(K_Y+E)E = 2$, $E^2 \le -2$, and $K_Y(K_Y+E) = 2$, so that the latter implies, once more by adjunction, that $|-(K_Y+E)|$ is a one-dimensional linear system, whose generic element is a smooth rational curve. This linear system contains precisely two smooth curves L', L'' (quadratically) tangent to E, and $4-E^2$ reducible curves, all of type L_1+L_2, where $L_1^2 = L_2^2 = -1$, $L_1L_2 = 1$, $L_1E = L_2E = 1$. In particular, it provides a conic bundle structure on Y and shows that Y can be regarded as the plane blown up at ≥ 6 points on a smooth conic (E is the strict transform of the conic) and at one more point outside the conic. We will assume that the blown up points are in general position subject to the above allocation with respect to the conic. The curves L' and L'' are called *supporting curves.*

Introduce the sets

$$\mathcal{E}(E) = \{E' \in \mathrm{Pic}(Y) \ : \ (E')^2 = -1,\ E'K_Y = -1,\ E'E > 0\} \ .$$

$$\mathcal{E}(E)^{\perp D} = \{E' \in \mathcal{E}(E) \ : \ E'D = 0\}, \quad D \in \mathrm{Pic}(Y) \ .$$

Suppose that (Y,E) is equipped with a real structure such that $\mathbb{R}Y \supset \mathbb{R}E \ne \emptyset$. Denote by F the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$. We also choose a conjugation invariant class $\varphi \in H_2(Y \setminus F, \mathbb{Z}/2)$.

Quadruples (Y,E,F,φ) as above are called *basic quadruples.*

3.2 Divisor classes

Let Σ be a smooth real surface. We denote by $\mathrm{Pic}^{\mathbb{R}}(\Sigma)$ the subgroup of $\mathrm{Pic}(\Sigma)$ formed by real divisor classes of Σ and denote by $\mathrm{Pic}^{\mathbb{R}}_{+}(\Sigma)$ the subsemigroup of $\mathrm{Pic}^{\mathbb{R}}(\Sigma)$ generated by effective real divisor classes. Let $E \subset \Sigma$ be a smooth real curve. Put $\mathrm{Pic}_{++}(\Sigma, E)$ to be the subsemigroup of $\mathrm{Pic}(\Sigma)$ generated by complex irreducible curves C such that $CE \geq 0$. The involution of complex conjugation $\mathrm{Conj} : \Sigma \to \Sigma$ naturally acts on $\mathrm{Pic}(\Sigma)$ and preserves $\mathrm{Pic}_{++}(\Sigma, E)$. Denote by $\mathrm{Pic}^{\mathbb{R}}_{++}(\Sigma, E)$ the disjoint union of the sets

$$\{D \in \mathrm{Pic}_{++}(\Sigma, E) \ : \ \mathrm{Conj}\ D = D\}$$

and

$$\big\{\{D_1, D_2\} \in \mathrm{Sym}^2(\mathrm{Pic}_{++}(\Sigma, E)) \ : \ \mathrm{Conj}\ D_1 = D_2\big\} \ .$$

For an element $\mathcal{D} \in \mathrm{Pic}^{\mathbb{R}}_{++}(\Sigma, E)$, define $[\mathcal{D}] \in \mathrm{Pic}_{++}(\Sigma, E)$ by

$$[\mathcal{D}] = \begin{cases} D, & \mathcal{D} = D, \ \text{a divisor class,} \\ D_1 + D_2, & \mathcal{D} = \{D_1, D_2\}, \ \text{a pair of divisor classes.} \end{cases}$$

For a element $\mathcal{D} \in \mathrm{Pic}^{\mathbb{R}}_{++}(\Sigma, E)$ and a vector $\beta \in \mathbb{Z}^\infty_+$, put

$$R_\Sigma(\mathcal{D}, \beta) = -[\mathcal{D}](K_\Sigma + E) + \|\beta\| - \begin{cases} 1, & \mathcal{D} = D, \ \text{a divisor class,} \\ 2, & \mathcal{D} = \{D_1, D_2\}, \ \text{a pair of divisor classes.} \end{cases}$$

3.3 Some notations

Let $\mathbb{Z}^\infty_+$ be the direct sum of countably many additive semigroups $\mathbb{Z}_+ = \{k \in \mathbb{Z} \mid k \geq 1\}$, labeled by the positive integer numbers, with the basis formed by the summand generators e_i, $i = 1, 2, ...$ For $\alpha = (\alpha_1, \alpha_2, ...) \in \mathbb{Z}^\infty_+$, put

$$\|\alpha\| = \sum_{i=1}^\infty \alpha_i, \quad I\alpha = \sum_{i=1}^\infty i\alpha_i, \quad I^\alpha = \prod_{i=1}^\infty i^{\alpha_i}, \quad \alpha! = \prod_{i=1}^\infty \alpha_i! \ .$$

For $\alpha, \beta \in \mathbb{Z}^\infty_+$, we write $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for any positive integer number i. For $\alpha^{(0)}, ..., \alpha^{(m)}, \alpha \in \mathbb{Z}^\infty_+$ such that $\alpha^{(0)} + ... + \alpha^{(m)} \leq \alpha$, put

$$\binom{\alpha}{\alpha^{(0)}, ..., \alpha^{(m)}} = \frac{\alpha!}{\alpha^{(0)}!...\alpha^{(m)}!(\alpha - \alpha^{(0)} - ... - \alpha^{(m)})!} \ .$$

Introduce also the semigroups

$$\begin{aligned} \mathbb{Z}^{\infty,\ \text{odd}}_+ &= \mathrm{Span}\{e_{2i+1} \ : \ i \geq 0\} \ , \\ \mathbb{Z}^{\infty,\ \text{even}}_+ &= \mathrm{Span}\{e_{2i} \ : \ i \geq 1\} \ , \\ \mathbb{Z}^{\infty,\ \text{odd}\cdot\text{even}}_+ &= \mathrm{Span}\{e_{4i+2} \ : \ i \geq 0\} \ . \end{aligned}$$

3.4 Families of real curves

Let (Y, E) be a real monic log-del Pezzo pair. An *admissible tuple* $(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$ consists of an element $\mathcal{D}$ in $\text{Pic}^{\mathbb{R}}_{++}(Y, E)$, vectors $\alpha, \beta^{\rm re}, \beta^{\rm im} \in \mathbb{Z}^\infty_+$ satisfying $I(\alpha + \beta^{\rm re} + 2\beta^{\rm im}) = [\mathcal{D}]E$, and a sequence $\boldsymbol{p}^\flat = \{p_{i,j} \ : \ i \geq 1, \ 1 \leq j \leq \alpha_i\}$ of $\|\alpha\|$ distinct real generic points on E. Denote by $V^{\mathbb{R}}_Y(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$ the closure in the linear system $|[\mathcal{D}]|$ of the family of real reduced curves C such that

(i) if $\mathcal{D} = D$, a divisor class, then $C \in |D|$ is an irreducible over $\mathbb{C}$ rational curve; if $\mathcal{D} = \{D_1, D_2\}$, a pair of divisor classes, then $C = C_1 \cup C_2$, where $C_1 \in |D_1|$, $C_2 \in |D_2|$ are distinct, irreducible, rational, complex conjugate curves;

(ii) $C \cap E$ consists of $\boldsymbol{p}^\flat$ and of $\|\beta^{\rm re} + 2\beta^{\rm im}\|$ other points: $\|\beta^{\rm re}\|$ of them real, and $2\|\beta^{\rm im}\|$ form pairs of complex conjugate points;

(iii) at each point of $C \cap E$, the curve C has one local branch, and the intersection multiplicities of C and E are described as follows:

- $(C \cdot E)(p_{i,j}) = i$ for all $i \geq 1$, $1 \leq j \leq \alpha_i$,
- for each $i \geq 1$, there are $\beta^{\rm re}_i$ real points $q \in (C \cap E) \setminus \boldsymbol{p}^\flat$ such that $(C \cdot E)(q) = i$;
- for each $i \geq 1$ there are $\beta^{\rm im}_i$ pairs q, q' of complex conjugate points of $C \cap E$ such that $(C \cdot E)(q) = (C \cdot E)(q') = i$.

If $D \in \text{Pic}^{\mathbb{R}}_{++}(Y, E)$ a divisor class, introduce also the variety $V_Y(D, \alpha, \beta, \boldsymbol{p}^\flat)$ which is the closure in $|D|$ of the family of complex reduced irreducible rational curves C such that $C \cap E$ consists of $\boldsymbol{p}^\flat$ and of $\|\beta\|$ other points, at each point of $C \cap E$, the curve C has one local branch, and the intersection multiplicities of C and E are as follows:

- $(C \cdot E)(p_{i,j}) = i$ for all $i \geq 1$, $1 \leq j \leq \alpha_i$,
- for each $i \geq 1$, there are β_i points $q \in (C \cap E) \setminus \boldsymbol{p}^\flat$ such that $(C \cdot E)(q) = i$.

Lemma 1 *If $\mathcal{D} = D$ is a divisor class and $V^{\mathbb{R}}_Y(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$ is nonempty, then $R_Y(\mathcal{D}, \beta^{\rm re} + 2\beta^{\rm im}) \geq 0$, and each component of $V^{\mathbb{R}}_Y(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$ has dimension $\leq R_Y(\mathcal{D}, \beta^{\rm re} + 2\beta^{\rm im})$. Moreover, a generic element C of any component of $V^{\mathbb{R}}_Y(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$ of dimension $R_Y(\mathcal{D}, \beta^{\rm re} + 2\beta^{\rm im})$ is an immersed curve, nonsingular along E. If, in addition, $E^2 \geq -3$, then C is nodal.*

Proof. If D is a multiple of a divisor class orthogonal to $K_Y + E$, then $V^{\mathbb{R}}_Y(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$ cannot be nonempty, since such a linear system contains only non-reduced curves. In the other case, the statement follows from [18, Proposition 2.1]. $\Box$

Suppose that $R_Y(\mathcal{D}, \beta^{\rm re} + 2\beta^{\rm im}) \geq 0$. Pick a set $\boldsymbol{p}^\sharp$ of $R_Y(\mathcal{D}, \beta^{\rm re} + 2\beta^{\rm im})$ generic points of $F \setminus E$ and denote by $V^{\mathbb{R}}_Y(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$ the set of curves $C \in V^{\mathbb{R}}_Y(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$ passing through $\boldsymbol{p}^\sharp$.

Lemma 2 *Assume that $V_Y^{\mathbb{R}}(\mathcal{D},\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat)$ is nonempty.*

(1) If $\mathcal{D}=D$, a divisor class, then $V_Y^{\mathbb{R}}(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ is a finite set of real immersed irreducible rational curves which are nonsingular along E.

(2) If $\mathcal{D}=\{D_1,D_2\}$, a pair of divisor classes, then $V_Y^{\mathbb{R}}(\mathcal{D},\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ is finite, and it is nonempty only if $\alpha=\beta^{\mathrm{re}}=0$, $R_Y(\mathcal{D},2\beta^{\mathrm{im}})=0$, and $\boldsymbol{p}^\flat=\boldsymbol{p}^\sharp=\emptyset$.

Proof. By Lemma 1 we have to show only that $R_Y(\mathcal{D},2\beta^{\mathrm{im}})=0$ is necessary for the nonemptyness of $V_Y^{\mathbb{R}}(\mathcal{D},0,0,\beta^{\mathrm{im}},\emptyset,\boldsymbol{p}^\sharp)$ with $\mathcal{D}=\{D_1,D_2\}$, and the proof of this fact literally coincides with the proof of [15, Lemma 3(2)]. $\square$

Lemma 3 *The only nonempty sets $V_Y^{\mathbb{R}}(\mathcal{D},\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat)$ for admissible tuples $(\mathcal{D},\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat)$ such that $R_Y(\mathcal{D},\beta^{re}+2\beta^{\mathrm{im}})=0$ are the following ones:*

(1) if $\mathcal{D}=D$ is a divisor class and $I(\alpha+\beta^{\mathrm{re}}+2\beta^{\mathrm{im}})=DE>0$,

- *(1i) $V_Y^{\mathbb{R}}(E',0,e_1,0,\emptyset)$ consists of one element, where E' is a real (-1)-curve crossing E;*
- *(1ii) $V_Y^{\mathbb{R}}(-(K_Y+E),0,e_2,0,\emptyset)$ consists of two elements L',L'', if L',L'' are both real;*
- *(1iii) $V_Y^{\mathbb{R}}(-(K_Y+E),e_1,e_1,0,\boldsymbol{p}^\flat)$ consists of one element;*
- *(1iv) $V_Y^{\mathbb{R}}(D,\alpha,0,0,\boldsymbol{p})$ consists of one element, if $(K_Y+E)D=-1$, $I\alpha=DE$;*

(2) if $\mathcal{D}=\{D_1,D_2\}$ is a pair of divisor classes and $I(\alpha+\beta^{\mathrm{re}}+2\beta^{\mathrm{im}})=[\mathcal{D}]E>0$,

- *(2i) $V_Y^{\mathbb{R}}(\{E_1',E_2'\},0,0,e_1,\emptyset)$ consists of one element, where E_1',E_2' are complex conjugate (-1)-curves crossing E;*
- *(2ii) $V_Y^{\mathbb{R}}(\{-(K_Y+E),-(K_Y+E)\},0,0,e_2,\emptyset)$ consists of one element $\{L',L''\}$, if L',L'' are complex conjugate;*

(3) if $I(\alpha+\beta^{\mathrm{re}}+2\beta^{\mathrm{im}})=[\mathcal{D}]E=0$,

- *(3i) $V_Y^{\mathbb{R}}(E',0,0,0,\emptyset)$ consists of one element, where E' of a real (-1)-curve disjoint from E;*
- *(3ii) $V_Y^{\mathbb{R}}(\{E_1',E_2'\},0,0,0,\emptyset)$ consists of one element, where E_1',E_2' are complex conjugate (-1)-curves disjoint from E.*

Proof. Straightforward. $\square$

3.5 Deformation diagrams and CH position

3.5.1 Deformation diagrams

Let (Y, E) be a monic log-del Pezzo pair such that Y and E are real and $\mathbb{R}E \neq \varnothing$. Denote by F the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$ and pick a conjugation-invariant class $\varphi \in H_2(\mathbb{R}Y \setminus F, \mathbb{Z}/2)$. Let $(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ be an admissible tuple, where $D \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$ is a divisor class and $R_Y(D, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}) > 0$. Pick a set $\widetilde{\boldsymbol{p}}^\sharp$ of $R_Y(D, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}) - 1$ generic real points of $F \setminus E$, a generic real point $p \in E \setminus \boldsymbol{p}^\flat$, and a smooth real algebraic curve germ Λ, crossing E transversally at p. Denote by $\Lambda^+ = \{p(t) \ : \ t \in (0, \varepsilon)\}$ a parameterized connected component of $\Lambda \setminus \{p\}$ with $\lim_{t \to 0} p(t) = p$. There exists $\varepsilon_0 > 0$ such that, for all $t \in (0, \varepsilon_0]$, the sets $V_Y(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \widetilde{\boldsymbol{p}}^\sharp \cup \{p(t)\})$ are finite, their elements remain immersed, nonsingular along E as t runs over the interval $(0, \varepsilon_0]$, and the closure in $V_Y(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ of the family

$$V = \bigcup_{t \in (0, \varepsilon_0]} V_Y(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \widetilde{\boldsymbol{p}}^\sharp \cup \{p(t)\}) \tag{2}$$

is a union of real algebraic arcs, disjoint for $t > 0$. This closure is called a *deformation diagram* of $(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \widetilde{\boldsymbol{p}}^\sharp, p)$, *cf.* [15, Section 3.3], and the real algebraic arcs under consideration are called *branches* of the deformation diagram. The elements of $V_Y(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \widetilde{\boldsymbol{p}}^\sharp \cup \{p(1)\})$ are called *leaves* of the deformation diagram, and the elements of $\overline{V} \setminus V$ are called *roots* of the deformation diagram.

Lemma 4 *Each connected component of a deformation diagram of* $(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \widetilde{\boldsymbol{p}}^\sharp, p)$ *contains exactly one root. Each root is either a generic member of an* $(R_Y(D, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}) - 1)$*-dimensional component of one of the families*

$$V_Y(D, \alpha + \theta_j, \beta^{\mathrm{re}} - \theta_j, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat \cup \{p\}, \widetilde{\boldsymbol{p}}^\sharp\}) \ ,$$

where j *is a natural number such that* $\beta^{\mathrm{re}}_j > 0$*, or a reducible curve having* E *as a component.*

Proof. The statement follows from [18, Proposition 2.6]. □

For any root ρ of a deformation diagram, the leaves belonging to the connected component of ρ is said to be *generated* by ρ.

3.5.2 CH position

Pick a divisor class $D_0 \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$ and put $N = \dim |D_0|$. Note that the set

$$\mathrm{Prec}(D_0) = \{D \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E) \ : \ D \text{ is a divisor class and } D_0 \geq D\}$$

is finite, and we have $\dim |D| \leq N$ for each $D \in \mathrm{Prec}(D_0)$. Furthermore, for each nonempty variety $V^{\mathbb{R}}_Y(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ with $D \in \mathrm{Prec}(D_0)$, we have

$$\|\alpha\| + R_Y(D, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}) \leq N \ .$$

Lemma 5 *(cf. [15, Lemma 10]) Let $D_0 \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$ be a divisor class with $N = \dim|D_0| > 0$. Then, there exists a sequence $\Lambda(D_0) = (\Lambda_i)_{i=1,...,N}$ of N disjoint smooth real algebraic arcs in Y, which are parameterized by $t \in [-1,1] \mapsto p_i(t) \in \Lambda_i$, such that $p_i(0) \in E$, the arcs Λ_i are transverse to E at $p_i(0)$, $i = 1,...,N$, and the following condition holds:*

for any admissible tuple $(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$, any disjoint subsets $J^\flat, J^\sharp \subset \{1,...,N\}$, any positive integer $k \le N$, and any sequence $\sigma = (\sigma_i)_{i=1,...,N}$ such that

(i) $D \in \mathrm{Prec}(D_0)$,

(ii) $R_Y(D, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}) > 0$,

(iii) $i < k < j$ *for all* $i \in J^\flat$, $j \in J^\sharp$,

(iv) the number of elements in $J^\sharp$ is equal to $R_Y(D, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}) - 1$,

(v) $\boldsymbol{p}^\flat = \{p_i(0) \ : \ i \in J^\flat\}$,

(vi) $\sigma_i = \pm 1$ *for any integer* $1 \le i \le N$,

the closure of the family

$$\bigcup_{t \in (0,1]} V_Y(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \widetilde{\boldsymbol{p}}^\sharp \cup \{p_k(\sigma_k t)\}) \ ,$$

where $\widetilde{\boldsymbol{p}}^\sharp = \{p_j(\sigma_j)\}_{j \in J^\sharp}$, is a deformation diagram of $(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \widetilde{\boldsymbol{p}}^\sharp, p_k(0))$.

Proof. Take a sequence $\widehat{\Lambda}_i$, $i = 1,...,N$, of disjoint smooth real algebraic arcs in Y, which are parameterized by $t \in [-1,1] \mapsto p_i(t) \in \Lambda_i$, such that $(p_i(0))_{i=1,...,N}$ is a generic sequence of points in E, and the arcs $\widehat{\Lambda}_i$ are transverse to E at $p_i(0)$, $i = 1,...,N$. We will inductively shorten these arcs in order to satisfy the condition required in Lemma.

Take an integer $1 \le k \le N$, and suppose that we have already constructed arcs $\Lambda_1,...,\Lambda_{k-1}$ parameterized respectively by intervals $[0, \varepsilon_i]$, $1 \le i < k$. There are finitely many admissible tuples $(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$, subsets $J^\flat, J^\sharp \subset \{1,...,N\}$, and sequences σ satisfying the restrictions (i)-(vi) above. Given such a datum $D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, J^\flat, J^\sharp, \sigma$, we take a small positive number ε_k such that the closure of the family

$$\bigcup_{t \in (0,\varepsilon_k]} V_Y(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \widetilde{\boldsymbol{p}}^\sharp \cup \{p_k(\sigma_k t)\}) \ ,$$

where $\widetilde{\boldsymbol{p}}^\sharp = \{p_i(\varepsilon_i)\}_{1 \le i < k}$, is a deformation diagram of $(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \widetilde{\boldsymbol{p}}^\sharp, p_k(0))$, and put

$$\Lambda_k(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, J^\flat, J^\sharp, \sigma) = \bigcup_{t \in [-\varepsilon_k, \varepsilon_k]} p_k(t).$$

Then, we define

$$\Lambda_k = \bigcap_{(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,J^\flat,J^\sharp,\sigma)} \Lambda_k(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,J^\flat,J^\sharp,\sigma).$$

It remains now to reparameterize by the interval $[-1,1]$ the arcs $\Lambda_1, \ldots, \Lambda_N$ obtained. $\square$

Take a divisor class $D_0 \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$ such that $N = \dim|D_0| > 0$ and a sequence of arcs $(\Lambda_i)_{i=1,\ldots,N}$ as in Lemma 5. Given a sequence $\sigma = (\sigma_i)_{i=1,\ldots,N}$ of ± 1 and two subsets $J^\flat, J^\sharp \subset \{1,\ldots,N\}$ such that $i < j$ for all $i \in J^\flat$, $j \in J^\sharp$, we say that the pair of point sequences

$$\boldsymbol{p}^\flat = \{p_i(0)\ :\ i \in J^\flat\}, \quad \boldsymbol{p}^\sharp = \{p_j(\sigma_j)\ :\ j \in J^\sharp\}$$

is *in D_0-CH position.* A pair of point sequences

$$(\boldsymbol{p}^\flat)' = \{p_i(0)\ :\ i \in (J^\flat)'\}, \quad (\boldsymbol{p}^\sharp)' = \{p_j(\sigma_j)\ :\ j \in (J^\sharp)'\}$$

in D_0-CH position is said to be a *predecessor* of a pair of point sequences

$$\boldsymbol{p}^\flat = \{p_i(0)\ :\ i \in J^\flat\}, \quad \boldsymbol{p}^\sharp = \{p_j(\sigma_j)\ :\ j \in J^\sharp\}$$

in D_0-CH position if $(J^\sharp)' = \{j \in J^\sharp : j > k\}$ for a certain integer k.

Let $(\mathcal{D},\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat)$ be an admissible tuple such that $\mathcal{D} = D$ is a divisor class. Choose a sequence $\boldsymbol{p}^\sharp$ of $R_Y(\mathcal{D},\beta^{\rm re}+2\beta^{\rm im})$ points in F, and assume that the pair of point sequences $\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp$ is in a D_0-CH position. Then, $(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ is called a *D_0-proper tuple.* The elements of $V^{\mathbb{R}}_Y(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ are called *interpolating curves* constrained by the D_0-proper tuple $(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$. We say that a D_0-proper tuple $(D',\alpha',(\beta^{\rm re})',(\beta^{\rm im})',(\boldsymbol{p}^\flat)',(\boldsymbol{p}^\sharp)')$ *precedes* a D_0-proper tuple $(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ if $R_Y(D',(\beta^{\rm re})'+2(\beta^{\rm im})') < R_Y(D,\beta^{\rm re}+2\beta^{\rm im})$ and the pair $(\boldsymbol{p}^\flat)',(\boldsymbol{p}^\sharp)'$ is a predecessor of $\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp$.

Lemma 6 *Let $D_0 \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$ be a divisor class, and let $(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ be a D_0-proper tuple such that $R_Y(D,\beta^{\rm re}+2\beta^{\rm im}) > 0$ and $\beta^{\rm im} \neq 0$. Then, $V^{\mathbb{R}}_Y(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp) = \emptyset$.*

Proof. Assume that $V^{\mathbb{R}}_Y(D,\alpha,\beta^{\rm re},\beta^{\rm im}\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp) \neq \emptyset$, and put $k = \min J^\sharp$, where $\boldsymbol{p}^\sharp = \{p_j(\sigma_j)\ :\ j \in J^\sharp\}$ (see Lemma 5). We obtain inductively a contradiction showing that $V^{\mathbb{R}}_Y(D',\alpha',(\beta^{\rm re})',(\beta^{\rm im})'(\boldsymbol{p}^\flat)',(\boldsymbol{p}^\sharp)') \neq \emptyset$ for a certain D_0-proper tuple $(D',\alpha',(\beta^{\rm re})',(\beta^{\rm im})',(\boldsymbol{p}^\flat)',(\boldsymbol{p}^\sharp)')$ that precedes $(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ and such that $R_Y(D',(\beta^{\rm re})'+2(\beta^{\rm im})') > 0$ and $(\beta^{\rm im})' \neq 0$.

Consider the degeneration of $C \in V^{\mathbb{R}}_Y(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ when $p_k \in \boldsymbol{p}^\sharp$ tends to E along the arc Λ_k. By [18, Proposition 2.6], the degenerate curve is either an irreducible interpolating curve constrained by a D_0-proper tuple that precedes $(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$, or of the form $E \cup C'$. In the latter case, the curve C' has a real component belonging to $V^{\mathbb{R}}_Y(D',\alpha',(\beta^{\rm re})',(\beta^{\rm im})',(\boldsymbol{p}^\flat)',(\boldsymbol{p}^\sharp)')$, where $(D',\alpha',(\beta^{\rm re})',(\beta^{\rm im})',(\boldsymbol{p}^\flat)',(\boldsymbol{p}^\sharp)')$ is a D_0-proper tuple which precedes $(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ and $R_Y(D',(\beta^{\rm re})'+2(\beta^{\rm im})')>0$, $(\beta^{\rm im})'\ne 0$. This statement follows from [18, Lemma 2.9] and the fact that any imaginary component of C' avoids $\boldsymbol{p}^\sharp$, and thus has a unique intersection point with E (see Lemma 3). The former lemma states that the intersection points of C with $E\setminus\boldsymbol{p}^\flat$ all come from the intersection points of C' with $E\setminus\boldsymbol{p}^\flat$, and that, in the deformation of $E\cup C'$ into C, each component of C' glues up with E via smoothing out one of its intersection points with $E\setminus\boldsymbol{p}^\flat$. $\Box$

3.6 Ordinary w-numbers

Let C be a real curve on a real smooth surface Σ, and let z be a real singular point of C such that all local branches of C at z are smooth. Denote by $s(C,z)$ the number of pairs of imaginary complex conjugate local branches of C at z, each pair being counted with the weight equal to the intersection number of the branches.

Lemma 7 *Let $C(t)$, $-\varepsilon<t<\varepsilon$, be a continuous family of real curves in Σ, and let z_0 be a real singular point of $C(0)$ such that all local branches of $C(0)$ at z_0 are smooth. Assume that for a certain neighborhood $U(z_0)\subset\Sigma$ of z_0 and a sufficiently small number $\varepsilon'>0$, the curves C_t, $-\varepsilon'<t<\varepsilon'$, are transversal to the boundary of $U(z_0)$, and the curves $C(t)\cap U(z_0)$, $-\varepsilon'<t<\varepsilon'$, admit simultaneous parametrizations by a continuous family of immesrions $\Delta_i(t)\to U(z_0)$, $i=1,\ldots,b(z_0)$, where $b(z_0)$ is the number of local branches of $C(0)$ at $z(0)$, and $\Delta_i(t)$, $-\varepsilon'<t<\varepsilon'$, is a continuous family of discs in $\mathbb{C}$. Then, $\sum_{z\in{\rm Sing}(C(t))\cap U(z_0)}s(C(t),z)$ does not depend on t.*

Proof. Straightforward. $\Box$

For an immersed real curve $C\subset\Sigma$, put $s(C)=\sum_{z\in{\rm Sing}(C)}s(C,z)$.

Let $(\mathcal{D},\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat)$ be an admissible tuple such that $\mathcal{D}=D\in{\rm Pic}^{\mathbb{R}}_{++}(Y,E)$ is a divisor class, and let $\boldsymbol{p}^\sharp$ be a generic set of $R_Y(\mathcal{D},\beta^{\rm re}+2\beta^{\rm im})$ points in $F\setminus E$. The set $V^{\mathbb{R}}_Y(\mathcal{D},\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ is finite and consists of immersed curves (see Lemma 2). We put

$$W_{Y,E,\varphi}(\mathcal{D},\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)=\sum_{C\in V^{\mathbb{R}}_Y(\mathcal{D},\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)}\mu_\varphi(C)\ , \qquad (3)$$

where

$$\mu_\varphi(C)=(-1)^{s(C)+C_{1/2}\circ\varphi}\ . \qquad (4)$$

and $C_{1/2}$ is the image of one of the halves of $\mathbb{P}^1\setminus\mathbb{R}P^1$ by the normalization map $\mathbb{P}^1\to C$ if C is an irreducible real curve, and one of the irreducible components of C if C is a pair of complex conjugate irreducible curves. The number $\mu_\varphi(C)$ is called (*modified*) *Welschinger sign.*

The proof of the following proposition literally coincides with the proof of [15, Proposition 11].

Proposition 8 *Let (Y, E, F, φ) be a basic quadruple. Fix a tuple $(D, \alpha, \beta^{\rm re}, \beta^{\rm im})$, where $D \in {\rm Pic}^{\mathbb R}_{++}(Y, E)$ is a divisor class, $\alpha, \beta^{\rm re} \in \mathbb{Z}_+^{\infty,\,{\rm odd}}$, and $\beta^{\rm im} \in \mathbb{Z}_+^{\infty}$ are such that $R_Y(D, \beta^{\rm re} + 2\beta^{\rm im}) > 0$. Choose two point sequences $\boldsymbol{p}^\flat$ and $\boldsymbol{p}^\sharp$ satisfying the following restrictions:*

(r1) the tuple $(D, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$ is admissible,

(r2) the number of points in $\boldsymbol{p}^\sharp$ is equal to $R_Y(D, \beta^{\rm re} + 2\beta^{\rm im})$,

(r3) the pair $(\boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$ is in D_0-CH position for some divisor class $D_0 \in {\rm Pic}^{\mathbb R}_{++}(Y, E)$, $D_0 \geq D$.

Then, the number $W_{Y,E,\varphi}(D, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$ does not depend on the choice of sequences $\boldsymbol{p}^\flat$ and $\boldsymbol{p}^\sharp$ subject to (r1)-(r3). $\square$

Proposition 9 *The only non-zero numbers $W_{Y,E,\varphi}(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \emptyset)$ for admissible tuples $(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$ such that $\mathcal{D} \in {\rm Pic}^{\mathbb R}_{++}(Y, E)$, $\alpha, \beta^{\rm re} \in \mathbb{Z}^{\infty,\,{\rm odd}}$, $\beta^{\rm im} \in \mathbb{Z}_+^{\infty}$, and*

$$I(\alpha + \beta^{\rm re} + 2\beta^{\rm im}) = [\mathcal{D}]E > 0, \quad R_Y(\mathcal{D}, \beta^{re} + 2\beta^{\rm im}) = 0 \ ,$$

are the following ones:

(1) if $\mathcal{D} = D$ is a divisor class,

(1i) $W_{Y,E,\varphi}(E', 0, e_1, 0, \emptyset, \emptyset) = (-1)^{E'_{1/2}\circ\varphi}$, where $E' \in \mathcal{E}(E)$ is real;

(1ii) $W_{Y,E,\varphi}(-(K_Y + E), e_1, e_1, 0, \boldsymbol{p}^\flat, \emptyset) = (-1)^{L_{1/2}\circ\varphi}$, where $L \in |-(K_Y + E)|$ is real, $\mathbb{R}L \subset F$;

(1iii) $W_{Y,E,\varphi}(D, \alpha, 0, 0, \boldsymbol{p}^\flat, \emptyset) = (-1)^{C_{1/2}\circ\varphi}$, where $-(K_Y+E)D = 1$, $I\alpha = DE$, $C \in V^{\mathbb R}_Y(D, \alpha, 0, 0, \boldsymbol{p}^\flat, \emptyset)$;

(2) if $\mathcal{D}$ is a pair of divisor classes,

(2i) if $E'_1, E'_2 \in \mathcal{E}(E)$ are complex conjugate, $E'_1E'_2 = 1$, then

$$W_{Y,E,\varphi}(\{E'_1, E'_2\}, 0, 0, e_1, \emptyset, \emptyset) = -(-1)^{E'_1\circ\varphi} \ ,$$

(2ii) if $E'_1, E'_2 \in \mathcal{E}(E)$ are disjoint complex conjugate, then

$$W_{Y,E,\varphi}(\{E'_1, E'_2\}, 0, 0, e_1, \emptyset, \emptyset) = (-1)^{E'_1\circ\varphi} \ ,$$

(2iii) $W_{Y,E,\varphi}(\{-(K_Y + E), -(K_Y + E)\}, 0, 0, e_2, \emptyset, \emptyset) = 1$, if L', L'' are complex conjugate.

Proof. Proposition 9 can easily be derived from Lemma 3. Notice only that, in case (2iii), $L'\circ\varphi\equiv 0 \mod 2$ since the linear system $|-(K_Y+E)|$, which contains L', contains also a real rational curve whose complex locus is divided into two halves by its real locus located in F. $\Box$

The numbers $W_{Y,E,\varphi}(\mathcal{D},\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\emptyset)$ in Proposition 9 do not depend on the choice of $\boldsymbol{p}^\flat$.

We skip $\boldsymbol{p}^\flat$ and $\boldsymbol{p}^\sharp$ in the notation of the numbers appearing in Propositions 8 and 9, and write $W_{Y,E,\varphi}(\mathcal{D},\alpha,\beta^{\rm re},\beta^{\rm im})$ for these numbers calling them *ordinary w-numbers.*

3.7 Formula for ordinary w-numbers

Theorem 2 *Let (Y,E,F,φ) be a basic quadruple.*

(1) For any divisor class $D\in\mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$ and vectors $\alpha,\beta^{\rm re}\in\mathbb{Z}_+^{\infty,\,\rm odd}$, $\beta^{\rm im}\in\mathbb{Z}_+^\infty$ such that $I(\alpha+\beta^{\rm re}+2\beta^{\rm im})=DE$, $R_Y(D,\beta^{\rm re}+2\beta^{\rm im})\ge 0$, and $\beta^{\rm im}\ne 0$, one has

$$W_{Y,E,\varphi}(D,\alpha,\beta^{\rm re},\beta^{\rm im})=0\ . \qquad (5)$$

(2) For any divisor class $D\in\mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$ and vectors $\alpha,\beta\in\mathbb{Z}_+^{\infty,\,\rm odd}$ such that $I(\alpha+\beta)=DE$ and $R_Y(D,\beta)>0$, one has

$$W_{Y,E,\varphi}(D,\alpha,\beta,0)=\sum_{j\ge1,\ \beta_j>0}W_{Y,E,\varphi}(D,\alpha+e_j,\beta-e_j,0)$$

$$+(-1)^{E_{1/2}\circ\varphi}\sum(-1)^{(I\beta^{(0)}+I\alpha^{(0)})(L_{1/2}\circ\varphi)}\cdot\frac{2^{\|\beta^{(0)}\|}}{\beta^{(0)}!}(l+1)\binom{\alpha}{\alpha^{(0)}\alpha^{(1)}...\alpha^{(m)}}\frac{(n-1)!}{n_1!...n_m!}$$

$$\times\prod_{i=1}^m\left(\binom{(\beta^{\rm re})^{(i)}}{\gamma^{(i)}}W_{Y,E,\varphi}(\mathcal{D}^{(i)},\alpha^{(i)},(\beta^{\rm re})^{(i)},(\beta^{\rm im})^{(i)})\right)\ , \qquad (6)$$

where L is any real curve in $|-(K_Y+E)|$ with $\mathbb{R}L\subset F$,

$$n=R_Y(D,\beta),\quad n_i=R_Y(\mathcal{D}^{(i)},(\beta^{\rm re})^{(i)}+2(\beta^{\rm im})^{(i)}),\ i=1,...,m\ ,$$

and the second sum in (6) is taken

- *over all integers $l\ge0$ and vectors $\alpha^{(0)}\le\alpha$, $\beta^{(0)}\le\beta^{\rm re}$;*
- *over all sequences*

$$(\mathcal{D}^{(i)},\alpha^{(i)},(\beta^{\rm re})^{(i)},(\beta^{\rm im})^{(i)}),\ 1\le i\le m\ , \qquad (7)$$

such that, for all $i=1,...,m$,

(1a) $\mathcal{D}^{(i)}\in\mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$, and $\mathcal{D}^{(i)}$ is neither the divisor class $-(K_Y+E)$, nor the pair $\{-(K_Y+E),-(K_Y+E)\}$,

(1b) $I(\alpha^{(i)}+(\beta^{\rm re})^{(i)}+2(\beta^{\rm im})^{(i)})=[\mathcal{D}^{(i)}]E$, *and* $R_Y(\mathcal{D}^{(i)},(\beta^{\rm re})^{(i)}+2(\beta^{\rm im})^{(i)})\ge 0$,

(1c) $\mathcal{D}^{(i)}$ *is a pair of divisor classes if and only if* $(\beta^{\rm im})^{(i)}\ne 0$,

(1d) if $\mathcal{D}^{(i)}$ *is a pair of divisor classes, then* $n_i=0$ *and* $\alpha^{(i)}=(\beta^{\rm re})^{(i)}=0$,

and

(1e) $D-E=\sum_{i=1}^m[\mathcal{D}^{(i)}]-(2l+I\alpha^{(0)}+I\beta^{(0)})(K_Y+E)$,

(1f) $\sum_{i=0}^m\alpha^{(i)}\le\alpha$, $\sum_{i=0}^m(\beta^{\rm re})^{(i)}\ge\beta$,

(1g) each tuple $(\mathcal{D}^{(i)},0,(\beta^{\rm re})^{(i)},(\beta^{\rm im})^{(i)})$ *with* $n_i=0$ *appears in (7) at most once,*

- *over all sequences*

$$\gamma^{(i)}\in\mathbb{Z}_+^{\infty,\,\rm odd},\quad \|\gamma^{(i)}\|=\begin{cases}1, & \mathcal{D}^{(i)}\ \text{is a divisor class},\\ 0, & \mathcal{D}^{(i)}\ \text{is a pair of divisor classes},\end{cases}\qquad i=1,...,m\ , \tag{8}$$

satisfying

(2a) $(\beta^{\rm re})^{(i)}\ge\gamma^{(i)}$, $i=1,...,m$, *and* $\sum_{i=1}^m\left((\beta^{\rm re})^{(i)}-\gamma^{(i)}\right)=\beta^{\rm re}-\beta^{(0)}$,

and the second sum in (6) is factorized by simultaneous permutations in the sequences (7) and (8).

(3) All ordinary w-numbers $W_{Y,E,\varphi}(D,\alpha,\beta,0)$, *where* $D\in{\rm Pic}^{\mathbb{R}}_{++}(Y,E)$ *is a divisor class and* $R_Y(D,\beta)>0$, *are recursively determined by the formula (6) and the initial conditions given by Proposition 9.*

Remark 10 *It is easy to verify that* $n-1=\sum_i n_i+\|\beta^{(0)}\|$ *(in the notation of Theorem 2).*

The proof of Theorem 2 literally coincides with the proof of [15, Theorem 1 and Corollary 14].

We present here an immediate consequence that will be used below.

Corollary 11 *Under the hypotheses of Theorem 2(2), assume in addition that* $F\setminus\mathbb{R}E$ *is disconnected,* $DE=0$, *and* $R_Y(D,0)\ge 2$. *Then*

$$W_{Y,E,\varphi}(D,0,0,0)=0\ .$$

Proof. This follows from Proposition 8: indeed, we may choose two of the points of $\boldsymbol{p}^\sharp$ in different components of $F\setminus\mathbb{R}E$ making the set $V^{\mathbb{R}}_Y(\mathcal{D},0,0,0,\emptyset,\boldsymbol{p}^\sharp)$ empty, since a real rational curve cannot have two one-dimensional real components. $\square$

3.8 Sided w-numbers

Let (Y, E, F, φ) be a basic quadruple. Suppose in addition that $F \setminus \mathbb{R}E$ splits into two connected components F_+ and F_-. In this case, (Y, E, F, φ) is called *dividing basic quadruple.*

Let $(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$ be an admissible tuple. Choose a sequence $\boldsymbol{p}^\sharp$ of $R_Y(\mathcal{D}, \beta^{\rm re} + 2\beta^{\rm im})$ points in F_+. Suppose that the pair of point sequences $\boldsymbol{p}^\flat, \boldsymbol{p}^\sharp$ is in a D_0-CH position with respect to some divisor class $D_0 \in {\rm Pic}_{++}(Y, E)$, $D_0 \geq [\mathcal{D}]$. Put

$$V^{\mathbb{R}}_{Y,F_+}(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp) = \{C \in V^{\mathbb{R}}_Y(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp) \ : \ {\rm card}(C \cap F_-) < \infty\} \ .$$

Clearly, if $\mathcal{D}$ is a pair of divisor classes, then

$$V^{\mathbb{R}}_{Y,F_+}(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp) = V^{\mathbb{R}}_Y(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp) \ .$$

Set

$$W^{\varepsilon}_{Y,F_+,\varphi}(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp) = \sum_{C \in V^{\mathbb{R}}_{Y,F_+}(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)} \mu^{\varepsilon}_{\varphi}(C), \quad \varepsilon = \pm \ , \tag{9}$$

where $\mu^+_\varphi(C) = \mu_\varphi(C)$ is defined by (4) and

$$\mu^-_\varphi(C) = (-1)^{s(C) + C_{1/2} \circ \varphi + {\rm card}(C_{1/2} \cap F_-)}. \tag{10}$$

Remark 12 *By Lemma 2(2), if $\mathcal{D} \in {\rm Pic}^{\mathbb{R}}_{++}(Y, E)$ is a pair of divisor classes, then*

$$W^+_{Y,F_+,\varphi}(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp) = 0$$

as long as $\alpha + \beta^{\rm re} > 0$ or $R_Y(\mathcal{D}, \beta^{\rm re} + 2\beta^{\rm im}) > 0$.

Proposition 13 *Let (Y, E, F, φ) be a dividing basic quadruple. Fix an admissible tuple $(D, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$, where $D \in {\rm Pic}^{\mathbb{R}}_{++}(Y, E)$ is a divisor class, $\alpha, \beta^{\rm re} \in \mathbb{Z}^{\infty,\,\rm even}_+$, $\beta^{\rm im} \in \mathbb{Z}^\infty_+$, and $R_Y(D, \beta^{\rm re} + 2\beta^{\rm im}) > 0$. Choose two point sequences $\boldsymbol{p}^\flat \subset \mathbb{R}E$ and $\boldsymbol{p}^\sharp \subset F_+$ satisfying the restrictions (r1)-(r3) of Proposition 8. Then, the numbers $W^{\pm}_{Y,F_+,\varphi}(D, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$ do not depend on the choice of sequences $\boldsymbol{p}^\flat$ and $\boldsymbol{p}^\sharp$ subject to (r1)-(r3).*

The proof of Proposition 13 is given in section 3.10.

Proposition 14 *Let (Y, E, F, φ) be a dividing basic quadruple, and let $(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$ be an admissible tuple such that $\alpha, \beta^{\rm re} \in \mathbb{Z}^{\infty,\,\rm even}_+$ and*

$$I(\alpha + \beta^{\rm re} + 2\beta^{\rm im}) = [\mathcal{D}]E > 0, \quad R_Y(\mathcal{D}, \beta^{re} + 2\beta^{\rm im}) = 0 \ .$$

(1) Assume that $W^{\pm}_{Y,F_+,\varphi}(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \emptyset) \neq 0$ and $\mathcal{D} = D$ is a divisor class. Then,

(1i) either $D = -(K_Y + E)$, $\alpha = 0$, $\beta^{\rm re} = e_2$, $\beta^{\rm im} = 0$, $\boldsymbol{p}^\flat = \emptyset$, and the supporting curves L', L'' are both real,

(1ii) or $-(K_Y + E)D = 1$, $I\alpha = DE$, and D is represented by a curve $C \in V^{\mathbb{R}}_{Y,F_+}(\mathcal{D}, \alpha, 0, 0, \boldsymbol{p}^\flat, \emptyset)$ with $\mathbb{R}C \subset \overline{F}_+$.

In the first case, $W^{\pm}_{Y,F_+,\varphi}(-(K_Y + E), 0, e_2, 0, \emptyset, \emptyset) = \lambda(-1)^{L'_{1/2}\circ\varphi}$, where λ is the number of supporting curves L', L'' whose real part is contained in $\overline{F}_+$. In the second case, $W^{\pm}_{Y,F_+,\varphi}(D, \alpha, 0, 0, \boldsymbol{p}^\flat, \emptyset) = (-1)^{C_{1/2}\circ\varphi}$.

(2) Assume that $W^{\pm}_{Y,F_+,\varphi}(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \emptyset) \neq 0$ and $\mathcal{D}$ is a pair of divisor classes. Then,

(2i) either $\mathcal{D} = \{E'_1, E'_2\}$, where $E'_1, E'_2 \in \mathcal{E}(E)$ are complex conjugate,

(2ii) or $\mathcal{D} = \{L', L''\}$ and the supporting curves L', L'' are complex conjugate.

In the first case,

$$\begin{aligned} W^{+}_{Y,F_+,\varphi}(\{E'_1, E'_2\}, 0, 0, e_1, \emptyset, \emptyset) &= (-1)^{E'_1\circ\varphi + E'_1\circ E'_2}, \\ W^{-}_{Y,F_+,\varphi}(\{E'_1, E'_2\}, 0, 0, e_1, \emptyset, \emptyset) &= (-1)^{E'_1\circ\varphi + \mathrm{card}(E'_1\cap F_-) + E'_1\circ E'_2} \ . \end{aligned}$$

In the second case,

$$W^{\pm}_{Y,F_+,\varphi}(\{-(K_Y + E), -(K_Y + E)\}, 0, 0, e_2, \emptyset, \emptyset) = 1 \ .$$

Proof. The statement can be easily derived from Lemma 3, taking into account that $L' \circ \varphi \equiv 0 \mod 2$ in (2ii) (*cf.*, the proof of Proposition 9). □

We skip $\boldsymbol{p}^\flat$ and $\boldsymbol{p}^\sharp$ in the notation of the numbers appearing in Propositions 13 and 14, and write $W^{\pm}_{Y,F_+,\varphi}(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im})$ for these numbers calling them *sided w-numbers.*

3.9 Sided w-numbers in deformation diagrams

Let (Y, E, F, φ) be a dividing basic quadruple, and let $(\mathcal{D}, \alpha, \beta, 0, \boldsymbol{p}^\flat)$ be an admissible tuple such that $\mathcal{D} = D$ is a divisor class and $R_Y(\mathcal{D}, \beta) > 0$. Choose a sequence $\boldsymbol{p}^\sharp$ of $R_Y(\mathcal{D}, \beta)$ points in F_+, and assume that the pair of point sequences $\boldsymbol{p}^\flat, \boldsymbol{p}^\sharp$ is in a D_0-CH position with respect to some divisor class $D_0 \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$.

Put $k = \min J^\sharp$, where $\boldsymbol{p}^\sharp = \{p_j(\sigma_j) \ : \ j \in J^\sharp\}$ (see Lemma 5), and denote by $\overline{V}$ a deformation diagram of $(D, \alpha, \beta, 0, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp \setminus \{p_k\}, p_k)$.

Lemma 15 *Let $\beta_j > 0$, and $C \in V^{\mathbb{R}}_{Y,F_+}(D, \alpha + e_j, \beta - e_j, 0, \boldsymbol{p}^\flat \cup \{p_k(0)\}, \boldsymbol{p}^\sharp \setminus \{p_k\})$ intersects E at $p_k(0)$ with multiplicity j. Then, the real leaves of $\overline{V}$ that are generated by the root C consist of two curves $C_1, C_2 \in V^{\mathbb{R}}_{Y,F_+}(D, \alpha, \beta, 0, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$, and $\mu^{\pm}_\varphi(C_i) = \mu^{\pm}_\varphi(C)$, $i = 1, 2$.*

Proof. Choose local coordinates x, y in a neighborhood of $p_k(0)$ so that $E = \{y = 0\}$, $F_+ = \{y > 0\}$, $\Lambda_k = \{(0,t)\ :\ t \ge 0\}$, and

$$C = \left\{ ay + bx^j + \sum_{m+jn>j} c_{mn}x^m y^n = 0 \right\}, \quad a, b \in \mathbb{R}^* .$$

Since $\mathrm{card}(\mathbb{R}C \cap F_-) < \infty$, the multiplicity j is even, and $ab < 0$. Hence, the root C has two real branches given in a neighborhood of $p_k(0)$ by (*cf.* [18, Formulas (22) and (23)])

$$C_i(t) = \left\{ ay + b(x+\tau)^j + \sum_{m+jn>j} O(\tau) \cdot (x+\tau)^m y^n = 0 \right\},$$

where $\tau = \left(-\frac{a}{b}\right)^{1/j} t^{1/j} + o(t^{1/j})$ for $i = 1$, and $\tau = -\left(-\frac{a}{b}\right)^{1/j} t^{1/j} + o(t^{1/j})$ for $i = 2$. For each curve $C_i(t)$, $i = 1, 2$, one has $\mu^{\pm}_{\varphi}(C_i) = \mu^{\pm}_{\varphi}(C)$. Indeed, the above local formula insures that the topology of the curves is preserved in a neighborhood of $p_k(0)$; outside of a neighborhood of $p_k(0)$, the equality required follows from Lemma 7. $\square$

Lemma 16 *Let $C = E \cup \check{C}$ be a root of $\overline{V}$ such that C generates at least one leaf belonging to $V^{\mathbb{R}}_{Y,F_+}(D, \alpha, \beta, 0, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$. Then, $\check{C}$ splits in* primary components *from the following list:*

(i) pairs of reduced complex conjugate components as described in Lemma 3(2);

(ii) real reduced components, whose all intersection points with E are real and have even multiplicity; each of these components belongs to $V^{\mathbb{R}}_{Y,F_+}(D', \alpha', \beta', 0, (\boldsymbol{p}^\flat)', (\boldsymbol{p}^\sharp)')$ for a certain D_0-proper tuple $(D', \alpha', \beta', 0, (\boldsymbol{p}^\flat)', (\boldsymbol{p}^\sharp)')$;

(iii) non-reduced components $s'L'$, $s''L''$, where $L', L'' \in |-(K_Y + E)|$ are the supporting curves, and, in addition, $s' = s''$ if L', L'' are complex conjugate;

(iv) non-reduced components $sL(z)$, where s is even, $z \in \boldsymbol{p}^\sharp \setminus \{p_k\}$, and L is the curve belonging to $|-(K_Y + E)|$ and passing through z;

(v) non-reduced components $iL(p_{ij})$, where i is even, $p_{ij} \in \boldsymbol{p}^\flat$, and $L(p_{ij})$ is the curve belonging to $|-(K_Y + E)|$ and passing through p_{ij}.

Moreover,

(1) pairwise intersections of distinct primary components are either empty, or transversal contained in $Y \setminus E$, and all reduced primary components are immersed and are nonsingular along E,

(2) *the intersection multiplicities of $\check{C}$ and E at the points $\check{C} \cap \boldsymbol{p}^\flat$ are encoded by a vector $\check{\alpha} \le \alpha$,*

(3) *there exists a conjugation invariant set $\boldsymbol{z} \subset \check{C} \cap E \setminus \boldsymbol{p}^\flat$ containing exactly one point of each irreducible component of $\check{C}$ and such that the intersection multiplicities of $\check{C}$ and E at the points of $\check{C} \cap E \setminus (\boldsymbol{p}^\flat \cup \boldsymbol{z})$ are encoded by the vector β.*

Proof. The statement follows from [18, Proposition 2.6]. The fact that each real reduced primary component of $\check{C}$ intersects E only at real points is guaranteed by Lemmas 3 and 6; these intersection points have even multiplicity due to the assumption that the C generates a leaf belonging to $V^{\mathbb{R}}_{Y,F_+}(D, \alpha, \beta, 0, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$. □

To describe the leaves of deformation diagrams with reducible roots, we use the deformation labels (DL1)-(DL9) introduced in [15, Section 3.3]. They are certain curves on toric surfaces, associated to Newton polygons of their defining polynomials. We recall here the formulas of deformation labels used below (some formulas are slightly modified by a conjugation-invariant coordinate change), where $\mathrm{cheb}_k(t) = \cos\arccos kt$ is the k-th Chebyshev polynomial and y_k is the only simple positive root of $\mathrm{cheb}_k(t) - 1$:

$(\mathrm{DL2})_j$ for an even $j > 0$, two deformation labels defined by the equations

$$\psi_1(x, y) = y^2 + 1 - y \cdot \mathrm{cheb}_j(x), \quad \psi_2(x, y) = \psi_1(x\sqrt{-1}, y) \ ,$$

$(\mathrm{DL3})_i$ for an even $i > 0$, a deformation label

$$(x - 1)(y^i - x) = 0 \ ,$$

$(\mathrm{DL5})_s$ for an even $s > 0$, two deformation labels

$$(x - 1)(x((y \pm 1)^s + 1) - 1) = 0 \ ,$$

$(\mathrm{DL6})_s$ for an even $s > 0$, a deformation label

$$\frac{y + x^2}{2y}\left(1 - \mathrm{cheb}_{s+1}\left(y_{s+1} - \frac{y}{2^{(s-1)/(s+1)}}\right)\right) - 1 = 0 \ ,$$

$(\mathrm{DL7})_s$ for an odd $s > 0$, two deformation labels

$$\frac{y + x^2}{2y}\left(\mathrm{cheb}_{s+1}\left(y_{s+1} + \frac{y\varepsilon_0}{2^{(s-1)/(s+1)}}\right) - 1\right) + 1 = 0, \quad \varepsilon_0 = \pm 1 \ ,$$

$(\mathrm{DL8})_s$ for $s > 0$, $s + 1$ pairs of complex conjugate deformation labels

$$1 + \frac{y + \sqrt{-1}x^2}{2y}\left(\mathrm{cheb}_{s+1}\left(\frac{y\varepsilon}{2^{(s-1)/(s+1)}} + y_{s+1}\right) - 1\right) = 0 \ ,$$

$$1 + \frac{y - \sqrt{-1}x^2}{2y}\left(\mathrm{cheb}_{s+1}\left(\frac{y\overline{\varepsilon}}{2^{(s-1)/(s+1)}} + y_{s+1}\right) - 1\right) = 0 \ ,$$

where $\varepsilon^{s+1} = 1$.

Lemma 17 *Let a curve $C = E \cup \check{C} \in |D|$ be such that the primary components of $\check{C}$ belong to the list (i)-(v) and satisfy conditions (1)-(3) in the statement of Lemma 16. Then, the curve C is a root of $\overline{V}$. It generates at least one leaf that belongs to $V^{\mathbb{R}}_{Y,F_+}(D,\alpha,\beta,0,\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ if and only if $\check{C}$ does not have a component L' or L'' which is real with the real point set contained in $\overline{F}_-$. If the set of leaves generated by C and belonging to $V^{\mathbb{R}}_{Y,F_+}(D,\alpha,\beta,0,\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ is non-empty, then it is in one-to-one correspondence with the following data:*

- *a set $\boldsymbol{z} \subset \check{C} \cap E \setminus \boldsymbol{p}^\flat$ satisfying the condition (3) of Lemma 16,*
- *a collection $\mathcal{DL}$ of deformation labels chosen as follows:*

  - a deformation label of type (DL2)$_j$ for each point $q \in \boldsymbol{z}'$ satisfying $(\check{C} \cdot E)_q = j$, where $\boldsymbol{z}' \subset \boldsymbol{z}$ consists of the points which do not lie on the primary components $s'L'$, $s''L''$, $iL(p_{ij})$, $sL(z)$ of $\check{C}$,

  - a deformation label of type (DL3)$_i$ for each primary component $iL(p_{ij})$ of $\check{C}$,

  - a deformation label of type (DL5)$_s$ for each primary component $sL(z)$ of $\check{C}$,

  - a deformation label of type (DL6)$_s$ or (DL7)$_s$ for a real primary component sL' (resp. sL'') of $\check{C}$, according as s is even or odd,

  - a pair of complex conjugate deformation labels of type (DL8)$_s$ for a pair of complex conjugate primary components sL', sL'' of $\check{C}$.

Proof. Since the primary components of $\check{C}$ belong to the list (i)-(v) and satisfy conditions (1)-(3) in the statement of Lemma 16, [18, Lemma 2.20] implies that C is a root of the deformation diagram $\overline{V}$, and the leaves of $\overline{V}$ are in one-to-one correspondence with the set of pairs $(\boldsymbol{z}, \mathcal{DP})$, where $\boldsymbol{z} \subset \check{C} \cap E \setminus \boldsymbol{p}^\flat$ is a set of points satisfying the condition (3) of Lemma 16, and $\mathcal{DP}$ is a conjugation-invariant collection of deformation patterns

$$\{\Psi_q \ : \ q \in \boldsymbol{z}'\}, \quad \{\Psi_{iL(p_{ij})} \ : \ iL(p_{ij}) \subset \check{C}\}, \quad \{\Psi_{sL(z)} \ : \ sL(z) \subset \check{C}\},$$
$$\{\Psi_{s'L'} \ : \ s'L' \subset \check{C}\}, \quad \{\Psi_{s''L''} \ : \ s''L'' \subset \check{C}\},$$

where $iL(p_{ij})$, $sL(z)$, $s'L'$, or $s''L''$ run over the corresponding primary components of $\check{C}$, and Ψ_* denote specific curves on toric surfaces introduced in [18, Section 2.5.3]. We consider these pairs $(\boldsymbol{z}, \mathcal{DP})$ in detail, prove that they induce leaves, belonging to $V^{\mathbb{R}}_{Y,F_+}(D,\alpha,\beta,0,\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ if and only if $\check{C}$ does not have a component L' or L'' which is real with the real point set contained in $\overline{F}_-$, and bring deformation patterns to the form of deformation labels.

A deformation of C into any leaf-curve $\widetilde{C} \in V_{Y,F_+}(D,\alpha,\beta,0,\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp(t))$, where $\boldsymbol{p}^\sharp(t) = (\boldsymbol{p}^\sharp \setminus \{p_k\}) \cup \{p_k(t)\}$, $t > 0$, can be described by a family of sections of $H^0(Y, \mathcal{O}_Y(D))$:

$$S\check{T}_\tau + \tau^\kappa T_\tau, \ \tau \in (\mathbb{C}, 0), \quad \tau^\kappa = t \ , \tag{11}$$

where $S, \check{T}_0, T_0$ are real, $S^{-1}(0) = E$, $\check{T}_0^{-1}(0) = \check{C}$. In addition, S and $\check{T}_0$ are negative in F_- (except possibly for a finite set), and

$$\begin{cases} S(p_k(t)) > 0,\ \check{T}_0(p_k(t)) < 0,\ T_0(p_k(t)) > 0 \quad \text{as } t > 0, \\ T_0(z) > 0 \quad \text{for all} \quad z \in \mathbb{R}E \setminus \boldsymbol{p}^\sharp\ . \end{cases} \tag{12}$$

Indeed, formula (11) follows from [18, Lemma 2.10]. Observe that S does not divide T_0 in view of $t = \tau^k$ (see (11)). Hence $T_0^{-1}(0)$ intersects E at finitely many points and with even multiplicities, since the same holds for each curve $C_\tau = \{S\check{T}_\tau + \tau^\kappa T_\tau = 0\}$, $\tau \ne 0$. This claim combined with the facts that $\text{card}(\mathbb{R}\check{C} \cap F_-) < \infty$, $\text{card}(\mathbb{R}\widetilde{C} \cap F_-) < \infty$, $(S\check{T}_\tau + \tau^\kappa T_\tau)(p_k(t)) = 0$ for all $t > 0$, and with the assumption $\alpha + \beta \in \mathbb{Z}_+^{\infty,\text{ even}}$ yields all sign conditions (12).

Given a point $q \in \boldsymbol{z}'$, the intersection multiplicity $j = (\check{C} \cdot E)_q$ is even by Lemma 16. Choose local real coordinates x, y in a neighborhood of q in Y so that $E = \{y = 0\}$, $q = (0,0)$, $F_- = \{y < 0\}$. Then in formula (11), we get $\widetilde{T}_0 = y - 2x^j + \text{h.o.t.}$, $T_0 = a + \text{h.o.t.}$, where $a > 0$ due to (12). Thus, by [18, Lemma 2.11], there are two real deformation patterns Ψ_q given by $y^2 - 2yP(x) + a = 0$ with $P(x) = x^j + \text{l.o.t.}$, and they can be brought to the form $(\text{DL}2)_j$ by a transformation

$$\psi(x, y) \mapsto \lambda\psi(\mu x, \nu y), \quad \lambda, \mu, \nu > 0\ . \tag{13}$$

Given a primary component $iL(p_{ij})$ of $\check{C}$ (with i even by Lemma 16), it has a unique deformation pattern $\Psi_{iL(p_{ij})}$ (see [18, Lemma 2.15]) which is real and can be brought to the form $(\text{DL}3)_i$ by transformation (13).

Given a primary component $sL(z)$ of $\check{C}$ (with s even by Lemma 16) and an intersection point $q \in L(z) \cap E$ belonging to $\boldsymbol{z}$, we are interested in deformation patterns that describe a deformation of C in a neighborhood of $L(z)$ such that the intersection point q of $sL(z)$ and E smoothes out, and the other intersection point of $sL(z)$ and E turns into an intersection point of multiplicity s of the deformed curve with E. Choose real coordinates x, y in a neighborhood of $L(z)$ so that $L(z) = \{y = 0\}$, $E = \{x + y^2 + xy = 0\}$, $q = (0,0)$, $z = (x_0, 0)$, and $L(z) \cap F_+ = \{(x, 0)\ :\ 0 < x < 1\}$. In particular, $0 < x_0 < 1$, since $z \in \boldsymbol{p}^\sharp \subset F_+$. By [18, Lemma 2.16(1)], deformation patterns for the pair $(sL(z), (0,0))$ are of the form $\Psi_{sL(z)} = \{h(x,y) = 0\}$, where

$$h(x, y) = xf(y) + a, \quad f(y) + a = (y + \xi)^s,\ \xi \in \mathbb{C}, \quad x_0 f(0) + a = 0\ ,$$

and $-a = T_0(z) > 0$ by (12). From this we easily derive that $\xi^s = -a\frac{1-x_0}{x_0} > 0$, obtaining two real deformation patterns that can be brought to the form $(\text{DL}5)_s$ via transformation (13).

If $\check{C} \supset s(L' \cup L'')$, where L', L'' are complex conjugate, then there are s pairs of complex conjugate deformation patterns for these primary components (see [18, Lemma 2.13]), which can be brought to the form $(\text{DL}8)_s$.

Let L' be real, $\mathbb{R}L' \subset \overline{F}_+$, and $\check{C} \supset sL'$. By (11) and (12), we can choose real coordinates x, y in a neighborhood of L' so that

$$\begin{cases} L' = \{y = 0\}, \quad S = y + x^2 + xy, \quad q = (0,0) = E \cap L', \\ F_+ = \{S > 0\}, \quad \widetilde{T}_0 = y^s((-1)^{s+1} + \text{h.o.t.}), \quad c = T_0(q) > 0 \ . \end{cases} \tag{14}$$

Substituting this data to the formulas of [18, Lemma 2.13], we obtain that the (complex) deformation patterns of sL' are given by the formula

$$\Psi_{sL'} = \{(y + x^2)f(y) + c\,(-1)^{s+1} = 0\} \ , \tag{15}$$

where

$$yf(y) + c\,(-1)^{s+1} = \frac{c\,(-1)^{s+1}}{2}\left(\mathrm{cheb}_{s+1}\left(\frac{\xi y}{(2^{s-1}c\,(-1)^{s+1})^{1/(s+1)}} + y_{s+1}\right) + 1\right) \ , \tag{16}$$

$\xi^{s+1} = 1$. If s is even, then there exists a unique real deformation pattern, and via transformation (13) preserving the terms $y + x^2$ in the above equation $S(x,y) = 0$ for E we can bring it to the form $(\mathrm{DL6})_s$. If s is odd, then there exist two real deformation patterns. Indeed, the equation for $\Psi_{sL'}$ can be rewritten as

$$x^2 = -y \cdot \frac{\mathrm{cheb}_{s+1}\left(\frac{\xi_0 y}{(2^{s-1}c)^{1/(s+1)}} + y_{s+1}\right) + 1}{\mathrm{cheb}_{s+1}\left(\frac{\xi_0 y}{(2^{s-1}c)^{1/(s+1)}} + y_{s+1}\right) - 1} \tag{17}$$

with $(2^{s-1}c)^{1/(s+1)} > 0$, $\xi_0 = \pm 1$. It is easy to bring them to the form $(\mathrm{DL7})_s$ via transformation of type (13).

Suppose that L' is real, $\mathbb{R}L' \subset \overline{F}_-$, and $\check{C}$ contains a primary component sL', $s > 0$. Then we are in the conditions of the preceding case with the only change that $F_- = \{S > 0\}$ in (14). One can easily see that all the real deformation labels $(\mathrm{DL6})_s$ and $(\mathrm{DL7})_s$ produce a one-dimensional branch in the domain $\{S > 0\} = F_-$, which is not possible for leaf-curves in $V^{\mathbb{R}}_{Y,F_+}(D, \alpha, \beta, 0, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$. $\square$

Introduce the following numbers:

- $\mu^+(\Psi) = (-1)^{s(\Psi)}$ and $\mu^-(\Psi) = (-1)^{s^-(\Psi)}$, where Ψ is a deformation label of type $(\mathrm{DL2})_j$, $s(\Psi)$ is the number of solitary nodes of Ψ, and $s^-(\Psi)$ is the number of solitary nodes lying in the domain $y > 0$;

- $\mu^+(\Psi) = \mu^-(\Psi) = 1$, where Ψ is a deformation label of type $(\mathrm{DL3})_i$ or $(\mathrm{DL5})_s$;

- $\mu^+(\Psi) = (-1)^{s(\Psi)}$ and $\mu^-(\Psi) = (-1)^{s^-(\Psi)}$, where Ψ is a deformation label of type $(\mathrm{DL3})_i$ or $(\mathrm{DL5})_s$, $s(\Psi)$ is the number of solitary nodes of Ψ, and $s^-(\Psi)$ is the number of solitary nodes lying in the domain $y + x^2 > 0$.

Let $\mu^\pm_{2,j}$, $\mu^\pm_{3,i}$, $\mu^\pm_{5,s}$, $\mu^\pm_{6,s}$, and $\mu^\pm_{7,s}$ be the sums of the numbers $\mu^\pm(\Psi)$ over all deformation labels of type $(\mathrm{DL2})_j$, $(\mathrm{DL3})_i$, $(\mathrm{DL5})_s$, $(\mathrm{DL6})_s$, and $(\mathrm{DL7})_s$, respectively.

Lemma 18 *We have*

$$\mu^{+}_{2,j} = 0, \quad \mu^{-}_{2,j} = \begin{cases} 0, & j \equiv 0 \mod 4, \\ 2, & j \equiv 2 \mod 4, \end{cases} \tag{18}$$

$$\mu^{\pm}_{3,i} = 1, \quad \mu^{\pm}_{5,s} = 2 \ , \tag{19}$$

$$\mu^{\pm}_{6,s} = 1, \quad \mu^{+}_{7,s} = 0, \quad \mu^{-}_{7,s} = \begin{cases} 2, & s \equiv 1 \mod 4, \\ 0, & s \equiv 3 \mod 4. \end{cases} \tag{20}$$

Proof. All the relations follow from a direct computation. □

Let $C = E \cup \check{C}$ be a root of $\overline{V}$ such that C generates a leaf $\widetilde{C} \in V^{\mathbb{R}}_{Y,F_+}(D, \alpha, \beta, 0, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$ corresponding to a pair $(\boldsymbol{z}, \mathcal{DL})$ (see Lemma 17). Introduce the following numbers:

- for each point $q \in \boldsymbol{z}'$ (where $\boldsymbol{z}' \subset \boldsymbol{z}$ consists of the points which do not lie on the primary components $s'L'$, $s''L''$, $iL(p_{ij})$, $sL(z)$ of $\check{C}$), put $\mu^{\pm}(C, q) = \mu^{\pm}_{2j}$, where $j = (\check{C} \cdot E)_q$;
- for each primary component $sL(z)$ of $\check{C}$, put $\mu^{\pm}(C, sL(z)) = \mu^{\pm}_{5,s}$;
- for each real primary component sL' of $\check{C}$, put $\mu^{\pm}(C, sL') = \mu^{\pm}_{6,s}$ or $\mu^{\pm}_{7,s}$ according as s is even or odd.

Lemma 19 *Let $C = E \cup \check{C}$ be a root of $\overline{V}$ such that C generates at least one leaf belonging to $V^{\mathbb{R}}_{Y,F_+}(D, \alpha, \beta, 0, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$, and let $\mathrm{Lf}(C)$ be the set of all such leaves. Then,*

$$\sum_{\widetilde{C} \in \mathrm{Lf}(C)} \mu^{\pm}_{\varphi}(\widetilde{C}) = (-1)^{E_{1/2} \circ \varphi} \cdot \mu^{\pm}_{\varphi}(\check{C}^{red}) \cdot 2^m \cdot M^{\pm}(C) \cdot \sum_{\boldsymbol{z}} \prod_{q \in \boldsymbol{z}'} \mu^{\pm}(C, q) \ , \tag{21}$$

where $\check{C}^{red}$ is the union of all reduced primary components of $\check{C}$ different from L', L'', the exponent m is the number of primary components $sL(z)$ of $\check{C}$, the factor $M^{\pm}(C)$ equals $(-1)^{(s'+s'')(L'_{1/2} \circ \varphi)} \mu^{\pm}(C, s'L') \mu^{\pm}(s''L'')$ if $s'L'$ and $s''L''$ are real primary components of $\check{C}$, and equals $s+1$ if $\check{C}$ contains a pair of complex conjugate primary components sL', sL'', and finally, $\boldsymbol{z}$ runs over all subsets of $\check{C} \cap E \setminus \boldsymbol{p}^\flat$ satisfying condition (3) of Lemma 16, and $\boldsymbol{z}' \subset \boldsymbol{z}$ consists of the points which do not lie on the primary components $s'L'$, $s''L''$, $iL(p_{ij})$, $sL(z)$ of $\check{C}$.

Proof. Let $\widetilde{C} \in \mathrm{Lf}(C)$. By [18, Lemma 2.9], singular points of $\widetilde{C}$ (regarded as a small deformation of C) appear in a neighborhood of $\mathrm{Sing}(C)$. Furthermore, the local branches do not glue up in local deformation of singular points in $\mathrm{Sing}(\check{C}^{red})$, of intersection points $q \in \boldsymbol{p}^\flat \cup (\check{C} \cap E) \setminus \boldsymbol{z}$, and of the intersection points of $\check{C}^{red}$ with the other primary components of $\check{C}$. In particular, first, local deformations of the intersection points of $\check{C}^{red}$ with other primary components of $\check{C}$ and of the points

$q \in \boldsymbol{p}^\flat \cup (\check{C} \cap E) \setminus \boldsymbol{z}$ do not bear solitary nodes, and, second, due to Lemma 7, the multiplicative contribution of $\mathrm{Sing}(\check{C}^{red})$ to $\mu^\pm_\varphi(\widetilde{C})$ is $\mu^\pm_\varphi(\check{C}^{red})$. Local deformations of the primary components $iL(p_{ij})$, $sL(z)$, $s'L'$, $s''L''$ of $\check{C}$ and of the points $q \in \boldsymbol{z}'$ are determined by the corresponding deformation labels so that the solitary nodes of $\widetilde{C}$, which appear in these deformations are in one-to-one correspondence with the solitary nodes of all deformation labels involved. Deformation labels of type $(\mathrm{DL3})_i$, $(\mathrm{DL5})_s$, and $(\mathrm{DL8})_s$ do not have solitary nodes. The solitary nodes of the other deformation labels, which belong to the domains indicated in the definition of numbers $\mu^-(\Psi)$, correspond precisely to the nodes of $\widetilde{C}$ belonging to F_+. It follows from the fact that, in the coordinates x, y in the equations of a deformation label Ψ, this domain defines an intersection of F_+ with a neighborhood of a point $q \in \boldsymbol{z}'$ or with a neighborhood of real primary components $s'L$", $s''L''$ of $\check{C}$.

Then, formula (21) immediately follows from Lemmas 16 and 17. $\square$

3.10 Formula for sided w-numbers

Theorem 3 *Let (Y, E, F, φ) be a dividing basic quadruple.*

(1) For a divisor class $D \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$ and vectors $\alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}} \in \mathbb{Z}^\infty_+$ such that $I(\alpha + \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}) = DE$ and $R_Y(D, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}) > 0$, one has

$$W^\pm_{Y,F_+,\varphi}(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}) = 0 \ , \tag{22}$$

provided that either $\alpha \notin \mathbb{Z}^{\infty,\,\mathrm{even}}_+$, or $\beta^{\mathrm{re}} \notin \mathbb{Z}^{\infty,\,\mathrm{even}}_+$, or $\beta^{\mathrm{im}} \neq 0$.

(2) For a divisor class $D \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$ and vectors $\alpha, \beta \in \mathbb{Z}^{\infty,\,\mathrm{even}}_+$ such that $I(\alpha + \beta) = DE$ and $R_Y(D, \beta) > 0$, one has:

(2i) If $(K_Y + E)D = 0$ or $(K_Y + E)D < -2$, then

$$W^+_{Y,F_+,\varphi}(D, \alpha, \beta, 0) = 0 \ . \tag{23}$$

(2ii) If $(K_Y + E)D = -1$, then

$$W^+_{Y,F_+,\varphi}(D, \alpha, \beta, 0) = 2^{\|\beta\|} W^+_{Y,F_+,\varphi}(D, \alpha + \beta, 0, 0) \ . \tag{24}$$

(2iii) If $(K_Y + E)D = -2$, then

$$W^+_{Y,F_+,\varphi}(D, \alpha, \beta, 0) = 2 \sum_{j \geq 2,\ \beta_j > 0} W^+_{Y,F_+,\varphi}(D, \alpha + e_j, \beta - e_j, 0)$$
$$+4^{n-1}(-1)^{E_{1/2}\circ\varphi} \cdot \sum (-1)^{I\alpha^{(0)} \cdot (L_{1/2}\circ\varphi)} \eta^+(2l) \binom{\alpha}{\alpha^{(0)}} \prod_{i=1}^m W^+_{Y,F_+,\varphi}(\mathcal{D}^{(i)}, 0, 0, e_1) \ , \tag{25}$$

where $L\in|-(K_Y+E)|$ *is real with* $\mathbb{R}L\subset F$, $n=R_Y(D,\beta)$,

$$\eta^+(2l)=\begin{cases}1, & \textit{if } l=0,\\ 1, & \textit{if the supporting curves } L',L'' \textit{ are real and either}\\ & \quad \mathbb{R}L'\subset\overline{F}_+,\ \mathbb{R}L''\subset\overline{F}_- \textit{ or } \mathbb{R}L'\subset\overline{F}_-,\ \mathbb{R}L''\subset\overline{F}_+\ ,\\ l+1, & \textit{if either the supporting curves } L',L'' \textit{ are imaginary,}\\ & \quad \textit{or } L',L'' \textit{ are real and } \mathbb{R}L'\cup\mathbb{R}L''\subset\overline{F}_+\ ,\\ 0, & \textit{if } l>0, \textit{the supporting curves } L',L'' \textit{ are real,}\\ & \quad \textit{and } \mathbb{R}L'\cup\mathbb{R}L''\subset\overline{F}_-,\end{cases} \tag{26}$$

and the second sum in (25) is taken over all integers $l\ge 0$, *vectors* $\alpha^{(0)}\le\alpha$, *and sequences of distinct tuples*

$$(\mathcal{D}^{(i)},0,0,e_1),\ 1\le i\le m\ , \tag{27}$$

such that

- *each* $\mathcal{D}^{(i)}\in\mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$ *is a pair of divisor classes that is different from* $(-(K_Y+E),-(K_Y+E))$ *and satisfies* $[\mathcal{D}^{(i)}]E=2$, $R_Y(\mathcal{D}^{(i)},2e_1)=0$,
- $D-E=\sum_{i=1}^m[\mathcal{D}^{(i)}]-(2l+I\alpha^{(0)}+I\beta)(K_Y+E)$;

the second sum in (25) is factorized by permutations of sequences (27).

(3) For any divisor class $D\in\mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$ *and vectors* $\alpha,\beta\in\mathbb{Z}_+^{\infty,\,\mathrm{even}}$ *such that* $I(\alpha+\beta)=DE$ *and* $R_Y(D,\beta)>0$, *one has*

$$W^-_{Y,F_+,\varphi}(D,\alpha,\beta,0)=2\sum_{j\ge2,\ \beta_j>0}W^-_{Y,F_+,\varphi}(D,\alpha+e_j,\beta-e_j,0)$$

$$+(-1)^{E_{1/2}\circ\varphi}\cdot\sum(-1)^{(I\alpha^{(0)}+\beta^{(0)})(L_{1/2}\circ\varphi)}\cdot\frac{4^{\|\beta^{(0)}\|}}{\beta^{(0)}!}\eta^-(l)\begin{pmatrix}\alpha\\ \alpha^{(0)}\alpha^{(1)}...\alpha^{(m)}\end{pmatrix}\frac{(n-1)!}{n_1!...n_m!}$$

$$\times\prod_{i=1}^m\left(\begin{pmatrix}(\beta^{\mathrm{re}})^{(i)}\\ \gamma^{(i)}\end{pmatrix}W^-_{Y,F_+,\varphi}(\mathcal{D}^{(i)},\alpha^{(i)},(\beta^{\mathrm{re}})^{(i)},(\beta^{\mathrm{im}})^{(i)})\right), \tag{28}$$

where $L\in|-(K_Y+E)|$ *is real with* $\mathbb{R}L\subset F$,

$$n=R_Y(D,\beta),\quad n_i=R_Y(\mathcal{D}^{(i)},(\beta^{\mathrm{re}})^{(i)}+2(\beta^{\mathrm{im}})^{(i)}),\ i=1,...,m\ ,$$

$$\eta^-(l)=\begin{cases}1, & \text{if } l=0,\\ l/2+1, & \text{if } l \text{ is even, } L',L'' \text{ are imaginary},\\ 0, & \text{if } l \text{ is odd, } L',L'' \text{ are imaginary},\\ 0, & \text{if } l>0,\ L',L'' \text{ are real},\\ & \quad \mathbb{R}L',\mathbb{R}L''\subset\overline{F}_-,\\ (l/2+1)(2-(-1)^{l/2}), & \text{if } l \text{ is even, } L',L'' \text{ are real},\\ & \quad \mathbb{R}L',\mathbb{R}L''\subset\overline{F}_+,\\ 2(l+(-1)^{(l-1)/2})(-1)^{L'_{1/2}\circ\varphi}, & \text{if } l \text{ is odd, } L',L'' \text{ are real},\\ & \quad \mathbb{R}L',\mathbb{R}L''\subset\overline{F}_+,\\ 1, & \text{if } l \text{ is even, } L',L'' \text{ are real},\\ & \quad \mathbb{R}L'\subset\overline{F}_+,\ \mathbb{R}L''\subset\overline{F}_-,\\ (1+(-1)^{(l-1)/2})(-1)^{L'_{1/2}\circ\varphi}, & \text{if } l \text{ is odd, } L',L'' \text{ are real},\\ & \quad \mathbb{R}L'\subset\overline{F}_+,\ \mathbb{R}L''\subset\overline{F}_-,\end{cases} \tag{29}$$

and the second sum in (28) is taken

- *over all integers* $l\ge 0$ *and vectors* $\alpha^{(0)}\le\alpha$, $\beta^{(0)}\le\beta^{\mathrm{re}}$;
- *over all sequences*

$$(\mathcal{D}^{(i)},\alpha^{(i)},(\beta^{\mathrm{re}})^{(i)},(\beta^{\mathrm{im}})^{(i)}),\ 1\le i\le m\ , \tag{30}$$

  such that, for all $i=1,...,m$,

  (3a) $\mathcal{D}^{(i)}\in\mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$, *and* $\mathcal{D}^{(i)}$ *is neither the divisor class* $-(K_Y+E)$, *nor the pair* $\{-(K_Y+E),-(K_Y+E)\}$,

  (3b) $I(\alpha^{(i)}+(\beta^{\mathrm{re}})^{(i)}+2(\beta^{\mathrm{im}})^{(i)})=[\mathcal{D}^{(i)}]E$, *and* $R_Y(\mathcal{D}^{(i)},(\beta^{\mathrm{re}})^{(i)}+2(\beta^{\mathrm{im}})^{(i)})\ge 0$,

  (3c) $\mathcal{D}^{(i)}$ *is a pair of divisor classes if and only if* $(\beta^{\mathrm{im}})^{(i)}\ne 0$,

  (3d) if $\mathcal{D}^{(i)}$ *is a pair of divisor classes, then* $n_i=0$, $\alpha^{(i)}=(\beta^{\mathrm{re}})^{(i)}=0$, *and* $(\beta^{\mathrm{im}})^{(i)}=e_1$,

  and

  (3e) $D-E=\sum_{i=1}^m[\mathcal{D}^{(i)}]-(l+I\alpha^{(0)}+I\beta^{(0)})(K_Y+E)$,

  (3f) $\sum_{i=0}^m\alpha^{(i)}\le\alpha$, $\sum_{i=0}^m(\beta^{\mathrm{re}})^{(i)}\ge\beta$,

  (3g) each tuple $(\mathcal{D}^{(i)},0,(\beta^{\mathrm{re}})^{(i)},(\beta^{\mathrm{im}})^{(i)})$ *with* $n_i=0$ *appears in (27) at most once,*

- *over all sequences*

$$\gamma^{(i)}\in\mathbb{Z}_+^{\infty,\,\mathrm{odd}\cdot\,\mathrm{even}},\quad i=1,...,m\ , \tag{31}$$

  satisfying

(3h) $\|\gamma^{(i)}\| = \begin{cases} 1, & \mathcal{D}^{(i)} \text{ is a divisor class,} \\ 0, & \mathcal{D}^{(i)} \text{ is a pair of divisor classes,} \end{cases} \quad i = 1, ..., m,$

(3i) $(\beta^{\mathrm{re}})^{(i)} \ge \gamma^{(i)}$, $i = 1, ..., m$, *and* $\sum_{i=1}^m \left((\beta^{\mathrm{re}})^{(i)} - \gamma^{(i)}\right) = \beta^{\mathrm{re}} - \beta^{(0)}$;

the second sum in (28) is factorized by simultaneous permutations in the sequences (30) and (31).

(4) All sided w-numbers $W^{\pm}_{Y,F_+,\varphi}(D, \alpha, \beta, 0)$, *where* $D \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$ *is a divisor class and* $R_Y(D, \beta) > 0$, *are recursively determined by the formulas (23), (24), (25), (28) and the initial conditions given by Proposition 14.*

Proof. We follow the main lines of the proof of the recursive formula in [15, Section 3].

Proof of (1). The statement is clear for $\alpha \notin \mathbb{Z}_+^{\infty,\,\mathrm{even}}$ or $\beta^{\mathrm{re}} \notin \mathbb{Z}_+^{\infty,\,\mathrm{even}}$, since the curves in count must have a non-empty one-dimensional part in F_- contrary to the definition of $V^{\mathbb{R}}_{Y,F_+}(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$. In the case $\beta^{\mathrm{im}} \ne 0$, the statement follows from Lemma 6.

Proof of (2i). If $(K_Y + E)D = 0$, then either $D^2 = -1$, $DE = 1$, or $D = -(K_Y+E)$, $DE = 2$, and in both situations, $V^{\mathbb{R}}_{Y,F_+}(D, \alpha, \beta, 0, \boldsymbol{p}^\flat) = \emptyset$. Indeed, in the former case, we have $DE = -DK_Y = 1$; in the latter case, the condition $R_Y(-(K_Y+E), \beta) > 0$ yields $\beta = 2e_1$, and both conclusions contradict the assumption $\beta \in \mathbb{Z}_+^{\infty,\,\mathrm{even}}$.

Let $(K_Y + E)D < -2$. The leaf-curves from $V^{\mathbb{R}}_{Y,F_+}(D, \alpha, \beta, 0, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$ generated by any reducible root-curve $C = E \cup \check{C}$ do not contribute to $W^+_{Y,F_+,\varphi}(D, \alpha, \beta, 0, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$. Indeed, $\check{C}$ must contain a reduced real primary component, since $(K_Y+E)(D-E) < 0$ and $K_Y + E$ vanishes on all imaginary or non-reduced primary components of $\check{C}$ (*cf.* Lemma 16). Hence, the total contribution of the leafs $\widetilde{C} \in \mathrm{Lf}(C)$ is zero in view of the factor $\mu^+(C, q) = 0$ (see formula (18)) in (21). Thus, by Lemma 15,

$$W^+_{Y,F_+,\varphi}(D, \alpha, \beta, 0, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp) = 2^{\|\beta\|} W^+_{Y,F_+,\varphi}(D, \alpha+\beta, 0, 0, (\boldsymbol{p}^\flat)', (\boldsymbol{p}^\sharp)') \ . \qquad (32)$$

However, $R_Y(D, 0) = -(K_Y + E)D - 1 > 0$, that is $(\boldsymbol{p}^\sharp)' \ne \emptyset$, and, as explained above, the right-hand side of (32) must vanish.

Proof of (2ii). Notice that $D - E$ is not effective, since $-(K_Y + E)(D - E) = -1$ and $-(K_Y+E)$ is nef. Hence, there are only irreducible root-curves, and the formula follows from Lemma 15.

Proof of Proposition 13 and assertions (2iii) and (3). All these statements follow by induction on $n = R_Y(D, \beta)$ from Lemmas 15 and 19. Proposition 14 serves as the base of induction.

In the right-hand side of formulas (25) and (28), the first sum runs over irreducible root-curves and the second sum runs over root-curves containing E. We only explain notations and coefficients occurring in the second sum:

- the vector $\alpha^{(0)}$ encodes the multiplicities of the primary components of type $iL(p_{ij})$,
- the vector $\beta^{(0)}$ encodes the multiplicities of the primary components of type $sL(z)$, and their multiplicative contribution amounts to $4^{\|\beta^{(0)}\|}$,
- the factors $\eta^+(2l)$, $\eta^-(l)$ are the sums of the contributions of the pairs of primary components $s'L' \cup s''L''$ (computed by (20)) over the range $s'+s''=2l$, respectively $s'+s''=l$,
- the vectors $\gamma^{(i)}$ encode the intersection multiplicities $j=(\check{C}\cdot E)_q$ of the points $q\in \boldsymbol{z}'$ (see Lemma 17) for the reducible root-curves $C=E\cup\check{C}$. $\square$

4 ABV formula over the reals

Let Y be a smooth rational surface, $E\subset Y$ a smooth rational curve. If the anti-canonical class $-K_Y$ is effective, positive on all curves different from E, and $K_YE=0$, we call the pair (Y,E) a *nodal del Pezzo pair*. (It follows from the adjunction formula that $E^2=-2$.) Notice that a nodal del Pezzo pair may be not monic log-del Pezzo, and vice versa. Throughout this section we assume that (Y,E) is a nodal del Pezzo pair.

An example of a nodal del Pezzo pair is provided by the plane blown up at a generic collection of ≤ 8 points subject to the condition that six of them belong to a conic.

A nodal del Pezzo pair (Y,E) is an almost Fano surface in the sense of [22, Section 4.1], and thus by [22, Theorem 4.2] we have the following Abramovich-Bertram-Vakil formula (briefly *ABV formula*):

$$GW_0(Y,D)=\sum_{m\ge 0}\binom{DE+2m}{m}N_Y(D-mE,0,(DE+2m)e_1)\ , \qquad (33)$$

where $D\in \mathrm{Pic}(Y)$ and $N_Y(D',0,(D'E)e_1)$ is the number of rational curves $C\in|D'|$ passing through a generic collection of $-D'K_Y-1$ points in $Y\setminus E$.

4.1 Deformation representation of ABV formula

ABV formula (33) can be represented geometrically. Let $\pi:\mathfrak{X}\to(\mathbb{C},0)$ be a proper holomorphic submersion of a smooth three-dimensional variety $\mathfrak{X}$ (with $(\mathbb{C},0)$ being understood as a disc germ), where each fiber $\mathfrak{X}_t, t\ne 0$, is a del Pezzo surface and the central fiber $Y=\mathfrak{X}_0$ contains a smooth rational curve E such that (Y,E) is a nodal del Pezzo pair.

Remark 20 *There is a natural isomorphism*

$$\mathrm{Pic}(\mathfrak{X}_t) \xrightarrow{\simeq} \mathrm{Pic}(Y), \quad t \ne 0 \ , \tag{34}$$

preserving the intersection form; for the sake of brevity we use the same symbol for corresponding divisor classes in $\mathfrak{X}_t$, $t \in (\mathbb{C}, 0)$. To distinguish linear systems themselves we use the notation $|D|_{\mathfrak{X}_t}$.

Let $D \in \mathrm{Pic}(\mathfrak{X}_t)$ be effective for $t \ne 0$ and satisfy $-K_{\mathfrak{X}_t}D - 1 \ge 0$. Pick $r = -K_{\mathfrak{X}_t}D - 1$ disjoint sections $z_i : (\mathbb{C}, 0) \to \mathfrak{X}$, $1 \le i \le r$, so that $\boldsymbol{p}^\sharp(0) = \{z_i(0),\ 1 \le i \le r\}$ is a generic point collection in $Y \backslash E$, and $\boldsymbol{p}^\sharp(t) = \{z_i(t),\ 1 \le i \le r\}$ is a generic point collection in $\mathfrak{X}_t$, $t \ne 0$. For each $t \in (\mathbb{C}, 0)$, denote by $V_t(D, \boldsymbol{p}^\sharp(t))$ the set of reduced irreducible rational curves $C \in |D|_{\mathfrak{X}_t}$ that pass through $\boldsymbol{p}^\sharp(t)$. It is well-known (see, for instance, [7]) that $V_t(D, \boldsymbol{p}^\sharp(t))$, $t \ne 0$, is finite, contains $GW_0(Y, D)$ elements, and each element is a nodal curve (*cf.* [16, Lemma 3]). By [18, Proposition 2.1], for each $m \ge 0$, the set $V_0(D - mE, \boldsymbol{p}^\sharp(0))$ is finite, and its elements are immersed curves crossing E transversally at $DE + 2m$ distinct points. Thus, we have a diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{C}}' & \stackrel{\nu'}{\longrightarrow} & \mathcal{C}' & \hookrightarrow & \mathfrak{X} \\ \downarrow \widetilde{\pi}' & & \downarrow & & \downarrow \pi \\ (\mathbb{C}, 0) \setminus \{0\} & = & (\mathbb{C}, 0) \setminus \{0\} & \hookrightarrow & (\mathbb{C}, 0) \end{array} \tag{35}$$

where $\mathcal{C}'$ is the union of $GW_0(Y, D)$ families of curves $C \in V_t(D, \boldsymbol{p}^\sharp(t))$, $t \in (\mathbb{C}, 0) \setminus \{0\}$, and $\widetilde{\mathcal{C}}'$ is its normalization. The following statement follows from [22, Theorem 4.2].

Proposition 21 *There exists a diagram*

$$\begin{array}{ccccc} \widetilde{\mathcal{C}} & \stackrel{\nu}{\longrightarrow} & \mathcal{C} & \hookrightarrow & \mathfrak{X} \\ \downarrow \widetilde{\pi} & & \downarrow & & \downarrow \pi \\ (\mathbb{C}, 0) & = & (\mathbb{C}, 0) & = & (\mathbb{C}, 0) \end{array} \tag{36}$$

which extends the diagram (35) so that

(1)
- *$\mathcal{C}$ is the closure of $\mathcal{C}'$ in $\mathfrak{X}$;*
- *the fiber over 0 of each component of $\mathcal{C}$ is $C_0 \cup mE$ for some $m \ge 0$, where $C_0 \in V_0(D, \boldsymbol{p}^\sharp(0))$;*
- *each curve $C_0 \cup mE$ with $m \ge 0$, $C_0 \in V_0(D - mE, \boldsymbol{p}^\sharp(0))$ appears as the fiber over 0 for exactly $\binom{DE+2m}{m}$ components of $\mathcal{C}$;*

(2)
- *$\widetilde{\mathcal{C}}$ is the union of $GW_0(Y, D)$ disjoint nonsingular surfaces;*
- *the fiber over 0 of each component of $\widetilde{\mathcal{C}}$ is either isomorphic to $\mathbb{P}^1$ with $\nu : \mathbb{P}^1 \to C_0 \in V_0(D, \boldsymbol{p}^\sharp(0))$ birational, or is a nodal reducible rational curve $\bigcup_{i=0}^m \mathbb{P}^1_{(i)}$ with some $m \ge 1$, $\mathbb{P}^1_{(i)} \simeq \mathbb{P}^1$ for all $i = 0, ..., m$, $\mathbb{P}^1_{(1)}, ..., \mathbb{P}^1_{(m)}$ disjoint from each other, $\mathbb{P}^1_{(0)}$ intersecting each $\mathbb{P}^1_{(1)}, ..., \mathbb{P}^1_{(m)}$ at one point, and such that $\nu : \mathbb{P}^1_{(0)} \to C_0 \in V_0(D - mE, \boldsymbol{p}^\sharp(0))$ is birational, $\nu : \mathbb{P}^1_{(i)} \to E$ is an isomorphism for all $i = 1, ..., m$;*

- *for each $C_0 \in V_0(D-mE,\boldsymbol{p}^\sharp(0))$, $m \ge 0$, there are exactly $\binom{DE+2m}{m}$ components of $\widetilde{\mathcal{C}}$ whose fiber $\bigcup_{i=0}^m \mathbb{P}^1_{(i)}$ over 0 covers C_0, and they differ from each other by the image of the m-tuple $\mathbb{P}^1_{(0)} \cap \bigcup_{i=1}^m \mathbb{P}^1_{(i)})$ in the $(DE+2m)$-tuple $C_0 \cap E$.*

If the families $\mathfrak{X}$, $\mathcal{C}'$, and $\widetilde{\mathcal{C}}'$ are defined over the reals, then so are the families $\mathcal{C}$ and $\widetilde{\mathcal{C}}$. □

4.2 Nodal degenerations

Let $\pi' : \mathfrak{X}' \to (\mathbb{C},0)$ be a holomorphic map of a smooth three-dimensional variety $\mathfrak{X}'$ such that the fibers $\mathfrak{X}'_\tau$, $\tau \in (\mathbb{C},0)\backslash\{0\}$, are del Pezzo surfaces, the central fiber $\mathfrak{X}'_0$ is a surface with one singular point z of type A_1 (node), the germ $(\mathfrak{X}',z)$ is isomorphic to $(\mathbb{C}^3,0)$ with coordinates x_1,x_2,x_3 in which the map π' is given by

$$\pi'(x_1,x_2,x_3) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2, \quad a_1a_2a_3 \ne 0 \ , \tag{37}$$

and π' is a submersion at each point of $\mathfrak{X}' \setminus \{z\}$. Such a family is called *nodal degeneration.*

Make the base change $\tau = t^2$, perform the blow up $\widetilde{\mathfrak{X}}' \to \mathfrak{X}'$ at z and obtain a family $\widetilde{\pi}' : \widetilde{\mathfrak{X}}' \to (\mathbb{C},0)$, whose fibers $\widetilde{\mathfrak{X}}'_t$, $t \ne 0$, are del Pezzo surfaces, and $\widetilde{\mathfrak{X}}'_0 = Y \cup Z$, where $Z \simeq (\mathbb{P}^1)^2$, $E = Y \cap Z$ is a smooth rational (-2)-curve in Y, and (Y,E) is a nodal del Pezzo pair. Here, E represents the class $C_1 + C_2$ in $\mathrm{Pic}(Z)$, C_1, C_2 being the generators of the two rulings of Z. We call the family $\widetilde{\pi}' : \widetilde{\mathfrak{X}}' \to (\mathbb{C},0)$ the *unscrew* of the nodal degeneration $\pi' : \mathfrak{X}' \to (\mathbb{C},0)$.

Contracting Z to E along the lines of one of the rulings (see [2]), say, generated by C_1, we get a family

$$\pi : \mathfrak{X} \to (\mathbb{C},0) \tag{38}$$

of smooth surfaces as in Section 4.1. The following lemma is straightforward.

Lemma 22 *The family of curves $\mathcal{C}'$ in (35) lifts to a family of curves in $\widetilde{\pi}' : \widetilde{\mathfrak{X}}' \to (\mathbb{C},0)$ as follows: if a component of $\mathcal{C}'$ closes up in $\mathfrak{X}$ with the central fiber $C'_m \cup mE$, $C'_m \in |D-mE|_Y$, then this component of $\mathcal{C}'$ closes up in $\widetilde{\mathfrak{X}}'$ with a central fiber $C'_m \cup C_1^{(DE+m)} \cup C_2^{(m)}$, where $C_2^{(m)}$ is the union of m lines in $|C_2|_Z$ attached to m intersection points of C'_m and E, and $C_1^{(DE+m)}$ is the union of $DE+m$ lines from $|C_1|_Z$ attached to the remaining points of $C'_m \cap E$.* □

Remark 23 *A family of plane quartics with the central fiber Q having one node z induces a family of del Pezzo surfaces of degree 2 degenerating into a nodal del Pezzo pair. In this setting, E is the exceptional divisor of the blow up of the node of the double cover of the plane ramified along the nodal quartic, the six pairs of intersecting (-1)-curves crossing E respectively cover the six lines in the plane passing though z and tangent to Q outside z, and, finally, the supporting curves L', L'' doubly cover the tangent lines to Q at the node z.*

4.3 Real versions of ABV formula

4.3.1 Ordinary and sided u-numbers

Let (Y, E) be a nodal del Pezzo pair such that Y and E are real, and $\mathbb{R}E \ne \emptyset$. Denote by F the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$ and pick a conjugation-invariant class $\varphi \in H_2(Y \setminus F, \mathbb{Z}/2)$. Let $D \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$. Choose a generic collection $\boldsymbol{p}^\sharp$ of $-K_Y D - 1$ points in $F \setminus E$.

By [22, Proposition 4.1(b)], the set $V_Y(D, \boldsymbol{p}^\sharp)$ of rational curves in the linear system $|D|$ which pass through the points of $\boldsymbol{p}^\sharp$ is finite and consists of immersed curves crossing E transversally at DE distinct points.

For any nonnegative integers k and l such that $k + 2l = DE$, define an *ordinary u-number* $U_{Y,F,\varphi}(D, ke_1, le_1, \boldsymbol{p}^\sharp)$ putting (*cf.* the definition of ordinary w-numbers in Section 3.6)

$$U_{Y,E,\varphi}(D, ke_1, le_1, \boldsymbol{p}^\sharp) = \sum_{C \in V^{\mathbb{R}}_Y(D, ke_1, le_1, \boldsymbol{p}^\sharp)} \mu_\varphi(C) \ , \tag{39}$$

where $V^{\mathbb{R}}_Y(D, ke_1, le_1, \boldsymbol{p}^\sharp) \subset V_Y(D, \boldsymbol{p}^\sharp)$ is formed by the curves intersecting $\mathbb{R}E$ in k points (and intersecting $E \setminus \mathbb{R}E$ in l pairs of complex conjugate points) and $\mu_\varphi(C)$ is defined by (4). If $F \setminus E$ splits into two components F_+ and F_-, the configuration $\boldsymbol{p}^\sharp$ lies in F_+, and DE is even, then define a *sided u-number* $U^{\pm}_{Y,F_+,\varphi}(D, 0, (DE/2)e_1, \boldsymbol{p}^\sharp)$, putting

$$U^{+}_{Y,F_+,\varphi}(D, 0, (DE/2)e_1, \boldsymbol{p}^\sharp) = \sum_{C \in V^{\mathbb{R}}_{Y,F_+}(D, 0, (DE/2)e_1, \boldsymbol{p}^\sharp)} \mu_\varphi(C) \ , \tag{40}$$

$$U^{-}_{Y,F_+,\varphi}(D, 0, (DE/2)e_1, \boldsymbol{p}^\sharp) = \sum_{C \in V^{\mathbb{R}}_{Y,F_+}(D, 0, (DE/2)e_1, \boldsymbol{p}^\sharp)} \mu^-_\varphi(C) \ , \tag{41}$$

where

$$V^{\mathbb{R}}_{Y,F_+}(D, 0, (DE/2)e_1, \boldsymbol{p}^\sharp) = \{C \in V^{\mathbb{R}}_Y(D, 0, (DE/2)e_1, \boldsymbol{p}^\sharp) \ : \ \mathrm{card}(C \cap F_-) < \infty\}$$

and $\mu^-_\varphi(C)$ is defined by (10).

We say that a quadruple (Y, E, F, φ) has *property* (R) if for any divisor class $D \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$ and for any connected component F_+ of $F \setminus E$, there exists a generic collection $\boldsymbol{p}^\sharp$ of $-K_Y D - 1$ points in F_+ (referred to as R_{D,F_+}*-collection* or R_D*-collection*) such that, for any $m \ge 0$ with $D - mE \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$, the following holds:

(R1) $U_{Y,F,\varphi}(D - mE, 0, ke_1, le_1, \boldsymbol{p}^\sharp) = 0$ whenever $l > 0$,

(R2) if $F \setminus E$ splits into two components and the intersection DE is even, then $U^{+}_{Y,F_+,\varphi}(D - mE, 0, (DE/2)e_1, \boldsymbol{p}^\sharp) = 0$.

Proposition 24 *Let (Y, E) be a nodal del Pezzo pair such that Y and E are real, and $\mathbb{R}E \neq \emptyset$. Denote by F the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$ and pick a conjugation-invariant class $\varphi \in H_2(Y \setminus F, \mathbb{Z}/2)$. Assume in addition that (Y, E) is monic log-del Pezzo.*

(1) Pick a divisor class $D_0 \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$ such that $\dim|D_0| > 0$. Let $D \in \mathrm{Prec}(D_0)$ be a divisor class, and let $\boldsymbol{p}^\sharp$ be a collection of $-K_Y D - 1$ points in D_0-CH position in $F \setminus E$. Then, for any nonnegative integers k and l such that $k + 2l = DE$, one has

$$U_{Y,E,\varphi}(D, ke_1, le_1, \boldsymbol{p}^\sharp) = W_{Y,E,\varphi}(D, 0, ke_1, le_1, \emptyset, \boldsymbol{p}^\sharp).$$

If $F \setminus E$ splits into two components F_+ and F_-, the collection $\boldsymbol{p}^\sharp$ is contained in F_+, and DE is even, then

$$U^{\pm}_{Y,F_+,\varphi}(D, 0, (DE/2)e_1, \boldsymbol{p}^\sharp) = W^{\pm}_{Y,F_+,\varphi}(D, 0, 0, (DE/2)e_1, \emptyset, \boldsymbol{p}^\sharp).$$

The numbers $U_{Y,E,\varphi}(D, ke_1, le_1, \boldsymbol{p}^\sharp)$ and $U^{\pm}_{Y,F_+,\varphi}(D, 0, (DE/2)e_1, \boldsymbol{p}^\sharp)$ do not depend on the choice of a collection $\boldsymbol{p}^\sharp$ in D_0-CH position.

(2) The quadruple (Y, E, F, φ) has property (R).

Proof. The equality of ordinary (respectively, sided) u- and w-numbers is tautological. The invariance of the u-numbers considered follows from the invariance of w-numbers; for the latter invariance see Propositions 8 and 13. $\square$

4.3.2 ABV formulas for Welschinger invariants, I

As in Section 4.1, let $\pi : \mathfrak{X} \to (\mathbb{C}, 0)$ be holomorphic submersion of a smooth three-dimensional variety $\mathfrak{X}$, where each fiber $\mathfrak{X}_t$, $t \neq 0$, is a del Pezzo surface and the central fiber $Y = \mathfrak{X}_0$ contains a smooth rational curve E such that (Y, E) is a nodal del Pezzo pair.

Suppose that $\mathfrak{X}$ possesses a real structure $\mathrm{Conj} : \mathfrak{X} \to \mathfrak{X}$ such that

$$\pi \circ \mathrm{Conj} = \mathrm{conj} \circ \pi \tag{42}$$

(where conj is the standard real structure on $(\mathbb{C}, 0)$). We get a family $\pi : {}^{\mathbb{R}}\mathfrak{X} = \pi^{-1}(\mathbb{R}, 0) \to (\mathbb{R}, 0) \hookrightarrow (\mathbb{C}, 0)$ of real surfaces, the fibers $\mathfrak{X}_t$, $t \in (\mathbb{R}, 0) \setminus \{0\}$, being real del Pezzo surfaces, and (Y, E) being a real nodal del Pezzo pair. Assume that $\mathbb{R}E \neq \emptyset$. Denote by F_0 the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$ and pick a conjugation-invariant class $\varphi_0 \in H_2(Y \setminus F_0, \mathbb{Z}/2)$. The family $\mathbb{R}\mathfrak{X} \to (\mathbb{R}, 0)$ is topologically trivial. We extend F_0 to a continuous family F_t of connected components of $\mathbb{R}\mathfrak{X}_t$, and extend φ_0 to a continuous family of classes $\varphi_t \in H_2(\mathbb{R}\mathfrak{X}_t \setminus F_t, \mathbb{Z}/2)$, $t \in (\mathbb{R}, 0)$.

Theorem 4 *(1) The isomorphism $\mathrm{Pic}(\mathfrak{X}_t) \xrightarrow{\sim} \mathrm{Pic}(Y)$ is conjugation invariant and induces an isomorphism $\mathrm{Pic}^{\mathbb{R}}(\mathfrak{X}_t) \xrightarrow{\sim} \mathrm{Pic}^{\mathbb{R}}(Y)$, $t \in (\mathbb{R}, 0)$.*

(2) For any real effective divisor class D on $\mathfrak{X}_t$, $t \ne 0$, one has

$$W(\mathfrak{X}_t, D, F_t, \varphi_t) = W(\mathfrak{X}_t, D + (DE)E, F_t, \varphi_t) \ .$$

(3) Assume that the quadruple (Y, E, F_0, φ_0) has property (R). Then, for any $t \in (\mathbb{R}, 0)$, $t \ne 0$, any divisor class $D \in \mathrm{Pic}^{\mathbb{R}}(\mathfrak{X}_t)$, and any R_D-collection $\boldsymbol{p}^\sharp \subset F_0 \setminus E$, the following equality holds:

$$W(\mathfrak{X}_t, D, F_t, \varphi_t) = \sum_{m \ge 0} (-1)^{m(E_{1/2} \circ \varphi_0)} \binom{DE + 2m}{m} U_{Y,E,\varphi_0}(D - mE, (DE+2m)e_1, 0, \boldsymbol{p}^\sharp) \ . \tag{43}$$

Proof. The first statement follows from $\mathrm{Pic}(\mathfrak{X}_t) = H^2(\mathfrak{X}_t; \mathbb{Z})$ and $\mathrm{Pic}^{\mathbb{R}}(\mathfrak{X}_t) = \mathrm{Ker}(1 + \mathrm{Conj}|^*_{\mathfrak{X}_t})$, $t \in (\mathbb{R}, 0)$.

For the second statement, without loss of generality, assume that $DE = -d < 0$ and choose a continuous family of collections $\boldsymbol{p}^\sharp_t \subset F_t$ of $-K_X D - 1$ distinct points so that $\boldsymbol{p}^\sharp_0$ is generic in $F_0 \setminus E$. We establish a one-to-one correspondence between the sets M_1 and M_2 of real rational curves in $|D|_{\mathfrak{X}_t}$ and $|D - dE|_{\mathfrak{X}_t}$, respectively, passing through $\boldsymbol{p}^\sharp_t$, such that the correspondence preserves the Welschinger signs. Indeed, by Proposition 21, degenerations of curves from M_1 are of type $C \cup (d+s)E$, $s \ge 0$, where $C \in |D - (d+s)E|_Y$ is a real rational curve passing through $\boldsymbol{p}^\sharp_0$; furthermore, to each such a curve C and a conjugation invariant subset $w \subset C \cap E$ of $d + s$ points there corresponds a unique curve in M_1, and its Welschinger sign coincides with that of C. Similarly, degenerations of curves from M_2 are of type $C \cup sE$, $C \in |D - (d+s)E|_Y$ as above, and to each subset $(C \cap E) \setminus w \subset C \cap E$ of s points there corresponds a unique curve in M_2, and its Welschinger sign coincides with that of C.

To prove the third statement, notice that, since the quadruple (Y, E, F_0, φ_0) has property (R), the degenerate real curves to consider are of type $C \cup mE$, $m \ge 0$, where $C \in V^{\mathbb{R}}_Y(D - mE, 0, (DE + 2m)e_1, 0, \boldsymbol{p}^\sharp)$, and the Welschinger sign of each real rational curve in $\mathfrak{X}_t$, $t \ne 0$, appearing in a deformation of $C \cup mE$ is $(-1)^{m(\frac{1}{2}E \circ \varphi)} \mu_\varphi(C)$. Thus, formula (43) follows from Proposition 21. $\square$

4.3.3 ABV formulas for Welschinger invariants, II

Assume that a nodal degeneration $\pi' : \mathfrak{X}' \to (\mathbb{C}, 0)$ possesses a real structure Conj which lifts the standard complex conjugation $\mathrm{conj} : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, and each real surface $\mathfrak{X}'_\tau$, $\tau \in (\mathbb{R}, 0)$, has a non-empty real part. Notice that the node z of $\mathfrak{X}'_0$ is real. The real structure Conj lifts to a real structure $\widetilde{\mathrm{Conj}}$ on the unscrew $\widetilde{\pi}' : \widetilde{\mathfrak{X}}' \to (\mathbb{C}, 0)$ of $\pi' : \mathfrak{X}' \to (\mathbb{C}, 0)$. The quadric $Z \subset \widetilde{\mathfrak{X}}'_0$ is real. We call the unscrew $\widetilde{\pi}' : \widetilde{\mathfrak{X}}' \to (\mathbb{C}, 0)$ *hyperbolic*, respectively *elliptic*, if $\mathbb{R}Z \simeq (S^1)^2$, respectively $\mathbb{R}Z \simeq S^2$ (we do not consider the case $\mathbb{R}Z = \emptyset$).

In the case of a hyperbolic unscrew, both rulings of Z are real. The contraction of Z along one of the rulings leads to a family of smooth real surfaces $\pi : \mathfrak{X} \to (\mathbb{C}, 0)$

(*cf.* Section 4.2). Thus, Theorem 4 applies and gives Welschinger invariants of the real del Pezzo surfaces $\mathfrak{X}_t$, $t \in (\mathbb{R},0)\setminus\{0\}$, via w-numbers of the real nodal del Pezzo pair (Y,E) by formula (43).

Theorem 5 *Assume that the unscrew* $\widetilde{\pi}' : \widetilde{\mathfrak{X}}' \to (\mathbb{C},0)$ *of a real nodal degeneration* $\pi' : \mathfrak{X}' \to (\mathbb{C},0)$ *is elliptic. Then, the following holds.*

(1) The isomorphism $\operatorname{Pic}(\widetilde{\mathfrak{X}}'_t) \xrightarrow{\sim} \operatorname{Pic}(Y)$, $t \neq 0$, *induces a monomorphism* $\operatorname{Pic}^{\mathbb{R}}(\widetilde{\mathfrak{X}}'_t) \to \operatorname{Pic}^{\mathbb{R}}(Y)$, $t \in (\mathbb{R},0)\setminus\{0\}$, *and the image of the latter is orthogonal to* $[E] \in \operatorname{Pic}^{\mathbb{R}}(Y)$.

(2) Suppose that $F\setminus E$ *is connected, where* F *is the component of* $\mathbb{R}Y$ *containing* $\mathbb{R}E$. *Let* F_t *be a component of* $\mathbb{R}\widetilde{\mathfrak{X}}'_t$ *merging to* F *as* $t \to 0$. *Choose a class* $\varphi \in H_2(\mathbb{R}Y\setminus F,\mathbb{Z}/2)$, *and denote by* φ_t *the class in* $H_2(\mathbb{R}\widetilde{\mathfrak{X}}'_t\setminus F_t,\mathbb{Z}/2)$, $t \in (\mathbb{R},0)\setminus\{0\}$, *which converges to* φ *as* $t \to 0$. *If the quadruple* (Y,E,F,φ) *has property (R), then for any* $t \in (\mathbb{R},0)$, $t \neq 0$, *any divisor class* $D \in \operatorname{Pic}^{\mathbb{R}}_+(\widetilde{\mathfrak{X}}'_t)$ *such that* $D^2 \geq 0$, *and any* R_D*-collection* $\boldsymbol{p}^\sharp \subset F\setminus E$, *one has*

$$W(\widetilde{\mathfrak{X}}'_t, D, F_t, \varphi_t) = U_{Y,E,\varphi}(D,0,0,\boldsymbol{p}^\sharp)\ . \tag{44}$$

(3) Suppose that $F\setminus E$ *splits into two components* F_+, F_-, *and* $\mathbb{R}\widetilde{\mathfrak{X}}'_t$ *contains two connected components* $F_{+,t}$ *and* $F_{-,t}$ *which merge to* F_+ *and* F_-, *respectively. Choose a class* $\varphi \in H_2(\mathbb{R}Y\setminus F,\mathbb{Z}/2)$, *and denote by* φ_t *be the class in* $H_2(\mathbb{R}\widetilde{\mathfrak{X}}'_t\setminus F_t,\mathbb{Z}/2)$, $t \in (\mathbb{R},0)\setminus\{0\}$, *which converge to* φ *as* $t \to 0$.

- *(i) If the quadruple* (Y,E,F,φ) *has property (R), then, for any* $t \in (\mathbb{R},0)$, $t \neq 0$, *any divisor class* $D \in \operatorname{Pic}^{\mathbb{R}}_+(\widetilde{\mathfrak{X}}'_t)$ *such that* $D^2 \geq 0$, *and any* R_{D,F_+}*-collection* $\boldsymbol{p}^\sharp \subset F_+$, *one has*

$$W(\widetilde{\mathfrak{X}}'_t, D, F_{+,t}, \varphi_t) = U^+_{Y,F_+,\varphi}(D,0,0,\boldsymbol{p}^\sharp)\ . \tag{45}$$

- *(ii) For any* $t \in (\mathbb{R},0)$, $t \neq 0$, *any divisor class* $D \in \operatorname{Pic}^{\mathbb{R}}_+(\widetilde{\mathfrak{X}}'_t)$ *such that* $D^2 \geq 0$, *and any generic configuration* $\boldsymbol{p}^\sharp \subset F_+$ *of* $-K_YD-1$ *points, one has*

$$W(\widetilde{\mathfrak{X}}'_t, D, F_{+,t}, F_{-,t}\cup\varphi_t) = U^-_{Y,F_+,\varphi}(D,0,0,\boldsymbol{p}^\sharp)\ . \tag{46}$$

Proof. The action of Conj in $\operatorname{Pic}(\widetilde{\mathfrak{X}}'_t) = H^2(\widetilde{\mathfrak{X}}'_t;\mathbb{Z})$ and $\operatorname{Pic}(Y) = H^2(Y;\mathbb{Z})$ does not commute with the natural (as in Remark 20) isomorphism $H^2(\widetilde{\mathfrak{X}}'_t;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$, but differ by a composition with the reflection in $[E] \in H^2(Y;\mathbb{Z})$. This implies the first statement.

Pick a divisor class $D \in \operatorname{Pic}^{\mathbb{R}}_+(\widetilde{\mathfrak{X}}'_t)$, $t \neq 0$, such that $D^2 \geq 0$, take disjoint sections $z_i : (\mathbb{R},0) \to \mathbb{R}\widetilde{\mathfrak{X}}'$, $1 \leq i \leq -K_YD-1$, and consider the limits that the real rational curves in $|D|_{\widetilde{\mathfrak{X}}'_t}$ passing through $\{z_i(t)\}_{1\leq i\leq -K_YD-1}$, $t \neq 0$, have in the central fiber $Y\cup Z$. These limits are of type $C'_m \cup C_1^{(DE+m)} \cup C_2^{(m)}$ (see Lemma 22).

To prove statements (2) and (3i), assume that the quadruple (Y,E,F,φ) has property (R), and that $\boldsymbol{p}^\sharp = \{z_i(0)\}_{1\leq i\leq -K_YD-1} \subset F\setminus E$ is an R_D-collection. Due to property (R), the components $C'_m \in |D - mE|_Y$, $m > 0$, of the limits of real rational curves in $|D|_{\widetilde{\mathfrak{X}}'_t}$ must have real intersection points with E. However, such curves C'_m cannot be completed to real curves in $|D|_Y$. Taking this into account, we prove statements (2) and (3i) in the same manner as Theorem 4(3).

To prove statement (3ii), put $\boldsymbol{p}^\sharp = \{z_i(0)\}_{1 \le i \le -K_Y D - 1}$, and notice that if the limit curve contains as a component a real rational curve $C \in |D - mE|_Y$, $m > 0$, then $C \cap \mathbb{R}E = \emptyset$. In the family $\{\widetilde{\mathfrak{X}}'_t\}_{t \in [0,1]}$, the surface $\widetilde{\mathfrak{X}}'_0 = Y \cup Z$ deforms so that F_+ glues up with a component Z_+ of $\mathbb{R}Z \setminus E$, whereas F_- glues up with the other component Z_- of $\mathbb{R}Z \setminus E$. In turn, each real rational curve $C \in |D - mE|_Y$ with $C \cap \mathbb{R}E = \emptyset$ can be completed up to a real curve on $Y \cup Z$ in 2^m ways, when attaching to each pair $z', z'' \in C \cap E$ of complex conjugate points either the pair $C'_1 \supset \{z'\}$, $C''_2 \supset \{z''\}$, or the pair $C''_1 \supset \{z''\}$, $C'_2 \supset \{z'\}$, where C'_1, C''_1 belong to one ruling of Z, and C'_2, C''_2 to the other. Observe that one of the pairs (C'_1, C''_2), (C''_1, C'_2) has a solitary node in Z_+, which contributes the factor (-1) to the Welschinger sign μ^-_φ of the corresponding deformed curve in $|D|_{\mathfrak{X}'_t}$, $t > 0$, whereas the other pair has a solitary node in Z_- that does not affect μ^-_φ. Hence, the total contribution $W(\widetilde{\mathfrak{X}}'_t, D, F_{+,t}, F_{-,t} \cup \varphi_t)$ of the curves coming from C is zero, which proves formula (46). $\Box$

Corollary 25 *Let (Y, E) be a real nodal del Pezzo pair such that $\mathbb{R}E$ divides a connected component F of $\mathbb{R}Y$ into two parts, F_+ and F_-. Let $D \in \mathrm{Pic}^{\mathbb{R}}(Y)$ be an effective divisor class such that $DE = 0$ and $D^2 \ge 0$. Let $\varphi \in H_2(Y \setminus F, \mathbb{Z}/2)$ be a conjugation invariant class. Then, the sided u-number $U^-_{Y,F_+,\varphi}(D, 0, 0, \boldsymbol{p}^\sharp)$ does not depend on the choice of a generic configuration $\boldsymbol{p}^\sharp \subset F_+$ of $-K_Y D - 1$ points.* $\Box$

Let $\pi' : \mathfrak{X}' \to (\mathbb{C}, 0)$ be a real nodal degeneration. Assume that its unscrew is hyperbolic; denote it by $\pi^h : \mathfrak{X}^h \to (\mathbb{C}, 0)$. The unscrew of the nodal degeneration obtained from the given one by the change of parameter $\tau \mapsto -\tau$ is elliptic; denote it by $\pi^e : \mathfrak{X}^e \to (\mathbb{C}, 0)$. Let (Y, E) be the real nodal del Pezzo pair that appears as a component of both $\mathfrak{X}^h_0$ and $\mathfrak{X}^e_0$. Let F be the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$, and let F^h_t, respectively, F^e_t, be the component (or the union of components) of $\mathbb{R}\mathfrak{X}^h_t$, respectively, $\mathbb{R}\mathfrak{X}^e_t$, $t \ne 0$, which merges to F as $t \to 0$. Let $\varphi^h_t \in H_2(\mathfrak{X}^h_t \setminus F^h_t, \mathbb{Z}/2)$, respectively, $\varphi^e_t \in H_2(\mathfrak{X}^e_t \setminus F^e_t, \mathbb{Z}/2)$, $t \ne 0$, be families of conjugation-invariant classes converging to the same class $\varphi \in H_2(Y \setminus F, \mathbb{Z}/2)$ as $t \to 0$.

Corollary 26 *Assume that the quadruple (Y, E, F, φ) has property (R). Let D be a real effective divisor class on Y such that $DE = 0$. Then,*

$$W(\mathfrak{X}^e_t, D, F^e_{+,t}, \varphi^e_t) = \sum_{m \in \mathbb{Z}} (-1)^m W(\mathfrak{X}^h_t, D - mE, F^h_t, \varphi^h_t), \quad t \ne 0 \ ,$$

where $F^e_{+,t}$ is any component of F^e_t.

Proof. This is an immediate consequence of Theorems 4(2,3) and 5(2,3i), and of the relation $\sum_{m\in\mathbb{Z}}(-1)^m\binom{2k}{m-k}=0$ which holds for any integer $k\neq 0$. $\square$

5 Positivity and asymptotics

5.1 Real del Pezzo surfaces of degree 2

The anticanonical linear system on a real del Pezzo surface X of degree 2 defines a double covering $X\to\mathbb{P}^2$ branched in a nonsingular real quartic curve $Q_X\subset\mathbb{P}^2$, and thus identifies X with a hypersurface defined in the weighted projective space $P^3(1,1,1,2)$ by equation $u^2=\varepsilon f_X(x,y,z)$, where f_X is a real defining polynomial of Q_X and $\varepsilon=\pm 1$. Therefore, as a topological space, $\mathbb{R}X$ is the result of gluing of two copies of $\mathbb{R}f_{X,\varepsilon}=\{p\in\mathbb{RP}^2\ :\ \varepsilon f_X(p)\geq 0\}$ along their common boundary, if this boundary is non-empty, and the disjoint union of two copies otherwise. Below we always choose the sign for f_X so that $\mathbb{R}f_{X,-}$ is non-orientable.

As is known, the real part of a real non-singular quartic is isotopic in $\mathbb{R}P^2$ either to the union of $0\leq q\leq 4$ null-homologous circles placed outside each other (denote this isotopy type by $\langle q\rangle$), or to a pair of null-homologous circles placed one inside the other (denote this isotopy type by $1\langle 1\rangle$). In accordance with this notation and the above sign-convention, the topological types of real del Pezzo surfaces X of degree 2 with $\mathbb{R}X\neq\emptyset$ are denoted below by $\langle 0\rangle^-$, $\langle q\rangle^\varepsilon$, $1\leq q\leq 4$, and $1\langle 1\rangle^\varepsilon$. For example, the topological type of the plane blown up at a real points and b pairs of complex conjugate points, $a+2b=7$, which we denote by $\mathbb{P}^2_{a,b}$, coincides with $\langle 4-b\rangle^-$.

For surfaces X of type $\langle q\rangle^\varepsilon$, $1\leq q\leq 4$, $\langle 0\rangle^-$, and $1\langle 1\rangle^+$, the choice of a connected component F of $\mathbb{R}X$ does not affect the computation of Welschinger invariants; indeed, for X of type $\langle 0\rangle^-$ the two connected components of $\mathbb{R}X$ are interchanged by the deck transformation of the above double covering, while for other types of X with disconnected $\mathbb{R}X$ such an independence follows from Theorem 1(2).

As to surfaces X of type $1\langle 1\rangle^-$, they have two connected components: one, which we denote $F^{\,o}$, is orientable, and the other one, $F^{\,no}$, is not.

Notice also that the 28 bitangents of Q_X lift into the 56 curves in X with self-intersection -1, and that the curves of the linear system $|-K_X|$ are the pull-backs of the straight lines in $\mathbb{P}^2$.

5.2 ABV-families

Any two real del Pezzo surfaces of degree 2 that have homeomorphic real parts are deformation equivalent in the class of such surfaces (see, for example, [6], Theorem 17.3), and, as a result, they have the same system of Welschinger invariants. Therefore, in the proof of Theorems 6 and 7, for each topological type, we pick a particular

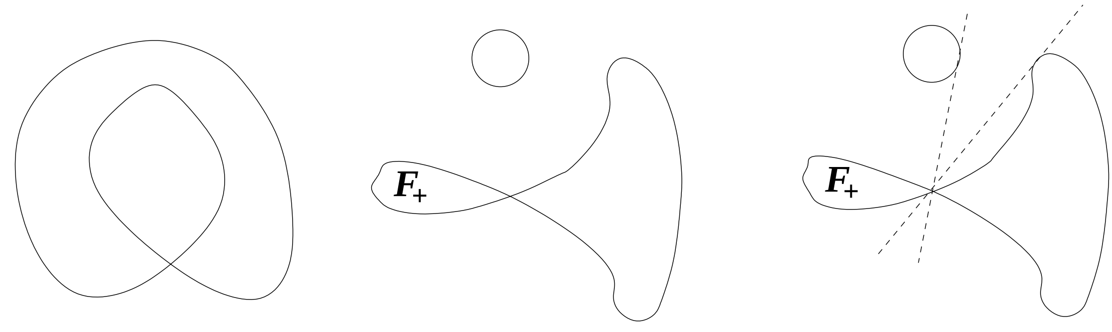

Figure 1: Nodal quartics

real del Pezzo surface, X, that we include into an appropriate family with a special fiber containing a real nodal del Pezzo pair, (Y, E).

If X is of type $\mathbb{P}^2_{a,b}$, $a + 2b = 7$, (or, equivalently, of type $\langle q \rangle^-$, $1 \le q \le 4$), we include X into a family $\mathfrak{X} \to (\mathbb{C}, 0)$, which is a holomorphic submersion possessing a real structure subject to (42) and whose central fiber is a real nodal del Pezzo pair (Y, E) (*cf.* Section 4.3.2). Namely, we specialize a conjugation-invariant set of 6 blown up points on a real conic C_2, E being the strict transform of C_2. We call $\mathfrak{X} \to (\mathbb{C}, 0)$ a *regular ABV-family* of X.

Real del Pezzo surfaces X of degree 2 of other topological types can be included into real nodal degenerations corresponding to nodal degenerations of quartics Q_X: if X is of type $1\langle 1 \rangle^+$, $1\langle 1 \rangle^-$, or $\langle 1 \rangle^+$, we degenerate Q_X into a nodal quartic shown in Figure 1(a); if X is of type $\langle 0 \rangle^-$, the quartic Q_X (having an empty real part) degenerates into a nodal quartic with a one-point real part; if X is of type $\langle q \rangle^+$, $2 \le q \le 4$, we degenerate Q_X into a real nodal quartic as shown in Figure 1(b). Then, for nodal degenerations of surfaces of types $1\langle 1 \rangle^+$ or $\langle 1 \rangle^+$, we take a hyperbolic unscrew and call the result a *hyperbolic ABV-family* of X. For nodal degenerations of surfaces of types $1\langle 1 \rangle^-$, $\langle 0 \rangle^-$, or $\langle q \rangle^+$, $2 \le q \le 4$, we take an elliptic unscrew and call the result an *elliptic ABV-family* of X.

5.3 F-compatible divisor classes

Let X be a real del Pezzo surface (which can be nodal). Denote by $\mathrm{bh} : \mathrm{Pic}^{\mathbb{R}}(X) \to H_1(\mathbb{R}X; \mathbb{Z}/2)$ the natural homomorphism which sends each real effective divisor class D that is represented by a real reduced curve, say C, to $[\mathbb{R}C \cap \mathbb{R}X] \in H_1(\mathbb{R}X, \mathbb{Z}/2)$ (cf. [3, 23]). If $\mathcal{F}$ is a union of some connected components of $\mathbb{R}X$, then denote by $\mathrm{bh}_{\mathcal{F}}$ the composition of bh with the projection $H_1(\mathbb{R}X; \mathbb{Z}/2) \to H_1(\mathcal{F}; \mathbb{Z}/2)$.

Let F be a connected component of $\mathbb{R}X$. We say that a real effective divisor class D on X is *F-compatible*, if $\mathrm{bh}_{\mathbb{R}X \setminus F}(D) = 0$. It is clear that if a real effective divisor class D is not F-compatible, then $W(X, D, F, \varphi)$ vanishes for any conjugation invariant class $\varphi \in H_2(X \setminus F, \mathbb{Z}/2)$.

Remark 27 *The F-compatibility condition holds for all real divisor classes if $H_1(\mathbb{R}X \setminus F, \mathbb{Z}/2) = 0$. Hence, the only cases with a non-trivial condition for del Pezzo surfaces of degree 2 are as follows: either X is of type $\langle 0\rangle^-$, or X is of type $1\langle 1\rangle^-$ and $F = F^o$. For example, $-K_X$ is not F-compatible in either of these two cases.*

We use below the following characterization of the F-compatibility condition for surfaces of types $\langle 0\rangle^-$ and $1\langle 1\rangle^-$. A surface X of type $\langle 0\rangle^-$ or $1\langle 1\rangle^-$ can be included into an elliptic ABV-family with the central fiber $Y \cup Z$, where Z is a quadric and Y is the plane blown up at three pairs of complex conjugate points on a real conic C_2 such that $\mathbb{R}C_2 \neq \emptyset$, and at one more real point, which, in the case of type $\langle 0\rangle^-$, is chosen in the orientable component of $\mathbb{RP}^2 \setminus \mathbb{R}C_2$, and, in the case of $1\langle 1\rangle^-$, is chosen in the non-orientable component of $\mathbb{RP}^2 \setminus \mathbb{R}C_2$. Consider the basis of $\mathrm{Pic}(Y)$ consisting of the pull-back L of a generic line and the exceptional divisors $E_1, ..., E_7$, where E_{2i}, E_{2i+1} are complex conjugate, $i = 1, 2, 3$, and E_1 is real. Since the components of $\mathbb{R}X$ for X of type $\langle 0\rangle^-$ are interchanged by an automorphism, we can choose any of them, and from now on our choice will be such that, in the above ABV-family, the chosen component F merges to the component of $\mathbb{R}Y \setminus E$ that does not contain $\mathbb{R}E_1$.

Following Theorem 5, let us identify $\mathrm{Pic}^{\mathbb{R}}(X)$ with a subgroup of $\mathrm{Pic}^{\mathbb{R}}(Y)$.

Proposition 28 *For a divisor class D on a surface X of type $\langle 0\rangle^-$ or $1\langle 1\rangle^-$ with $D \in \mathrm{Pic}^{\mathbb{R}}(X)$, $D = dL - d_1E_1 - ... - d_7E_7$ as a divisor on Y, we have*

- *$d_{2i} = d_{2i+1}$, $i = 1, 2, 3$, and $2d = d_2 + ... + d_7$,*
- *D is F-compatible if and only if*

(FC1) in the case of type $\langle 0\rangle^-$, the number $d_1 = DE_1$ is even,

(FC2) in the case of type $1\langle 1\rangle^-$ and $F = F^o$, the numbers $d = DL$ and $d_1 = DE_1$ are even.

Proof. Straightforward. □

We say that a divisor class $D \in \mathrm{Pic}^{\mathbb{R}}(Y)$ is *F_+-compatible* if it satisfies conditions (FC1) and (FC2) introduced in Proposition 28.

5.4 Main results

Theorem 6 *Let X be a real del Pezzo surface of degree 2 with a non-empty real point set $\mathbb{R}X \neq S^2$, and let F be a connected component of $\mathbb{R}X$. If D is an F-compatible nef and big divisor class on X, then*

$$W(X, D, F, [\mathbb{R}X \setminus F]) > 0 \ . \tag{47}$$

In particular, through any collection of $-K_XD-1$ points of F, one can trace a real rational curve $C\in|D|$.

Furthermore,

$$\log W(X,nD,F,[\mathbb{R}X\setminus F])=-K_XD\cdot n\log n+O(n),\quad n\to+\infty. \tag{48}$$

The proof of Theorem 6 is presented in Sections 5.5-5.8. Our strategy is to degenerate X into a nodal del Pezzo pair and to apply Theorems 4 and 5 expressing Welschinger invariants in terms of w-numbers. The obtained nodal del Pezzo pairs are monic log-del Pezzo, which allows us to compute and estimate the above w-numbers using Theorems 2 and 3.

Remark 29 Theorem 6 implies that the same positivity and log-asymptotic statements hold for all real del Pezzo surfaces of degree ≥ 3 and thus, in particular, it covers all the cases studied in [11, 13, 14, 15, 20] (notice that the proof given in [15, Section 4.1.1] for $\mathbb{P}^2_{2,2}$ contains a gap). Indeed, one can turn any real del Pezzo surface of degree ≥ 3 into a del Pezzo surface of degree 2 blowing up an appropriate number of real points, and then apply Theorem 6 to the divisors D which are disjoint from the exceptional divisors of the blow up.

The following table contains the values of Welschinger invariants $W(X,D,F,\varphi)$ for $D=-K_X$ or $-2K_X$ and $\varphi=0$ or $\varphi=\varphi_F=[\mathbb{R}X\setminus F]$ (for surfaces of types $\langle 2\rangle^+$, $\langle 3\rangle^+$ or $\langle 4\rangle^+$, the invariants do not depend on the choice of F among the components of $\mathbb{R}X$).

D	φ	$\langle 4\rangle^-$	$\langle 3\rangle^-$	$\langle 2\rangle^-$	$\langle 1\rangle^-$	$1\langle 1\rangle^+$	$1\langle 1\rangle^-, F^{no}$	$1\langle 1\rangle^-, F^{o}$	$\langle 1\rangle^+$	$\langle 2\rangle^+$	$\langle 3\rangle^+$	$\langle 4\rangle^+$
$-K_X$	0	8	6	4	2	2	0	0	0	-2	-4	-6
$-K_X$	φ_F	8	6	4	2	2	4	0	0	2	4	6
$-2K_X$	0	224	128	64	24	32	0	0	8	0	0	0
$-2K_X$	φ_F	224	128	64	24	32	48	24	8	24	40	72

The zero and negative values deserve a special comment.

- In the table above, the original Welschinger invariants ($\varphi=0$) take negative values or vanish for the multi-component del Pezzo surfaces. This reflects a general phenomenon stated below in Theorem 8.
- The vanishing of $W(-K_X,\langle 1\rangle^+)$ appears to be a special feature of such del Pezzo surfaces (see Theorem 7). Furthermore, this vanishing is "sharp": if the only oval of a real plane quartic of type $\langle 1\rangle$ is convex, then there is no real tangent through a point inside the oval, and hence there are no real rational curves $C\in|-K_X|$ at all.

Theorem 7 *Let X be a real del Pezzo surface of degree 2 with $\mathbb{R}X=S^2$. Then,*

(i) for any real effective divisor class D on X, we have $W(X,D,\mathbb{R}X,0)\ge 0$;

(ii) the big and nef real effective divisor classes D on X such that $W(X, D, \mathbb{R}X, 0) > 0$ form a subsemigroup in $\mathrm{Pic}(X)$*; this subsemigroup contains $-mK_X$ with $m \ge 2$ and all the divisor classes D' and $D' - K_X$, where D' is big, nef, and disjoint from a pair of complex conjugate (-1)-curves;*

(iii) if a big and nef real effective divisor class D on X satisfies $W(X, D, \mathbb{R}X, 0) > 0$, then

$$\log W(X, nD, \mathbb{R}X, 0) = -K_X D \cdot n \log n + O(n), \quad n \to +\infty; \tag{49}$$

(iv) if a big and nef real effective divisor class D on X satisfies $D^2 \le 2$, then $W(X, D, \mathbb{R}X, 0) = 0$ as long as $-K_X D \ne 4$.

The proof is given in Section 5.9 and is based on the same ideas as the proof of Theorem 6. Notice that Theorem 7 (iv) implies the following statement: for a real Del Pezzo surface X of degree 2 with $\mathbb{R}X = S^2$ there are infinitely many nef and big real divisors D such that $W(X, D, \mathbb{R}X, 0) = 0$. Indeed, represent X as an ellipsoid blown up at 3 pairs of complex conjugate points and choose basis $L_1, L_2, E_1, ..., E_6$ of $\mathrm{Pic}(X)$, where L_1, L_2 are generators of the ellipsoid and $E_1, ..., E_6$ are the exceptional divisors of the blow up; then, each divisor $D = m(L_1 + L_2) - n(E_1 + ... + E_6)$, where $m^2 - 3n^2 = 1$ and $m \ne 7$, is real and nef, and it satisfies $D^2 = 2$ and $-K_X D = 4m - 6n \ne 4$.

Theorem 8 *Let X be a real del Pezzo surface of degree 2 with disconnected real point set, let F and F' be two distinct connected components of $\mathbb{R}X$, and let $\varphi \in H_2(X \setminus (F \cup F'); \mathbb{Z}/2)$ be a conjugation invariant class. Then, $W(X, D, F, \varphi) = 0$ for any big and nef real effective divisor class D on X such that $-K_X D \ge 3$.*

Proof. Straightforward from (23) and (45). □

The vanishing statement given by Theorem 8 was also proved by E. Brugallé and N. Puignau (cf. [4, Proposition 3.3]).

5.5 Auxiliary statements

Let $\pi : \mathfrak{X} \to (\mathbb{C}, 0)$ be a proper holomorphic submersion of a smooth three-dimensional variety $\mathfrak{X}$ (with $(\mathbb{C}, 0)$ being understood as a disc germ), where each fiber $\mathfrak{X}_t, t \ne 0$, is a del Pezzo surface of degree 2 and the central fiber $Y = \mathfrak{X}_0$ contains a smooth rational curve E such that (Y, E) is a monic log-del Pezzo pair. In what follows we identify the Picard groups of the fibers as in Remark 20.

Lemma 30 *Let $X = \mathfrak{X}_t$ for some $t \ne 0$, and let $D \in$* $\mathrm{Pic}(X)$.

(i) If D is big and X-nef, then $-K_X D > 1$, and the linear system $|D|_X$ contains an irreducible rational curve.

(ii) The divisor class D is X-nef if and only if its intersection with any (-1)-curve on X is non-negative. In this case $D^2 \ge 0$. The divisor class D is Y-nef if and only if its intersection with E and any (-1)-curve on Y is non-negative. If D is Y-nef then it is X-nef.

(iii) If D is nonzero and X-nef, and satisfies $D^2 = 0$, then $D = kD''$, where D' is primitive (i.e., not multiple of another divisor). Furthermore, $-K_X D' = 2$, $\dim |D'|_X = 1$, and a generic element of $|D'|_X$ is a smooth connected rational curve. If $D'E \ge 0$, then $|D'|_Y$ is one-dimensional with a smooth, connected, rational curve as a generic element. Furthermore, if $D'E > 1$, then $D' = -(K_Y + E)$.

(iv) If $D \in \mathrm{Pic}(Y, E)$ a Y-nef and big divisor, satisfying $R_Y(D, 0) > 0$. Then the divisor class $D' = D - E - \sum_{E' \in \mathcal{E}(E)^{\perp D}} E'$ is Y-nef and satisfies $D'E' = 0$ for all $E' \in \mathcal{E}(E)^{\perp D}$; furthermore, if $D' \neq 0$, then D' is presented by the union of curves different from E and crossing E positively.

(v) If D is big and X-nef such that $\mathcal{E}(D, E) \neq \emptyset$, then $D - mE$ with $m > 0$ cannot be represented by an irreducible curve in Y.

Proof. It is known that big and nef divisors on del Pezzo surfaces are effective and can be represented by irreducible rational curves (see, for instance, [8, Theorems 3, 4, and Remark 3.1.4]). Hence $-K_X D > 0$. In the case when $D = -K_X$ or when $-K_X - D$ is effective, the inequality $-K_X D > 1$ can easily be verified. If $-K_X - D$ is not effective, then $-K_X D > 1$ due to $\dim |-K_X| = 2$.

Statement (ii) on the X-nefness (respectively, Y-nefness) follows from the fact that the effective cone in $\mathrm{Pic}(X)$ (respectively, $\mathrm{Pic}(Y)$) is generated by (-1)-curves (respectively, by (-1)-curves and E).

If D is Y-nef, then $DE' \ge 0$ for all (-1)-curves E' on Y, and $DE \ge 0$. Any (-1)-curve E'' in X degenerates either into a (-1)-curve of Y, or into a curve $E + E'$ with a (-1)-curve E' on Y, and hence in both the cases $DE'' \ge 0$.

The nonnegativity of D^2 in statement (ii) and the part of statement (iii), concerning divisors and linear systems on X, follow, for instance, from [8, Theorems 3, 4, and Remark 3.1.4]. In particular, if D' is primitive and satisfies $(D')^2 = 0$, then a general curve in $|D'|_X$ is non-singular, rational.

Let $D'E \ge 0$ in statement (iii). Suppose that $D'E' = 0$ for some (-1)-curve $E' \in \mathcal{E}(E)$. Then we can blow down E' and reduce the degenerating family to a family of del Pezzo surfaces, which immediately yields that $\dim |D'|_Y = 1$ as well as the fact that a generic element of $|D'|_Y$ can be chosen to be a smooth rational curve. Suppose that $D'E' > 0$ for all $E' \in \mathcal{E}(E)$. By Proposition 21, a general curve $C_t \in |D'|_{\mathfrak{X}_t}$, $t \neq 0$, degenerates into a curve $C_0 + mE \in |D'|_Y$ with some $m \ge 0$ and $C_0 \not\supset E$. If m were positive, we would have $(C_0)^2 = (D')^2 - 2D'E - 2m^2 \le -2$, and, in view of $-K_Y D' = 2$ (comes from genus formula) and $-K_Y C \ge -1$ for all irreducible curves $C \neq E$, we would get C_0 consisting of components with negative self-intersection, a contradiction to $\dim |D'|_Y \ge 1$.

Let $D'E > 1$ in statement (iii), then $-(K_Y+E)D' = -K_YD' - D'E = 2 - D'E \leq 0$, which in view of the nefness of $-(K_Y+E)$, yields $D' = -(K_Y+E)$.

In view statement (iii), to prove (iv) it is enough to check that $D-E$ non-negatively crosses each (-1)-curve of Y. If $\mathcal{E}(E)^{\perp D} = \emptyset$, then this immediately follows from the fact that $EE' = 1$ for all $E' \in \mathcal{E}(E)$.

In the case of $\mathcal{E}(E)^{\perp D} \neq \emptyset$, we have $D'E' < 0$ for all $E' \in \mathcal{E}(E)^{\perp D}$, and hence $D - mE$ with $m > 0$ cannot be represented by an irreducible curve in Y: any curve in $|D - mE|_Y$ must contain all $E' \in \mathcal{E}(E)^{\perp D}$ as components, and such a component cannot be unique, otherwise D would not be big. This proves (v). Furthermore, formula (33) and statement (i) yield

$$N_Y(D, 0, (DE)e_1) = GW_0(X, D) > 0 \ . \tag{50}$$

Since $R_Y(D,0) > 0$, computing $N_Y(D, 0, (DE)e_1)$ via a sequence of formulas (66) from [18] written in the form

$$N_Y(D, 0, je_1, (DE-j)e_1) = N_Y(D, 0, (j+1)e_1, (DE-j-1)e_1) + S_j^{\mathbb{C}} \ , \tag{51}$$

$$j = 0, ..., DE \ ,$$

where $S_j^{\mathbb{C}}$ stands for the second sum in the right-hand side of the cited formula, and $N_Y(D, 0, (DE+1)e_1, -e_1)$ is zero by definition, we get $S_0^{\mathbb{C}} + ... + S_{D'E}^{\mathbb{C}} = GW_0(X, D) > 0$. That means the divisor $D - E$ is effective, and the divisor class $D' = D - E - \sum_{E' \in \mathcal{E}(E)^{\perp D}} E'$ is represented by a curve C', whose all components are disjoint from the (-1)-curves $E' \in \mathcal{E}(E)^{\perp D}$ and intersect with E. Notice, first, that C' does not contain (-1)-curves disjoint from E, and hence $D'E'' \geq 0$ for all (-1)-curves E'' with $E''E = 0$, and, second, $(D')^2 \geq 0$, since otherwise, C' would contain a (-1)-curve crossing E and disjoint from the other components of C' and from $E' \in \mathcal{E}(E)^{\perp D}$, contrary to the definition of $\mathcal{E}(E)^{\perp D}$. Altogether this yields the required statement. $\Box$

Remark 31 *Notice that Lemma 30 can be applied to all ABV-families introduced in section 5.2. Namely, over $\mathbb{C}$ we can contract the quadric surface in the central fiber of the family along one of the rulings and thus obtain a family exactly as in Lemma 30.*

The following two claims will be used in the proof of the asymptotic statements in Theorems 6 and 7.

Lemma 32 *Let $\{a_n\}_{n \geq 0}$ be a sequence of positive numbers, $a_0 = 1$, and let $0 \leq f(n) \leq n$ an integral-valued function. If*

- *either*

$$a_{n+1} \geq \lambda a_{f(n)} a_{n-f(n)}, \quad \text{for all } n \geq n_0 \text{ and some } \lambda > 0 \ , \tag{52}$$

- *or*

$$a_n \geq \lambda a_{f(n)} a_{n-f(n)}, \quad \textit{for all } n \geq n_0 \textit{ and some } \lambda > 0 \ , \tag{53}$$

then there exist $\xi, \eta > 0$ *such that* $a_n \geq \xi\eta^n$ *for all* $n \geq n_0$.

Proof. Straightforward induction on n with ξ, η found from the equations

$$\lambda\xi = \eta, \quad \xi\eta^{n_0} = a_0$$

in the first case, and the equations

$$\lambda\xi = 1, \quad \xi\eta^{n_0} = a_{n_0}$$

in the second case. $\square$

Lemma 33 *Asymptotic relations (48) and (49) follow from*

$$\log W(X, nD, \mathbb{R}X, [\mathbb{R}X \setminus F]) \geq -K_X D \cdot n \log n + O(n), \quad n \to +\infty \ .$$

Proof. Straightforward from

$$\begin{aligned} \log |W(X, nD, \mathbb{R}X, [\mathbb{R}X \setminus F])| &\leq \log GW_0(X, nD) \\ &= -K_X D \cdot n \log n + O(n), \quad n \to +\infty \ , \end{aligned}$$

(see [12, Theorem 1]). $\square$

5.6 Non-negativity of w-numbers

In each of the families in Section 5.2, the central fiber coincides with or contains a real del Pezzo surface Y of degree 2 with a smooth real rational curve $E \subset Y$ whose nonempty real part $\mathbb{R}E$ lies in some connected component F of $\mathbb{R}Y$.

Lemma 34 *If* Y *appears in a regular or hyperbolic ABV-family (described in Section 5.2), then, for any* $D' \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$ *such that* $-K_Y D' \geq 1$ *and, for any* $\alpha, \beta \in \mathbb{Z}_+^{\infty,\mathrm{odd}}$ *such that* $I(\alpha + \beta) = D'E$*, we have*

$$W_{Y,E,0}(D', \alpha, \beta, 0) \geq 0 \ . \tag{54}$$

If Y *appears in an elliptic ABV-family described in Section 5.2, then, for any component* F_+ *of* $F \setminus \mathbb{R}E$*, for any* $D' \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$ *such that* $-K_Y D' \geq 2$ *and* $D'E$ *is even, and for any* $\alpha, \beta \in \mathbb{Z}_+^{\infty,\mathrm{even}}$ *such that* $I(\alpha + \beta) = D'E$*, we have*

$$W^-_{Y,F_+,[\mathbb{R}Y \setminus F]}(D', \alpha, \beta, 0) \geq 0 \ . \tag{55}$$

Proof. The right-hand side of (54) can be recursively computed via formula (6), whereas the right-hand side of (55) via formula (28). Notice that the coefficients in both these formulas are non-negative (for instance, values of $\eta_-(l)$ given in (29) are all non-negative). Hence (54) and (55) would immediately follow if the initial values given in Propositions 9 and 14 are non-negative. In the case of X of type $\mathbb{P}^2_{a,b}$, $a+2b=7$, this is so, since $\mathcal{E}(E)$ consists of $2b$ pairs of disjoint complex conjugate lines and of $6-2b$ pairs of intersecting real lines. In the case of X of type $1\langle 1\rangle^+$, $\langle 0\rangle^-$, or $1\langle 1\rangle^-$ this is so, since the corresponding real nodal plane quartic curve Q_Y has no real lines passing through the node z of Q_Y and tangent to Q_Y at a point $z'\ne z$ (*cf.* Remark 23). In the case of X of type $\langle q\rangle^+$, $2\le q\le 4$, this is so, since we may assume that the tangency points of the real tangents to Q_Y, passing through z, lift to Y as solitary nodes in $\mathbb{R}Y\setminus F$ or in F_- (see Figure 1(c)). $\square$

5.7 Positivity and asymptotics statements for surfaces of types $\mathbb{P}^2_{a,b}$, $a+2b=7$, and $1\langle 1\rangle^+$

For a real del Pezzo surface X as in the title of this section, the real part $\mathbb{R}X$ is connected. We put $F=\mathbb{R}X$, $\varphi=0$, and consider an ABV-family for X as in Section 5.2. Following Theorem 4(1), we identify $\mathrm{Pic}^{\mathbb{R}}(X)$ and $\mathrm{Pic}^{\mathbb{R}}(Y)$, where Y is the central fiber of that ABV-family. Furthermore, we restrict ourselves to the case $DE\ge 0$.

5.7.1 Positivity

From formula (43) and inequality (54) it follows that

$$W(X,D,\mathbb{R}X,0)\ge W(X,D-mE,\mathbb{R}X,0)\quad\text{for all } m\ge 0\ . \tag{56}$$

Indeed, the both terms are sums of non-negative w-numbers, and all the w-numbers occurring in the development of the right-hand side appear in the development of the left-hand side with non-smaller coefficients:

$$\binom{DE+2m+2k}{m+k}\ge\binom{DE+2m+2k}{k}\quad\text{for all } k\ge 0\ .$$

We will prove inequality (47) for all real big and X-nef divisors $D\in\mathrm{Pic}(X)$, assuming $DE\ge 0$ and using induction on $\rho(D)=-(K_X+E)D$.

Observe that $\rho(D)>0$, since D is big and $|-K_X-E|$ defines a conic bundle. Suppose that $\rho(D)=1$, or, equivalently, $R_Y(D,0)=0$. From formula (43) and inequality (54), we get

$$W(X,D,\mathbb{R}X,0)\ge W_{Y,E,0}(D,0,(DE)e_1,0)\ , \tag{57}$$

and then, applying formula (6) DE times, we end up with

$$W_{Y,E,0}(D,0,(DE)e_1,0)\ge W_{Y,E,0}(D,(DE)e_1,0,0)=1\ ,$$

the latter equality coming from Proposition 9(1iii) and inequality

$$DE = -K_Y D - \rho(D) = -K_X D - 1 \overset{\text{Lemma 30(i)}}{>} 0 \ .$$

Assume that $\rho(D) > 1$, or, equivalently,

$$R_Y(D, 0) > 0 \ . \tag{58}$$

Suppose that $\mathcal{E}(E)^{\perp D} = \emptyset$. Then, $D - E$ is Y-nef and satisfies $(D - E)^2 \geq 0$ (see Lemma 30(ii)).

If $(D - E)^2 > 0$, then $D - E$ is big and X-nef, and $\rho(D - E) = \rho(D) - 2 < \rho(D)$. Thus, by the induction assumption and (56)

$$W(X, D, \mathbb{R}X, 0) \geq W(X, D - E, \mathbb{R}X, 0) > 0 \ .$$

If $(D - E)^2 = 0$, then by Lemma 30(iii), $D - E = kD''$ with $k \geq 1$ and a primitive $D'' \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$ such that $D''E > 0$, $(D'')^2 = 0$, $\dim |D''|_Y = -K_Y D'' - 1 = 1$, and the linear system $|D''|_Y$ contains a real, rational, smooth curve. If $k = 1$, then $W(X, D - E, \mathbb{R}X, 0) = 1$, and again (47) follows. If $k \geq 2$, then $R_Y(D, (DE)e_1) = -K_Y D - 1 = -kK_Y D'' - 1 = 2k - 1$, and we get

$$\begin{aligned} W(X, D, \mathbb{R}X, 0) &\overset{(43)\ \&\ (54)}{\geq} W_{Y,E,0}(D, 0, (DE)e_1, 0) \\ &\overset{(6)\ \&\ (54)}{\geq} W_{Y,E,0}(D, (k-2)e_1, k((D''E) - 1)e_1, 0) \\ &\overset{(6)\ \&\ (54)}{\geq} (W_{Y,E,0}(D'', 0, (D''E)e_1, 0))^k \overset{\text{Lemma 30(iii)}}{=} 1 \ . \end{aligned} \tag{59}$$

Suppose now that $\mathcal{E}(E)^{\perp D} \neq \emptyset$. By Lemma 30(v) and Theorem 4(3), we have

$$W(X, D, \mathbb{R}X, 0) = W_{Y,E,0}(D, 0, (DE)e_1) \ . \tag{60}$$

By Lemma 30(iv), the divisor class $D' = D - E - \sum_{E' \in \mathcal{E}(E)^{\perp D}} E'$ is Y-nef, and hence, by Lemma 30(ii), is also X-nef.

Assume that $(D')^2 > 0$. Since $\rho(D') = \rho(D) - 2$, we have $W(X, D', \mathbb{R}X, 0) > 0$, which due to $\mathcal{E}(E)^{\perp D'} \supset \mathcal{E}(E)^{\perp D} \neq \emptyset$ yields (*cf.* (60))

$$W_{Y,E,0}(D', 0, (D'E)e_1, 0) = W(X, D', \mathbb{R}X, 0) > 0 \ .$$

Appropriately applying formula (6) and using (54), we obtain

$$\begin{aligned} W(X, D, \mathbb{R}X, 0) &= W_{Y,E,0}(D, 0, (DE)e_1, 0) \\ &\geq W_{Y,E,0}(D, (s-1)e_1, (DE - s + 1)e_1, 0) \\ &\geq W_{Y,E,0}(D', 0, (D'E)e_1, 0) > 0 \ , \end{aligned} \tag{61}$$

where $s = \mathrm{card}(\mathcal{E}(E)^{\perp D})$ (notice that $s = -K_Y D + K_Y D' \leq -K_Y D - 1$ and $s - 1 = (D - E - D')E - 1 = DE + 1 - D'E \leq DE$).

Assume that $(D')^2 = 0$. If $D' = 0$, then the relations (61) transform to

$$W(X, D_1, \mathbb{R}X, 0) \ge \prod_{\mathcal{D}_1} W_{Y,E,0}(\mathcal{D}_1, 0, e_1, 0) \prod_{\mathcal{D}_2} W_{Y,E,0}(\mathcal{D}_2, 0, 0, e_1) > 0,$$

where $\mathcal{D}_1$ runs over the real elements of $\mathcal{E}(E)^{\perp D}$, and $\mathcal{D}_2$ runs over the pairs of complex conjugate elements in $\mathcal{E}(E)^{\perp D}$. If $D' \ne 0$, by Lemma 30(iii), $D' = kD''$ with one-dimensional linear system $|D''|_Y$ represented by a smooth real rational curve C'', and the relations (61) transform to

$$\begin{aligned} W(X, D, \mathbb{R}X, 0) &\ge W_{Y,E,0}(D, le_1, (DE - l)e_1, 0) \\ &\ge (W_{Y,E,0}(D'', 0, (D''E)e_1, 0))^k = 1 \ , \end{aligned} \tag{62}$$

where $l = -K_Y D - 2 - k$. Note that

$$k = (-K_Y D'')^{-1}(-K_Y D - s) \le (-K_Y D - 1)/2 \le -K_Y D - 2$$

and $l \le DE$, the latter inequality coming from the relations

$$DE = D'E - 2 + E \cdot \sum_{E' \in \mathcal{E}(E)^{\perp D}} E' = D'E - 2 + s \ge k - 2 + s,$$

$$l = -K_Y D - 2 - k = -K_Y D' - K_Y E - K_Y \cdot \sum_{E' \in \mathcal{E}(E)^{\perp D}} E' - 2 - k = k - 2 + s.$$

5.7.2 Asymptotics

Let $D \in \mathrm{Pic}(X)$ be a real, big and X-nef divisor. By Theorem 4(2) we can suppose that D is Y-nef. We prove the asymptotic relation (48) by induction on $\tau(D) = \min\{DE' \ : \ E' \in \mathcal{E}(E)\}$.

Let $\tau(D) = 0$, or, equivalently $\mathcal{E}(E)^{\perp D} \ne \emptyset$. Put $s = \mathrm{card}(\mathcal{E}(E)^{\perp D})$. By Lemma 30(iv), there exists an integer $m_0 \ge 1$ such that the divisors $D'_m = mD - E - \sum_{E' \in \mathcal{E}(E)^{\perp D}} E'$ are big and Y-nef and satisfy $D'_m E > 0$ for all integers $m \ge m_0$. Note also that by (47) and (60), one has

$$W_{Y,E,0}(D'_m, 0, (D'_m E)e_1, 0) > 0, \quad m \ge m_0 \ .$$

Put $\widetilde{D} = D'_{m_0}$ and $\widetilde{s} = \mathrm{card}(\mathcal{E}(E)^{\perp \widetilde{D}})$. Again there exists an integer $m_1 \ge 1$ such that $\widetilde{D}'_{m_1}$ is big and Y-nef and satisfy

$$\widetilde{D}'_m E > 0, \quad W_{Y,E,0}(\widetilde{D}'_m, 0, (\widetilde{D}'_m E)e_1, 0) > 0 \quad \text{for all integers } m \ge m_1 \ .$$

For any integer $n \ge 2$, we have decompositions

$$\begin{cases} \widetilde{D}'_{nm_1} - E = \widetilde{D}'_{m_1 i(n)} + \widetilde{D}'_{(n-i(n))m_1} + \sum_{E' \in \mathcal{E}(E)^{\perp D}} E', \\ -K_Y \widetilde{D}'_{nm_1} - 2 = (-K_Y \widetilde{D}'_{m_1 i(n)} - 1) + (-K_Y \widetilde{D}'_{(n-i(n))m_1} - 1) + s, \end{cases} \quad i(n) = \left[\frac{n}{2}\right],$$

and inequality $\widetilde{D}'_{nm_1}E \geq \tilde{s}$ coming from the first relation. (Note that $\tilde{s} \geq s$ and, moreover, if we choose $m_0 \geq 3$ then $\mathcal{E}(E)^{\perp \widetilde{D}} = \mathcal{E}(E)^{\perp D}$ and $\tilde{s} = s$.) By Proposition 9, the product of all the terms $W_{Y,E,0}(\mathcal{D}, 0, \beta^{\rm re}, \beta^{\rm im})$ with $\mathcal{D}$ combined from $E' \in \mathcal{E}(E)^{\perp D}$ equals 1, and hence by formula (6) and inequality (54), one has

$$W_{Y,E,0}(\widetilde{D}'_{nm_1}, 0, (\widetilde{D}'_{nm_1}E)e_1, 0) \geq W_{Y,E,0}(\widetilde{D}'_{nm_1}, \tilde{s}e_1, (\widetilde{D}'_{nm_1}E - \tilde{s})e_1, 0)$$

$$\geq \frac{1}{2}(-K_Y\widetilde{D}'_{nm_1} - 2)!(\widetilde{D}'_{m_1 i(n)}E)(\widetilde{D}'_{(n-i(n))m_1}E)$$

$$\times \frac{W_{Y,E,0}(\widetilde{D}'_{m_1 i(n)}, 0, (\widetilde{D}'_{m_1 i(n)}E)e_1, 0)}{(-K_Y\widetilde{D}'_{m_1 i(n)} - 1)!} \cdot \frac{W_{Y,E,0}(\widetilde{D}'_{(n-i(n))m_1}, 0, (\widetilde{D}'_{(n-i(n))m_1}E)e_1, 0)}{(-K_Y\widetilde{D}'_{(n-i(n))m_1} - 1)!}.$$

There exists $\lambda > 0$ such that, for all integers $n \geq 2$, one has

$$\frac{(\widetilde{D}'_{m_1 i(n)}E)(\widetilde{D}'_{(n-i(n))m_1}E)}{2(-K_Y\widetilde{D}'_{nm_1} - 1)} = \frac{(m_1 i(n)\widetilde{D}E + 2 - \tilde{s})((n-i(n))m_1\widetilde{D}E + 2 - \tilde{s})}{2(-nm_1K_Y\widetilde{D} - 1 - \tilde{s})} > \lambda \ .$$

Hence, the sequence

$$a_n = \frac{W_{Y,E,0}(\widetilde{D}'_{nm_1}, 0, (\widetilde{D}'_{nm_1}E)e_1, 0)}{(-K_Y\widetilde{D}'_{nm_1} - 1)!}, \quad n \geq 1 \ ,$$

satisfies the relation (52). Thus, by Lemma 32, one has

$$\log W_{Y,E,0}(\widetilde{D}'_{nm_1}, 0, (\widetilde{D}'_{nm_1}E)e_1, 0) \geq$$

$$-K_Y\widetilde{D}'_{m_1} \cdot n\log n + O(n) = -K_XD \cdot m_0m_1n\log n + O(n), \quad n \to +\infty \ . \qquad (63)$$

Observe that

$$(n+1)m_0m_1D - E = D'_{(n+1)m_0m_1} + \sum_{E' \in \mathcal{E}(E)^{\perp D}} E' \ ,$$

$$D'_{(n+1)m_0m_1+j} - E = D'_{m_0m_1+j} + D'_{nm_0m_1} + \sum_{E' \in \mathcal{E}(E)^{\perp D}} E', \quad 0 \leq j < m_0m_1 \ ,$$

$$D'_{nm_0m_1} = \widetilde{D}'_{nm_1} + nm_1(E + \sum_{E' \in \mathcal{E}(E)^{\perp D}} E') \ ,$$

and hence, applying formula (6) and inequality (54) as above and omitting positive integer coefficients, we obtain

$$W_{Y,E,0}(((n{+}1)m_0m_1{+}j)D, 0, ((n{+}1)m_0m_1{+}j)(DE)e_1, 0)$$

$$\geq W_{Y,E,0}(D'_{(n+1)m_0m_1+j}, 0, (D'_{(n+1)m_0m_1+j}E)e_1, 0)$$

$$\geq W_{Y,E,0}(D'_{nm_0m_1}, 0, (D'_{nm_0m_1}E)e_1, 0)$$

$$\geq W_{Y,E,0}(D'_{nm_0m_1} - E - \sum_{E' \in \mathcal{E}(E)^{\perp D}} E', 0, (D'_{nm_0m_1}E + 2 - s)e_1, 0) \geq \ \ldots$$

$$\geq W_{Y,E,0}(D'_{nm_0m_1} - nm_1(E + \sum_{E' \in \mathcal{E}(E)^{\perp D}} E'), 0, (D'_{nm_0m_1}E + nm_1(2-s))e_1, 0)$$

$$= W_{Y,E,0}(\widetilde{D}_{nm_1}, 0, (\widetilde{D}_{nm_1}E)e_1, 0) \quad \text{for all integers } n \geq 2,\ 0 \leq j < m_0m_1 \ .$$

These inequalities, together with (60) and (63), imply

$$\begin{aligned} \log W(X, nD, \mathbb{R}X, 0) &= \log W_{Y,E,0}(nD, 0, n(DE)e_1, 0) \\ &\geq -K_X D \cdot n \log n + O(n), \quad n \to +\infty \ . \end{aligned}$$

and hence (48) by Lemma 33.

Now suppose that $\tau(D) > 0$. By Lemma 30(ii), $D - E$ is Y-nef, $(D-E)^2 \geq 0$ and $\tau(D-E) = \tau(D) - 1$.

If $(D-E)^2 > 0$, then by (56) and the induction assumption

$$\begin{aligned} \log W(X, nD, \mathbb{R}X, 0) \geq \log W(X, n(D-E), \mathbb{R}X, 0) = \\ -K_X(D-E) \cdot n \log n + O(n) = -K_X D \cdot n \log n + O(n), \end{aligned}$$

which as above implies (48).

If $(D-E)^2 = 0$, then by Lemma 30(iii), $D - E = kD''$, where $k \geq 1$, D'' is a primitive Y-nef divisor represented by a real smooth rational curve crossing E, and $\dim|D''| = 1$. Consider the divisor $D'_2 = 2D - E$. It is Y-nef and satisfies $D'_2E = 2DE + 2 \geq 2$. It follows from formula (6), inequality (54), decompositions

$$D'_2 - E = 2kD'', \quad -K_Y D'_2 - 2 = 2k(-K_Y D'' - 1) + 2(k-1)$$

and inequality $D'_2E = 2kD''E - 2 \geq 2k - 2$ that

$$W_{Y,E,0}(D'_2, 0, (D'_2E)e_1, 0) \geq W_{Y,E,0}(D'_2, 2(k-1)e_1, (D'_2E - 2(k-1))e_1, 0)$$

$$\geq W_{Y,E,0}(D'', 0, (D''E)e_1, 0)^{2k} = 1 \ .$$

In the same way from decompositions

$$\begin{cases} D + nD'_2 - E = kD'' + nD'_2, \\ -K_Y(D + nD'_2) - 2 = k(-K_Y D'' - 1) + (-nK_Y D'_2 - 1) + (k-1) \end{cases} \tag{64}$$

and

$$\begin{cases} (n+1)D'_2 - E = 2kD'' + [n/2]D'_2 + [(n+1)/2]D'_2, \\ -(n+1)K_Y D'_2 - 2 = 2k(-K_Y D'' - 1) + (-[n/2]K_Y D'_2 - 1) \\ \qquad\qquad + (-[(n+1)/2]K_Y D'_2 - 1) + 2k \end{cases} \tag{65}$$

for all $n \geq 2$, we obtain

$$W_{Y,E,0}(D + nD'_2, 0, (D + nD'_2E)e_1, 0) \geq W_{Y,E,0}(nD'_2, 0, n(D'_2E)e_1, 0)$$

and

$$W_{Y,E,0}((n+1)D'_2, 0, (n+1)(D'_2E)e_1, 0) \geq \frac{1}{2}(-(n+1)K_Y D'_2 - 2)! \cdot (D'_2E)^2$$

$$\times \left[\frac{n}{2}\right]\cdot\left[\frac{n+1}{2}\right]\cdot\frac{W_{Y,E,0}([\frac{n}{2}]D'_2,0,[\frac{n}{2}](D'_2E)e_1,0)}{(-[\frac{n}{2}]K_YD'_2-1)!}\cdot\frac{W_{Y,E,0}([\frac{n+1}{2}]D'_2,0,[\frac{n+1}{2}](D'_2E)e_1,0)}{(-[\frac{n+1}{2}]K_YD'_2-1)!}\ .$$

Since there exists $\lambda > 0$ such that

$$\frac{[n/2]\cdot[(n+1)/2](D'_2E)^2}{2(-(n+1)K_YD'_2-1)} \geq \lambda > 0 \quad \text{for all } n \geq 2,$$

by Lemma 32 we get

$$\log W_{Y,E,0}((2n+j+1)D,0,(2n+j+1)(DE)e_1,0) \geq$$
$$\log W_{Y,E,0}(D+nD'_2,0,((D+nD'_2)E)e_1,0) \geq \log W_{Y,E,0}(nD'_2,0,n(D'_2E)e_1,0) \geq$$
$$-K_YD'_2\cdot n\log n + O(n) = -K_XD\cdot 2n\log n + O(n)$$

which in view of (56), (57), and hence (48) by Lemma 33.

5.8 Positivity and asymptotics statement for surfaces of types $\langle 0\rangle^-$, $1\langle 1\rangle^-$, and $\langle q\rangle^+$, $2 \leq q \leq 4$

Let X be a real del Pezzo surface as in the title, and let F be any of the connected components of $\mathbb{R}X$. We include X into an elliptic ABV-family as described in section 5.2. If X is of type $\langle 0\rangle^-$, we assume, in addition, that F is chosen as specified in section 5.3.

The central fiber of that ABV-family contains a real nodal del Pezzo pair (Y,E). Observe that, for all divisors $D \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$, the intersection number DE is even, and, in addition, if X is of type $\langle q\rangle^+$, $2 \leq q \leq 4$, then D^2 and DK_Y are even as well. Denote by $\hat{F}$ the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$, and by F^+ the component of $\hat{F} \setminus \mathbb{R}E$ to which merges the component F of $\mathbb{R}X$.

Following Theorem 5, we identify $\mathrm{Pic}^{\mathbb{R}}(X)$ with a subgroup of $\mathrm{Pic}^{\mathbb{R}}(Y)$.

5.8.1 Positivity

Let $D \in \mathrm{Pic}^{\mathbb{R}}(X)$ be X-nef, big, and F-compatible. Then $DE = 0$, and by formula (46), one has

$$W(X,D,F,[\mathbb{R}X \setminus F]) = W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{F}]}(D,0,0,0)\ . \qquad (66)$$

Thus, to prove (47) it is sufficient to show that for any big, Y-nef, and F_+-compatible divisor class $D' \in \mathrm{Pic}^{\mathbb{R}}(Y)$, one has

$$W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{F}]}(D',0,(D'E/2)e_2,0) > 0\ . \qquad (67)$$

In what follows, we prove this inequality by induction on $\rho(D') = -(K_Y+E)D'$.

As we know, $\rho(D') = -(K_Y+E)D' > 0$, since D' is a big X-nef divisor and $\dim|-K_X-E| = 1$. If $\rho(D') = 1$, then it follows from Remark 27 and Proposition

28 that either X is of type $\langle 0\rangle^-$, or X is of type $1\langle 1\rangle^-$ and $F=F^{no}$, so that due to (28) and inequality (55) one has

$$W_{Y,F_+,[\mathbb{R}Y\setminus\hat F]}(D',0,(D'E/2)e_2,0)\ge 2^{D'E/2}W_{Y,F_+,[\mathbb{R}Y\setminus\hat F]}(D',(D'E/2)e_2,0,0)>0\ ,$$

where the latter (strict) inequality follows from the fact that the real part of the unique rational curve $C\in|D'|_Y$ quadratically tangent to E at $D'E/2$ generic real points lies in $\overline{F}_+$.

Let $\rho(D')\ge 2$, which is equivalent to $R_Y(D',0)>0$. By Lemma 30(iv), the divisor $D''=D'-E-\sum_{E'\in\mathcal{E}(E)^{\perp D'}}E'$ is Y-nef, and it is easy to see that D'' is F_+-compatible. Clearly, $\rho(D'')\le\rho(D')-2$. Since $D''E$ is even, we have $\mathrm{card}(\mathcal{E}(E)^{\perp D'})=2s$, $1\le s\le 3$.

So, if $(D'')^2>0$, then formula (28), inequality (55), the induction assumption, and the relations

$$2s=(D'-E-D'')E\le(D'-E)E-2=D'E,\quad D''E/2=D'E/2-s+1$$

result in

$$\begin{aligned}&W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat F]}(D',0,(D'E/2)e_2,0)\ge 2^sW^-_{Y,F_+,[\mathbb{R}Y\setminus\hat F]}(D',se_2,(D'E/2-s)e_2,0)\\ &\ge 2^s\textstyle\prod_{\mathcal{D}}W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat Y]}(\mathcal{D},0,0,e_1)\cdot(D''E)\cdot W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat F]}(D'',0,(D''E/2)e_2,0)>0\ ,\end{aligned}$$

where $\mathcal{D}$ runs over all elements $\mathcal{D}\in\mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$ combined from $E'\in\mathcal{E}(E)^{\perp D'}$.

If $D''=0$ (which is relevant only if $\mathcal{E}(E)^{\perp D'}\ne\emptyset$), we get by the same arguments the same expression but without $(D''E)\cdot W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat F]}(D'',0,(D''E)/2,0)$ in the very end, which again leads to the required positivity.

Thus, it remains to treat the case $D''\ne 0$ and $(D'')^2=0$. Then, since $D''E$ is positive and even, Lemma 30(iii) implies that $D''=-k(K_Y+E)$, $k\ge 1$.

If X is of type $1\langle 1\rangle^-$ and $F=F^{no}$, then the both $L',L''\in|-K_Y-E|_Y$ (see Section 3.1) are real, $\mathbb{R}L'\cup\mathbb{R}L''\subset\overline{F}_+$. Using

$$R_Y(D',(D'E/2)e_2)=-(K_Y+E)D'+D'E/2-1=2+(k+s-1)-1=k+s\ ,\quad (68)$$

and applying formula (28) and inequality (55), we derive that

$$\begin{aligned}W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat F]}(D',0,(D'E/2)e_2,0)&=W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat F]}(D',0,(k+s-1)e_2,0,0)\\ &\ge 2^{k+s-1}W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat F]}(D',(k+s-1)e_2,0,0)\\ &\ge 2^{k+s-1}\eta^-(k)\prod_{\mathcal{D}}W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat Y]}(\mathcal{D},0,0,e_1)>0\ ,\end{aligned}$$

where the latter expression corresponds to the summand in the second sum of the right-hand side of (28), in which the product runs over pairs of divisors $\mathcal{D}\in\mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$, combined out of the lines $E'\in\mathcal{E}(E)^{\perp D'}$, the parameters in the condition (3e) of Theorem 3 are chosen to be $\alpha^{(0)} = \beta^{(0)} = 0$, $l = k = (D-E)E/2$, and the value of $\eta^-(k)$ is given by formula (29):

$$\eta^-(k) = \begin{cases} (k/2+1)(2-(-1)^{k/2}), & k \text{ is even,} \\ 2k + 2(-1)^{(k-1)/2}, & k \text{ is odd} \end{cases}$$

If X is either of type $\langle 0\rangle^-$, or of type $1\langle 1\rangle^-$ with $F = F^{\,o}$, or of type $\langle q\rangle^+$, $2 \le q \le 4$, then k is even by the F_+-compatibility condition (*cf.* Proposition 28, conditions (FC1) and (FC2)). Now from (68), formula (28), and inequality (55), we derive

$$\begin{aligned} W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{F}]}(D',0,(D'E/2)e_2,0) &= W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{F}]}(D',0,(k+s-1)e_2,0,0) \\ &\ge 2^{k/2+s-1} W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{F}]}(D',(k/2+s-1)e_2,(k/2)e_2,0,0) \\ &\ge 2^{k/2+s-1}4^{k/2}\prod_{\mathcal{D}} W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{Y}]}(\mathcal{D},0,0,e_1) > 0\ , \end{aligned}$$

where the latter expression corresponds to the summand in the second sum in the right-hand side of (28), matching the parameter values $\alpha^{(0)} = 0$, $\beta^{(0)} = (k/2)e_2$, and $l = 0$ in the condition (3e) in Theorem 3.

5.8.2 Asymptotics

Let $D \in \mathrm{Pic}^{\mathbb{R}}(X)$ be X-nef, big, and F-compatible. In particular, $DE = 0$, which by Lemma 30(ii) yields that D is Y-nef, and hence $W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{Y}]}(D,0,0,0) > 0$ (see (67)). Since $-K_Y D > 1$ (see Lemma 30(i)), formula (28) for $W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{F}]}(D,0,0,0)$ reads

$$W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{F}]}(D,0,0,0) = 2W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{F}]}(D-E,0,e_2,0)\ , \qquad (69)$$

thus,

$$W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{F}]}(D-E,0,e_2,0) > 0\ .$$

Again, using formula (28) and non-negativity statement (55), we obtain

$$W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{F}]}(nD-E,0,e_2,0) \ge 2(-nK_Y D-2)!$$

$$\times \frac{W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{F}]}([\frac{n}{2}]D-E,0,e_2,0)}{(-[\frac{n}{2}]K_Y D-1)!}\cdot\frac{W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{F}]}([\frac{n+1}{2}]D-E,0,e_2,0)}{(-[\frac{n+1}{2}]K_Y D-1)!}$$

for all $n \ge 2$, and hence the sequence

$$a_n = \frac{W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{F}]}(nD-E,0,e_2,0)}{(-nK_Y D)!}, \quad n \ge 1\ ,$$

satisfies (53) with

$$\lambda = \inf_{n>3}\frac{[n/2]\cdot[(n+1)/2]K_Y D}{n(-nK_Y D-1)} > 0\ .$$

So, from Lemma 32 we derive

$$\liminf_{n\to\infty}\frac{\log W^-_{Y,F_+,[\mathbb{R}Y\setminus\hat{Y}]}(nD-E,0,e_2,0)}{n\log n}\ge -K_YD=-K_XD\ ,$$

and hence in view of (56), (69) applied to nD, and of Lemma 33, we obtain the desired relation (48).

5.9 Proof of Theorem 7

Take a real quadric surface in $\mathbb{P}^3$ with a spherical real point set and blow up this surface at three pairs of complex conjugate points. The resulting del Pezzo surface X is of degree 2 and of type $\langle 1\rangle^+$. We have a natural basis $L_1, L_2, E_1, \ldots, E_6$ in $\mathrm{Pic}(X)$, where:

- L_1 and L_2 are complex conjugate, $L_1^2 = L_2^2 = 0$, and $L_1L_2 = 1$,
- $E_i^2 = -1$, and $L_1E_i = L_2E_i = 0$ for $1 \le i \le 6$,
- $E_iE_j = 0$ for $1 \le i < j \le 6$,
- E_{2i-1}, E_{2i} are complex conjugate for $i = 1, 2, 3$.

Any real big effective divisors D in X can be represented as

$$D=d(L_1+L_2)-d_1(E_1+E_2)-d_2(E_3+E_4)-d_3(E_5+E_6),\quad d>0,\quad d_1,d_2,d_3\ge 0\ ,$$

in particular, D^2 is even.

Consider a degeneration of X in an elliptic ABV-family into a real nodal del Pezzo pair (Y, E) as in Section 5.2; this family corresponds to a degeneration of the real plane quartic Q_X (of type $\langle 1\rangle$) into the nodal quartic Q_Y. From Theorem 5(1) and relation $E^2 = -2$, we immediately derive that the divisor class E is either $\pm(L_1 - L_2)$, or $\pm(E_{2i-1} - E_{2i})$, $i = 1, 2, 3$. Since Q_Y does not have real tangents passing through the node (except for tangent lines at the node), there is no real (-1)-curve on Y crossing E, and we are left with the only option $E = \pm(L_1 - L_2)$ (we can assume that $E = L_1 - L_2$).

Let $D \in \mathrm{Pic}(X)$ be an X-nef and big real divisor. By Theorem 5(2) and non-negativity statement (54), one obtains

$$W(X,D,\mathbb{R}X,0)=W_{Y,E,0}(D,0,0,0)\ge 0\ ,$$

proving statement (i). If $W(X,D,\mathbb{R}X,0) = W_{Y,E,0}(D,0,0,0) > 0$, then formula (6) applied to $W_{Y,E,0}(D,0,0,0)$ must contain in the right-hand side a summand

$$c\cdot W_{Y,E,0}(D^{(1)},0,e_1,0)\cdot W_{Y,E,0}(D^{(2)},0,e_1,0)$$

with a positive integer c, and

$$D^{(1)}, D^{(2)} \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y,E), \quad D^{(1)}+D^{(2)} = D-E, \quad W_{Y,E,0}(D^{(i)},0,e_1,0) > 0,\ i=1,2\ . \tag{70}$$

Here $D^{(1)}, D^{(2)}$ cannot be represented by (-1)-curves, since $(D-E)^2 = D^2-2 \ge 0$, and complex conjugate (-1)-curves, crossing E, are disjoint.

Assume now that $D^2 \le 2$. Then $(D-E)^2 = D^2-2 \le 0$, which in the case of $W(X,D,\mathbb{R}X,0) > 0$ leaves the only option $D^{(1)} = D^{(2)} = D'$, where $(D')^2 = 0$ and $\dim|D'|_Y = 1$; hence $-K_YD' = 2$ and $-K_XD = -2K_YD' = 4$ as asserted in statement (iv).

Assume now that $D_1, D_2 \in \mathrm{Pic}^{\mathbb{R}}(X)$ are X-nef and big, and satisfy $W(X,D_i,\mathbb{R}X,0) > 0$, $i=1,2$. Show that $W(X,D_1+D_2,\mathbb{R}X,0) > 0$ in agreement with statement (ii). As we have seen above, there are $D_i^{(j)} \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$, $i,j=1,2$, such that

$$D_i^{(j)}E = 1, \quad W_{Y,E,0}(D_i^{(j)},0,e_1,0) > 0, \quad i,j=1,2\ ,$$
$$D_i^{(1)} + D_i^{(2)} = D_i - E, \quad i=1,2\ .$$

Appropriately applying formula (6), we obtain the required inequality from

$$W(X,D_1+D_2,\mathbb{R}X,0) \ge c_1W_{Y,E,0}(D_1^{(1)}+D_2,0,e_1,0)W_{Y,E,0}(D_1^{(2)},0,e_1,0)\ ,$$

and from

$$\begin{aligned} &W_{Y,E,0}(D_1^{(1)}+D_2,0,e_1,0) \ge W_{Y,E,0}(D_1^{(1)}+D_2,e_1,0,0)\\ &\quad\ge c_{2j}W_{Y,E,0}(D_1^{(1)},0,e_1,0)W_{Y,E,0}(D_2^{(1)},0,e_1,0)W_{Y,E,0}(D_2^{(2)},0,e_1,0) > 0\ , \end{aligned} \tag{71}$$

where c_1, c_{21}, c_{22} are some positive integers. If $D \in \mathrm{Pic}^{\mathbb{R}}(X)$ is X-nef and big, and is disjoint with a pair of complex conjugate (-1)-curves on X, then, as noticed in Remark 29, we can regard D as a real divisor on a surface of type $\mathbb{P}^2_{1,3}$, whose Welschinger invariants are positive by Theorem 6. If $D \in \mathrm{Pic}^{\mathbb{R}}(X)$ is X-nef and big with $W(X,D,\mathbb{R}X,0) > 0$, then

$$\begin{aligned} W(X,D-K_X,\mathbb{R}X,0) &= W_{Y,E,0}(D-K_X,0,0,0)\\ &\ge c_0\cdot W_{Y,E,0}(D^{(1)},0,e_1,0)\cdot W_{Y,E,0}(D^{(2)}-K_X,0,e_1,0)\\ &\ge 2c_0\cdot W_{Y,E,0}(D^{(1)},0,e_1,0)\cdot W_{Y,E,0}(D^{(2)},0,e_1,0)\cdot W_{Y,E,0}(-K_X-E,0,2e_1,0) > 0\ . \end{aligned}$$

To complete the proof of statement (ii), we have to show the positivity of $W(X,-2K_X,\mathbb{R}X,0)$: a direct application of Theorem 5(2) and formula (6) gives $W(X,-2K_X,\mathbb{R}X,0) = 8$.

Let $D \in \mathrm{Pic}^{\mathbb{R}}(X)$ be an X-nef and big divisor such that $W(X,D,\mathbb{R}X,0) > 0$, and let $D^{(1)}, D^{(2)} \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$ be as in (70). By (71), $W_{Y,E,0}(D^{(1)}+mD,0,e_1,0) > 0$ for all $m \ge 0$. Hence, first, for any $n \ge 1$,

$$\begin{aligned} W(X,(n+2)D,\mathbb{R}X,0) &= W_{Y,E,0}((n+2)D,0,0,0)\\ &\ge c_0W_{Y,E,0}(D^{(2)},0,e_1,0)W_{Y,E,0}(D^{(1)}+(n+1)D,0,e_1,0)\\ &\ge W_{Y,E,0}(D^{(1)}+(n+1)D,0,e_1,0)\ , \end{aligned}$$

and further on,

$$W_{Y,E,0}(D^{(1)}+(n+1)D,0,e_1,0)\ge W_{Y,E,0}(D^{(1)}+(n+1)D,e-1,0,0)$$

$$\frac{(-K_Y(D^{(1)}+(n+1)D)-2)!}{2}\cdot W_{Y,E,0}(D^{(2)},0,e_1,0)$$
$$\times\frac{W_{Y,E,0}(D^{(1)}+[\frac{n}{2}]D,0,e_1,0)}{(-K_Y(D^{(1)}+[\frac{n}{2}]D)-1)!}\cdot\frac{W_{Y,E,0}(D^{(1)}+[\frac{n+1}{2}]D,0,e_1,0)}{(-K_Y(D^{(1)}+[\frac{n+1}{2}]D)-1)!}\ .$$

Then, the sequence

$$a_n=\frac{W_{Y,E,0}(D^{(1)}+nD,0,e_1,0)}{(-K_Y(D^{(1)}+nD))!},\quad n\ge 0\ ,$$

satisfies (52) in Lemma 32 with

$$\lambda=\inf_{n\ge 3}\frac{W_{Y,E,0}(D^{(2)},0,e_1,0)\cdot(-K_Y(D^{(1)}+[\frac{n}{2}]D))\cdot(-K_Y(D^{(1)}+[\frac{n+1}{2}]D))}{2(-K_Y(D^{(1)}+(n+1)D))\cdot(-K_Y(D^{(1)}+(n+1)D)-1)}>0\ .$$

Hence (49) follows. □

6 Monotonicity

Lemma 35 *(cf. [14, Lemma 7.6]) Let D_1,D_2 be X-nef and big real divisor classes on a del Pezzo surface X of type $\mathbb{P}^2_{a,b}$, $a+2b=7$. If D_2-D_1 is effective, then D_2-D_1 can be decomposed into a sum $E^{(1)}+...+E^{(k)}$, where $E^{(i)}$ is either a real (-1)-curve, or a pair of disjoint complex conjugate (-1)-curves, $i=1,...,k$, and, moreover, each real divisor $D^{(i)}=D_1+\sum_{j\le i}E^{(j)}$ is X-nef and big, and satisfies $D^{(i)}E^{(i+1)}>0$, $i=0,...,k-1$.*

Proof. It is well known that the effective cone in Pic(X) is generated by (-1)-curves. It is easy to verify that two complex conjugate (-1)-curves in X intersect in at most one point, and if they intersect, then their sum is linearly equivalent to a pair of real (-1)-curves, thus, D_2-D_1 can be decomposed into a sum $E^{(1)}+...+E^{(k)}$, where $E^{(i)}$ is either a real (-1)-curve, or a pair of disjoint complex conjugate (-1)-curves, $i=1,...,k$. We show that a suitable reordering of $E^{(1)},...,E^{(k)}$ ensures the X-nefness and bigness of $D^{(i)}$ together with $D^{(i)}E^{(i+1)}>0$, $i=0,...,k-1$. The divisor $D^{(0)}=D_1$ is X-nef and big. Suppose now that $D^{(i)}$ is X-nef and big for some $0\le i<k$. If $i=k-1$, then $D^{(k)}=D_2$ is X-nef and big, and furthermore

$$D^{(k-1)}E^{(k)}=(D_2-E^{(k)})E^{(k)}=D_2E^{(k)}-(E^{(k)})^2>0\ .$$

If $i\le k-2$, then there exists $i<j\le k$ such that $D^{(i)}E^{(j)}>0$. Indeed, otherwise, we would have

- either all $E^{(i+1)},...,E^{(k)}$ orthogonal to each other and to $D^{(i)}$, and thus, $D_2E^{(j)}<0$, $i<j\le k$ contrary to the X-nefness of D_2,

- or we would have some $i<j<j'\le k$ such that $E^{(j)}E^{(j')}>0$, but then $\dim|E^{(j)}+E^{(j')}|>0$, contradicting to the bigness of $D^{(i)}$.

So, we may assume that $D^{(i)}E^{(i+1)}>0$. Then $D^{(i+1)}=D^{(i)}+E^{(i+1)}$ is X-nef and big:

$$\begin{array}{rcl} D^{(i+1)}E^{(i+1)} & = & D^{(i)}E^{(i+1)}+(E^{(i+1)})^2\ge 0\ , \\ (D^{(i+1)})^2 & = & (D^{(i)})^2+2D^{(i)}E^{(i+1)}+(E^{(i+1)})^2\ge(D^{(i)})^2>0. \quad \Box \end{array}$$

Theorem 9 *Let D_1,D_2 be X-nef and big divisor classes on a real del Pezzo surface X of type $\mathbb{P}^2_{a,b}$, $a+2b=7$, $0\le b\le 2$, such that D_2-D_1 is effective. Then*

$$W(X,D_2,\mathbb{R}X,0)\ge W(X,D_1,\mathbb{R}X,0)\ . \tag{72}$$

Proof. By Lemma 35, we should only consider the case of $E^*=D_2-D_1$ either a real (-1)-curve, or a pair of disjoint complex conjugate (-1)-curves. Let λ be the number of irreducible components of E^*. We can assume that E^* consists of λ exceptional divisors of the blow up $X\to\mathbb{P}^2$. Specializing $6-\lambda$ other exceptional divisors so that their blow-downs and the blow-downs of the components of E^* appear on a real conic, we degenerate X in a regular ABV-family into a real nodal del Pezzo pair (Y,E) where E is the strict transform of the above plane conic. Since $D_2E=D_1E+\lambda>D_1E$, we have $\binom{D_2E+2m}{m}\ge\binom{D_1E+2m}{m}$ for all $m\ge 0$, and hence using the non-negativity statement (54) and formula (43) for the both sides of (72), we reduce the problem to establishing inequality

$$W_{Y,E,0}(D,\alpha,\beta+\lambda e_1,0)\ge W_{Y,E,0}(D-E^*,\alpha,\beta,0) \tag{73}$$

for all divisors $D\in\mathrm{Pic}^{\mathbb{R}}_{++}(Y,E)$ such that $DE\ge\lambda$, and for all vectors $\alpha,\beta\in\mathbb{Z}^{\infty,\mathrm{odd}}_+$ such that $I(\alpha+\beta)=DE-\lambda$. We prove (73) by induction on $R_Y(D,\beta+\lambda e_1)$. The case $R_Y(D,\beta+\lambda e_1)<\lambda$ is trivial, since then $R_Y(D-E^*,\beta)=R_Y(D,\beta+\lambda e_1)-\lambda<0$. If $R_Y(D,\beta+\lambda e_1)=\lambda$ and, respectively, $R_Y(D-E^*,\beta)=0$, the only relevant case is that of Proposition 9(1iii) with $D-E^*$ playing the role of D and $\beta=0$, in which case by (54), formula (6), and Proposition 9(1iii) we have

$$W_{Y,E,0}(D,\alpha,\lambda e_1,0)\ge W_{Y,E,0}(D,\alpha+\lambda e_1,0,0)=1=W_{Y,E,0}(D-E^*,\alpha,0,0)\ .$$

If $R_Y(D,\beta+\lambda e_1)=R_Y(D-E^*,\beta)+\lambda>\lambda$, we compute both sides of (73) by formula (6) and compare them using the induction assumption (in the sequel we shortly write $\mathrm{RHS}(6)_l$ and $\mathrm{RHS}(6)_r$ for the right-hand side of (6) expressing the left and the right terms of (73) respectively). So, for the summands in the first sum in $\mathrm{RHS}(6)_l$ and $\mathrm{RHS}(6)_r$ the induction assumption yields

$$W_{Y,E,0}(D,\alpha+e_j,\beta-e_j+\lambda e_1,0)\ge W_{Y,E,0}(D-E^*,\alpha+e_j,\beta-e_j,0)\ .$$

For the second sum in RHS$(6)_l$ and RHS$(6)_r$ we perform the following comparison. Let

$$S_r = c\cdot\frac{2^{\|\beta^{(0)}\|}}{\beta^{(0)}!}\cdot\frac{(n-1-\lambda)!}{n_1!...n_m!}\cdot\prod_{i=1}^m\binom{(\beta^{\rm re})^{(i)}}{\gamma^{(i)}}W_{Y,E,0}(\mathcal{D}^{(i)},\alpha^{(i)},(\beta^{\rm re})^{(i)},(\beta^{\rm im})^{(i)})$$

be a summand in the second sum of RHS$(6)_r$, where $n = R_Y(D,\beta+\lambda e_1)$ and $n_i = R_Y(\mathcal{D}^{(i)},(\beta^{\rm re})^{(i)}+2(\beta^{\rm im})^{(i)})$, $1\le i\le m$. Notice that $m\ge 1$, since $DE^*>0$, and hence there is $\mathcal{D}^{(i)}$ such that $[\mathcal{D}^{(i)}]E^*>0$. Pick $\mathcal{D}^{(j)}$ with $[\mathcal{D}^{(j)}]E^*>0$ and associate with S_r the following summand S_l in the second sum of RHS$(6)_l$:

- if $[\mathcal{D}^{(j)}]\ne-\lambda(K_Y+E)-E^*$ (in which case $\mathcal{D}^{(j)}$ is a real divisor), then

$$S_l = c\cdot\frac{s^r_j}{s^l_j}\cdot\frac{2^{\|\hat\beta^{(0)}\|}}{\hat\beta^{(0)}!}\cdot\frac{(n-1)!}{\hat n_1!...\hat n_m!}\cdot\prod_{i=1}^m\binom{(\hat\beta^{\rm re})^{(i)}}{\gamma^{(i)}}W_{Y,E,0}(\hat{\mathcal{D}}^{(i)},\alpha^{(i)},(\hat\beta^{\rm re})^{(i)},(\beta^{\rm im})^{(i)}),\tag{74}$$

  where $\hat\beta^{(0)}=\beta^{(0)}$, $\hat n_i = R_Y(\hat{\mathcal{D}}^{(i)},(\hat\beta^{\rm re})^{(i)}+2(\beta^{\rm im})^{(i)})$,

$$\hat{\mathcal{D}}^{(i)}=\begin{cases}\mathcal{D}^{(i)}, & i\ne j,\\ \mathcal{D}^{(j)}+E^*, & i=j,\end{cases}\quad\text{and}\quad(\hat\beta^{\rm re})^{(i)}=\begin{cases}(\beta^{\rm re})^{(i)}, & i\ne j,\\ (\beta^{\rm re})^{(j)}+\lambda e_1, & i=j,\end{cases}$$

  s^r_j counts how many times the tuple $(\mathcal{D}^j,\alpha^{(j)},(\beta^{\rm re})^{(j)},\gamma^{(j)})$ occurs in the list $\{(\mathcal{D}^{(i)},\alpha^{(i)},(\beta^{\rm re})^{(i)},\gamma^{(i)})\}_{i=1}^m$, and s^l_j counts how many times the tuple $(\hat{\mathcal{D}}^j,\alpha^{(j)},(\hat\beta^{\rm re})^{(j)},\gamma^{(j)})$ occurs in the list $\{(\hat{\mathcal{D}}^{(i)},\alpha^{(i)},(\hat\beta^{\rm re})^{(i)},\gamma^{(i)})\}_{i=1}^m$;

- if $D^{(j)}=-\lambda(K_Y+E)-E^*$, then

$$S_l = c\cdot\frac{2^{\|\hat\beta^{(0)}\|}}{\hat\beta^{(0)}!}\cdot\frac{(n-1)!}{\hat n_1!...\hat n_m!}\cdot\prod_{\substack{1\le i\le m\\ i\ne j}}\binom{(\hat\beta^{\rm re})^{(i)}}{\gamma^{(i)}}W_{Y,E,0}(\hat{\mathcal{D}}^{(i)},\alpha^{(i)},(\hat\beta^{\rm re})^{(i)},(\beta^{\rm im})^{(i)})\ ,\tag{75}$$

  where $\hat\beta^{(0)}=\beta^{(0)}+\lambda e_1$, $(\hat\beta^{\rm re})^{(i)}=(\beta^{\rm re})^{(i)}$, $\hat n_i=n_i$, and $\hat{\mathcal{D}}^{(i)}=\mathcal{D}^{(i)}$ for $1\le i\le m$, $i\ne j$, and $\hat n_j=0$.

It is easy to verify (again using the induction assumption) that

$$S_r\le\begin{cases}S_l\cdot\frac{s^l_j\hat n_j}{s^r_j(n-1)}, & \text{if }\lambda=1,\\ S_l\cdot\frac{s^l_j\hat n_j(\hat n_j-1)}{s^r_j(n-1)(n-2)}, & \text{if }\lambda=2,\end{cases}\qquad\text{in (74)}\ ,$$

and

$$S_r\le\begin{cases}S_r\cdot\frac{\hat\beta^{(0)}_1}{2(n-1)}, & \text{if }\lambda=1,\\ S_r\cdot\frac{\hat\beta^{(0)}_1(\hat\beta^{(0)}_1-1)}{4(n-1)(n-2)}, & \text{if }\lambda=2,\end{cases}\qquad\text{in (75)}\ .$$

Since $n-1=\sum_j s^l_j\hat n_j+\|\hat\beta^{(0)}\|$ (*cf.* Remark 10), we conclude that the total value of the terms in the second sum in RHS$(6)_r$ associated with a given summand S_l in the second sum in RHS$(6)_l$ does not exceed S_l, which completes the proof. $\square$

7 Mikhalkin's congruence

Theorem 10 *For any X-nef and big divisor class D on a surface X of type $\mathbb{P}^2_{7,0}$, one has*

$$W(X, D, \mathbb{R}X, 0) \equiv GW_0(X, D) \mod 4 \ .$$

Proof. Degenerating X in a regular ABV-family into a real nodal del Pezzo pair (Y, E) and using formulas (33), (43), we reduce the question to the congruence

$$W_{Y,E,0}(D, 0, (DE)e_1, 0) \equiv N_Y(D, 0, (DE)e_1) \mod 4$$

for all divisors $D \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$.

In fact, a more general statement holds: for any divisor class $D \in \mathrm{Pic}^{\mathbb{R}}_{++}(Y, E)$ and any vectors $\alpha, \beta \in \mathbb{Z}^\infty_+$ such that $I(\alpha + \beta) = DE$, one has

$$W_{Y,E,0}(D, \alpha, \beta, 0) \equiv I^\beta N_Y(D, \alpha, \beta) \mod 4 \quad \text{if } \beta \in Z^{\infty,\,\mathrm{odd}}_+, \tag{76}$$

and

$$I^\beta \cdot N_Y(D, \alpha, \beta) \equiv 0 \mod 4 \quad \text{if } \beta \notin \mathbb{Z}^{\infty,\,\mathrm{odd}}_+ \ , \tag{77}$$

where the numbers $N_Y(D, \alpha, \beta)$ are the degrees of varieties $V_Y(D, \alpha, \beta, \boldsymbol{p}^\flat)$ (here $\boldsymbol{p}^\flat = \{p_{i,j} \ : \ i \geq 1, \ 1 \leq j \leq \alpha_i\}$ are sequences of $\|\alpha\|$ distinct generic points on E) introduced in Section 3.4. The proof literally coincides with that of [15, Theorem 5] and uses induction on $R_Y(D, \beta)$ and the recursive formulas [18, Formula (66)] and (6). $\square$

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