MODIFICATION SYSTEMS AND INTEGRATION IN THEIR CHOW GROUPS

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Abstract. We introduce a notion of integration on the category of proper birational maps to a given variety X, with value in an associated Chow group. Applications include new birational invariants; comparison results for Chern classes and numbers of nonsingular birational varieties; 'stringy' Chern classes of singular varieties; and a zeta function specializing to the topological zeta function.

In its simplest manifestation, the integral gives a new expression for Chern-Schwartz-MacPherson classes of possibly singular varieties, placing them into a context in which a 'change-of-variable' formula holds.

Contents

1. Introduction

Our goal in this paper is the introduction of a technique for the study of intersection theoretic invariants in the birational class of a given variety X. Assuming resolution of singularities we introduce an 'integral', defined over the category $\mathcal{C}_X$ of *modifications* of (that is, proper and birational maps onto) X, with values in the limit $A_*\mathcal{C}_X$ of the Chow groups (with rational coefficients) of the modifications. We will refer to the category $\mathcal{C}_X$ as the *modification system* of X. The push-forward of the integral to a variety Z is called its *manifestation* in Z. If two varieties X and Y admit a modification from a common source then the corresponding systems $\mathcal{C}_X$ and $\mathcal{C}_Y$ can be identified, in the sense that they share their main structures; for example, their Chow groups are isomorphic. It is thus natural to look for invariants of X which can be expressed in terms of $\mathcal{C}_X$: such invariants have, via the identification of the latter with $\mathcal{C}_Y$, a natural manifestation as invariants of Y. Such invariants must really be reflections of invariants of the birational class of X.

We find that *Chern classes* are such an invariant. More precisely, we can evaluate the total (homology) Chern class of the tangent bundle of a nonsingular variety X as the manifestation in X of an integral $\int_{\mathcal{C}_X} 1\!\!1(0)\, d\mathfrak{c}_X$ over $\mathcal{C}_X$ (Proposition 4.2). By the mechanism described above, the Chern class of X has a natural manifestation in the Chow group of any variety Y sharing its modification system, even if no morphism exists between X and Y. (This operation does *not* agree with the naive process of pulling back the class from X to a common modification and pushing-forward to Y; it is considerably subtler.)

In the proper generality, the integral is defined over any constructible subset $\mathcal{S}$ of a system, and applies to a divisor $\mathcal{D}$ of the system (these notions are defined in §§2.6 and 2.7; constructible subsets and Cartier divisors of X provide important examples). The 'integral' is defined by intersection-theoretic means in §3, and is shown (§4) to satisfy additivity with respect to $\mathcal{S}$, a change-of-variables formula with respect to proper birational maps $Y \to X$, and the 'normalization' property with respect to Chern classes mentioned above.

The mere existence of such an operation triggers several applications: precise comparison results for Chern classes and numbers of birational varieties, new birational invariants for nonsingular varieties, 'stringy' Chern classes of singular varieties, invariants of singularities as contributions to a zeta function. These topics are discussed in §6.

For example, the relation with Chern classes implies immediately that Chern numbers $c_1^i \cdot c_{n-i}$ of complete nonsingular n-dimensional varieties in the same K-equivalence class coincide. This is pointed out in §6.2; Theorem 6.1 and Corollary 6.2 give a blueprint to obtain identities involving Chern numbers of birational varieties regardless of their K-equivalence.

Stringy Chern classes may be defined by taking the lead from the normalization mentioned above: since $\int_{\mathcal{C}_X} 1\!\!1(0)\, d\mathfrak{c}_X$ recovers $c(TX) \cap [X]$ when X is nonsingular, we can define the *stringy* Chern class of X, for (possibly) singular varieties X, to be the identity manifestation of $\int_{\mathcal{C}_X} 1\!\!1(0)\, d\mathfrak{c}_X$ (and again, the definition of these objects through an integral provides us simultaneously with a consistent choice of manifestations for all varieties birational to X). This stringy Chern class extends to more general singular varieties the notion introduced in [Alu], Corollary 1.2, for varieties admitting a crepant resolution.

There is an important subtlety hidden in the stringy notion given in this paper: if X is singular, our integral depends on the consistent choice of a *relative canonical divisor* for modifications $\pi : V \to X$ from a nonsingular source, and there is more than one possible such choice. Using Kähler differentials leads to what we call the 'Ω flavor' of the integral; this provides a stringy notion which is defined for arbitrary singular varieties. In birational geometry it is customary to use a notion arising from the double dual of $\Omega_X^{\dim X}$, which leads to the 'ω flavor' of our integral. The corresponding stringy Chern class is defined for $\mathbb{Q}$-Gorenstein varieties admitting at least one log resolution with log discrepancies greater than zero, that is, with log terminal singularities. If X is complete, the degree of its ω-stringy Chern class agrees with Victor Batyrev's stringy Euler number ([Bat99b], with $\Delta_X = 0$; the stringy Euler number for a Kawamata pair (X, Δ_X) is the degree of $\int_{\mathcal{C}_X} 1\!\!1(-\Delta_X)\, d\mathfrak{c}_X$). (A notion of stringy Chern classes apparently equivalent to our 'ω flavor' has been developed simultaneously and independently by Tommaso de Fernex, Ernesto Lupercio, Thomas Nevins, and Bernardo Uribe, [dFLNU].)

An example comparing the two flavors is given in §7.7. The delicate issue of the definition of the integral (and hence of the stringy classes) over varieties with non log-terminal singularities is discussed in §8, but we fall short of a definition in this case.

There is a different way to extract invariants for possibly singular varieties S from the integral defined here: one may embed S in a nonsingular ambient variety X, then take the integral of the divisor 0 *over the constructible set* $\mathcal{S}$ *of* $\mathcal{C}_X$ *determined by* S. We prove (Theorem 5.1) that (in characteristic 0) the identity manifestation of this integral agrees with a known invariant, that is, the Chern-Schwartz-MacPherson class of S. In fact, there is a very tight connection between the identity manifestation of any integral and MacPherson's natural transformation, discussed in §5. In particular, the stringy Chern classes of a singular variety X can be written as explicit linear combinations of Chern-Schwartz-MacPherson classes of subvarieties of X, as the image through MacPherson's natural transformation of a specific constructible function $I_X(0, \mathcal{C}_X)$ on X. This constructible function appears to us as a much more basic invariant of X than the stringy Euler number or Chern class obtained from it; it would be worth studying it further.

It should be noted that the theory of Chern-Schwartz-MacPherson classes is *not* an ingredient of the integral introduced here; thus, the result mentioned above provides us with a candidate for a possible alternative treatment of Chern-Schwartz-MacPherson classes (in the Chow group with rational coefficient), relying solely on canonical resolution of singularities. The original definition of MacPherson in [Mac74] relied on transcendental invariants; other approaches, such as the one in [Ken90], appear to require generic smoothness or be otherwise bound to characteristic zero for more fundamental reasons.

(Canonical) resolution of singularities is essential for our approach, in two respects. First, the integral is defined by pushing forward a weighted $\mathbb{Q}$-linear combination of classes, defined in a nonsingular variety suitably 'resolving' the data (§3.3, §3.5); this exists by virtue of resolution of singularities. Second, the key independence on the choice of resolving variety (Claim 3.5, Theorem 3.7) is a technical exercise relying on the factorization theorem of [AKMW02], which uses resolution of singularities (cf. Remark (3), p. 533, in [AKMW02]). Resolution of singularities appears to be the only obstruction to extending the integral defined here to positive characteristic.

Our guiding idea in this paper is an attempt to mimic some of the formalism of *motivic integration* (as developed by Jan Denef and François Loeser) in a setting carrying more naturally intersection-theoretic information. While motivic integration is not used in the paper, the reader may notice several points of contact; in particular, the degree of our integral agrees with an expression which arises naturally in that context (see especially Remark 5.6 and Claim 5.7). This observation was at the root of our previous work along these lines, [Alu].

We found the excellent survey [Vey] particularly useful as a guide to motivic integration and its applications, and refer to this reference in several place. We note that a different and powerful approach to 'motivic Chern classes' has appeared in the recent work [BSY]. Also, Shoji Yokura has examined inverse systems of algebraic varieties for intersection-theoretic purposes, in [Yok03].

The most powerful approach to date to defining 'stringy' characteristic classes for singular variety appears to be the one by Lev Borisov and Anatoly Libgober ([BL03] and [BL]). While we have not checked this too carefully, it is essentially inevitable that the (ω flavor of the) stringy Chern class introduced here should agree with a suitable specialization of the *elliptic orbifold class* introduced in [BL], Definition 3.2. It is worth mentioning that the factorization theorem of [AKMW02] is also at the root of the approach of Borisov and Libgober, as it reduces the key independence to the analysis of the behavior of the invariants through blow-ups. Accordingly, our Theorem 3.7 could likely be derived as a corollary of Theorem 3.5 in [BL].

In this connection we would like to suggest that a notion of 'elliptic integration' could be defined using Borisov-Libgober's elliptic classes analogously to what we have done here for Chern classes. It would be very interesting if an analog of Theorem 5.1 were to hold for such a notion, linking it to a functorial theory generalizing the theory of Chern-Schwartz-MacPherson classes and MacPherson's natural transformation (maybe the theory proposed in [BSY]?). Lev Borisov informs me that he and Libgober have entertained the idea of studying relations of their work with other characteristic classes for singular varieties, such as Chern-Mather classes.

The inverse limit of the modification system of a variety X exists as a ringed space, and Heisuke Hironaka calls it the *Zariski space* of X ([Hir64], Chapter 0, §6). We prefer to work throughout with the inverse system of modifications, while systematically taking limits of structures (Chow groups, etc.) associated to it. As modification systems are closely related to Hironaka's *voûte étoilée,* Matilde Marcolli has suggested that we could name the operation defined here *celestial integration.* We will show due restraint and use this poetic term very sparingly. In any case we warmly thank her, as well as Ettore Aldrovandi and Jörg Schürmann, for extensive discussions on the material presented here.

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2. Modification systems

2.1. We work over any algebraically closed field k over which canonical resolution of singularities à la Hironaka and the factorization theorem of [AKMW02] hold.

In this section we introduce the objects ('modification systems') over which the integration will take place. Roughly speaking, a modification system consists of the collection of varieties mapping properly and birationally onto a fixed variety X. A modification system carries a number of natural structures, all obtained by taking direct or inverse limits of corresponding structures on the individual varieties in the collection: thus the *Chow group* of a modification system is the inverse limit of the Chow groups (under push-forward); a *divisor* is an element of the direct limit of the groups of Cartier divisors (under pull-back); and so on.

The key advantage of working with modification system is that, thanks to embedded resolution, the information carried by many such structures can in fact be represented in terms of a divisor with normal crossings and nonsingular components on one nonsingular modification of X. Thus, for example, arbitrary subschemes of X may be described in terms of divisors of the corresponding modification system (cf. Remark 2.9).

2.2. The main sense in which the quick description given in §2.1 is imprecise is that working with the *varieties* mapping to X is inadequate; it is necessary to work with the proper, birational *maps* themselves.

Definition 2.1. Let X be an irreducible variety over k. The *modification system* $\mathcal{C}_X$ of X is the category whose objects are the proper birational maps $\pi : V \to X$, and morphisms $\alpha : \pi_1 \to \pi_2$ are commutative diagrams of proper birational maps

$$\begin{tikzcd} V_1 \arrow[rr, "\alpha"] \arrow[dr, "\pi_1"'] & & V_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

We often (but not always) denote by V_π the source of π. If π and π' are objects of a modification system $\mathcal{C}_X$, we say that π *dominates* π' if there is a morphism $\pi \to \pi'$ in $\mathcal{C}_X$.

Lemma 2.2. *Every pair π_1, π_2 in $\mathcal{C}_X$ is dominated by some object π of $\mathcal{C}_X$. In fact, π may be chosen so that its source V_π is nonsingular, and its exceptional divisor has normal crossings and nonsingular components in V_π.*

Proof. In fact, modification systems have products: if π_1, π_2 are in $\mathcal{C}_X$, the component W of $V_{\pi_1} \times_X V_{\pi_2}$ dominating V_{π_1} and V_{π_2} makes the diagram

$$\begin{tikzcd} & W = V_\pi \arrow[dl, "\alpha_1"'] \arrow[dd, "\pi"] \arrow[dr, "\alpha_2"] & \\ V_1 \arrow[dr, "\pi_1"'] & & V_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

commute, with π proper and birational and satisfying the evident universal property.

This implies the first assertion. The second assertion follows from embedded resolution of singularities. $\square$

2.3. Modification systems are inverse systems under the ordering $\pi \geq \pi' \iff \pi$ dominates π'. We want to identify systems 'with the same limit'; in our context, this translates into the following.

Definition 2.3. Two systems $\mathcal{C}_X$, $\mathcal{C}_Y$ are *equivalent* if there are objects π_X in $\mathcal{C}_X$ and π_Y in $\mathcal{C}_Y$ with isomorphic source; that is, if there exists a variety V and proper birational maps $\pi_X : V \to X$ and $\pi_Y : V \to Y$:

$$\begin{tikzcd} & V \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

The transitivity of this notion follows from Lemma 2.2.

Lemma 2.4. *If $\mathcal{C}_X$ and $\mathcal{C}_Y$ are equivalent, then every object ρ of $\mathcal{C}_Y$ is dominated by an object σ whose source is the source of an object of $\mathcal{C}_X$.*

Proof. This follows immediately from Lemma 2.2: since $\mathcal{C}_X$ and $\mathcal{C}_Y$ are equivalent, there are objects π_X, π_Y in $\mathcal{C}_X$, $\mathcal{C}_Y$ resp., with a common source V; any object σ dominating both π_Y and ρ:

$$\begin{tikzcd}
 & & W \arrow[dl] \arrow[dr] \arrow[ddrr, bend right=20, "\sigma"'] & & \\
 & V \arrow[dl, "\pi_X"] \arrow[drrr, "\pi_Y"'] & & V_\rho \arrow[dr, "\rho"] & \\
X & & & & Y
\end{tikzcd}$$

satisfies the requirement. $\square$

2.4. A proper birational morphism $\rho : Y \to X$ determines a covariant functor $\mathcal{C}_Y \to \mathcal{C}_X$ by composition: $\pi \mapsto \rho \circ \pi$; $\mathcal{C}_X$ and $\mathcal{C}_Y$ are then trivially equivalent systems.

Via this functor, the category $\mathcal{C}_Y$ is realized as a *full* subcategory of $\mathcal{C}_X$. Indeed, every morphism $\rho \circ \pi_1 \to \rho \circ \pi_2$ in $\mathcal{C}_X$ is given by a proper birational map α such that $\rho \circ \pi_1 = \rho \circ \pi_2 \circ \alpha$; but then π_1 and $\pi_2 \circ \alpha$ agree on a nonempty open set (on which they are isomorphisms), hence they agree everywhere. In other words, every morphism $\rho \circ \pi_1 \to \rho \circ \pi_2$ is induced by a morphism $\pi_1 \to \pi_2$ in $\mathcal{C}_Y$.

In particular, any object $\rho : V \to X$ of $\mathcal{C}_X$ determines a modification system equivalent to $\mathcal{C}_X$ and equal to a 'copy' of $\mathcal{C}_V$; we will denote this system by $\mathcal{C}_\rho$. Two systems $\mathcal{C}_X$, $\mathcal{C}_Y$ are equivalent if and only if they both contain a copy of a third system $\mathcal{C}_V$.

As mentioned in §2.1, standard structures can be defined on modification systems by taking limits of the same structures on the (sources of the) objects in the system. The foregoing considerations imply that any functorial structure defined in this fashion will be preserved under equivalence of systems.

We will explicitly need very few such structures: Chow groups, divisors, constructible sets. These are presented in the next few sections.

2.5. For V a variety, we denote by A_*V the Chow group of V, *tensored with* $\mathbb{Q}$ (rational coefficients appear to be necessary for the integral introduced in §3). The modification system $\mathcal{C}_X$ determines the inverse system of Chow groups A_*V_π (under push-forwards), as π ranges over the objects of $\mathcal{C}_X$.

Definition 2.5. The *Chow group* of $\mathcal{C}_X$ is the limit

$$A_*\mathcal{C}_X := \varprojlim_{\pi \in Ob(\mathcal{C}_X)} A_*V_\pi \quad .$$

That is, an element of the Chow group of $\mathcal{C}_X$ is the choice of a class a_π in A_*V_π for each object π of $\mathcal{C}_X$, with the condition that if $\alpha : \pi_1 \to \pi_2$ is a morphism then $\alpha_*(a_{\pi_1}) = a_{\pi_2}$. We say that a_π is the π-*manifestation* of this element of $A_*\mathcal{C}_X$.

If X is complete, we may define the *degree* of a class $a \in A_*\mathcal{C}_X$,

$$\deg a$$

as the degree of (the zero-dimensional component of) any manifestation $A_\pi \in A_*V_\pi$ of a; indeed, this is clearly independent of π.

Next, we observe that equivalent systems have isomorphic Chow groups.

Lemma 2.6. *For all objects ρ of $\mathcal{C}_X$, $A_*\mathcal{C}_\rho \cong A_*\mathcal{C}_X$.*

Proof. If σ is an object of $\mathcal{C}_\rho$ then $\rho\circ\sigma$ is an object of $\mathcal{C}_X$. The identity homomorphism $A_*V_{\rho\circ\sigma} \to A_*V_\sigma$ induces compatible homomorphisms $A_*\mathcal{C}_X \to A_*V_\sigma$ for all σ; and hence a homomorphism $A_*\mathcal{C}_X \to A_*\mathcal{C}_\rho$.

Lemma 2.2 gives the inverse homomorphism. More precisely, for π in $\mathcal{C}_X$ there exists a π' dominating both π and ρ:

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] \arrow[dd, "\pi'"] & \\ V_\rho \arrow[dr, "\rho"'] & & V_\pi \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

Push-forwards yield a homomorphism $A_*\mathcal{C}_\rho \to A_*W \to A_*V_\pi$, which is easily seen to be independent of the chosen π' dominating π and ρ, and to satisfy the necessary compatibility, giving a homomorphism $A_*\mathcal{C}_\rho \to A_*\mathcal{C}_X$ by the universal property of the inverse limit.

Checking that the two compositions are the identity is equally straightforward. $\square$

Corollary 2.7. *Equivalent systems have isomorphic Chow groups.*

Proof. Indeed, equivalent systems $\mathcal{C}_X, \mathcal{C}_Y$ contain a copy of a third system $\mathcal{C}_V$, cf. §2.4. By Lemma 2.6, $A_*\mathcal{C}_X \cong A_*\mathcal{C}_V \cong A_*\mathcal{C}_Y$. $\square$

Note however that these isomorphisms are not canonical, as they depend on the choice of a common source of objects in $\mathcal{C}_X$ and $\mathcal{C}_Y$. A canonical isomorphism is available if the common source is chosen; for example, for a given proper birational map $\pi : Y \to X$.

2.6. Divisors are arguably the most important structure within a modification system. The group of divisors of $\mathcal{C}_X$ is the direct limit of the groups of Cartier divisor of sources of objects of $\mathcal{C}_X$. Explicitly:

Definition 2.8. Let $\mathcal{C}_X$ be a modification system. A *divisor* $\mathcal{D}$ of $\mathcal{C}_X$ is a pair (π, D_π), where π is an object of $\mathcal{C}_X$ and D_π is a Cartier divisor on the source V_π of π, modulo the equivalence relation:

$$(\pi', D_{\pi'}) \sim (\pi'', D_{\pi''}) \iff \text{for all } \pi = \pi' \circ \alpha' = \pi'' \circ \alpha'' \text{ in } \mathcal{C}_X$$
$$\text{dominating } \pi' \text{ and } \pi'',\ \alpha'^* D_{\pi'} = \alpha''^* D_{\pi''} \text{ in } V_\pi.$$

That is: $D_{\pi'}$ and $D_{\pi''}$ determine the same divisor of $\mathcal{C}_X$ if they agree after pull-backs.

Remark 2.9.
- Divisors of X determine divisors of $\mathcal{C}_X$ by pull-backs. In fact, every proper subscheme Z of X determines a divisor of $\mathcal{C}_X$, namely (π, E), where $\pi : \widetilde{X} \to X$ is the blow-up along Z, and E is the exceptional divisor.
- In particular, if X is smooth then every canonical divisor of X determines a divisor of $\mathcal{C}_X$. If $\mathcal{C}_X$ and $\mathcal{C}_Y$ are equivalent, the corresponding canonical divisors need not agree; if they do, then X and Y are in the same *K-equivalence*

class. This notion has been thoroughly studied by Chin-Lung Wang (for $\mathbb{Q}$-Gorenstein varieties), see [Wan03].

- As we are assuming that resolution of singularities hold, every divisor of $\mathcal{C}_X$ admits a representation (π, D) in which the source V_π of π is nonsingular, and D is supported on a divisor with normal crossings and nonsingular components.
- Divisors act on the Chow group: if $\mathcal{D}$ is a divisor and $a \in A_*\mathcal{C}_X$, for every object ρ we can find a dominating $\pi = \rho \circ \alpha$ such that $\mathcal{D}$ is represented by (π, D_π), and set
$$(\mathcal{D} \cdot a)_\rho := \alpha_*(D_\pi \cdot a_\pi) \in A_* V_\rho \quad ;$$
the projection formula guarantees that this is independent of the chosen π, and that the resulting classes satisfy the compatibility needed to define an element $\mathcal{D} \cdot a \in A_*\mathcal{C}_X$.
- Equivalent systems have isomorphic divisor groups. Indeed, if $\mathcal{C}_X$ and $\mathcal{C}_Y$ are equivalent, and (π, D_π) represents a divisor on $\mathcal{C}_X$, then (by Lemma 2.4) we may assume that V_π is a source of an object of $\mathcal{C}_Y$, so that D_π represents a divisor on $\mathcal{C}_Y$; and, conversely, divisors of $\mathcal{C}_Y$ determine divisors of $\mathcal{C}_X$.

  As in §2.5, these identifications are canonical once a common source is chosen.

2.7. A second main character in the definition of the integral is the notion of constructible subset.

Definition 2.10. A *constructible subset* $\mathcal{S}$ of a modification system $\mathcal{C}_X$ is a pair (π, S_π), where π is an object of $\mathcal{C}_X$ and S_π is a constructible subset of V_π, modulo the equivalence relation:
$$(\pi', S_{\pi'}) \cong (\pi'', S_{\pi''}) \iff \text{for all } \pi = \pi' \circ \alpha' = \pi'' \circ \alpha'' \text{ in } \mathcal{C}_X$$
$$\text{dominating } \pi' \text{ and } \pi'',\ {\alpha'}^{-1}(S_{\pi'}) = {\alpha''}^{-1}(S_{\pi''}) \text{ in } V_\pi.$$

That is: S' and S'' determine the same subset of $\mathcal{C}_X$ if they agree after preimages through the system.

Remark 2.11.

- As for divisors, note that the constructible subsets of a system are in one-to-one correspondence with constructible subsets of equivalent systems.
- By resolution of singularities, every constructible subset of a system may be represented by a pair (π, S_π) where π has nonsingular source, and S_π is obtained by taking unions and complements of nonsingular hypersurfaces meeting with normal crossings.
- One can take unions or intersections of constructible subsets, by performing these operations in a V_π in which all terms admit representatives.

We say that $\mathcal{S}$ is *closed, locally closed, etc.* if it is represented by a closed, locally closed, etc. set.

2.8. More structures could be considered easily, such as maps from a modification system to a variety, or constructible functions on a modification system, etc. We will only have fleeting encounters with such notions, and the reader should have no difficulties filling in appropriate definitions as needed.

3. Definition of the integral

3.1. *Caveat on terminology.* In this section we introduce the 'integration' operation on a modification system. As the reader may now expect the appearance of a measure, a special class of functions, and other ingredients of a good theory of integration, we hasten to warn that none will be given here.

Our guiding idea comes from *motivic integration* (see e.g., Eduard Looijenga's Bourbaki survey, [Lo02]). A motivic integral is obtained from a suitable measure on the arc space of a variety X; the objects to which it is applied are in the form $\mathbb{L}^{-\alpha}$, where α is a constructible function on the space. In practice, most applications are to the case in which α is the order function of a divisor on X; and resolution of singularities allows one to further restrict attention to functions arising from a divisor with normal crossings and nonsingular components. In this case the integral can be computed explicitly, bypassing motivic measure entirely: see for example Alastair Craw's "user-friendly formula" (Theorem 2.15 in [Cra] or §3.6 in [Vey]).

Our definition is motivated by such formulas. As pointed out in §2, most structures of importance in the context of modification systems are encoded by normal crossing divisors, so we feel free to cut the middle man and offer the 'user-friendly' analog as our definition. It would be interesting to interpret our definition in terms of a measure (be it on the modification system, or perhaps on its arc space), but this does not appear to be necessary for applications.

One advantage of this approach is that it comes with a built-in change-of-variables formula. We pay the price for this benefit by having to prove independence of the choice of representative for the divisor. In motivic integration the analogous formula is proved to agree with the definition based on a measure, so the corresponding independence is automatic; while the change-of-variables formula requires an argument.

3.2. Let X be an irreducible variety, let $\mathcal{D}$ be a divisor in the modification system $\mathcal{C}_X$ of X (cf. §2.6), and let $\mathcal{S}$ be a constructible subset of $\mathcal{C}_X$ (cf. §2.7). The rest of this section is devoted to the definition of an element

$$\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X \in A_*\mathcal{C}_X \quad .$$

While the definition is rather transparent, proving that it does not depend on the various choices will require a certain amount of technical work. Simple properties of this definition, and applications, will be discussed in later sections and will not involve the more technical material in the present one.

We have to define an element

$$\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X$$

of $A_*\mathcal{C}_X$, and this is equivalent to defining all of its manifestations in A_*V_π, as π ranges over the objects of $\mathcal{C}_X$. In order to streamline the exposition, we will begin by assuming that $\mathcal{S}$ is closed, and that $\mathcal{D}$, $\mathcal{S}$, and the *relative canonical divisor* are in a particularly favorable position in V_π; the definition in this case is given in §3.3. The definition for all objects of $\mathcal{C}_X$ is given in §3.5, and the extension to constructible subsets $\mathcal{S}$ is completed in §3.11.

The notion of relative canonical divisor requires a discussion. If $\pi : V \to X$ is a birational morphism of nonsingular varieties, we will denote by K_π the divisor of the jacobian of π; so if K_X is a canonical divisor of X, then $\pi^* K_X + K_\pi$ is a canonical divisor of V. This notion behaves well with respect to composition of maps, in the sense that if $\alpha : W \to V$ is a birational morphism of nonsingular varieties, then

$$(*) \qquad K_{\pi\circ\alpha} = K_\alpha + \pi^* K_\pi \quad .$$

We must call the attention of the reader to the fact that there are several ways to extend this notion to the case in which X may be *singular.* Necessary requirements from our viewpoint are that

(1) the notion agrees with the one recalled above in the nonsingular case;
(2) if $\pi : V \to X$ is a proper birational morphism, with V nonsingular, then there exists a π' dominating π and such that $K_{\pi'}$ is a divisor with normal crossings and nonsingular components in $V_{\pi'}$; and
(3) (*) holds for $W \xrightarrow{\alpha} V \xrightarrow{\pi} X$, with V and W nonsingular, assuming K_π is a divisor.

Note that we are not requiring that K_π be a divisor *for all* π. This may lead to some confusion as we will nevertheless stubbornly refer to any K_π as a relative canonical *divisor,* since it determines a divisor in the system (cf. Remark 2.9).

One simple possibility, which has the advantage of working without further assumptions on X, is the following. If $\pi : V \to X$ is a birational morphism of n-dimensional varieties, there is an induced morphism of sheaves of Kähler differentials

$$\pi^* \Omega^n_X \to \Omega^n_V \quad .$$

If V is nonsingular, so that Ω^n_V is locally free, the image of this morphism may be written as $\Omega^n_V \otimes \mathcal{I}$ for an $\mathcal{O}_V$-ideal sheaf $\mathcal{I}$. By composing with a sequence of blow-ups $\rho : V' \to V$ we may ensure that the ideal $\mathcal{I}'$ in $\mathcal{O}_{V'}$ corresponding to $\pi' = \pi \circ \rho$ is principal, thus $\mathcal{I}' = \mathcal{O}_{V'}(-K_{\pi'})$ for a Cartier divisor $K_{\pi'}$ on V'. In fact, by the same token $K_{\pi'}$ may be assumed to be a divisor with normal crossings and nonsingular component, as promised.

This notion of K_π gives what we call the 'Ω flavor' of the integral. A second possibility, which is more natural in the context of birational geometry, will be mentioned later (§6.5) and gives the 'ω flavor'.

It should be noted that, for X singular, the value of the integral will depend on the chosen notion of relative canonical divisor, cf. Example 7.7; but the basic set-up and properties do not depend on this choice, so we will not dwell further on this important point until later sections. By requirement (1) above, the integral is univocally determined if the base X is nonsingular.

3.3. Let π be an object of $\mathcal{C}_X$, K_π be the relative canonical divisor of π, and let $\mathcal{D} = (\pi, D)$, $\mathcal{S} = (\pi, S)$. *We assume that V_π is nonsingular, K_π is a divisor, and there exists a divisor E of V_π, with normal crossings and nonsingular components E_j, $j \in J$, such that*

$$D + K_\pi = \sum_{j\in J} m_j E_j \quad ,$$

with $m_j > -1$*; further, we assume that* S *is* V_π *or the union of a collection of components* $\{E_\ell\}_{\ell\in L}$ *of* E*, and we let* $\mathbb{J}_S$ *be the whole family of subsets of* J*, if* $S = V_\pi$*, or the subfamily of those subsets which meet* L *if* $S = \cup_{\ell\in L} E_\ell$*.*

In other words, we essentially require π to be a 'log resolution' for the relevant data. We will say that π (or, more loosely, V_π) *resolves* $\mathcal{D}$, $\mathcal{S}$ if it satisfies these assumptions.

Remark 3.1. The condition that no m_j be ≤ -1 is undesirable, but necessary for the present set-up. Ideally one would like to replace this with the weaker request that m_j be $\neq -1$ for all j, which suffices for the expression in Definition 3.2 to make sense. But this would come at the price of having certain manifestations of the integral remain undefined; more importantly, the argument given here does not suffice to prove that this would lead to consistent definitions in the presence of multiplicities ≤ -1.

This issue is discussed further and illustrated with an example in §8.

Definition 3.2. If π resolves $\mathcal{D}$, $\mathcal{S}$, then the manifestation of $\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X$ in A_*V_π is defined to be

$$\left(\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X\right)_\pi := c(TV_\pi(-\log E)) \cdot \left(\sum_{I\in\mathbb{J}_S} \prod_{i\in I} \frac{E_i}{1+m_i}\right) \cap [V_\pi]$$

Here $TV_\pi(-\log E)$ is the dual of the bundle of differential forms with logarithmic poles along E. Its Chern class serves as a shorthand for a longer expression, cf. Lemma 3.8, (1).

Remark 3.3.
- The expression given in Definition 3.2 is supported on S; it should be viewed as the push-forward to V_π of an element of A_*S.
- If E_j is a component which does not belong to S, and for which $m_j = 0$, then the given expression is independent of whether E_j is counted or not in E (exercise!).

3.4. The expression given in Definition 3.2 can be written in several alternative ways, some of which are rather suggestive, and sometimes easier to apply.

For example, for $I \subset J$ let E_I equal the intersection $\cap_{i\in I} E_i$; this is a nonsingular subvariety of V_π since E is a divisor with normal crossings and nonsingular components. Also, denote by $E^{\overline{I}}$ be the divisor $\sum_{i\notin I} E_i$, as well as its restriction to subvarieties of V_π; note that $E^{\overline{I}}$ intersects the subvariety E_I along a divisor with normal crossings and nonsingular components.

Then the given expression of $\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X$ in A_*V_π equals

$$\sum_{I\in\mathbb{J}_S} \frac{c(TE_I(-\log E^{\overline{I}})) \cap [E_I]}{\prod_{i\in I}(1+m_i)} \quad .$$

If $S = V_\pi$, so that $\mathbb{J}_S$ is the whole family of subsets of J, then trivial manipulations show that the class equals a linear combination of the Chern classes of the subvarieties E_I:

$$\sum_{I\subset J} (-1)^{|I|} \prod_{i\in I} \frac{m_i}{1+m_i} \cdot c(TE_I) \cap [E_I] \quad .$$

Equally trivial manipulations show that the same class can be written as a weighted average of log-twists of the Chern class of V_π:

$$\frac{1}{\prod_{j\in J}(1+m_j)}\sum_{I\subset J} m_I\cdot c(TV_\pi(-\log E^I))\cap[V_\pi]$$

where $m_I=\prod_{i\in I} m_i$, and $E^I=\sum_{i\in I}E_i$; as $\sum_{I\subset J} m_I=\prod_{j\in J}(1+m_j)$, the top-dimensional term in this expression equals $[V_\pi]$, as it should.

3.5. We have to define the manifestation of $\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\,d\mathfrak{c}_X$ in A_*V_ρ, for arbitrary objects ρ of $\mathcal{C}_X$. By principalization and embedded resolution of singularities, for any divisor $\mathcal{D}$ and closed subset $\mathcal{S}$, every object ρ in $\mathcal{C}_X$ is dominated by an object π resolving $\mathcal{D}$, $\mathcal{S}$.

Definition 3.4. Let $\mathcal{D}$ be a divisor of $\mathcal{C}_X$, and let $\mathcal{S}$ be a closed subset of $\mathcal{C}_X$. For arbitrary ρ in $\mathcal{C}_X$, let α be a proper birational map such that $\pi=\rho\circ\alpha$ resolves $\mathcal{D}$, $\mathcal{S}$. Then the manifestation of $\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\,d\mathfrak{c}_X$ in A_*V_ρ is defined to be

$$\left(\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\,d\mathfrak{c}_X\right)_\rho := \alpha_*\left(\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\,d\mathfrak{c}_X\right)_\pi \quad .$$

Of course we have to prove that this expression does not depend on the choice of a resolving π dominating ρ, and we note that this will also immediately imply that the manifestations do define an element of the inverse limit $A_*\mathcal{C}_X$.

To establish the independence on the choice, we have to prove that if $\pi_1=\rho\circ\alpha_1$ and $\pi_2=\rho\circ\alpha_2$ both resolve $\mathcal{D}$, $\mathcal{S}$, then the two push-forwards $\alpha_{i*}\left(\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\,d\mathfrak{c}_X\right)_{\pi_i}$ to A_*V_ρ, $i=1,2$, coincide. This is our next task.

By the factorization theorem of [AKMW02], there exists an object π dominating both π_1 and π_2:

$$\begin{tikzcd}
 & V_\pi \arrow[dl, "\beta_1"'] \arrow[dr, "\beta_2"] & \\
V_{\pi_1} \arrow[dr, "\alpha_1"'] & & V_{\pi_2} \arrow[dl, "\alpha_2"] \\
 & V_\rho \arrow[d, "\rho"'] & \\
 & X &
\end{tikzcd}$$

and such that β_i decomposes as a sequence of maps

$$\begin{tikzcd}
V_\pi \arrow[r, "\gamma_1"] \arrow[rr, bend right, "\beta_i"'] & \cdots \arrow[r, "\gamma_r"] & V_{\pi_i}
\end{tikzcd}$$

with each γ_k a sequence of blow-ups followed by a sequence of blow-downs. Further, the centers of these blow-ups may be chosen to intersect the relevant divisors with normal crossings.

We should note that [AKMW02] assumes the varieties to be complete; we can reduce to this case by working in the modification system $\mathcal{C}_{\overline{X}}$ of a completion $\overline{X}$ of X, and then taking the inverse image of X throughout the system.

Claim 3.5. *If π_1 and π_2 resolve $\mathcal{D}$, $\mathcal{S}$, then so do all the intermediate stages in the resolution. In particular, so does π; further,*

$$\left(\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X\right)_{\pi_i} = \beta_{i*}\left(\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X\right)_{\pi}$$

for $i = 1, 2$.

Claim 3.5 immediately implies the sought independence on the choices. Its proof will occupy us for the next several subsections. Here we simply remark that the fine print in [AKMW02] (specifically part (6) of Theorem 0.3.1) yields the first part of the claim, as it guarantees that the inverse images of the distinguished normal crossing divisors in V_{π_i} are normal crossing divisors; it is easily checked that all multiplicities remain > -1 throughout the resolution. What remains to be proved is the claimed compatibility between the manifestations in V_π and V_{π_i}; by the recalled structure of the β_i's, it suffices to prove this in the particular case in which β_1, β_2 are blow-ups along nonsingular centers meeting the relevant divisors in V_{π_1}, V_{π_2} with normal crossings.

3.6. The upshot of the preceding considerations is that the independence of Definition 3.4 on the choices follows from the minimalist case of a particularly favorable blow-up, that is, the precise statement given below. Notation:

- V is a nonsingular irreducible variety;
- $E = \sum m_j E_j$ is a normal crossing divisor with nonsingular components E_j, $j \in J$, in V;
- $\alpha : W \to V$ is the blow-up of V along a nonsingular subvariety B of codimension d, meeting E with normal crossings;
- F_0 is the exceptional divisor of the blow-up, and F_j is the proper transform of E_j, $j \in J$; let $J' = J \cup \{0\}$;
- $m_0 = (d-1) + \sum_{E_j \supset B} m_j$;

Remark 3.6.
- Note that $F = \sum_{j\in J'} m_j F_j$ is a divisor with normal crossings and nonsingular components (since B meets E with normal crossings). Also, note that $F - \alpha^{-1}E = (d-1)F_0 = K_\alpha$. This is engineered to match the rôle of the normal crossing divisor vis-a-vis the given divisor $\mathcal{D}$ of the modification system, cf. §3.3: if $\mathcal{D}$ is represented by D_V on V and D_W on W, $\pi : V \to X$ is proper and birational, and

$$D_V + K_\pi = \sum_{j\in J} m_j E_j$$

as in §3.3, then

$$D_W + K_{\pi\circ\alpha} = \alpha^{-1}(D_V + K_\pi) + K_\alpha = \alpha^{-1}(\sum_{j\in J} m_j E_j) + (d-1)F_0 = \sum_{j\in J'} m_j F_j$$

as needed in order for W to again satisfy the assumptions given in §3.3.
- The hypothesis that B meets E with normal crossings implies that at most d components E_j contain B. Hence $m_0 > -1$ if all $m_j > -1$ for $E_j \supset B$, guaranteeing that the assumption on multiplicities specified in §3.3 is satisfied in W if and only if it is satisfied in V.

Lastly, we must deal with $\mathcal{S}$:

- $\mathbb{J}$ is either the whole family of subsets of J, or the family of subsets of J having nonempty intersection with a fixed $L \subset J$;
- $\mathbb{J}'$ is the whole family of subsets of J' in the first case; in the second case, $\mathbb{J}'$ depends on whether any of the divisors E_ℓ for $\ell \in L$ contains the center B of blow-up:
- if none of the E_ℓ contains B, then $\mathbb{J}' = \mathbb{J}$;
- if some of the E_ℓ contain B, then $\mathbb{J}' = \mathbb{J} \cup \{I \,|\, I \subset J', 0 \in I\}$.

This messy recipe encodes a rather simple situation. On V, $\mathcal{S}$ is represented by either V itself or by a union $\sum_{\ell \in L} E_\ell$, as required in §3.3. In the first case, $\mathcal{S}$ is represented by W on W; in the second case, it is represented by either the union $\sum_{\ell \in L} F_\ell$, if no E_ℓ contains B, or by $\sum_{\ell \in L \cup \{0\}} F_\ell$ if some E_ℓ do contain B. The prescription follows the fate of the distinguished families $\mathbb{J}$, $\mathbb{J}'$ through this predicament.

We are finally ready to state the main result, which will complete the proof of Claim 3.5, and hence of the independence of Definition 3.4 on the choices.

Theorem 3.7. *With notation as above,*

$$\alpha_* \left(c(TW(-\log F)) \cap \sum_{I \in \mathbb{J}'} \prod_{i \in I} \frac{F_i}{1+m_i} \right) = c(TV(-\log E)) \cap \sum_{I \in \mathbb{J}} \prod_{i \in I} \frac{E_i}{1+m_i} \quad .$$

3.7. We prove Theorem 3.7 in §3.8 and §3.9. The argument essentially amounts to careful bookkeeping, but is not completely straightforward. We collect a few necessary preliminaries in this subsection.

Let V be a nonsingular variety, B a nonsingular subvariety of codimension d, $\alpha : W \to V$ the blow-up of V along B, and let F be the exceptional divisor.

Lemma 3.8. *The following hold in A_*V.*

(1) *Let E be a divisor with normal crossings and nonsingular components E_j, $j \in J$, in V. Then*
$$c(TV(-\log E)) = \frac{c(TV)}{\prod_{j \in J}(1+E_j)} \quad .$$
(2) $\alpha_*(c(TW) \cap [W]) = c(TV) \cap [V] + (d-1) \cdot c(TB) \cap [B]$.
(3) $\alpha_*(c(TF) \cap [F]) = d \cdot c(TB) \cap [B]$.
(4) $\alpha_* \left(\dfrac{c(TW)}{(1+F)} \cap [W] \right) = c(TV) \cap [V] - c(TB) \cap [B]$.
(5) *Let E_j, $j \in J$, be nonsingular hypersurfaces of V meeting with normal crossings, and let F_j be the proper transform of E_j in W. Assume at least one of the E_j contains B. Then*
$$\alpha_* \left(\frac{c(TW)}{(1+F)\prod_{j \in J}(1+F_j)} \cap [W] \right) = \frac{c(TV)}{\prod_{j \in J}(1+E_j)} \cap [V] \quad .$$

Proof. (1) The equivalent statement $c(\Omega^1_V(\log E)) = c(\Omega^1_V)/\prod_{j \in J}(1-E_j)$ follows by a residue computation, see [Sil96], 3.1.

(2) This follows from Theorem 15.4 in [Ful84].

(3) Let $\underline{\alpha}$ be the projection $F \to B$, so that the class $\alpha_*(c(TF) \cap [F])$ is the push-forward to A_*V of the class $\underline{\alpha}_*(c(TF) \cap [F]) \in A_*B$. The exceptional divisor is identified with the projectivization of the normal bundle N_BV; therefore its tangent bundle fits in the Euler sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \underline{\alpha}^* N_B V \otimes \mathcal{O}(1) \longrightarrow TF \longrightarrow \underline{\alpha}^* TB \longrightarrow 0 \quad .$$

Hence

$$c(TF) = c((\underline{\alpha}^* N_B V \otimes \mathcal{O}(1))/\mathcal{O}) \cdot \underline{\alpha}^* c(TB) \quad ,$$

and by the projection formula

$$\underline{\alpha}_*(c(TF) \cap [F]) = c(TB) \cap \underline{\alpha}_*(c((\underline{\alpha}^* N_B V \otimes \mathcal{O}(1))/\mathcal{O}) \cap [F]) \quad .$$

Since $\underline{\alpha}$ has relative dimension $(d-1)$ over B, and $(\underline{\alpha}^* N_B V \otimes \mathcal{O}(1))/\mathcal{O}$ has rank $(d-1)$, only the top Chern class of this bundle survives the push-forward through $\underline{\alpha}$. This class may be evaluated using [Ful84], Example 3.2.2, yielding the stated result.

(4) This can also be easily deduced from Theorem 15.4 in [Ful84]; or write

$$\frac{c(TW)}{1+F} \cap [W] = c(TW) \cap [W] - c(TF) \cap [F]$$

and apply (2) and (3).

(5) If E_j does not contain B, then F_j is the pull-back of E_j; by the projection formula all such terms can be factored out of both sides of the identity, so we may assume without loss of generality that E_j contains B for *all* $j \in J$, with $J \neq \emptyset$. Let $J = \{1, \dots, r\}$, with $r \geq 1$.

The formula is immediate if B is a hypersurface, so we may assume B has codimension > 1. Then F_r is the blow-up of E_r along B; and the other hypersurfaces E_j, $j < r$, cut out a divisor with normal crossings and nonsingular components along E_r, containing B. Also note that $\frac{1}{1+F_r} = 1 - \frac{F_r}{1+F_r}$, and $c(TW)\frac{F_r}{1+F_r} \cap [W] = c(TF_r) \cap [F_r]$. If $r \geq 2$,

$$\begin{aligned} &\frac{c(TW)}{(1+F)\prod_{1 \leq j \leq r}(1+F_j)} \cap [W] \\ &\qquad = \frac{c(TW)}{(1+F)\prod_{1 \leq j < r}(1+F_j)} \cap [W] - \frac{c(TF_r)}{(1+F)\prod_{1 \leq j < r}(1+F_j)} \cap [F_r] \end{aligned}$$

and the needed equality follows if it is known for smaller, nonempty J. Thus, we are reduced to proving

$$\alpha_* \left(\frac{c(TW)}{(1+F)(1+\widetilde{E})} \cap [W] \right) = \frac{c(TV)}{1+E} \cap [V]$$

for any nonsingular hypersurface E containing the center B of the blow-up, where $\widetilde{E}$ denotes the proper transform of E. For this, rewrite the left-hand-side as

$$\begin{aligned} &c(TW) \left(1 - \frac{F}{1+F} - \frac{\widetilde{E}}{1+\widetilde{E}} + \frac{F\widetilde{E}}{(1+F)(1+\widetilde{E})} \right) \cap [W] \\ &\qquad = c(TW) \cap [W] - c(TF) \cap [F] - c(T\widetilde{E}) \cap [\widetilde{E}] + c(T(F \cap \widetilde{E})) \cap [F \cap \widetilde{E}] \end{aligned}$$

and use (2) and (3) to compute the push-forward:

$$\begin{aligned}&(c(TV)\cap[V]+(d-1)\cdot c(TB)\cap[B])-d\cdot c(TB)\cap[B]-(c(TE)\cap[E]\\&+(d-2)\cdot c(TB)\cap[B])+(d-1)\cdot c(TB)\cap[B]=c(TV)\cap[V]-c(TE)\cap[E]\quad,\end{aligned}$$

agreeing with the right-hand-side:

$$c(TV)\cap[V]-c(TE)\cap[E]=c(TV)\cap\left(1-\frac{E}{1+E}\right)\cap[V]=\frac{c(TV)}{1+E}\cap[V]$$

and concluding the proof. □

Remark 3.9. In characteristic 0, parts (2) and (3) are (even more) immediate from the functoriality of Chern-Schwartz-MacPherson classes.

3.8. In this subsection we prove Theorem 3.7 under the hypothesis that $\mathbb{J}$ is the whole family of subsets of J. In this case, and using (1) in Lemma 3.8, the statement to prove is

$$\alpha_*\left(\frac{c(TW)}{(1+F_0)\prod_{j\in J}(1+F_j)}\left(1+\frac{F_0}{M+d}\right)\prod_{j\in J}\left(1+\frac{F_j}{1+m_j}\right)\cap[W]\right)$$
$$=\frac{c(TV)}{\prod_{j\in J}(1+E_j)}\prod_{j\in J}\left(1+\frac{E_j}{1+m_j}\right)\cap[V]$$

with $M=\sum_{E_j\supset B}m_j$.

First, observe that if E_j does *not* contain B, then $F_j=\alpha^*E_j$; by the projection formula, all factors involving such components can be factored out. Thus we may assume that all E_j contain B, without loss of generality.

Second, with this additional assumption we can prove a stronger statement, not binding M: we claim that, with M an indeterminate,

$$(*)\quad \alpha_*\left(\frac{c(TW)}{(1+F_0)\prod_{j\in J}(1+F_j)}\left(1+\frac{F_0}{M+d}\right)\prod_{j\in J}\left(1+\frac{F_j}{1+m_j}\right)\cap[W]\right)$$
$$=\frac{c(TV)}{\prod_{j\in J}(1+E_j)}\prod_{j\in J}\left(1+\frac{E_j}{1+m_j}\right)\cap[V]-\frac{M-\sum_{j\in J}m_j}{(M+d)\prod_{j\in J}(1+m_j)}c(TB)\cap[B]$$

if all E_j contain B.

If $d=1$, that is, B is itself a hypersurface of V, then α is an isomorphism and (*) is immediately verified (note that in this case J consists of at most one element; and if E_1 contains B then $F_0=B$, $F_1=0$ in $W\cong V$; the statement follows from the identity $\frac{1}{M+1}=\frac{1}{1+m_1}-\frac{M-m_1}{(M+1)(1+m_1)}$).

Therefore, we may assume $d\geq 2$. Formula (*) is then proven by induction on the size of J. If $J=\emptyset$, the statement is

$$\alpha_*\left(\frac{c(TW)}{(1+F_0)}\left(1+\frac{F_0}{M+d}\right)\cap[W]\right)=c(TV)\cap[V]-\frac{M}{M+d}c(TB)\cap[B]\quad.$$

To prove this, rewrite the left-hand-side as

$$\alpha_*\left(c(TW)\cdot\left(1-\frac{M+d-1}{M+d}\cdot\frac{F_0}{1+F_0}\right)\cap[W]\right)\quad;$$

distributing and using (2) and (3) from Lemma 3.8:

$$c(TV)\cap[V]+(d-1)\cdot c(TB)\cap[B]-\frac{(M+d-1)d}{M+d}\cdot c(TB)\cap[B]$$

gives the stated result.

If $J=\{1,\dots,r\}$ with $r\geq 1$, split off the F_r term from the left-hand-side of (*):

$$\frac{c(TW)}{(1+F_0)\prod_{j=1}^{r}(1+F_j)}\left(1+\frac{F_0}{M+d}\right)\prod_{j=1}^{r}\left(1+\frac{F_j}{1+m_j}\right)\cap[W]$$

equals

$$\frac{c(TW)}{(1+F_0)\prod_{j=1}^{r-1}(1+F_j)}\left(1+\frac{F_0}{M+d}\right)\prod_{j=1}^{r-1}\left(1+\frac{F_j}{1+m_j}\right)\cap[W]$$
$$-\frac{m_r}{1+m_r}\frac{c(TW)}{(1+F_0)\prod_{j=1}^{r-1}(1+F_j)}\frac{F_r}{1+F_r}\left(1+\frac{F_0}{M+d}\right)\prod_{j=1}^{r-1}\left(1+\frac{F_j}{1+m_j}\right)\cap[W]$$

Now note that $c(TW)\frac{F_r}{1+F_r}\cap[W]=c(TF_r)\cap[F_r]$; that F_r is the blow-up of E_r along B (since $E_r\supset B$, and $d\geq 2$); and that the other components cut E_r, F_r along a divisor with normal crossings. In other words, the induction hypothesis may be applied to both summands in this expression. Care must be taken for the rôle of M in the second summand: as the codimension of B in F_r is $d-1$, the denominator $M+d$ must be viewed as $(M+1)+(d-1)$. Therefore, applying the induction hypothesis evaluates the push-forward as

$$\frac{c(TV)}{\prod_{j=1}^{r-1}(1+E_j)}\prod_{j=1}^{r-1}\left(1+\frac{E_j}{1+m_j}\right)\cap[V]-\frac{M-\sum_{j=1}^{r-1}m_j}{(M+d)\prod_{j=1}^{r-1}(1+m_j)}c(TB)\cap[B]$$
$$-\frac{m_r}{1+m_r}\frac{c(TE_r)}{\prod_{j=1}^{r-1}(1+E_j)}\prod_{j=1}^{r-1}\left(1+\frac{E_j}{1+m_j}\right)\cap[E_r]+\frac{m_r}{1+m_r}\frac{(M+1)-\sum_{j=1}^{r-1}m_j}{(M+d)\prod_{j=1}^{r-1}(1+m_j)}c(TB)\cap[B]$$

and now (*) follows by reabsorbing the E_r term and performing trivial algebraic manipulations:

$$\frac{c(TV)}{\prod_{j=1}^{r-1}(1+E_j)}\left(1-\frac{m_r}{1+m_r}\frac{E_r}{1+E_r}\right)\prod_{j=1}^{r-1}\left(1+\frac{E_j}{1+m_j}\right)\cap[V]$$
$$-\left(\frac{M-\sum_{j=1}^{r-1}m_j}{(M+d)\prod_{j=1}^{r-1}(1+m_j)}-\frac{m_r}{1+m_r}\frac{M+1-\sum_{j=1}^{r-1}m_j}{(M+d)\prod_{j=1}^{r-1}(1+m_j)}\right)c(TB)\cap[B]$$
$$=\frac{c(TV)}{\prod_{j=1}^{r}(1+E_j)}\prod_{j=1}^{r}\left(1+\frac{E_j}{1+m_j}\right)\cap[V]-\frac{M-\sum_{j=1}^{r}m_j}{(M+d)\prod_{j=1}^{r}(1+m_j)}c(TB)\cap[B]$$

as needed. This proves (*).

Setting $M=\sum_{E_j\supset B}m_j=\sum_{j\in J}m_j$ in (*) concludes the proof of Theorem 3.7 for $\mathbb{J}=$ the whole family of subsets of J. $\square$

3.9. Now assume that $\mathbb{J}$ consists of the subsets of J having nonempty intersection with a given $L \subset J$. Let $J = \{1, \dots, r\}$, and $L = \{1, \dots, s\}$. As the proof of Theorem 3.7 in this case uses essentially the same techniques as those employed in §3.8, we provide fewer details.

The statement to prove depends on whether some of the E_ℓ, $\ell \in L$, contain B or not. Using Lemma 3.8 (1), the claim can be rewritten as follows:

(†) *If none of the E_ℓ contains B for $\ell \in L$,* then

$$\alpha_* \left(\frac{c(TW)}{(1+F_0)\prod_{j\in J}(1+F_j)} \left(1 + \frac{F_0}{M+d}\right) \sum_{\ell=1}^{s} \frac{F_\ell}{1+m_\ell} \prod_{i>\ell} \left(1 + \frac{F_i}{1+m_i}\right) \cap [W] \right)$$
$$= \frac{c(TV)}{\prod_{j\in J}(1+E_j)} \sum_{\ell=1}^{s} \frac{E_\ell}{1+m_\ell} \prod_{i>\ell} \left(1 + \frac{E_i}{1+m_i}\right) \cap [V]$$

(††) *If some of the E_ℓ contain B for $\ell \in L$,* then

$$\alpha_* \left(\frac{c(TW)}{(1+F_0)\prod_{j\in J}(1+F_j)} \frac{F_0}{M+d} \prod_{j\in J} \left(1 + \frac{F_j}{1+m_j}\right) \cap [W] \right.$$
$$\left. + \frac{c(TW)}{(1+F_0)\prod_{j\in J}(1+F_j)} \sum_{\ell=1}^{s} \frac{F_\ell}{1+m_\ell} \prod_{i>\ell} \left(1 + \frac{F_i}{1+m_i}\right) \cap [W] \right)$$
$$= \frac{c(TV)}{\prod_{j\in J}(1+E_j)} \sum_{\ell=1}^{s} \frac{E_\ell}{1+m_\ell} \prod_{i>\ell} \left(1 + \frac{E_i}{1+m_i}\right) \cap [V]$$

Proof of (†). By the projection formula we may factor out all terms corresponding to components not containing B; in particular, we may assume $L = \emptyset$, and the needed formula becomes

$$\alpha_* \left(\frac{c(TW)}{(1+F_0)\prod_{j\in J}(1+F_j)} \left(1 + \frac{F_0}{M+d}\right) \prod_{j\in J} \left(1 + \frac{F_j}{1+m_j}\right) \cap [W] \right)$$
$$= \frac{c(TV)}{\prod_{j\in J}(1+E_j)} \prod_{j\in J} \left(1 + \frac{E_j}{1+m_j}\right) \cap [V]$$

with all E_j containing B. This is precisely the formula proved in §3.8. □

Proof of (††). By the projection formula we may, once more, assume that all E_j contain B.

Consider first the terms in the $\sum$ with $\ell \geq 2$, on the left-hand-side:

$$\frac{c(TW)}{(1+F_0)\prod_{j\in J}(1+F_j)} \frac{F_\ell}{1+m_\ell} \prod_{i>\ell} \left(1 + \frac{F_i}{1+m_i}\right) \cap [W] \quad , \quad 2 \leq \ell \leq s \quad .$$

We claim that each of these terms pushes forward to the corresponding term in the $\sum$ on the right-hand-side:

$$\frac{c(TV)}{\prod_{j\in J}(1+E_j)} \frac{E_\ell}{1+m_\ell} \prod_{i>\ell} \left(1 + \frac{E_i}{1+m_i}\right) \cap [V]$$

(note: this is not so for the $\ell = 1$ term!, cf. Claim 3.10). To verify this, we argue as we did in §3.8. The formula is clear if B has codimension 1; if B has larger codimension, then F_ℓ is the blow-up of E_ℓ along B (since all E_j contain B), and an induction on the number of factors reduces the verification to proving that

$$\alpha_* \left(\frac{c(TW)}{(1+F_0)\prod_{j\in J}(1+F_j)} \cap [W] \right) = \frac{c(TV)}{\prod_{j\in J}(1+E_j)} \cap [V]$$

if $J \neq \emptyset$ (this is where the hypothesis $\ell \geq 2$ is used). This is proved in Lemma 3.8, (5).

Clearing the terms with $\ell \geq 2$ from both sides of (††), we are reduced to proving that

$$\alpha_* \left(\frac{c(TW)}{(1+F_0)\prod_{j\in J}(1+F_j)} \frac{F_0}{M+d} \prod_{j\in J} \left(1 + \frac{F_j}{1+m_j}\right) \cap [W] \right.$$
$$\left. + \frac{c(TW)}{(1+F_0)\prod_{j\in J}(1+F_j)} \frac{F_1}{1+m_1} \prod_{i>1} \left(1 + \frac{F_i}{1+m_i}\right) \cap [W] \right)$$
$$= \frac{c(TV)}{\prod_{j\in J}(1+E_j)} \frac{E_1}{1+m_1} \prod_{i>1} \left(1 + \frac{E_i}{1+m_i}\right) \cap [V]$$

and this is implied by the following explicit computation:

Claim 3.10. *If all E_j contain B, $j = 1, \dots, r$, then:*

$$\alpha_* \left(\frac{c(TW)}{(1+F_0)\prod_{j=1}^r(1+F_j)} \frac{F_1}{1+m_1} \prod_{i>1} \left(1 + \frac{F_i}{1+m_i}\right) \cap [W] \right)$$
$$= \frac{c(TV)}{\prod_{j=1}^r(1+E_j)} \frac{E_1}{1+m_1} \prod_{i>1} \left(1 + \frac{E_i}{1+m_i}\right) \cap [V] - \frac{1}{\prod_{j=1}^r(1+m_j)} \cdot c(TB) \cap [B] \quad ,$$

$$\alpha_* \left(\frac{c(TW)}{(1+F_0)\prod_{j=1}^r(1+F_j)} \frac{F_0}{M+d} \prod_{j=1}^r \left(1 + \frac{F_j}{1+m_j}\right) \cap [W] \right)$$
$$= \frac{1}{\prod_{j=1}^r(1+m_j)} \cdot c(TB) \cap [B] \quad .$$

Proof of the Claim. The first formula is clear if $d = 1$ (note that in this case $r = 1$ necessarily, since the E_j meet with normal crossings).

If $d \geq 1$ then each F_j is the blow-up of E_j along B. If $r > 1$, splitting off the last factor and using

$$-\frac{1}{(1+m_1)\cdots(1+m_{r-1})} + \frac{m_r}{1+m_r} \frac{1}{(1+m_1)\cdots(1+m_{r-1})} = -\frac{1}{(1+m_1)\cdots(1+m_r)}$$

shows that the general case follows from the case $r = 1$:

$$\alpha_* \left(\frac{c(TW)}{(1+F_0)(1+F_1)} \frac{F_1}{1+m_1} \cap [W] \right)$$
$$= \frac{c(TV)}{(1+E_1)} \frac{E_1}{1+m_1} \cap [V] - \frac{1}{(1+m_1)} \cdot c(TB) \cap [B] \quad .$$

Now this is equivalent to

$$\alpha_*\left(\frac{c(TF_1)}{(1+F_0)}\cap[F_1]\right) = c(TE_1)\cap[V]-c(TB)\cap[B] \quad ,$$

which follows from (4) in Lemma 3.8 (as F_1 is the blow-up of E_1 along B).

For the second formula, note that, for distinct $i_1,\dots,i_m$,

$$\alpha_*\left(\frac{c(TW)\cdot F_0\cdot F_{i_1}\cdots F_{i_m}}{(1+F_0)\cdot(1+F_{i_1})\cdots(1+F_{i_m})}\cap[W]\right) = (d-m)\cdot c(TB)\cap[B] \quad :$$

indeed, the intersection of the corresponding divisors $E_{i_1},\dots,E_{i_m}$ is nonsingular, of dimension $\dim V-m$ (since the divisor $\sum E_j$ has normal crossings), and $F_0\cdot F_{i_1}\cdots F_{i_m}$ is the class of the exceptional divisor of its blow-up along B; so this follows from Lemma 3.8, (3).

Therefore, rewriting the left-hand-side of the stated formula as

$$\alpha_*\left(\frac{1}{M+d}\frac{c(TW)\cdot F_0}{(1+F_0)}\prod_{j=1}^r\left(1-\frac{m_j}{1+m_j}\frac{F_j}{1+F_j}\right)\cap[W]\right)$$

and expanding the $\prod$ gives

$$\frac{1}{M+d}\left(d-(d-1)\sum\frac{m_j}{1+m_j}+(d-2)\sum\frac{m_jm_k}{(1+m_j)(1+m_k)}-\cdots\right)c(TB)\cap[B]$$

We leave to the reader the pleasant task of proving that this expression equals the right-hand-side of the stated formula. □

This concludes the proof of (††), and hence of Theorem 3.7. □

3.10. At this stage the integral

$$\int_{\mathcal{S}}1\!\!1(\mathcal{D})\,d\mathfrak{c}_X\in A_*\mathcal{C}_X$$

is defined for all *closed* subsets $\mathcal{S}$ of X and all $\mathcal{D}$. The definition for *locally closed* subsets is now forced upon us: set

$$\int_{\mathcal{S}}1\!\!1(\mathcal{D})\,d\mathfrak{c}_X := \int_{\mathcal{S}_1}1\!\!1(\mathcal{D})\,d\mathfrak{c}_X-\int_{\mathcal{S}_2}1\!\!1(\mathcal{D})\,d\mathfrak{c}_X$$

if $\mathcal{S}$ is the complement of a closed subset $\mathcal{S}_1$ in a closed subset $\mathcal{S}_2$.

Of course we have to show that this is independent of the choices of the closed subsets $\mathcal{S}_i$: that is, if $\mathcal{S}_i$, $\mathcal{T}_i$ are closed in $\mathcal{C}_X$, and

$$\mathcal{S}_1-\mathcal{S}_2=\mathcal{T}_1-\mathcal{T}_2$$

as subsets of $\mathcal{C}_X$, we have to prove that

$$\int_{\mathcal{S}_1}1\!\!1(\mathcal{D})\,d\mathfrak{c}_X-\int_{\mathcal{S}_2}1\!\!1(\mathcal{D})\,d\mathfrak{c}_X=\int_{\mathcal{T}_1}1\!\!1(\mathcal{D})\,d\mathfrak{c}_X-\int_{\mathcal{T}_2}1\!\!1(\mathcal{D})\,d\mathfrak{c}_X \quad .$$

By comparing both sides with the intersection with $\mathcal{S}_1\cap\mathcal{T}_1$, we may assume that $\mathcal{S}_1\subset\mathcal{T}_1$, and hence that $\mathcal{T}_1=\mathcal{S}_1\cup\mathcal{T}_2$ and $\mathcal{S}_2=\mathcal{S}_1\cap\mathcal{T}_2$. The needed equality is then the one arising from

$$(\mathcal{S}_1\cup\mathcal{T}_2)-\mathcal{T}_2=\mathcal{S}_1-(\mathcal{S}_1\cap\mathcal{T}_2) \quad ,$$

that is, the following form of 'inclusion-exclusion':

Lemma 3.11. *If $\mathcal{S}$, $\mathcal{T}$ are closed in $\mathcal{C}_X$, and $\mathcal{D}$ is a divisor, then*

$$\int_{\mathcal{S}\cup\mathcal{T}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X = \int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X + \int_{\mathcal{T}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X - \int_{\mathcal{S}\cap\mathcal{T}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X \quad .$$

Proof. The formula is clear if $\mathcal{S}$ or $\mathcal{T}$ equal $\mathcal{C}_X$, so we may assume that both are proper closed subsets.

It is enough to verify the statement for manifestations over a π in $\mathcal{C}_X$ resolving $\mathcal{D}$ and $\mathcal{S}$, $\mathcal{T}$, $\mathcal{S}\cap\mathcal{T}$, and hence $\mathcal{S}\cup\mathcal{T}$. We may assume all are combinations of components of a normal crossing divisor $\sum_{j\in J} E_j$: that is, the subsets have ideals locally generated by $\prod_{j\in L_S} e_j$, $\prod_{j\in L_T} e_j$, $\prod_{j\in L_S\cap L_T} e_j$, $\prod_{j\in L_S\cup L_T} e_j$ respectively, where e_j denotes a local generator for E_j. Note that the ideal of $\mathcal{S}\cap\mathcal{T}$ in V_π must be the sum of the ideals of $\mathcal{S}$ and $\mathcal{T}$, to wit

$$(\prod_{j\in L_S} e_j) + (\prod_{j\in L_T} e_j) = (\prod_{j\in L_S\cap L_T} e_j) \quad ;$$

this implies that if $s\in L_S - L_T$ and $t\in L_T - L_S$, then the corresponding components E_s, E_t have empty intersection.

Now denote by $\mathbb{J}_{\mathcal{S}}$, $\mathbb{J}_{\mathcal{T}}$, $\mathbb{J}_{\mathcal{S}\cap\mathcal{T}}$, $\mathbb{J}_{\mathcal{S}\cup\mathcal{T}}$ the corresponding families of subsets $I\subset J$. It suffices to prove that the individual contributions of each $I\subset J$ to the expressions defining the integrals (as in Definition 3.2) satisfy the relation in the statement.

The only case in which this is not trivially true is when $I\in\mathbb{J}_{\mathcal{S}}$ and $I\in\mathbb{J}_{\mathcal{T}}$, and hence $I\in\mathbb{J}_{\mathcal{S}\cup\mathcal{T}}$, but $I\notin\mathbb{J}_{\mathcal{S}\cap\mathcal{T}}$. In this case I contains indices $s\in L_S$, $t\in L_T$, such that $s\notin L_T$, $s\notin L_S$; as observed above, this implies that $E_s\cap E_t=\emptyset$. But then $\prod_{i\in I} E_i = 0$, and hence the contribution of I to all the integrals is zero. So the statement is verified in this case as well, concluding the proof. ☐

3.11. Finally, constructible sets are finite disjoint unions of locally closed subsets. If $\mathcal{S}=\amalg_{k\in K}\mathcal{S}_k$, with $\mathcal{S}_k$ locally closed in $\mathcal{C}_X$, and $\mathcal{D}$ is a divisor in $\mathcal{C}_X$, we define

$$\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X := \sum_{k\in K}\int_{\mathcal{S}_k} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X \quad .$$

If $\mathcal{S}_k$ is represented by S_k in V_ρ, this element of A_*V_ρ is in fact the image of a class in $A_*(\cup_k\overline{S}_k)$, cf. Remark 3.3.

The integral $\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X$ ought to be independent of the ambient system $\mathcal{C}_X$ containing $\mathcal{S}$, but we only know instances of this principle. For example, if X is nonsingular and $\mathcal{S}$ is represented by (id, S), with S a subvariety of X, then the manifestation of $\int_{\mathcal{S}} 1\!\!1(0)\, d\mathfrak{c}_X$ in X resides naturally in A_*S, and is independent of X, at least in characteristic 0: indeed, we will see (§5.1) that it equals the Chern-Schwartz-MacPherson class of S. Our argument will however rely on the theory of these classes; we feel that there should be a more straightforward justification, in the style of the computations performed in this section.

4. Techniques of (celestial) integration

4.1. *Inclusion-exclusion.* The notion

$$\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X$$

introduced in §3 is additive in $\mathcal{S}$ by definition (cf. §3.11), and along the way we have had to prove a simple form of inclusion-exclusion, that is, Lemma 3.11. This implies full inclusion-exclusion, that is: if $\mathcal{S} = \mathcal{S}_1 \cap \cdots \cap \mathcal{S}_r$, then

$$\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1 < \cdots < i_s} \int_{\mathcal{S}_{i_1} \cup \cdots \cup \mathcal{S}_{i_s}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X \quad .$$

This is useful for explicit computations. For example, assume $\mathcal{S}$ is represented by (X, S), with S a subvariety of X; then S may be written as the intersection of hypersurfaces $S_1, \dots, S_r$, and inclusion-exclusion reduces the computation of the integral over $\mathcal{S}$ to the integral over constructible sets represented by hypersurfaces of X.

4.2. *Change-of-variables formula.* Let $\rho : Y \to X$ be a proper birational map, with relative canonical divisor K_ρ. Here we need that for proper birational maps $\pi : V \to Y$, with V nonsingular, $K_{\rho\circ\pi} = \pi^* K_\rho + K_\pi$; for example Y could be nonsingular, cf. (3) in §3.2.

Recall that $\mathcal{C}_X$ and $\mathcal{C}_Y$ are then equivalent, and that consequently corresponding notions of divisors and constructible subsets coincide, cf. Remarks 2.9 and 2.11; and $A_*\mathcal{C}_X \cong A_*\mathcal{C}_Y$ by Lemma 2.6. So the integral of a divisor $\mathcal{D}$ over a constructible subset $\mathcal{S}$ is defined in both modification systems, and lands in the same target.

Theorem 4.1. *For all $\mathcal{S}$ and $\mathcal{D}$, and denoting by K_ρ the divisor in $\mathcal{C}_X$ represented by (ρ, K_ρ),*

$$\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X = \int_{\mathcal{S}} 1\!\!1(\mathcal{D} + K_\rho)\, d\mathfrak{c}_Y \quad .$$

Proof. It is enough to prove the equality of manifestations in a variety V resolving $\mathcal{D}$ and $\mathcal{S}$:

$$\begin{tikzcd} V \arrow[r, "\pi_Y"] \arrow[rr, bend right, "\pi_X"'] & Y \arrow[r, "\rho"] & X \end{tikzcd}$$

and this is clear from Definition 3.2, since

$$\mathcal{D} + K_{\pi_X} = (\mathcal{D} + K_\rho) + K_{\pi_Y}$$

as divisors in the modification system, by (3) in §3.2. $\square$

4.3. *Chern classes.* If X is nonsingular and D is a nonsingular hypersurface of X, let $\mathcal{D}$ be the corresponding divisor of $\mathcal{C}_X$, represented by (id, D). Then

$$\left(\int_{\mathcal{C}_X} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X\right)_{id} = c(TX) \cap [X] - \frac{1}{2} \cdot c(TD) \cap [D] \quad .$$

Indeed, the identity id already resolves $\mathcal{D}$, $\mathcal{C}_X$; applying Definition 3.2 gives

$$\left(\int_{\mathcal{C}_X} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X\right)_{id} = \frac{c(TX)}{(1+D)} \cdot \left(1 + \frac{D}{2}\right) \cap [X] = c(TX) \cdot \left(1 - \frac{1}{2} \cdot \frac{D}{1+D}\right) \cap [X]$$

with the given result. In particular,

$$\left(\int_{\mathcal{C}_X} 1\!\!1(0)\, d\mathfrak{c}_X\right)_{id} = c(TX) \cap [X] \quad :$$

that is, the identity manifestation of the integral of $1\!\!1(0)$ realizes a 'Chern class measure' on the variety. This observation is extended readily to *nonsingular* subvarieties of X:

Proposition 4.2. *Assume that X is nonsingular, and $\mathcal{S} \subset \mathcal{C}_X$ is represented by (id, S). Then*

$$\left(\int_{\mathcal{S}} 1\!\!1(0)\, d\mathfrak{c}_X\right)_{id} = c(TS) \cap [S] \quad .$$

Proof. Let $\pi : \widetilde{X} \to X$ be the blow-up of X along S, with exceptional divisor E. Then π resolves 0, $\mathcal{S}$, and $K_\pi = (d-1)E$, with $d = \operatorname{codim} S$. Applying Definition 3.2 and Lemma 3.8, (1) gives

$$\left(\int_{\mathcal{S}} 1\!\!1(0)\, d\mathfrak{c}_X\right)_{\pi} = \frac{c(T\widetilde{X})}{(1+E)} \cap \frac{E}{d} = \frac{c(TE) \cap [E]}{d} \quad ;$$

from which the statement follows, by Lemma 3.8, (3). □

We extend this remark to *singular* subvarieties in §5.

It is worth noting that the class in Proposition 4.2 is only one manifestation of the integral. Manifestations in varieties mapping to X amount to specific lifts of the total Chern class of X. For example, if B is a nonsingular subvariety and $\pi : Y \to X$ is the blow-up of X along B, with exceptional divisor E, then

$$\left(\int_{\mathcal{C}_X} 1\!\!1(0)\, d\mathfrak{c}_X\right)_{\pi} = c(TY) \cap [Y] - \frac{d-1}{d} \cdot c(TE) \cap [E]$$

according to Definition 3.2 (in one of the forms listed in §3.4). Manifestations in other varieties are obtained as push-forwards from the manifestation in a blow-up (cf. §3.5). Explicit examples can be found in Example 7.4.

5. Relation with Chern-Schwartz-MacPherson classes

5.1. Let X be a nonsingular variety. In §4.3 we have observed that the symbol $d\mathfrak{c}_X$ behaves as a 'Chern class measure' in the identity manifestation of the integral defined in §3, with respect to *nonsingular* subvarieties S. That is:

$$\left(\int_{\mathcal{S}} 1\!\!1(0)\, d\mathfrak{c}_X\right)_{id} = c(TS) \cap [S]$$

if $\mathcal{S}$ is the constructible subset of $\mathcal{S}$ represented by (id, S). It is natural to ask what class

$$\left(\int_{\mathcal{S}} 1\!\!1(0)\, d\mathfrak{c}_X\right)_{id}$$

computes if S is not required to be nonsingular.

Theorem 5.1. *In characteristic 0*

$$\left(\int_{\mathcal{S}} 1\!\!1(0)\, d\mathfrak{c}_X\right)_{id} = c_{\mathrm{SM}}(S) \in A_* X \quad ,$$

the Chern-Schwartz-MacPherson class of S.

The Chern-Schwartz-MacPherson class generalizes in a beautifully functorial fashion the Chern class of the tangent bundle of a nonsingular variety. The reader is addressed to [Mac74] for the original work of Robert MacPherson (over $\mathbb{C}$), and to [Ken90] for a discussion over arbitrary algebraically closed fields of characteristic 0. An equivalent notion had been defined earlier by Marie-Hélène Schwartz, cf. [BS81] for results comparing the two definitions.

The characteristic 0 restriction in the statement of Theorem 5.1 is due to the fact that a good theory of Chern-Schwartz-MacPherson classes does not seem to be available in other contexts. Also, we have used the factorization theorem for birational maps in the definition of the integral, and this relies on resolution of singularities. In fact, we will work over $\mathbb{C}$ in this section for simplicity of exposition, although everything can be extended without difficulty to arbitrary algebraically closed fields of characteristic 0.

Theorem 5.1 will follow from a more general result, linking the integral to *MacPherson's natural transformation.*

5.2. A quick reminder of related notions is in order.

A *constructible function* on a variety X is a linear combination of characteristic functions of closed subvarieties: $\sum_{Z\subset X} = m_Z 1\!\!1_Z$, where $m_Z \in \mathbb{Z}$ and $1\!\!1_Z(p) = 1$ if $p \in Z$, 0 otherwise. Thus a subset of X is constructible if and only if its characteristic function is.

Constructible functions form a group $F(X)$. Taking Euler characteristic of fibers makes the assignment $X \to F(X)$ a covariant functor under proper maps. More explicitly, if $f : X_1 \to X_2$ is a proper map, and Z is a subvariety of X_1, then the function $f_*(1\!\!1_Z)$ defined by $p \mapsto \chi(f^{-1}(p) \cap Z)$ is constructible; extending by linearity defines a push-forward $f_* : F(X_1) \to F(X_2)$.

MacPherson proved that there exists a unique natural transformation c_* from the functor F to a homology theory—in this paper we use Chow group with $\mathbb{Q}$-coefficients, denoted A_*, which also grants us the luxury of using constructible functions with $\mathbb{Q}$-coefficients—such that if S is nonsingular, then $c_*(1\!\!1_S) = c(TS) \cap [S] \in A_*S$. For arbitrary constructible $S \subset X$, $c_*(1\!\!1_S) \in A_*X$ is the class denoted $c_{\mathrm{SM}}(S)$ in §5.1.

An immediate application of the functoriality of the notion shows that, if the ambient X is complete, then the degree of $c_{\mathrm{SM}}(S)$ agrees with the topological Euler characteristic $\chi(S)$.

5.3. Let X be a nonsingular variety over an algebraically closed field of characteristic 0, and let $\mathcal{D}$, $\mathcal{S}$ resp. be a divisor and a constructible set in $\mathcal{C}_X$. If S is represented by (π, S), and $p \in X$, denote by $\mathcal{S}_p$ the constructible set represented by $(\pi, S \cap \pi^{-1}(p))$.

Definition 5.2. We define a function $I_X(\mathcal{D}, \mathcal{S}) : X \to \mathbb{Q}$ by

$$I_X(\mathcal{D}, \mathcal{S})(p) := \text{the degree of } \left(\int_{\mathcal{S}_p} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X \right)_{id} .$$

(Recall that the manifestation on the right-hand-side may be viewed as a class in $A_*p = \mathbb{Q}$, cf. §3.11, so its degree is well-defined even if X is not complete.)

The main result of this section is that the identity manifestation of the integral defined in §3 corresponds to $I_X(\mathcal{D}, \mathcal{S})$ via MacPherson's transformation:

Theorem 5.3. *The function* $I_X(\mathcal{D},\mathcal{S})$ *is constructible, and*

$$\left(\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X\right)_{id} = c_*(I_X(\mathcal{D},\mathcal{S})) \quad .$$

This result shows that, at the level of the identity manifestation, the function $I_X(\mathcal{D},\mathcal{S})$ contains at least as much information as the integral of $1\!\!1(\mathcal{D})$ over $\mathcal{S}$. We could in fact define a constructible function *of the system* $\mathcal{C}_X$, that is, an element of the *inverse limit* of the groups of constructible functions through the system (with push-forward as defined in §5.2), by setting

$$\mathcal{I}_X(\mathcal{D},\mathcal{S})_\pi := I_{V_\pi}(\mathcal{D}+K_\pi,\mathcal{S}) \quad .$$

Indeed, the following 'change-of-variables' formula holds: if $\pi' = \pi\circ\alpha$, then

$$\alpha_* I_{V_{\pi'}}(\mathcal{D}+K_{\pi'},\mathcal{S}) = I_{V_\pi}(\mathcal{D}+K_\pi,\mathcal{S})$$

(exercise!) The naturality of c_* yields a homomorphism from the group of constructible functions of the system to $A_*\mathcal{C}_X$, and the image of the 'celestial' constructible function $\mathcal{I}_X(\mathcal{D},\mathcal{S})$ through this homomorphism is the celestial integral.

Shoji Yokura has studied the *direct* limit of the groups of constructible functions of an inverse system of varieties, and associated Chern classes, in [Yok03].

5.4. The proof of Theorem 5.3, given in §5.5, relies on two lemmas. We will use the following notation: if E is a divisor with components E_j, $j\in J$, and $I\subset J$, then E_I° denotes the complement of $\cup_{i\notin I}E_i$ in $\cap_{i\in I}E_i$.

Lemma 5.4. *Let* V *be a nonsingular variety, and let* E *a divisor with normal crossings and nonsingular components* E_j, $j\in J$. *Then for all* $I\subset J$

$$c_{\rm SM}(E_I^\circ) = \left(c(TV(-\log E))\cdot\prod_{i\in I}E_i\right)\cap[V] \quad .$$

Proof. Denote by E_I the intersection $\cap_{i\in I}E_i$. Then E_I is nonsingular since E has normal crossings, and its normal bundle in V has Chern class $\prod_{i\in I}(1+E_i)$, hence (using Lemma 3.8, (1))

$$\left(c(TV(-\log E))\cdot\prod_{i\in I}E_i\right)\cap[V] = \frac{c(TE_I)}{\prod_{i\notin I}(1+E_i)}\cap[E_I] \quad ,$$

and we have to show that this equals

$$c_{\rm SM}(E_I^\circ) = c_*(1\!\!1_{E_I^\circ}) = c_{\rm SM}(E_I) - c_{\rm SM}(E_I\cap(\cup_{i\notin I}E_i))$$

Now observe that $E_I\cap(\cup_{i\notin I}E_i)$ is a divisor with normal crossings in E_I; the needed formula follows then immediately from (*) in §2.2 of [Alu99] (top of p. 4002). □

Lemma 5.5. *Let* V, W *be nonsingular varieties;* $\alpha: W\to V$ *be a proper birational map;* E *a divisor with normal crossings and nonsingular components* E_j, $j\in J$, *in* V; F *a divisor with normal crossings and nonsingular components* F_k, $k\in K$, *in* W; *and* m_j, n_k *integers such that*

$$\sum_{k\in K} n_k F_k = K_\alpha + \sum_{j\in J} m_j E_j \quad .$$

Finally, let S be a constructible subset of V. Then

$$\sum_{I\subset J}\frac{\chi(E_I^\circ\cap S)}{\prod_{i\in I}(1+m_i)}=\sum_{I\subset K}\frac{\chi(F_I^\circ\cap\alpha^{-1}(S))}{\prod_{i\in I}(1+n_i)}\quad.$$

Proof. By additivity of Euler characteristics we may assume that S is closed. The factorization theorem of [AKMW02] reduces the statement to the case of a blow-up along a nonsingular center meeting E with normal crossings, which is worked out for the universal Euler characteristic in Proposition 2.5 in [Alu]. □

Remark 5.6. With $S=$ a point, and $E=0$, Lemma 5.5 states that

$$1=\sum_{I\subset K}\frac{\chi(F_I^\circ\cap\alpha^{-1}(p))}{\prod_{i\in I}(1+n_i)}\quad;$$

with $S=V$ and $E=0$ again, the statement is that

$$\chi(V)=\sum_{I\subset K}\frac{\chi(F_I^\circ)}{\prod_{i\in I}(1+n_i)}\quad.$$

These formulas have been known for a long time—they were first proved by methods of p-adic integration in [DL92] (Theorem 6.1), and François Loeser informs me that he and Jan Denef knew them as early as 1987; and that while aware of the implication for Chern-Schwartz-MacPherson classes, they did not mention it for lack of applications at the time. These formula were later recovered (again by Denef and Loeser) by using motivic integration (see for example the survey [DL01], §4.4.3).

In fact, this is the 'point of contact with motivic integration' mentioned in the introduction:

Claim 5.7. *Let X be complete. With notations as in §3.3, and E_I° as above,*

$$\deg\int_{\mathcal{C}_X}1\!\!1(\mathcal{D})\,d\mathfrak{c}_X=\sum_{I\subset J}\frac{\chi(E_I^\circ)}{\prod_{i\in I}(1+m_i)}\quad.$$

This follows immediately from the definition and from Lemma 5.4, since the degree of c_{SM} equals the Euler characteristic. This formula allows us to relate invariants introduced by using our integral with other invariants arising naturally from considerations in motivic (and/or p-adic) integration, such as the *stringy Euler number,* cf. §7 of [Vey].

5.5. *Proof of Theorem 5.3.*

Let $\pi: V\to X$ be an object of $\mathcal{C}_X$ resolving $\mathcal{D}$ and $\mathcal{S}$, cf. §3.3. Thus there is a divisor E with normal crossings and nonsingular components E_j, $j\in J$, in V, such that

$$D+K_\pi=\sum m_jE_j$$

and (π,D) represents $\mathcal{D}$; and $\mathcal{S}$ is represented by (π,S), where $S=V$ or $S=\cup_{\ell\in L}E_\ell$ for some $L\subset J$.

First, we are going to show that

$$(\star)\qquad I_X(\mathcal{D},\mathcal{S})=\pi_*\left(\sum_{I\subset J}\frac{1\!\!1_{E_I^\circ\cap S}}{\prod_{i\in I}(1+m_i)}\right)\quad,$$

with push-forward of constructible functions defined as in §5.2; in particular, this shows that $I_X(\mathcal{D},\mathcal{S})$ is constructible.

In order to show ($\star$), evaluate the right-hand-side at a $p \in X$:

$$\pi_*\left(\sum_{I\subset J}\frac{1\!\!1_{E_I^\circ\cap S}}{\prod_{i\in I}(1+m_i)}\right)(p)=\sum_{I\subset J}\frac{\chi((E_I^\circ\cap S)\cap\pi^{-1}(p))}{\prod_{i\in I}(1+m_i)}$$

By Lemma 5.5, this may be evaluated after replacing π with an object dominating it and resolving $\mathcal{D}$ and $\mathcal{S}_p$ (represented by $S\cap\pi^{-1}(p)$, cf. §5.3); that is, we may assume that $S\cap\pi^{-1}(p)$ is a collection of components of E, indexed by $L_p\subset J$. Let $\mathbb{J}_{S_p}$ be the family of subsets of J meeting L_p. Then we may rewrite the right-hand-side of ($\star$) as

$$\pi_*\left(\sum_{I\subset J}\frac{1\!\!1_{E_I^\circ\cap S}}{\prod_{i\in I}(1+m_i)}\right)(p)=\sum_{I\in\mathbb{J}_{S_p}}\frac{\chi(E_I^\circ)}{\prod_{i\in I}(1+m_i)}\quad.$$

Now using Lemma 5.4 and the fact that the degree of the Chern-Schwartz-MacPherson class agrees with the Euler characteristic, this equals the degree of

$$\sum_{I\in\mathbb{J}_{S_p}}\frac{c_{\mathrm{SM}}(E_I^\circ)}{\prod_{i\in I}(1+m_i)}=c(TV(-\log E))\cdot\left(\sum_{I\in\mathbb{J}_{S_p}}\prod_{i\in I}\frac{E_i}{1+m_i}\right)\cap[V]=\left(\int_{\mathcal{S}_p}1\!\!1(\mathcal{D})\,d\mathfrak{c}_X\right)_\pi .$$

Finally, the degree is preserved after push-forward, so this equals $I_X(\mathcal{D},\mathcal{S})(p)$, concluding the proof of ($\star$).

Now apply c_* to both side of ($\star$), and use Lemma 5.4 again. As in §3.3, denote by $\mathbb{J}_S$ the family of subsets of J if $S=V$, and the subfamily of subsets meeting L otherwise. This gives

$$\begin{aligned}c_*(I_X(\mathcal{D},\mathcal{S}))&=c_*\pi_*\left(\sum_{I\subset J}\frac{1\!\!1_{E_I^\circ\cap S}}{\prod_{i\in I}(1+m_i)}\right)=\pi_*\left(\sum_{I\subset J}\frac{c_*(1\!\!1_{E_I^\circ\cap S})}{\prod_{i\in I}(1+m_i)}\right)\\&=\pi_*\left(\sum_{I\in\mathbb{J}_S}\frac{c_{\mathrm{SM}}(E_I^\circ)}{\prod_{i\in I}(1+m_i)}\right)=\pi_*\left(c(TV(-\log E))\cdot\sum_{I\in\mathbb{J}_S}\prod_{i\in I}\frac{E_i}{1+m_i}\right)\\&=\pi_*\left(\int_{\mathcal{S}}1\!\!1(\mathcal{D})\,d\mathfrak{c}_X\right)_\pi=\left(\int_{\mathcal{S}}1\!\!1(\mathcal{D})\,d\mathfrak{c}_X\right)_{id}\quad,\end{aligned}$$

concluding the proof of Theorem 5.3. □

5.6. Theorem 5.3 implies Theorem 5.1. To see this, assume $S\subset X$ is constructible. Then $I_X(0,\mathcal{S})(p)=0$ if $p\notin S$, since then $\mathcal{S}_p=\emptyset$; if $p\in S$ and X is nonsingular then

$$I_X(0,\mathcal{S})(p)=\text{degree of }\left(\int_{\mathcal{S}_p}1\!\!1(0)\,d\mathfrak{c}_X\right)_{id}=1$$

because then $\mathcal{S}_p$ is represented by (id,p), so the integral computes $c(Tp)\cap[p]$, see §4.3. This shows that $I_X(0,\mathcal{S})=1\!\!1_S$ if X is nonsingular, and hence

$$\left(\int_{\mathcal{S}}1\!\!1(0)\,d\mathfrak{c}_X\right)_{id}=c_*(1\!\!1_S)=c_{\mathrm{SM}}(S)$$

by Theorem 5.3.

Theorem 5.1 implies in particular that the identity manifestation of the integral of $1\!\!1(0)$ over a subvariety S of a nonsingular X is independent of the ambient variety X. We do not know to what extent integrals are independent of the ambient variety, cf. §3.11.

5.7. Given the close connection between the integral defined in §3 and MacPherson's natural transformation, we feel that a more thorough study of modification systems ought to yield a novel approach to the theory of Chern-Schwartz-MacPherson classes. The left-hand-side of the formula in Theorem 5.1 could be taken as the *definition* of the class, and a good change-of-variable formula for arbitrary proper maps should amount to the naturality of this notion. With this (hypothetical) set-up, a proof of resolution of singularity in positive characteristic would imply an automatic upgrade of the theory of Chern-Schwartz-MacPherson classes in that context.

In any case, if S is any subvariety of a nonsingular variety X, Theorem 5.1 affords many new manifestations of the Chern-Schwartz-MacPherson class of S: for example, if $\pi : V \to X$ is any proper birational map, then

$$\left(\int_{\mathcal{S}} 1\!\!1(0)\, d\mathfrak{c}_X\right)_\pi$$

is a distinguished lift of $c_{\mathrm{SM}}(S)$ in A_*V. These manifestations surely inherit good functoriality properties from $c_{\mathrm{SM}}(S)$, and it would be interesting to explore these properties.

6. Applications

6.1. The mere existence of an integral satisfying the properties in §4 has some immediate applications. For example, assume that X and Y are nonsingular complete birational Calabi-Yau varieties. Let V be any resolution of indeterminacies of a birational map between X and Y:

$$\begin{tikzcd} & V \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[rr, dashed] & & Y \end{tikzcd}$$

Then $K_{\pi_X} = K_{\pi_Y}$, hence by change of variables (§4.2)

$$\int_{\mathcal{S}} 1\!\!1(0)\, d\mathfrak{c}_X = \int_{\mathcal{S}} 1\!\!1(K_{\pi_X})\, d\mathfrak{c}_V = \int_{\mathcal{S}} 1\!\!1(K_{\pi_Y})\, d\mathfrak{c}_V = \int_{\mathcal{S}} 1\!\!1(0)\, d\mathfrak{c}_Y$$

for all $\mathcal{S}$. Applying to $\mathcal{S}$ = the whole system, and using that the integral of $1\!\!1(0)$ evaluates the total Chern class (§4.3), shows that the Chern classes of X and Y agree *as elements of* $A_*\mathcal{C}_X \cong A_*\mathcal{C}_Y$.

The same argument shows that any two nonsingular complete birational varieties in the same K-equivalence class (cf. Remark 2.9) have the same total Chern class in their (equivalent) modification systems. This clarifies the main result of [Alu].

6.2. The advantage of the more through investigation developed here over the work in [Alu] is that we can now move away from the hypothesis that the varieties are in the same K-class; in fact, this can be done in several ways. For example, assume we have a resolution of indeterminacies of a birational map between two varieties X and Y, as above:

$$\begin{tikzcd} & V \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[rr, dashed] & & Y \end{tikzcd}$$

with π_X and π_Y proper and birational. The varieties X and Y need not be complete, or nonsingular. The relative differentials of π_X, π_Y determine divisors $\mathcal{K}_{\pi_X}$, $\mathcal{K}_{\pi_Y}$ of the equivalent modification systems $\mathcal{C}_X$, $\mathcal{C}_Y$. Let $\mathcal{D}_X$, $\mathcal{D}_Y$ be any divisors such that $\mathcal{K}_{\pi_X} + \mathcal{D}_X = \mathcal{K}_{\pi_Y} + \mathcal{D}_Y$.

Theorem 6.1. *With these notations, and for all constructible subsets $\mathcal{S}$:*

$$\int_{\mathcal{S}} 1\!\!1(\mathcal{D}_X)\, d\mathfrak{c}_X = \int_{\mathcal{S}} 1\!\!1(\mathcal{D}_Y)\, d\mathfrak{c}_Y \quad .$$

The proof is again an immediate application of change-of-variables.

Numerical consequences may be extracted from this formula. Recalling that divisors act on Chow groups of modification systems (Remark 2.9):

Corollary 6.2. *With notations as above:*

$$(c_1(X) - \mathcal{D}_Y)^i \cdot \int_{\mathcal{S}} 1\!\!1(\mathcal{D}_X)\, d\mathfrak{c}_X = (c_1(Y) - \mathcal{D}_X)^i \cdot \int_{\mathcal{S}} 1\!\!1(\mathcal{D}_Y)\, d\mathfrak{c}_Y$$

for all $i \geq 0$.

Proof. Indeed: $c_1(X) - \mathcal{D}_Y = c_1(V) + \mathcal{K}_{\pi_X} - \mathcal{D}_Y = c_1(V) + \mathcal{K}_{\pi_Y} - \mathcal{D}_X = c_1(Y) - \mathcal{D}_X$. □

Specializing to the identity manifestation and taking degrees, in the particular case in which X and and Y are nonsingular complete varieties *in the same K-equivalence class,* taking $\mathcal{D}_X = \mathcal{D}_Y = 0$, and $\mathcal{S}$ = the whole modification system, this gives the equality:

$$(*) \qquad c_1(X)^i \cdot c_{n-i}(X) = c_1(Y)^i \cdot c_{n-i}(Y)$$

for all $i \geq 0$, with $n = \dim X$.

For $i = 0$ this is the well-known equality of Euler characteristics of varieties in the same K-class (see for example [Bat99a]); for $i = 1$ it can be derived as a consequence of the equality of Hodge numbers, which follows from the change of variable formula in motivic integration, and Theorem 3 in ([LW90]). The equality for all i should also be a very particular case of the fact that complex elliptic genera are preserved through K-equivalence, cf. [Wan03]; and [BL03], where this is byproduct of the definition of elliptic genera of singular varieties (answering a fundamental question raised by Burt Totaro, [Tot00], p. 758).

Even for varieties in the same K-equivalence class, Corollary 6.2 is substantially stronger than $(*)$. For example, let S_X, S_Y be subvarieties of X, Y resp., such that $\pi_X^{-1}(S_X) = \pi_Y^{-1}(S_Y)$. Then

$$\deg(c_1(X)^i \cdot c_{\mathrm{SM}}(S_X)) = \deg(c_1(Y)^i \cdot c_{\mathrm{SM}}(S_Y)) \quad :$$

again take $\mathcal{D}_X = \mathcal{D}_Y = 0$; and note that S_X, S_Y represent the same constructible subset in the modification system, then apply Theorem 5.1.

More generally, judicious choices for $\mathcal{D}_X$, $\mathcal{D}_Y$ may express interesting data on Y in terms of data on X; the simplest example is probably

$$\chi(Y) = \deg \int_{\mathcal{C}_X} 1\!\!1(\mathcal{K}_{\pi_Y} - \mathcal{K}_{\pi_X})\, d\mathfrak{c}_X \quad .$$

It is often possible to express $\mathcal{D}_X$, $\mathcal{D}_Y$ in terms of divisors arising from subschemes of X, Y, and Corollary 6.2 may be used to derive the equality of certain combinations of Chern numbers of X, Y, and of these subschemes; in fact, Theorem 6.1 should simply be viewed as a notationally convenient way to encode a large number of such identities. In general, these identities tend to appear rather complicated, a lesson also learned through the work of Lev Borisov and Anatoly Libgober, and Chin-Lung Wang. A very simple prototypical situation is presented in Example 7.5.

6.3. Some of the information exploited in §6.2 is also captured by the following invariant. For a nonsingular X, consider the set of classes

$$\mathrm{Can}(X) := \left\{ \int_{\mathcal{C}_X} 1\!\!1(\mathcal{K})\, d\mathfrak{c}_X \right\} \subset A_*\mathcal{C}_X$$

as $\mathcal{K}$ ranges over the divisors of $\mathcal{C}_X$ obtained by pulling back the effective canonical divisors of X.

This is a birational invariant of complete nonsingular varieties. Indeed, so is $\Gamma(X, \Omega^{\dim X}(X))$, and the change-of-variable formula ensures that the integrals of corresponding divisors coincide. Taking degrees of the classes in $\mathrm{Can}(X)$ one obtains a subset of $\mathbb{Z}$ which is likewise a birational invariant, and may be amenable to calculation. For example:

Proposition 6.3. *Let X be a nonsingular complete algebraic variety, and assume that X is birational to a Calabi-Yau manifold Y. Then*

$$\deg \mathrm{Can}(X) = \{\chi(Y)\} \quad .$$

Proof. Indeed $\mathrm{Can}(X) = \mathrm{Can}(Y)$ must be the single class $\int_{\mathcal{C}_Y} 1\!\!1(0)\, d\mathfrak{c}_Y$, whose identity manifestation is the total Chern class of Y by §4.3. □

6.4. **Zeta function.** For $\mathcal{D}$ a divisor of $\mathcal{C}_X$, and m a variable, we can consider the formal expression

$$Z(\mathcal{D}, m) := \int_{\mathcal{C}_X} 1\!\!1(m\mathcal{D})\, d\mathfrak{c}_X \in A_*\mathcal{C}_X[m]$$

(see §8 for a discussion of issues arising in letting the multiplicities be variables). This 'celestial zeta function' is a very interesting object, which deserves further study. For example:

Proposition 6.4. *Assume X is complete, and $\mathcal{D}$ is the divisor corresponding to the zero-scheme of a section f of a line bundle on X. Then the degree of $Z(\mathcal{D}, m)$ equals the topological zeta function of f.*

We are referring here to the topological zeta function of [DL92], see also §6 in [Vey], and we are abusing the terminology since classically the topological zeta function is defined for $f : M = \mathbb{C}^n \to \mathbb{C}$; this case can be recovered by compactifying M to $X = \mathbb{P}^n$, then taking the integral over the constructible subset represented by (id, M)).

Proposition 6.4 is proved easily by arguing as in §5.5 to relate $Z(\mathcal{D}, m)$ to a combination of Euler characteristics of subsets in the relevant normal crossing divisor in a resolution, thereby matching the expression in §6.6 of [Vey].

The connection with the topological zeta function hints that the poles of $Z(\mathcal{D}, m)$ carry interesting information; a version of the *monodromy conjecture* ([Vey], §6.8) can be phrased in terms of $Z(\mathcal{D}, m)$. For an explicit computation of Z, see Example 7.6.

6.5. **Stringy Chern classes.** The identity manifestation of the class

$$\int_{\mathcal{C}_X} 1\!\!1(0)\, d\mathfrak{c}_X$$

generalizes the total Chern class of the tangent bundle to possibly singular X. We loosely refer to this class as the *stringy* Chern class of X, for reasons explained below.

There actually are different interpretations of this formula, depending on the notion used to define the relative canonical divisor, cf. §3.2, and they lead to different classes. One important alternative to the possibility presented in §3.2, applicable to $\mathbb{Q}$-Gorenstein varieties, is to let ω_X be the *double-dual* of the sheaf Ω^n_X (where $n = \dim X$). This is a divisorial sheaf, corresponding to a Weil divisor $\widetilde{K}_X$; concretely, $\widetilde{K}_X$ may be realized as the closure in X of a canonical divisor of the nonsingular part of X. The $\mathbb{Q}$-Gorenstein property amounts to the requirement that a positive integer multiple $r\widetilde{K}_X$ of $\widetilde{K}_X$ is Cartier. If $\pi : V \to X$ is a proper birational map, we can formally set $\widetilde{K}_\pi$ to be a ('fractional') divisor such that $r\widetilde{K}_\pi = rK_V - \pi^*(r\widetilde{K}_X)$. This definition satisfies the properties mentioned in §3.2, hence it leads to an alternative notion of integration in the modification system of X.

In practice, the procedure sketched here assigns well-defined multiplicities $\in \mathbb{Q}$ to the components of the exceptional locus of π, giving the input needed for the definition of the integral. For more technical and contextual information on the construction of ω_X, see for example [Rei87].

The class $(\int_{\mathcal{C}_X} 1\!\!1(0)\, d\mathfrak{c}_X)_{id}$ obtained from this notion has some right to be called the *stringy Chern class* of X, following current trends in the literature (see e.g. [Vey], §7.7); if X is complete then the degree of its zero-dimensional component equals the *stringy Euler number* of X, by Claim 5.7. If X admits a crepant resolution, the stringy Chern class of X is simply the image in X of the Chern class of any such resolution.

One difficulty with this notion is that it allows for the possibility that some of the multiplicities m_i in Definition 3.2 may be ≤ -1, in which case our integral is simply undefined; this may occur if the singularities of X are not log-terminal. This annoying restriction may be circumvented in certain cases (cf. §8) but appears to be necessary for the time being.

Example 7.7 illustrates a clear-cut case in which the classes obtained from the two notions of relative canonical divisor considered here differ.

The choice of a notion for relative canonical divisors determines a constructible function

$$I_X(0, \mathcal{C}_X)$$

as in Definition 5.2. By Theorem 5.3, the corresponding stringy class is the image via MacPherson's natural transformation of this constructible function; which should hence be called the *stringy* constructible function on X. (The analogous notion for a Kawamata pair (X, Δ_X) would be the ω flavor of $I_X(-\Delta_X, \mathcal{C}_X)$.)

¿From our perspective these functions are more fundamental than their incarnation as stringy Euler numbers or Chern classes: the information carried by a stringy constructible function amounts to a list of strata of X, and coefficients associated to these strata, from which the invariants can be reconstructed by taking corresponding linear combinations of the invariants of the strata. An alternative viewpoint would associate to the stringy function a corresponding *stringy characteristic cycle* in the cotangent bundle of a nonsingular ambient variety containing X. It is natural to guess that stringy characteristic cycles admit a natural, intrinsic description.

It would be interesting to provide alternative computations of the stringy functions (or characteristic cycles), possibly in terms similar to those describing other invariants such as the local Euler obstruction (which corresponds to the Chern-Mather class under MacPherson's transformation). It would also be interesting to compare the stringy class(es) to other notions of Chern classes for singular varieties, such as Fulton's or Fulton-Johnson's (cf. [Ful84], Example 4.2.6).

7. Examples

We include here a few explicit examples of computations of the integral introduced in this paper.

7.1. In §4.3 we have seen that if $S \subset X$ are nonsingular, and $\mathcal{S}$ is represented by (S, id), then $\left(\int_{\mathcal{S}} 1\!\!1(0)\, d\mathfrak{c}_X\right)_{id}$ computes the Chern class of S.

More generally, if $\mathcal{D}$ is represented by (id, D), with D a nonsingular hypersurface intersecting S transversally, then

$$\left(\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X\right)_{id} = c(TS) \cap [S] - \frac{1}{2} \cdot c(T(D \cap S)) \cap [D \cap S] \quad .$$

Indeed, the blow-up of X along S resolves $\mathcal{D}$, $\mathcal{S}$, and the formula is obtained easily from Definition 3.2 and Lemma 3.8. Note that this shows that

$$\left(\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X\right)_{id} = \left(\int_{\mathcal{C}_S} 1\!\!1(\mathcal{D}_S)\, d\mathfrak{c}_S\right)_{id} \quad ,$$

where $\mathcal{D}_S$ denotes the subset of $\mathcal{C}_S$ represented by $(id, D \cap S)$; that is, the left-hand-side is independent of the ambient nonsingular variety X in this case (cf. §3.11).

7.2. An alternative (and more powerful) way to view the same computation is through the use of the function $I_X(\mathcal{D}, \mathcal{S})$ introduced in §5.3.

In the same situation (X nonsingular, $\mathcal{D}$ represented by (id, D) with $D \subset X$ a nonsingular hypersurface), let $p \in X$; for fun, consider a multiple $m\mathcal{D}$ of D. If

$p \notin D$, then $I_X(m\mathcal{D}, \mathcal{C}_X)(p) = I_X(0, \mathcal{C}_X) = 1$; if $p \in D$, by definition we can compute $I_X(m\mathcal{D}, \mathcal{C}_X)(p)$ as the degree of

$$\frac{c(T\widetilde{X})}{(1+E)(1+\widetilde{D})}\left(\frac{[E]}{m+n} + \frac{[E \cup \widetilde{D}]}{(1+m)(m+n)}\right)$$

where $\widetilde{X}$ is the blow-up of X along p, E is the exceptional divisor, $\widetilde{D}$ is the proper transform of D, and $n = \dim X$. A straightforward computation evaluates this as $\frac{1}{1+m}$, and hence

$$I_X(m\mathcal{D}, \mathcal{C}_X) = 1\!\!1_X - \frac{m}{1+m} 1\!\!1_D$$

as should be expected.

Now if $\mathcal{S}$ is represented by (id, S), then

$$I_X(m\mathcal{D}, \mathcal{S}) = 1\!\!1_S \cdot I_X(m\mathcal{D}, \mathcal{C}_X) = 1\!\!1_S - \frac{m}{1+m} 1\!\!1_{S\cap D} \quad ,$$

and by Theorem 5.3

$$\left(\int_{\mathcal{S}} 1\!\!1(m\mathcal{D})\, d\mathfrak{c}_X\right)_{id} = c_{\mathrm{SM}}(S) - \frac{m}{1+m} \cdot c_{\mathrm{SM}}(D \cap S) \quad ,$$

generalizing the formula of Example 7.1 to a constructible S with arbitrary singularities and intersecting D as it wishes.

7.3. As we observed in Remark 2.9, every subscheme of X determines a divisor in $\mathcal{C}_X$, which may be integrated. For example, let X be nonsingular, and let $Z \subset X$ be a nonsingular subvariety of codimension d; and let $\mathcal{Z}$ be the divisor corresponding to Z, and $m\mathcal{Z}$ the m-multiple of this divisor. Then

$$\left(\int_{\mathcal{C}_X} 1\!\!1(m\mathcal{Z})\, d\mathfrak{c}_X\right)_{id} = c(TX) \cap [X] - \frac{m}{d+m} \cdot c(TZ) \cap [Z] \quad .$$

Indeed, $\mathcal{Z} = (\pi, E)$, where $\pi : \widetilde{X} \to X$ is the blow-up along Z, and E is the exceptional divisor; as $K_\pi = (d-1)E$, Definition 3.2 gives the π manifestation as

$$\frac{c(T\widetilde{X})}{1+E} \cdot \left([\widetilde{X}] + \frac{[E]}{m+d}\right) \quad ,$$

from which the stated formula is straightforward (use Lemma 3.8).

Of course there are divisors of $\mathcal{C}_X$ which do not correspond to subschemes of X. For example, suppose $D \subset X$ is a nonsingular hypersurface containing a nonsingular subvariety Z of codimension d, and let $\mathcal{D}$, $\mathcal{Z}$ be the divisors of $\mathcal{C}_X$ corresponding to D, Z. Then $\mathcal{D} - \mathcal{Z}$ is not represented in X, even as a subscheme (if $d > 1$). It is however represented by a divisor in the blow-up along Z (in fact, as the proper transform of D) and one computes easily that

$$\left(\int_{\mathcal{C}_X} 1\!\!1(\mathcal{D} - \mathcal{Z})\, d\mathfrak{c}_X\right)_{id} = c(TX) \cap [X] - \frac{1}{2} \cdot c(TD) \cap [D] + \frac{1}{2d} \cdot c(TZ) \cap [Z] \quad .$$

7.4. The Chern class of $\mathbb{P}^2$ manifests itself in $\mathbb{P}^1 \times \mathbb{P}^1$ as

$$[\mathbb{P}^1 \times \mathbb{P}^1] + \frac{3}{2}[L_1] + \frac{3}{2}[L_2] + 3[\mathbb{P}^0] \quad ,$$

where L_1 and L_2 denote lines in the two rulings.

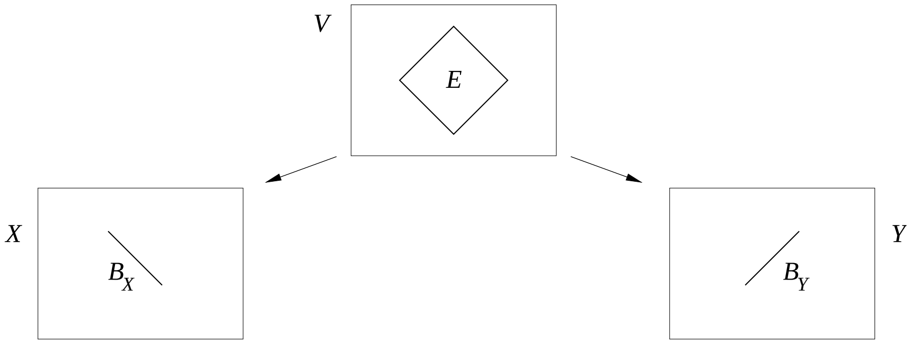
This is obtained by considering the projection to $\mathbb{P}^2$ from a point p of a nonsingular quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$: the blow-up of Q at p resolves $0, \mathcal{C}_{\mathbb{P}^2}$, and the computation in this blow-up is straightforward.

Similarly, the Chern class of $\mathbb{P}^1 \times \mathbb{P}^1$ manifests itself in $\mathbb{P}^2$ as

$$[\mathbb{P}^2] + \frac{5}{2}[\mathbb{P}^1] + 4[\mathbb{P}^0] \quad .$$

The denominators in these two expressions imply the (otherwise evident, in this case) fact that the birational isomorphisms between $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ does not extend to a regular map in either direction.

7.5. A simple situation illustrating Theorem 6.1 consists of a birational morphism resolved by a blow-up and a blow-down:



Assume that X and Y are nonsingular and complete, $B_X \subset X$, $B_Y \subset Y$ are nonsingular subvarieties, and $V = B\ell_{B_X} X = B\ell_{B_Y} Y$, with E the exceptional divisor for both blow-ups. Let d_X, d_Y be the codimension of B_X, B_Y respectively. By Theorem 6.1, and using the same notation for subvarieties of X, Y and for the corresponding divisors of the modification systems,

$$\int_{\mathcal{C}_X} 1\!\!1((d_Y - 1)B_X + \mathcal{D})\, d\mathfrak{c}_X = \int_{\mathcal{C}_Y} 1\!\!1((d_X - 1)B_Y + \mathcal{D})\, d\mathfrak{c}_Y$$

for any divisor $\mathcal{D}$. For example, representing $\mathcal{D}$ by $(1 - d_Y)E$ in the blow-up gives

$$\int_{\mathcal{C}_X} 1\!\!1(0)\, d\mathfrak{c}_X = \int_{\mathcal{C}_Y} 1\!\!1((d_X - d_Y)B_Y)\, d\mathfrak{c}_Y \quad ;$$

from which, evaluating the right-hand-side (using Example 7.3) and taking degrees:

$$\chi(X) = \chi(Y) + \frac{d_Y - d_X}{d_X}\chi(B_Y) \quad ,$$

which is of course easy to check otherwise. With the same choice of $\mathcal{D}$, applying Corollary 6.2 with $i = 1$ and taking degrees gives a slightly more mysterious identity for other Chern numbers. With $n = \dim X = \dim Y$, and denoting $c_{n-1}(X)$, etc. for $c_{n-1}(TX) \cap [X]$, etc., one finds that

$$c_1(X) \cdot c_{n-1}(X) + (d_Y - d_X) \left(\frac{d_X + 1}{2} \chi(B_X) + \frac{1}{d_X} c_1(N_{B_X} X) \cdot c_{n-d-1}(B_X) \right)$$

must equal

$$c_1(Y) \cdot c_{n-1}(Y) + \frac{d_Y - d_X}{d_X} c_1(Y) \cdot c_{r-1}(B_Y) \quad .$$

Making other choices for $\mathcal{D}$, and varying i, one easily gets a large number of such identities.

If $d_X = d_Y$ the Chern numbers $c_1^i \cdot c_{n-i}$ for X and Y coincide (X and Y are in the same K-equivalence class, in this case, cf. §6.2).

If $d_Y = 1$, then V is isomorphic to Y, and the identities compare Chern numbers of a variety X and of its blow-up along a subvariety B_X.

7.6. **Zeta function.** Let X be a complete nonsingular surface, $D \subset X$ a nonsingular curve, and $\mathcal{D}$ represented by (id, D). Then the identity manifestation of $Z(\mathcal{D}, m)$ is

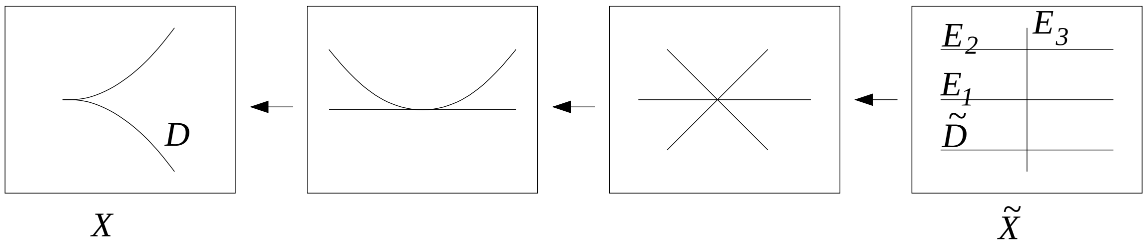
$$\frac{c(TX)}{1 + D} \left([X] + \frac{[D]}{1 + m} \right)$$

and hence

$$\deg Z(\mathcal{D}, m) = \chi(S) + \frac{m}{1 + m} (K_S \cdot D + D^2)$$

where K_S, $\chi(S)$ are the canonical divisor and Euler characteristic of S. By the adjunction formula, the 'interesting term' is a multiple of the Euler characteristic of D.

Now assume that D has a single singular point, consisting of an ordinary cusp. The data $\mathcal{D}$, $\mathcal{C}_X$ is resolved by $\pi : \widetilde{X} \to X$, obtained by a sequence of three blow-ups.



The pull-back of mD to $\widetilde{X}$ is the divisor

$$m\widetilde{D} + (1 + 2m)E_1 + (2 + 3m)E_2 + (4 + 6m)E_3 \quad ,$$

where $\widetilde{D}$ is the proper transform of D, and E_1, E_2, E_3 are (proper transforms of) the successive exceptional divisors.

Applying Definition 3.2 (in one of the equivalent forms in §3.4), the π manifestation of $Z(\mathcal{D},m)$ is

$$\begin{aligned} &c(T\widetilde{X})\cap[\widetilde{X}]-\frac{m}{1+m}\cdot c(T\widetilde{D})\cap[D]\\ &\quad-\frac{1+2m}{2+2m}c(TE_1)\cap[E_1]-\frac{2+3m}{3+3m}c(TE_2)\cap[E_2]-\frac{4+6m}{5+6m}c(TE_3)\cap[E_3]\\ &\qquad+\left(\frac{m}{1+m}\cdot\frac{4+6m}{5+6m}+\frac{1+2m}{2+2m}\cdot\frac{4+6m}{5+6m}+\frac{2+3m}{3+3m}\cdot\frac{4+6m}{5+6m}\right)[p]\end{aligned}$$

where $[p]$ is the class of a point. Taking degrees, we get

$$\deg Z(\mathcal{D},m)=\chi(S)+\frac{m}{1+m}(K_S\cdot D+D^2)-\frac{12m}{5+6m}\quad .$$

Comparing with the nonsingular case, we can think of

$$-\frac{12m}{5+6m}$$

as the contribution of the cusp to the zeta function. The fact that this term has a pole at $-5/6$ is a trivial instance of the 'monodromy conjecture', see §6.8 in [Vey].

7.7. **Stringy classes.** We consider the poster example illustrating the distinction between Ω^n_X and ω_X (§3.2, §6.5), cf. [Rei87], §1.8, and [Vey], §7.5.

Let M be a nonsingular variety, and $X\subset M$ a subvariety with the following property: there is a nonsingular subvariety B of X such that the proper transform $\widetilde{X}$ of X in the blow-up $\widetilde{M}$ of M along B is nonsingular, and meets the exceptional divisor E transversally.

$$\begin{array}{ccccc}\widetilde{X} & \xrightarrow{j} & \widetilde{M} & \hookleftarrow & E\\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\overline{\pi}} & & \downarrow\\ X & \xrightarrow{i} & M & \hookleftarrow & B\end{array}$$

Let $n=\dim X$, and let d be the codimension of B in X; thus, $d-1$ is the dimension of the (nonsingular) fibers of $\widetilde{X}\cap E$ over $X\cap B$.

Claim 7.1. *(Ω flavor.) With K_π defined as in §3.2,*

$$K_\pi=(d-1)E\cdot\widetilde{X}\quad .$$

To see this, note that $i^*\Omega^n_M\to\Omega^n_X$ is surjective, hence the image of $\pi^*\Omega^n_X$ in $\Omega^n_{\widetilde{X}}$ is the same as the image of $\pi^*i^*\Omega^n_M=j^*\overline{\pi}^*\Omega^n_M$. At a point of $\widetilde{X}$ along E, by hypothesis we may assume $\widetilde{X}$ has local coordinates $\widetilde{x}_1,\dots,\widetilde{x}_n$, part of a system of coordinates of $\widetilde{M}$ (the other coordinates of $\widetilde{M}$ being 0 along $\widetilde{X}$), mapping to M according to $x_1=\widetilde{x}_1$, $x_2=\widetilde{x}_2\widetilde{x}_1,\dots,x_d=\widetilde{x}_d\widetilde{x}_1$, $x_{d+1}=\widetilde{x}_{d+1},\dots,x_n=\widetilde{x}_n$. Here $\widetilde{x}_1=0$ is the local equation of E in $\widetilde{M}$. With these coordinates, the only element of the evident basis of Ω^n_M surviving in $\Omega^n_{\widetilde{X}}$ is $dx_1\wedge\cdots\wedge dx_n$, which maps to $\widetilde{x}_1^{d-1}\,d\widetilde{x}_1\wedge\cdots\wedge d\widetilde{x}_n$. Thus the image of $\pi^*\Omega^n_X$ in $\Omega^n_{\widetilde{X}}$ is $\Omega^n_{\widetilde{X}}\otimes\mathcal{O}(-(d-1)E)$, and the claim follows.

For the computation of the fancier $\widetilde{K}_\pi$, we will assume further that X is a hypersurface in M. Let d,n be as above, with $d\geq 2$, and let k be the multiplicity of X along B.

Claim 7.2. *(ω flavor.) With $\widetilde{K}_\pi$ defined as in §6.5,*

$$\widetilde{K}_\pi = (d-k)E \cdot \widetilde{X} \quad .$$

Indeed, as $\widetilde{K}_X$ agrees with the canonical divisor on the nonsingular part of X, whose complement has codimension ≥ 2 in X, the adjunction formula works as usual: $\widetilde{K}_X = (i^* K_M + X) \cdot X$. Thus $\widetilde{K}_X$ is Cartier, and

$$\widetilde{K}_\pi = K_{\widetilde{X}} - j^*\pi^*(K_M + X) = K_{\widetilde{X}} - (K_{\widetilde{M}} - dE + \widetilde{X} + kE) \cdot \widetilde{X} = (d-k)E \cdot \widetilde{X}$$

(cf. [Rei87], p. 350).

Note that the multiplicity of this divisor is ≤ -1 for $k \geq d+1$; this corresponds to the case in which the singularity is not log-terminal. From our perspective, this limits computations of integrals using this divisor to the case $k < (d+1)$ (cf. §8).

The two 'stringy' Chern classes corresponding to the two flavors of relative canonical divisors are respectively as follows:

Proposition 7.3.

$$(\Omega) \qquad c_{\mathrm{SM}}(X) + \frac{1}{d} \cdot \left(\frac{(1-k)^{d+1} - 1}{k} + 1 \right) \cdot c(TB) \cap [B]$$

$$(\omega) \qquad c_{\mathrm{SM}}(X) + \frac{1}{d+1-k} \cdot \left(\frac{(1-k)^{d+1} - 1}{k} + k \right) \cdot c(TB) \cap [B] \qquad k < d+1$$

Proof. Since $\widetilde{X}$ resolves the data for the integral this is a straightforward application of the definition, which we leave to the reader. To obtain the given form (in terms of the Chern-Schwartz-MacPherson class of X), use Theorem 5.3. $\square$

The conventional stringy Euler number of X is the degree of the ω stringy Chern class, hence

$$\chi(X) + \frac{1}{d+1-k} \cdot \left(\frac{(1-k)^{d+1} - 1}{k} + k \right) \cdot \chi(B)$$

by the second formula.

Remark 7.4. For this special class of hypersurfaces one can compute that

$$(1 - (1-k)^{d+1}) = \chi \quad ,$$

the Euler characteristic of the Milnor fiber of X at any of its singularities, while $1 - (1-k)^d = \mathrm{Eu}$, the *local Euler obstruction* of [Mac74]. It follows that

$$k = \frac{\mathrm{Eu} - \chi}{\mathrm{Eu} - 1} \quad ,$$

giving a more intrinsic flavor to the coefficients appearing in Proposition 7.3. For example, the first class may be rewritten

$$(\Omega) \qquad c_{\mathrm{SM}}(X) + \frac{1}{d} \cdot \frac{(\chi - 1)\mathrm{Eu}}{\chi - \mathrm{Eu}} \cdot c(TB) \cap [B] \quad .$$

Taking degrees in Proposition 7.3 gives explicit formulas for the corresponding stringy Euler characteristics.

The second formula in Proposition 7.3 shows that there is no direct generalization of the ω-stringy class to the non-log-terminal case. On the other hand, the Ω flavor offers an alternative which conveys essentially the same information (at least for this simple-minded class of examples), and generalizes to arbitrary singular varieties.

8. Negative multiplicities

8.1. According to the definition given in §3, $\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X$ is simply *undefined* if the multiplicities of the components of the relevant divisors in a resolving variety happen to be ≤ -1. It would be desirable to weaken this restriction, and we devote this final section to some musing on this issue. This is not unimportant: for example, the invariant introduced in §6.3 is vacuous if the variety has no effective canonical divisors; allowing integration over non-effective divisors would enhance its scope. Also, allowing negative coefficients would extend the range of the definition of stringy invariants to certain non-log terminal singularities.

The key Definition 3.2, that is, the manifestation of an integral on a resolving variety, may be formally computed so long as none of the multiplicities equals -1; this would therefore appear to be a more natural requirement than the stated one (that is, $m_j > -1$ for all i) for a resolving variety. The difficulty with adopting the same definition with $m_j \neq -1$ is purely in the proof of independence on the choice of resolving variety: while $m_j \neq -1$ may be satisfied for two chosen resolving varieties, it may fail at some stage in the sequence of varieties connecting them through the use of the factorization theorem. In other words, the first sentence in Claim 3.5 does not hold as stated for this more relaxed definition. This difficulty is also raised in a very similar context by Willem Veys, [Vey], §8.1, Question 1.

One would like to show that if two varieties resolve the data $\mathcal{D}$, $\mathcal{S}$ with all multiplicities $\neq -1$, then the resulting expressions agree after push-forward. In particular, this would answers affirmatively Veys' question. We note that allowing multiplicities ≤ -1 may cause some manifestations of the integral to remain undefined; in such cases the integral would not exist as an element of the Chow group $A_*\mathcal{C}_X$, although it may still carry useful information. The condition $m_j > -1$ adopted in §3 avoids this problem in the simplest way.

8.2. An example will highlight the difficulty, and will clarify how this may be circumvented in certain cases.

Example 8.1. The quadric cone $X \subset \mathbb{P}^4$ with equation $xy = zw$ is singular at one point. This singularity may be resolved in several ways: blowing up along the plane $x = z = 0$ or along the plane $x = w = 0$ produces two 'small' resolutions X_-, X_+ in which the singular point is replaced by a $\mathbb{P}^1$. Both these resolutions are dominated by the blow-up $\widetilde{X}$ of X at the vertex; in this blow-up the singularity is replaced by a copy E of $\mathbb{P}^1 \times \mathbb{P}^1$. The birational morphism from X_- to X_+ is a classical example of *flop*.

$$\begin{tikzcd}
 & \widetilde{X} \arrow[dl] \arrow[dr] \arrow[dd, "\pi"'] & \\
X_- \arrow[dr, "\pi_-"'] & & X_+ \arrow[dl, "\pi_+"] \\
 & X &
\end{tikzcd}$$

The equation $x = 0$ cuts out a Cartier divisor D on X, consisting of the union of the planes $x = z = 0$, $x = w = 0$ (these components are not themselves Cartier divisors). We are interested in integrating the divisor $-2\mathcal{D}$ represented by $(id, -2D)$.

The relative canonical divisor (ω flavor) is 0 in both X_- and X_+, since these are isomorphic to X away from the distinguished $\mathbb{P}^1$'s. The divisor D pulls back via π_- to the union D^- of two nonsingular divisors D_1^-, D_2^- meeting transversally, so we can formally apply Definition 3.2 and write

$$\left(\int_{\mathcal{C}_X} 1\!\!1(-2\mathcal{D})\, d\mathfrak{c}_X\right)_{\pi_-} = c(TX_-(\log D^-)) \cap \left([X_-] + \frac{[D_1^-]}{1-2} + \frac{[D_2^-]}{1-2} + \frac{[D_1^- \cap D_2^-]}{(1-2)^2}\right)$$

This can be evaluated easily, and pushes forward to

$$[X] + [D]$$

in $[X]$. An entirely analogous expression may be written in X_+, with the same push-forward to X. This is as expected from Claim 3.5.

However, the proof given in §3 does not work in this case. The blow-up/blow-down through $\widetilde{X}$ does give a factorization of the flop. The inverse image $\pi^{-1}(D)$ consists of three components meeting with normal crossings: two components D_1, D_2 dominating the two components of D, and the exceptional divisor E. The relative canonical divisor is E (as shown in Claim 7.2), so

$$-2\pi^{-1}(D) + \widetilde{K}_\pi = -2D_1 - 2D_2 - E \quad .$$

Thus, E appears with the forbidden multiplicity -1; the corresponding formal application of Definition 3.2:

$$\begin{aligned} c(T\widetilde{X}(\log \pi^{-1}D)) \cap \Bigg([\widetilde{X}] &+ \frac{[D_1]}{1-2} + \frac{[D_2]}{1-2} + \frac{[E]}{1-1} \\ &+ \frac{[D_1 \cap D_2]}{(1-2)^2} + \frac{[D_1 \cap E]}{(1-2)(1-1)} + \frac{[D_2 \cap E]}{(1-2)(1-1)} + \frac{[D_1 \cap D_2 \cap E]}{(1-2)^2(1-1)} \Bigg) \end{aligned}$$

appears hopelessly nonsensical.

8.3. We are now going to illustrate on this example how some information may be extracted from such 'meaningless' expressions in certain cases. In an earlier version of this paper we in fact claimed that the same approach used on this example can be applied in the general case; that appears to have been overly optimistic. I am grateful to Lev Borisov and Wim Veys for pointing out problems with the original argument.

The idea is to view the multiplicity of D as a variable m. The basic formula guaranteeing independence on the resolution becomes an equality of *rational functions* with coefficients in the Chow group. The proof of Claim 3.5 goes through verbatim in this context, and shows that the corresponding expression is well defined as a rational function; therefore, so must be the expression obtained by specializing m to the needed multiplicity.

In Example refflop, the π_- manifestation of the integral of $m\mathcal{D}$ is

$$\left(\int_{\mathcal{C}_X} 1\!\!1(m\mathcal{D})\, d\mathfrak{c}_X\right)_{\pi_-} = c(TX_-(\log D^-)) \cap \left([X_-] + \frac{[D_1^-]}{1+m} + \frac{[D_2^-]}{1+m} + \frac{[D_1^- \cap D_2^-]}{(1+m)^2}\right)$$

The π manifestation makes sense as a rational function in m:

$$c(T\widetilde{X}(\log \pi^{-1}D)) \cap \Bigg([\widetilde{X}] + \frac{[D_1]}{1+m} + \frac{[D_2]}{1+m} + \frac{[E]}{2+m} \\ + \frac{[D_1\cap D_2]}{(1+m)^2} + \frac{[D_1\cap E]}{(1+m)(2+m)} + \frac{[D_2\cap E]}{(1+m)(2+m)} + \frac{[D_1\cap D_2\cap E]}{(1+m)^2(2+m)} \Bigg)$$

and an explicit computation evaluates this as

$$1 + \frac{3+2m}{1+m}\,([D_1]+[D_2]) + \frac{3+m}{2+m}\,[E] \\ + \frac{(2+m)(4+3m)}{(1+m)^2}\,[D_1\cap D_2] - \frac{3+m}{1+m}\,[E]^2 + \frac{(2+m)(3+m)}{(1+m)^2}\,[P]$$

where $[P]$ is the class of a point.

The many cancellations clearing several of the factors of $(2+m)$ at denominator may appear surprising at first, but they are forced by the strong constraints imposed on the situation by the geometry of the flop: this class must push-forward *as a rational function* to the π_- and π_+ manifestations, which have no such factors; hence only the terms that are collapsed by the push-forwards are allowed to have poles at $m=-2$. In this example, the only such term is E (as E^2 and all other terms survive push-forward to X_- and X_+). Pushing forward to X gives the identity manifestation:

$$\left(\int_{\mathcal{C}_X} \mathbb{1}(m\mathcal{D})\, d\mathfrak{c}_X\right)_{id} = [X] + \frac{3+2m}{1+m}[D] + \frac{(2+m)(4+3m)}{(1+m)^2}[L] + \frac{(3+m)(2+m)}{(1+m)^2}[p] \quad ,$$

where L is the class of a line. For $m=-2$, this agrees (of course) with the class given in Example 8.1. For $m=0$ we obtain the (ω flavor of the) *stringy Chern class* of X, according to the definition given in §6.5 (and in agreement with Proposition 7.3):

$$[X] + 3[D] + 8[L] + 6[P] \quad .$$

As a last comment on this example, note that the situation is rather different concerning the Ω flavor of the relative canonical divisor: this is $2E$ for π (by Claim 7.1), while it is *not a divisor* for π_- and π_+: for these maps, Ω is tightly wrapped around the distinguished $\mathbb{P}^1$. In particular, the Ω flavor of $\int_{\mathcal{C}_X} \mathbb{1}(m\mathcal{D})\, d\mathfrak{c}_X$ cannot be computed by applying Definition 3.2 to X_- or X_+, as these do not resolve the data. It may be computed by working in $\widetilde{X}$.

8.4. A similar approach should work if, as in Example 8.1, the key data comes from a divisor on X, for this guarantees that the multiplicities on the two resolutions can be compatibly promoted to variables, and Claim 3.5 shows that the corresponding rational functions have the same push-forward in X. In fact, this attack to the question is not novel: see Remark 3.11 in [BL03] for a very similar approach; and the idea of attaching variable terms to problematic multiplicities has been championed by Veys with success in important cases ([Vey04], [Vey03]).

We hope that the flexibility gained by considering the whole modification system will allow us to assign compatible data in a more general situation.

The conclusion to be drawn from the preceding considerations is that the best setting in which to define the integrals considered in this paper may in fact be a 'decoration' of the Chow group of a modification system $\mathcal{C}_X$ by variables attached to all divisors of $\mathcal{C}_X$. The integral of a divisor should properly be considered as a rational function with coefficients in the Chow group. The poles and 'residues' of such rational functions may carry valuable information, as suggested by the connection with topological zeta functions encountered in §6.

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