ON A K3 SURFACE WHICH IS A BALL QUOTIENT

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1. INTRODUCTION.

For any positive integer m we introduce a hermitian form with variables $z = (z_1, z_2, z_3)$ by setting:

$$H_m(z) := z_1\bar{z}_1 + z_2\bar{z}_2 - (\zeta + \bar{\zeta}) z_3\bar{z}_3 \qquad \text{where}$$

$$\zeta = \exp(2\pi\sqrt{-1}/m) \quad .$$

By SU_m we denote the group of (3,3)-matrices of determinant 1 which are unitary with respect to H_m. SU_m operates naturally over the domain in $P_2(\mathbb{C}) : (z_1, z_2, z_3)$ given by

$$B_m := \{(z_1, z_2, z_3) ; H_m(z) < 0\} \quad .$$

Obviously we have isomorphisms
$SU_m \cong SU(2,1)$, $B_m \cong \{(x,y) \in \mathbb{C}^2 ; x\bar{x} + y\bar{y} < 1\}$; every reasonable quotient of B_m can be called a ball quotient. So far the integer m did not make any distinction. But now we observe the subgroup, denoted by Γ_m, of SU_m consisting of elements whose entries are integers of the cyclotomic field $\mathbb{Q}(\zeta)$.

By Borel, Harish-Chandra [BH] we know that, at least for $m = 5,7,8,12$, the group Γ_m operates properly discontinuously on B_m and the quotient B_m/Γ_m is compact. But, in this note, we restrict ourselves to the case where m is prime i.e. we assume that m is either 5 or 7. Now the principal ideal generated by $1-\zeta$ is prime and we denote it by $\mathfrak{p}$. We further denote by Γ'_m resp. Γ''_m the subgroups of Γ_m consisting of matrices which are congruent to the identity matrix modulo $\mathfrak{p}$ resp. $\mathfrak{p}^2$. By using a result of Terada [T], Yamazaki and Yoshida showed in [YY] that B_5/Γ'_5 is the del Pezzo surface of degree 5 (which is unique up to isomorphism) and the natural map $B_5/\Gamma''_5 \longrightarrow B_5/\Gamma'_5$ (or rather its composition with the blow down to $P_2(\mathbb{C})$) is one of the abelian coverings of $P_2(\mathbb{C})$ observed by Hirzebruch [H]; the branching locus in the del Pezzo surface is the union of ten exceptional curves of the 1st kind. The main purpose of this note is to prove the following:

<u>THEOREM 1</u>. <u>The quotient</u> B_7/Γ'_7 <u>is a smooth</u> K3 <u>surface with Picard number</u> 20. <u>The branching locus in</u> B_7/Γ'_7 <u>of</u> $B_7/\Gamma''_7 \longrightarrow B_7/\Gamma'_7$ <u>is a normal crossing of</u> 28 <u>smooth rational curves, which are transitively permuted by the natural action of</u> $\Gamma_7/\Gamma'_7 \cong PGL(2,\mathbb{Z}/7\mathbb{Z})$.

Since B_7/Γ'_7 is a K3 surface, the self-intersection number is equal to -2 for every smooth rational curve on it.

$|\Gamma'_7/\Gamma''_7| = 7^5$ and the ramification index along each of the 28 rational curves is generically equal to 7 . On the other hand Γ''_7 operates freely on B_7 . Since B/Γ'_7 is smooth, these imply that Γ'_7 is generated by unitary reflections of order 7, as the similar statement is mentioned in [YY] for $m = 5$. The way in which the 28 curves intersect each other is naturally described by the finite geometry associated with the group of order 336 . See for this Section 5.

Now the question arises: Is Γ''_7 a maximal subgroup of Γ_7 acting freely over B_7 ? The answer is negative. For Inoue [I] and Livné [L] construct a cyclic covering, denoted here by $\widetilde{S(7)}$, of $S(7)$ which is a free quotient of the ball; $S(7)$ is Shioda's elliptic modular surface associated with the principal congruence subgroup $\Gamma(7)$ of $SL(2,\mathbb{Z})$.

PROPOSITION 2. *The abelian covering in Theorem 1 can be factorized as follows:*

$$\begin{array}{ccc} B_7/\Gamma''_7 & \longrightarrow & B_7/\Gamma'_7 \\ \downarrow & & \uparrow \\ \widetilde{S(7)} & \longrightarrow & S(7) \end{array}$$

where the vertical coverings are both of degree 7^2 .

The corresponding factorization for the case $m = 5$ is described in Ishida [Is] by using the theory of torus imbedding.

The author was not able to construct this ball quotient by using any line arrangement in $P_2(\mathbb{C})$. In fact this is not found in the list of Höfer [Hö]. Nonetheless the surface B_7/Γ_7' is described as the double covering branched over a plane sextic curve. One can see further that B_7/Γ_7'' is a Galois covering of $P_2(\mathbb{C})$ ramified over the curve and six other lines (Section 5).

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2. Inoue-Livné's cyclic coverings.

In this section we recall Inoue-Livné's construction of cyclic coverings of the elliptic modular surfaces $S(m)$ associated with $\Gamma(m) \subseteq SL(2,\mathbb{Z})$. We will mainly follow the presentation of [L]. We begin with a rough description of $S(m)$. For every elliptic curve or more precisely 1-dimensional complex torus, the m-torsion points form a free module of rank 2 over the finite ring $\mathbb{Z}/m\mathbb{Z}$, this is a symplectic module since there is the natural skew-symmetric pairing on it, coming from the intersection form. An m-leveled structure of the torus is a symplectic isomorphism of $(\mathbb{Z}/m\mathbb{Z})^2$ (with the obvious symplectic form) onto this module. Now the surface $S(m)$ is explained to be some *"good"* compactification of the family of m-leveled 1-dimensional complex tori. This is an elliptic surface for which every singular fiber is of type I_m i.e. it is a cycle of m rational curves of self-intersection number -2. Note that, for such a fiber, its smooth locus has the natural group structure $\cong \mathbb{C}^* \times (\mathbb{Z}/m\mathbb{Z})$; so we have also m^2 m-torsion points of it. Thus the loci of m-torsion points are exactly the m^2 (disjoint) global sections, from which we obtain an abelian automorphism group $\cong (\mathbb{Z}/m\mathbb{Z})^2$ called the translations of $S(m)$. The symplectic group $SL(2,\mathbb{Z}/m\mathbb{Z})$ operates also naturally over $S(m)$, so that we have the action of the semi-direct product $(\mathbb{Z}/m\mathbb{Z})^2 \times SL(2,\mathbb{Z}/m\mathbb{Z})$ which coincides with the full automorphism group $\mathrm{Aut}(S(m))$ except for $m = 4$. ($S(4)$ is a K3 surface and the elliptic fibration is not unique for it). Now let $D_i (i = 1,2,...,m^2)$ be the global sections and set $\Sigma = [\cup_i D_i]$ where $[C]$ denotes the associated line bundle for any divisor C on the surface. Livné showed that, for an integer $d > 0$, there is a line bundle Δ such that $d\Delta = \Sigma$ $(d\Delta := \Delta^{\otimes d})$ if and only if d devides $NUM(m/2)$ (the numerator of $m/2$). Now, if $d\Delta = \Sigma$, then we can construct a cyclic covering of $S(m)$ of degree d branched exactly over $\cup_i D_i$ in the following way. Note that the natural covering map $\Delta \longrightarrow \Sigma$ is of degree d and it ramifies just along the zero-sections. By the definition of Σ it is clear that there is a section $S \hookrightarrow \Sigma$ whose zero locus is $\cup_i D_i$ and that such a section is unique up to non-zero constants. The inverse image of the section by the map $\Delta \longrightarrow \Sigma$, together with the restriction of this map to it, is now the desired cyclic covering, which Livné denotes by $S_d(m)_\Delta$. In [I], [L] it is shown by computing the Chern numbers c_1^2, c_2 and using Yau [Ya] that the universal covering of $S_d(m)_\Delta$ is isomorphic to the unit ball $U := \{(x,y) \in \mathbb{C}^2 ;\ x\bar{x} + y\bar{y} < 1\}$ if and only if (m,d) is one of the pairs $(5,5)$, $(7,7)$, $(8,4)$, $(9,3)$, $(12,2)$. Note that we always have $d = \mathrm{NUM}(m/6)$ in these cases. Livné proved that, if d divides $\mathrm{NUM}(m/6)$, there exists a unique Δ such that $d\Delta = \Sigma$ and that all elements of $\mathrm{Aut}(S(m))$ are liftable to $S_d(m)_\Delta$, which is then denoted simply by $S_d(m)$ in [L]. In case where $d = \mathrm{NUM}(m/6)$, we write $\widetilde{S(m)}$ for $S_d(m)$ in accordance with the notation of the introduction. Now we assume that m is one of the numbers $5,7,8,9,12$. The fundamental group of $\widetilde{S(m)}$, denoted by $\pi_1(m)$, is a discrete subgroup of $PSU(2,1)$ acting freely on U and $\widetilde{S(m)} = U/\pi_1(m)$. We have the natural exact sequences:

$$(2.1) \qquad 1 \longrightarrow \pi_1(m) \longrightarrow N(m) \longrightarrow \mathrm{Aut}(\widetilde{S(m)}) \longrightarrow 1$$

$$(2.2) \qquad 1 \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow \mathrm{Aut}(\widetilde{S(m)}) \longrightarrow \mathrm{Aut}(S(m)) \longrightarrow 1$$

where $d = \mathrm{NUM}(m/6)$ and $N(m)$ is the normalizer of $\pi_1(m)$ in $PSU(2,1)$. $N(m)$ is sufficiently big to be described by suitable generators and relations.

PROPOSITION 2.1 (Livné [L]). Assume first that $m \neq 5$. Then $N(m)$ is uniquely characterized as the subgroup of $PSU(2,1)$ generated by A, Y, Z satisfying the following conditions:

(1) $A^2 = 1$, $Z^m = 1$, $AZ = ZA$.

(2) for A and Z the fixed point sets (in U) are 1-dimensional and they are perpendicular.

(3) if one sets $X = Z^{-1}Y$, then one obtains $(AX)^3 = 1$, $X^3 = Y^2$ and $Y^{4d} (= X^{6d}) = 1$.

(4) the fixed set (in U) of $Y^2 = X^3$ is 1-dimensional and Y, X, Z act on it as the generator of the triangle group of type $(2,3,m)$.

To characterize $N(5)$ in this way one needs only to replace the condition (4) by the following:

(4)' Y, X, Z induce the action of the icosahedral group on the fixed line in $P_2(\mathbb{C})$ of $Y^2 = X^3$, which is however entirely outside U.

From the argument of [L] it is also clear how to describe the homomorphism $N(m) \longrightarrow Aut(\widetilde{S(m)})$ or $N(m) \longrightarrow Aut(S(m))$ explicitly. We set

$$C := X^{-1} A X \qquad B := Z C Z^{-1} \quad . \tag{2.3}$$

PROPOSITION 2.2. The natural mapping $N(m) \longrightarrow Aut(S(m)) \cong (\mathbf{Z}/m\mathbf{Z})^2 \times SL(2,\mathbf{Z}/m\mathbf{Z})$ maps AB, CB to the base of the abelian group $(\mathbf{Z}/m\mathbf{Z})^2$. The images of X,Y,Z by the same map generate $SL(2,\mathbf{Z}/m\mathbf{Z})$. The image of the commutator $[AB,CB] = (ABC)^2$ by $N(m) \longrightarrow Aut(\widetilde{S(m)})$, which should lie in $Ker(Aut(\widetilde{S(m)}) \longrightarrow Aut(S(m)) \cong \mathbf{Z}/d\mathbf{Z}$, generates this cyclic group.

Note that A,B,C are conjugate to each other, which implies that $A^2 = B^2 = C^2 = 1$. One can deduce also $X^3 = Y^2 = ABC$. For various identities satisfied by A,B,C,X,Y,Z see [L]. A more elegant description of $N(7) \longrightarrow Aut(S(7))$ is given in the next section.

3. Matrix representation of N(7)

Let $\zeta = \exp(2\pi\sqrt{-1}/7)$ and check the identity

$$\left(\zeta+\zeta^{-1}\right)\left(\zeta^2+\zeta^{-2}\right)\left(\zeta^3+\zeta^{-3}\right) = 1$$

which shows in particular that $\zeta^i + \zeta^{-i}$ $(i \not\equiv 0 \text{ mod. } 7)$ are units in the algebraic number field $\mathbf{Q}(\zeta)$. We set

$$A = \begin{pmatrix} \frac{\zeta^3+\zeta^{-3}}{\zeta^2+\zeta^{-2}} & & \frac{-(\zeta^2-\zeta^{-2})}{\zeta^2+\zeta^{-2}} \\ & -1 & \\ \frac{-(\zeta-\zeta^{-1})}{\zeta^2+\zeta^{-2}} & & \frac{-(\zeta^3+\zeta^{-3})}{\zeta^2+\zeta^{-2}} \end{pmatrix}$$

$Y = Y_0 Y_1 \qquad Z = \left[\zeta^{-2}, \zeta^4, \zeta^{-2}\right]$ where $Y_0 = \left[\zeta, -\zeta^3, -\zeta^3\right]$ and

$$Y_1 = \begin{pmatrix} 1 & & \\ & \frac{1}{\zeta^3-\zeta^{-3}} & \frac{1}{\zeta-\zeta^{-1}} \\ & \frac{1}{\zeta^2-\zeta^{-2}} & \frac{-1}{\zeta^3-\zeta^{-3}} \end{pmatrix} .$$

We have let $[a_1, a_2, a_3]$ denote the diagonal matrix with non-zero entries a_1, a_2, a_3 . One can directly check that all matrices are unitary with respect to the hermitian matrix $[1, 1, -(\zeta+\bar{\zeta})]$ (this is H_7 in the introduction) and that all conditions in Proposition 2.1 with $m = 7$ are fulfilled by these A, Y, Z (when one fixes an identification of the domains B_7 and U). Check also that the matrices are all of determinant 1. Thus A, Y, Z above generate a subgroup of

SU_7 (see the introduction) which is isomorphic to $N(7) \subseteq PSU(2,1)$. ($\mathbf{Q}(\zeta)$ does not contain any primitive third root of unity). Therefore we are allowed to regard $N(7)$ as this subgroup of SU_7. Now we call a matrix in $N(7)$ <u>integral</u> if its entries are integers in $\mathbf{Q}(\zeta)$. In this sense A, Z are clearly integral while $Y_1, Y, X = Z^{-1}Y$ are not. We can further check that the matrices B, C introduced by (2.3) are integral. This suggests that we have enough integral elements in $N(7)$. We want to prove that the subgroup of integral elements is of finite index in $N(7)$. Note that a matrix is integral if and only if it maps the following $\mathcal{O}$-lattice into itself:

$$L := \mathcal{O}e_1 + \mathcal{O}e_2 + \mathcal{O}e_3$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where $\mathcal{O}$ is the ring of integers of $\mathbf{Q}(\zeta)$. The discrete subgroup Γ_7 in the introduction is just $\{M \in SU_7 \; ; \; ML \subseteq L\}$; so what to prove is that $[N(7) : N(7) \cap \Gamma_7] < \infty$. For this purpose we introduce another $\mathcal{O}$-lattice which is mapped into itself by $N(7)$. We set simply

$$\tilde{L} := \mathcal{O}e_1 + \mathcal{O}\tilde{e}_2 + \mathcal{O}e_3$$

where $\tilde{e}_2 = Y_1 e_3$. Since $Y_1^2 = [1,-1,-1]$, we have $Y_1\tilde{e}_2 = -e_3$, which shows that $Y_1\tilde{L} \subseteq \tilde{L}$. Now $Y_0\tilde{L} \subseteq \tilde{L}$ proves $Y\tilde{L} \subseteq \tilde{L}$. $Z\tilde{L} \subseteq \tilde{L}$ is equally trivial. $A\tilde{e}_2 + \tilde{e}_2 \in L$ and $L \subseteq \tilde{L}$ prove also $A\tilde{L} \subseteq \tilde{L}$. Now we call a matrix M , $\tilde{L}$-integral (resp. L-integral) if and only if $M\tilde{L} \subseteq \tilde{L}$ (resp. $ML \subseteq L$). The L-integrality coinsides with the original integrality. But we introduced this terminology to avoid confusions, since we have to compare the two integralities. The group $N(7)$ is $\tilde{L}$-integral while the group Γ_7 is L-integral. Now we observe the following inclusion:

$$(3.1) \qquad \mathfrak{p}\tilde{L} \subseteq L \subseteq \tilde{L} \quad .$$

This implies in particular that the kernel of the natural map $N(7) \longrightarrow \mathrm{Aut}(\tilde{L}/\mathfrak{p}\tilde{L})$ is L-integral i.e. it is contained in $\Gamma_7 \cap N(7)$. This proves that $\Gamma_7 \cap N(7)$ is of finite index in $N(7)$. Since both B_7/Γ_7 and $B_7/N(7)$ are compact, the intersection is also of finite index in Γ_7 . By the same reasoning we see that the congruence subgroup Γ_7' in the introduction is $\tilde{L}$-integral. But we do not know a priori that Γ_7' is contained in $N(7)$. For this we have the following key lemma:

<u>LEMMA 3.1</u>. $\quad N(7) = \{M \in SU_7 \; ; \; M\tilde{L} \subseteq \tilde{L}\}$.

The idea of the proof is very simple. By an algebro-geometric argument [L],[I] we know the Euler number c_2 for the quotient $B_7/\pi_1(7) = \widetilde{S(7)}$. Since we know also $|N(7)/\pi_1(7)| = |\mathrm{Aut}(\widetilde{S(7)})| = 2^4 \cdot 3 \cdot 7$, we can compute the volume of $B_7/N(7)$. On the other hand we can also compute the volume for the quotient of B_7 by the arithmetic group $\{M \in SU_7 \; ; \; M\tilde{L} \subseteq \tilde{L}\}$ by reducing the problem to the calculation of the Tamagawa number in the well known way. Comparing these two volumes proves the lemma.

Now, to the group $N(7)$, we give the matrix representation with respect to the $\mathcal{O}$-base $\{e_1, \tilde{e}_2, e_3\}$ of $\tilde{L}$. The invariant hermitian matrix is then of the following form:

$$\tilde{H} = \begin{pmatrix} 1 & & \\ & -(\zeta+\zeta^{-1}) & (\zeta-\zeta^{-1})^{-1} \cdot (\zeta^2+\zeta^{-2})^{-1} \\ & -(\zeta-\zeta^{-1})^{-1} \cdot (\zeta^2+\zeta^{-2})^{-1} & \zeta+\zeta^{-1} \end{pmatrix}$$

Since $\tilde{H}$ is not diagonal, we have to make it clear in which sense a matrix M is called unitary with respect to $\tilde{H}$; M is unitary if and only if ${}^t\overline{M}\tilde{H}M = \tilde{H}$. Since H is not integral, it is not appropriate for the study of the reduction homomorphism $N(7) \longrightarrow \mathrm{Aut}(\tilde{L}/\mathfrak{p}\tilde{L})$. It is in fact more natural to consider the antihermitian matrix $\hat{H} = (\zeta-\zeta^{-1})(\zeta^2+\zeta^{-2})\tilde{H}$, which is now of the form:

$$\left(\begin{array}{c|cc} (\zeta-\zeta^{-1})\cdot(\zeta^{2}+\zeta^{-2}) & & \\ \hline & \zeta^{3}-\zeta^{-3} & 1 \\ & -1 & \zeta^{3}-\zeta^{-3} \end{array}\right) .$$

We have ${}^{t}\overline{M}\,\hat{H}\,M = \hat{H}$ for $M \in N(7)$. Now we will consider everything modulo $\mathfrak{p}$. $\hat{H}$ passes to the following degenerate skew - symmetric matrix:

$$\left(\begin{array}{c|cc} & & \\ \hline & & 1 \\ & -1 & \end{array}\right) .$$

This implies that the image of $N(7) \longrightarrow \mathrm{Aut}(\tilde{L}/\mathfrak{p}\tilde{L})$ lies the following matrix group over the finite field $\mathcal{O}/\mathfrak{p} \cong \mathbb{Z}/7\mathbb{Z}$:

$$G := \left\{ \begin{pmatrix} 1 & e & f \\ & a & b \\ & c & d \end{pmatrix} \;;\; \begin{array}{l} e,f \in F_7 \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,F_7) \end{array} \right\}$$

where we have used the abreviation F_7 for $\mathbb{Z}/7\mathbb{Z}$ and the identification $\tilde{L}/\mathfrak{p}\tilde{L} = (F_7)^3$. This group is obviously isomorphic to the semi-direct product $(F_7)^2 \times SL(2,F_7)$. By direct computation we see that the images of X,Y,Z generate $SL(2,F_7)$ and those of AB , CB the abelian normal subgroup $(F_7)^2$. This shows that $N(7) \longrightarrow G$ is nothing other than the natural homomorphism $N(7) \longrightarrow \mathrm{Aut}(S(7))$. Now we want to show the usefulness of this description. We begin by proving the following:

<u>LEMMA 3.2</u>. The congruence subgroup Γ_7' contains the kernel of $N(7) \longrightarrow G$.

Suppose that M is in the kernel. Then $M\tilde{e}_2 - \tilde{e}_2 \in \mathfrak{p}\tilde{L} \subseteq L$, which shows that M induces the trivial mapping on $\tilde{L}/L(\cong F_7)$. Since we have the inclusion $L/\mathfrak{p}\tilde{L} \subseteq \tilde{L}/\mathfrak{p}\tilde{L}$, M also induces the identity on $L/\mathfrak{p}\tilde{L}$. On the other hand we have the exact sequence

$$0 \longrightarrow \mathfrak{p}\tilde{L}/\mathfrak{p}L(\cong \tilde{L}/L) \longrightarrow L/\mathfrak{p}L \longrightarrow L/\mathfrak{p}\tilde{L} \longrightarrow 1 \quad .$$

Thus for the proof of the lemma it suffices to show that this splits M-equivariantly. Recall that M preserves the hermitian form H_7 which induces a non-degenerate quadratic form on $L/\mathfrak{p}L \cong (F_7)^3$. With respect to this, the 1-dimensional space $\mathfrak{p}\tilde{L}/\mathfrak{p}L$ is not isotropic. The orthogonal complement gives the desired splitting.

Now the group Γ_7' , and hence also $\Gamma_7 \cap N(7)$, are the full inverse images of subgroups of G by $N(7) \longrightarrow G$.

PROPOSITION 3.3. The subgroups of G corresponding to Γ_7' , $\Gamma_7 \cap N(7)$ are

$$G_0 := \left\{ \left(\begin{array}{c|cc} 1 & e & f \\ \hline & a & b \\ & c & d \end{array}\right) \in G \; ; \quad b = f = 0 \quad a = d = 1 \right\}$$

$$G_1 := \left\{ \left(\begin{array}{c|cc} 1 & e & f \\ \hline & a & b \\ & c & d \end{array}\right) \in G \; ; \quad b = 0 \right\} .$$

Suppose that $M \in N(7)$ and (m_{ij}) is the matrix representation of M with respect to $\{e_1, \tilde{e}_2, e_3\}$. We have already seen that $m_{21} \equiv m_{31} \equiv 0 \bmod. \mathfrak{p}$. Thus we always have $Me_1 \in L$. We have also the equivalence $Me_3 \in L \Longleftrightarrow m_{23}\tilde{e}_2 \in L \Longleftrightarrow m_{23} \equiv 0 \bmod. \mathfrak{p}$. By the identity $(\zeta - \zeta^{-1})\tilde{e}_2 = e_2 + (\zeta^3 + \zeta^{-3})e_3$, we have $Me_2 + (\zeta^3 + \zeta^{-3})Me_3 = (\zeta - \zeta^{-1})M\tilde{e}_2 \in \mathfrak{p}\tilde{L} \subseteq L$, which implies the equivalence $Me_2 \in L \Longleftrightarrow Me_3 \in L$. Thus $M \in N(7) \cap \Gamma_7 \Longleftrightarrow ML \subseteq L \Longleftrightarrow m_{23} \equiv 0$. The correspondence of Γ_7' and G_0 is less trivial. We see at once the necessity of $m_{13} \equiv m_{23} \equiv 0$, $m_{22} \equiv m_{33} \equiv 1 \bmod.\mathfrak{p}$ for M to belong to Γ_7' .

But, once this condition is assumed, then it follows from ${}^t\bar{M}\,\hat{H}\,M = \hat{H}$ that $m_{23} \equiv m_{21} \equiv 0 \bmod. \mathfrak{p}^2$, from which one can deduce $Me_i - e_i \in \mathfrak{p}L$ for $i = 1,3$. Note that $m_{22} \equiv 1 \bmod. \mathfrak{p}$ implies $M\tilde{e}_2 - \tilde{e}_2 \in L$. By this and the identity $Me_2 - e_2 = (\zeta - \zeta^{-1}) \times (M\tilde{e}_2 - \tilde{e}_2) + (\zeta^3 + \zeta^{-3}) \cdot (Me_3 - e_3)$ we conclude $Me_2 - e_2 \in \mathfrak{p}L$. This proves the sufficiency.

COROLLARY 3.4. The ball quotient B_7/Γ_7' can be identified with the quotient of $S(7)$ by an abelian subgroup of the form

$$G_0 = \left\{ \begin{pmatrix} 0 \\ a \end{pmatrix} \times \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} \quad ; \quad a,b \in F_7 \right\}$$

in $\mathrm{Aut}(S(7)) \cong (F_7)^2 \times SL(2,F_7)$.

4. Explicit description of the quotient

By Corollary 3.4 it suffices to describe $S(7)/G_0$. We first have to see how G_0 acts on $S(7)$. Recall that the set of global sections has the structure of the affine plane $(F_7)^2$ coming from the affine structure of fibers. The translation part of G_0 , which we denote by T_0 , acts on each fiber as a translation of order 7 .

$$T_0 = \left\{ \begin{pmatrix} 0 \\ a \end{pmatrix} ; \quad a \in F_7 \right\} .$$

We will first divide S(7) by T_0 and then divide it by

$$G_0/T_0 \cong \left\{ \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} ; \quad b \in F_7 \right\} .$$

For every regular fiber its quotient by T_0 is isomorphic to itself. The action of T_0 over singular fibers is a little more subtle. We know that all of them are of type I_7 . But, with respect to the action of T_0 , there are two types of them. We say a singular fiber to be of the first type if T_0 maps every irreducible component to itself. We call it of the second type if T_0 permutes the irreducible components transitively. Now we note that, for every singular fiber and every irreducible component of it, there are exactly seven global sections intersecting the component. They form an affine line on the affine plane $\cong (F_7)^2$ of global sections. In this way we can associate seven parallel affine lines with every singular fiber of S(7) . We can determine the type of the fiber by deciding whether the action of T_0 is parallel to the seven lines or not. By this consideration it is now clear that only three of the 24 singular fibers of S(7) are of the first type and the remaining 21 are of the second type.

(There are 8 different directions on the plane $(F_7)^2$).
On the fibers of the second type T_0 acts still freely while it has fixed points exactly where the irreducible components of fibers of the first type cross two by two. Thus the quotient $S(7)/T_0$, which has the induced elliptic fibration, has 21 singular fibers of type I_1 and three singular fibers each of which consists of seven rational curves intersecting cyclically at seven singlar points. The singular points of $S(7)/T_0$ are of the same singularity, which we want to describe now. At every fixed point $p \in S(7)$ of T_0 we can choose a local coordinate system (x,y) so that $p = (0,0)$ and that the projection to the base curve is given by $z = xy$ where z is a local coordinate near the image of p . Thus the equations $x = 0$, $y = 0$ define the irreducible components intersecting at p . We can also assume that the action of T_0 is generated by

$$(x,y) \longrightarrow (\zeta x, \zeta^{-1} y) \quad .$$

(Recall that T_0 acts trivially on the base). The singularity to be described is now

$$\xi \cdot \eta = z^7$$

where $\xi = x^7$, $\eta = y^7$.

Next we want to divide $S(7)/T_0$ by G_0/T_0. This cyclic group acts near the singularity above by sending z to ζz, leaving ξ and η invariant. Thus the singularity disappears by the final quotient formation. It is also clear that G_0/T_0 acts on the base curve outside the three points over which the fibers of the first type lie. We have thus almost proved the following

PROPOSITION 4.1. The quotient $B_7/\Gamma_7' \cong S(7)/G_0$ is a non-singular elliptic surface with three singular fibers of type I_7 and three singular fibers of type I_1. It has exactly seven (disjoint) global sections. The base curve is rational; in particular the surface is a K3 surface.

Recall that the base curve of $S(7)$ is of genus 3 [L]. (It is in fact the famous quartic curve of Klein). The Euler number is therefore equal to -4. By the above description of the action of G_0/T_0 over the curve, we see that the base curve of $S(7)/G_0$ is of Euler number

$$3 + (-4-3)/7 = 2 \quad .$$

This implies in particular that $K = (p_g - 1)F$ and that $1 + p_g = c_2/12$, where K, F are the canonical class and the class of fibers. Since c_2 is obviously equal to 24, we get the last assertion of the proposition.

REMARK: The group Γ_7/Γ_7' , which is of order 2×168 , is much larger than the automorphism group of the elliptic fibration. Since Γ_7/Γ_7' acts naturally on the K3 surface, this implies that there are more elliptic fibrations of the same type. But the 28 rational curves, which are the global sections and irreducible components of fibers of type I_7 , are permuted among themselves by the group action, since they are the ramification loci of $B_7 \longrightarrow B_7/\Gamma_7'$. Thus there must exist a finite geometry concerning the group $\Gamma_7/\Gamma_7' \cong PGL(2,F_7) \cong SO(3,F_7)$ which explains all combinatorial data describing the intersection behavior of the 28 curves. This is one of the subjects of the next section.

For the complete proof of Proposition 1 in the introduction we still have to show that the congruence subgroup Γ_7'' is contained in $\pi_1(7) = \mathrm{Ker}(N(7) \longrightarrow \mathrm{Aut}(\widetilde{S(7)}))$. For the proof we have to use the following lemma:

LEMMA 4.2. Let $M \in N(7)$ and (m_{ij}) the matrix representation of M with respect to the base $\{e_1, \tilde{e}_2, e_3\}$. Then $M \in \pi_1(7)$ if and only if $m_{12} \equiv m_{13} \equiv 0 \mod. \mathfrak{p}$, $m_{21} \equiv m_{31} \equiv m_{32} \equiv m_{23} \equiv 0 \mod. \mathfrak{p}^2$ and $m_{11} \equiv m_{22} \equiv m_{33} \equiv 1 \mod. \mathfrak{p}^2$.

We can only describe the procedure of proof. We first consider what the image of $N(7)$ by $N(7) \longrightarrow \mathrm{Aut}(\tilde{L}/\mathfrak{p}^2\tilde{L})$. As before we identify $\tilde{L}/\mathfrak{p}^2\tilde{L}$ with $(\mathcal{O}/\mathfrak{p}^2)^3$ and regard the image as a matrix group over the finite ring $\mathcal{O}/\mathfrak{p}^2 = F_7 + F_7\pi$

with $\pi^2 = 0$, where π is the residue class of $\zeta - \zeta^{-1}$ in $\mathcal{O}/\mathfrak{p}^2$. The antihermitian matrix corresponding to $\hat{H}$ is

$$\check{H} := \begin{pmatrix} 2\pi & & \\ & 3\pi & 1 \\ & -1 & 3\pi \end{pmatrix} .$$

The induced conjugation of $\mathcal{O}/\mathfrak{p}^2$ is $a + b\pi \longleftrightarrow a - b\pi$. The image is then in the group

$$\tilde{G} := \left\{ M \in M_{(3,3)}(\mathcal{O}/\mathfrak{p}^2) \ ; \ {}^t\overline{M}\,\check{H}\,M = \check{H} \right\} .$$

We have namely the natural mapping

$$N(7) \longrightarrow \tilde{G} .$$

Now recall that $[AB,CB] = (ABC)^2$ and that both $[AB,(ABC)^2]$, $[CB,(ABC)^2]$ are mapped to the identity by $N(7) \longrightarrow \mathrm{Aut}(\widetilde{S(7)})$. Their images in $\tilde{G}$ generate an abelian normal subgroup, which we denote by N . Now we immediately check that $\mathrm{Aut}(\widetilde{S(7)}) \cong \tilde{G}/N$ and that every element of N is of the form

$$\begin{pmatrix} 1 & a\pi & b\pi \\ & 1 & \\ & & 1 \end{pmatrix} \quad a,b \in F_7 .$$

This proves the lemma.

The proof of $\Gamma_7'' \subseteq \pi_1(7)$ is now not hard and the details are left to the reader.

5. Some finite geometry and elementary description of the surface

For brevity we denote $B_7/\Gamma_7' \cong S(7)/G_0$ by S in this section. S has an elliptic fibration over $P_1(\mathbb{C})$, with respect to which there are six singular fibers, three of type I_7 and three of type I_1. S has exactly seven global sections. Every irreducible component of a fiber of type I_7 intersects exactly one global section, which we associate with the component. In this way the seven irreducible components of the fiber are in one-to-one correspondence with the seven global sections. Now note that the seven components form a cycle by their intersection behavior; they give therefore a cyclic ordering to the global sections through the correspondence. Since we have three fibers of type I_7, we obtain three cyclic orderings of the seven sections which generate the same subgroup of order 7 in the permutation group. The fact that they are different determines combinatorially the intersection behavior of the 28 rational curves in the remark of Section 4. (Note that there are only $(7-1)/2 = 3$ cyclic orderings belonging to one and the same permutation group of order 7). We are now able to speak about the combinatorial automorphism group of the set of the 28 rational curves, which we denote by A^c.

PROPOSITION 5.1. We have the isomorphisms

$A^c \cong \Gamma_7/\Gamma_7' \cong PGL(2,F_7)$.

The proof is divided into several steps. We begin by noting that Γ_7 / Γ_7' is naturally isomorphic to the special orthogonal group associated with the quadratic form $x_1^2 + x_2^2 - 2x_3^2$ over $\mathcal{O}/\mathfrak{p} = F_7$. Since 2 is a square, the second isomorphism of the proposition follows from the following proposition due essentially to Cayley and Klein:

PROPOSITION 5.2. Let F_q be the finite field of q elements and $SO(3,F_q)$ the special orthogonal group associated with the quadratic form $x_1^2 + x_2^2 - x_3^2$ over F_q . Then we have the isomorphism $PGL(2,F_q) \cong SO(3,F_q)$.

We let $GL(2,F_q)$ act over the two 2-dimensional vector spaces (x,y) , (ξ,η) in the same way. Next we note that $x\xi - y\eta$, $x\eta + y\xi$, $x\xi + y\eta$ form a base of the space of bihomogeneous polynomials in $(x,y;\xi,\eta)$ of bidegree $(1,1)$ which are invariant by the involution $(x,y) \longleftrightarrow (\xi,\eta)$. Since $GL(2,F_q)$ naturally acts on this space, it induces linear transformations between the three polynomials. Note also that $x\eta - y\xi$ is a relative invariant of $GL(2,F_q)$. Now the identity

$$(x\xi - y\eta)^2 + (x\eta + y\xi)^2 - (x\xi + y\eta)^2 = (x\eta - y\xi)^2$$

shows that we have a homomorphism

$$\left\{ M \in GL(2,F_q) \ ; \ \det(M) = \pm 1 \right\} \longrightarrow SO(3,F_q)$$

whose kernel is obviously $\{\pm I\}$. Thus we have the injective homomorphism

$$\alpha : PGL(2,F_q) \longrightarrow SO(3,F_q) \quad .$$

To show that this is onto, we observe the mapping

$$J : P_1(F_q) \times P_1(F_q) /\sim \longrightarrow P_2(F_q) \ : \ (x_1,x_2,x_3)$$

induced by $x_1 = x\xi - y\eta$, $x_2 = x\eta + y\xi$, $x_3 = x\xi + y\eta$, where $\sim$ denotes the involution above. Since the diagonal Δ in $P_1(F_q) \times P_1(F_q)$ is defined by $x\eta - y\xi = 0$. The image of Δ is the conic

$$C := \left\{ (x) \in P_2(F_q) \ ; \ x_1^2 + x_2^2 - x_3^2 = 0 \right\} \quad .$$

The image of J minus C is the domain

$$D_+ := \left\{ (x) \in P_2(F_q) \ ; \ x_1^2 + x_2^2 - x_3^2 \text{ is a square } \neq 0 \right\} .$$

We set also

$$D_- := \left\{ (x) \in P_2(F_q) \ ; \ x_1^2 + x_2^2 - x_3^2 \text{ is not a square } \right\} .$$

The two rulings of $P_1(F_q) \times P_1(F_q)$ are transposed by $\sim$ and they are both mapped by J to the family of tangents of the conic C. Now any element of $SO(3,F_q) = PSO(3,F_q)$ induces a projecitve transformation of $C \cong P_1(F_q)$, an element of $PGL(2,F_q)$. That this is mapped to the element of $SO(3,F_q)$ by α is also evident. This proves the surjectivity of α. We have proved the following:

<u>LEMMA 5.3</u>. $|C| = q+1$, $|D_+| = q(q+1)/2$, $|D_-| = q(q-1)/2$. From a point of D_+ one can draw exactly two tangents to the conic C while there is no tangent of C passing through any point of D_-.

Now, for $(x) = (x_1,x_2,x_3) \in P_2(F_q)$, let $P(x)$ be the polar line $\left\{(y)\ ;\ x_1y_1 + x_2y_2 - x_3y_3 = 0\right\}$. If (x) is not on C, then there exists a unique involutive element of $SO(3,F_q)$ which fixes (x) and every point on $P(x)$. We denote this involution by $I(x)$. By the isomorphism α we can consider that $I(x)$ acts on $P_1(F_q) \cong C$ and hence also on $P_1(F_q) \times P_1(F_q)$ in such a way that J is equivariant with respect to it. It is clear that

$P(x) \cap \mathrm{Im}\,J = (P(x) \cap D_+) \cup (P(x) \cap C)$ is the image of the set $\left\{(z, I(x)z)\ ;\ z \in P_1(F_q)\right\}$. Thus it follows from Lemma 5.3 the following:

<u>LEMMA 5.4</u>. For $(x) \in D_+$, we have

$$|P(x) \cap D_+| = |P(x) \cap D_-| = (q-1)/2 \quad \text{and}$$

$$|P(x) \cap C| = 2 . \qquad \text{For} \quad (x) \in D_- , \text{ we have}$$

$$|P(x) \cap D_+| = |P(x) \cap D_-| = (q+1)/2 \quad .$$

It is well known that, for different (x), (y) of $P_2(F_q) - C$, $I(x)$ and $I(y)$ commute if and only if (y) is on $P(x)$ (or equivalently (x) is on $P(y)$).

<u>DEFINITION</u>: $G(D_+)$ denote the graph whose vertices are the elements of D_+ and in which two vertices (x), (y) are connected by a segment if and only if $I(x)$ and $I(y)$ commute.

It is obvious that the automorphism group of $G(D_+)$ is isomorphic to $SO(3,F_q)$. Thus, for the complete proof of Proposition 5.1, it suffices to prove that for $q = 7$, $G(D_+)$ is dual to the graph of the 28 rational curves on S . We begin with the following lemma:

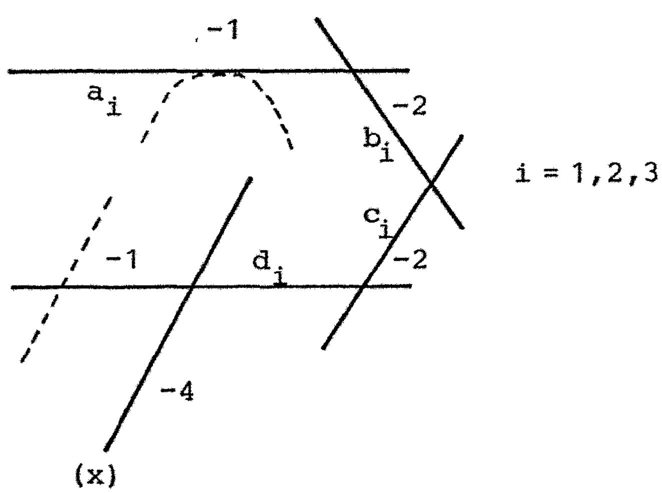
<u>LEMMA 5.5</u>. Assume that -1 is not a square in F_q . Then, among $(x),(y),(z) \in P_2(F_q) - C$ such that $I(x)$, $I(y)$, $I(z)$ commute, odd number of them belong to D_- ; in particular $G(D_+)$ contains no 3-cycle.

Note that the determinant modulo $(F_q^{\times})^2$ is an invariant for any quadratic form over F_q. This implies that for $x,y,z \in (F_q)^3 : (x_1,x_2,x_3)$ which are orthogonal and non-isotropic with respect to $x_1^2 + x_2^2 - x_3^2$, the product of norms of them is not a square. The lemma follows from this.

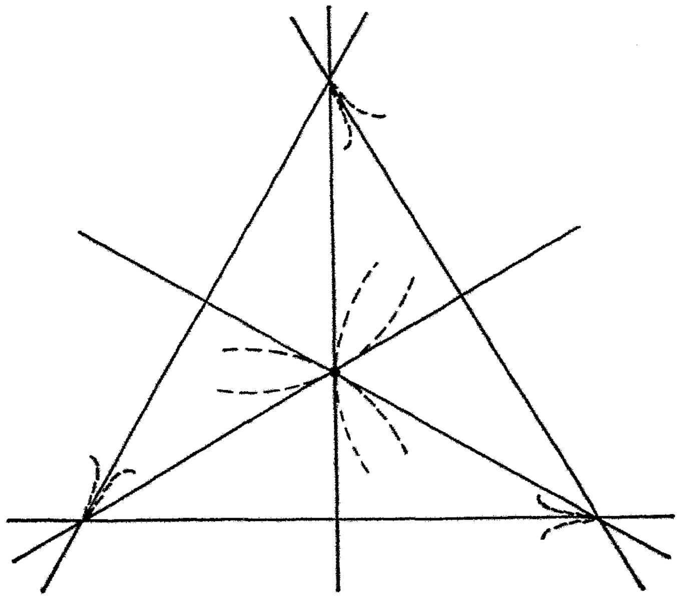
Now we assume that $q = 7$ and we fix a point p_0 of the conic C. We denote by L the tangent line at p_0 of C. From every $(x) \in L - \{p_0\}$ we can draw the other tangent to C, the contact point of which we associate with (x). In this way we obtain the canonical isomorphisms between the two affine lines $L - \{p_0\} \cong C - \{p_0\}$ $(\cong F_q)$. This shows that the translation group of $L - \{p_0\}$ is naturally extended to a subgroup of $SO(3,F_q)$, which we denote by T. Next we show that $D_+ - L$ decomposes into several disjoint cycles in $G(D_+)$. By Lemma 5.4, for $(x) \in D_+ - L$, there are exactly three elements adjacent to (x) in the graph, of which however one is on L. Now, since T acts on $D_+ - L$, we have the following alternative: Either $D_+ - L$ decomposes into seven 3-cycles which are permuted by T, or the T-orbits are exactly the (three) cycles in $D_+ - L$. The first case is impossible by Lemma 5.5. Now we call the elements of $L - \{p_0\}$ the <u>global sections</u> and the cycles in $D_+ - L$ the <u>singular fibers</u> (of type I_7). We have also seen above that the elements in a fiber of type I_7 are in one-to-one correspondence with the global sections by the intersection relation. Now let AF be the affine automorphism group of $L_0 - \{p_0\}$, in which T

is a normal subgroup. Since the factor group AF/T permutes the three T-orbits in $D_+ - L_0$ evenly, and since it permutes the three cyclic orderings in $L_0 - \{p_0\}$ in the same way, different singular fibers give different orderings to the set of global sections. This proves Proposition 5.1 completely.

Independent of the description of the quotient $S(7)/G_0$ given in the previous section, we will now directly prove the existence and uniqueness of a K3 surface with the desired elliptic fibration. This also gives an elementary description of the surface. We first suppose that S is such an elliptic surface. We fix a global section, denoted (x) , as the zero for all fibers. Then there must be an involution of S inducing on each fiber $a \longleftrightarrow -a$. This combinatorially corresponds to I(x) and we call it also I(x) . On the surface S/I(x) we have three configurations of the following form coming from fibers of type I_7 :



where a_i, b_i, c_i, d_i are the names for the curves, the dots denote the branching (= fixed point) locus of $I(x)$ outside (x) and $-1, -2, -4$ the self-intersection numbers, Now we blow down first a_i, d_i $(i = 1,2,3)$ and then the images of $b_i (i = 1,2,3)$. Then we obtain a (smooth) P_1 - bundle over P_1 , which is however not relatively minimal since the image of (x) is an exceptional curve of the 1st kind. Therefore, if we blow it down, we get $P_2(\mathbb{C}) : (x_1, x_2, x_3)$. We can assume that the coordinates are so chosen that the image of (x) is $(1,1,1)$ and that a_i and b_i are mapped on the point $x_j = x_k = 0$ $(\{i,j,k\} = \{1,2,3\})$. Then the image of c_i is the line $x_j = x_k$ and the remaining six global sections are two by two mapped onto the lines $x_1 = 0,\ x_2 = 0,\ x_3 = 0$. The branching locus ... is mapped birationally onto a curve which has D_4 at $(1,1,1)$ and A_4 at each vertex of the triangle $x_1 \cdot x_2 \cdot x_3 = 0$. We denote this curve by D . The fibers of S correspond to the lines passing through $(1,1,1)$. Since D intersects the general lines exactly three times outside the triple point $(1,1,1)$, D is a sextic curve. The surface S is thus described as (the minimal desingularization of) the double covering of $P_2(\mathbb{C})$ branched over D . Note also that the tangents of D at the cusps are directed as the following indicates:



Such a sextic curve is unique and it is in fact given by the equation

$$\left(x_1^2x_2 + x_2^2x_3 + x_3^2x_1 - 3x_1x_2x_3\right)^2 -$$

$$- 4x_1x_2x_3(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) = 0 \qquad .$$

The details are left to the reader.

REFERENCES

[BH] *A. Borel and Harish-Chandra;* Arithmetic subgroups of algebraic groups, Ann. of Math. 75 (1962), 485-535.

[H] *F. Hirzebruch;* Arrangements of lines and algebraic surfaces, Progress in Math. 36 , 113-140, Birkhäuser (1983).

[Hö] *Th. Höfer;* Ballquotienten als verzweigte Überlagerungen algebraischer Flächen, Dissertation Bonn (1985).

[I] *M. Inoue;* Some surface of general type with positive indices (Preprint).

[Is] *M. Ishida;* Hirzebruch's examples of surfaces of general type with $c_1^2 = 3c_2$, Algebraic Geometry, Springer Lecture Notes 1016, 412-431.

[L] *R.A. Livné;* On certain covers of the universal elliptic curves, Thesis to Harvard University (1981).

[T] *T. Terada;* Fonctions hypergeometriques F_1 et fonctions automorphes I, J. Math. Soc. Japan 35 (1983), 451-475.

[Ya] *S.T. Yau;* On Calabi's conjecture and some new results in algebraic geometry, Nat. Acad. Sci. USA, 74 (1977), 1798-1799.

[YY] *T. Yamazaki and M. Yoshida;* On Hirzebruch's examples of surfaces with $c_1^2 = 3c_2$, Math. Ann. 266 (1984), 421-431.