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CHARACTERISTIC CLASSES OF SYMMETRIC PRODUCTS OF COMPLEX QUASI-PROJECTIVE VARIETIES

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Abstract. We prove a generating series formula for the characteristic classes of symmetric powers of complexes of mixed Hodge modules on the corresponding symmetric products of a complex quasi-projective variety. As a special case, we obtain a generating series formula for the Brasselet–Schürmann–Yokura (intersection) homology Hirzebruch classes of symmetric products. Moreover, after a suitable re-normalization procedure, we recover as a corollary Ohmoto's generating series formula for the rationalized MacPherson homology Chern classes of symmetric products.

Contents

1. Introduction

Some of the most interesting examples of orbifolds are the symmetric products of algebraic varieties. The *n-th symmetric product* of a space X is defined by

$$X^{(n)} := \overbrace{X \times \cdots \times X}^{n \ times} / \Sigma_n,$$

i.e., the quotient of the product of n copies of X by the natural action of the symmetric group on n elements, Σ_n.

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The standard approach for computing invariants $\mathcal{I}(X^{(n)})$ of symmetric products is to encode the respective invariants of *all* symmetric products in a *generating series*, i.e., an expression of the form

$$S_{\mathcal{I}}(X) := \sum_{n \geq 0} \mathcal{I}(X^{(n)}) \cdot t^n,$$

provided $\mathcal{I}(X^{(n)})$ can be defined for all n. This is analogous to the zeta function of a variety over a finite field. The aim is to calculate such an expression solely in terms of invariants of X, so $\mathcal{I}(X^{(n)})$ is just the coefficient of t^n in the resulting expression.

There is a well-known formula due to Macdonald [Mac] for the generating series of the topological Euler characteristic. A class version of this formula was recently obtained by Ohmoto in [O] for the Chern classes of MacPherson [M]. Moonen [Mo] obtained generating series for the arithmetic genus of symmetric products of a projective manifold and, more generally, for the Baum–Fulton–MacPherson Todd classes of symmetric products of any projective variety. In [Za], Hirzebruch and Zagier obtained such generating series for the signature and L-classes of symmetric products of rational homology manifolds. Also, Borisov–Libgober [BL] computed generating series for the Hirzebruch χ_y-genus and, more generally, for elliptic genus of symmetric products of smooth compact varieties. Generating series for the mixed Hodge numbers of complexes of mixed Hodge modules on symmetric products of (possibly singular) quasi-projective varieties have been recently obtained in [MSa] by relating symmetric group actions on exterior products to the theory of lambda rings (e.g., see [Yau]), but see also [MSS] for an alternative approach.

In this paper we assume that X is a (possibly singular) complex quasi-projective variety, so its symmetric products $X^{(n)}$ are quasi-projective varieties as well.

The invariants of symmetric products considered in this paper are the homology Hirzebruch classes $T_{y*}(X^{(n)})$ of Brasselet–Schürmann–Yokura [BSY] (also see [SY, Sch2, Yo]) and, for X pure-dimensional, the intersection Hirzebruch classes $IT_{y*}(X^{(n)})$ studied by Cappell–Maxim–Shaneson [CMS].

For any (pure-dimensional) complex algebraic variety Z the classes $T_{y*}(Z)$ and $IT_{y*}(Z)$ are defined as the images of certain distinguished elements by a natural transformation

$$T_{y*} : K_0(\mathrm{MHM}(Z)) \to H^{BM}_{ev}(Z) \otimes \mathbb{Q}[y^{\pm 1}],$$

where $K_0(\mathrm{MHM}(Z))$ is the Grothendieck group of the abelian category of algebraic mixed Hodge modules on Z [Sa2], and $H^{BM}_{ev}(-)$ denotes the Borel–Moore homology in even degrees. More precisely, by building on Saito's functors (cf. [Sa2])

$$gr^F_p DR : D^b\mathrm{MHM}(Z) \to D^b_{coh}(Z) \tag{1}$$

(for $D^b_{coh}(Z)$ the bounded derived category of sheaves of $\mathcal{O}_Z$-modules with coherent cohomology sheaves), one first defines a motivic Chern class transformation MHC_y as follows: the transformations $gr^F_p DR$ induce functors on the level of Grothendieck groups, and we let

$$\mathrm{MHC}_y : K_0(\mathrm{MHM}(Z)) \to K_0(D^b_{\mathrm{coh}}(Z)) \otimes \mathbb{Z}[y^{\pm 1}] = K_0(\mathrm{Coh}(Z)) \otimes \mathbb{Z}[y^{\pm 1}] \tag{2}$$

be given by

$$[\mathcal{M}] \mapsto \sum_{i,p} (-1)^i [\mathcal{H}^i(gr^F_{-p} DR(\mathcal{M}))] \cdot (-y)^p. \tag{3}$$

(This is well-defined since $gr^F_p DR(\mathcal{M}) = 0$ for almost all p and $\mathcal{M}$ fixed.) Then the Hirzebruch class transformation $T_{y_*} : K_0(\mathrm{MHM}(Z)) \to H^{BM}_{ev}(Z) \otimes \mathbb{Q}[y^{\pm 1}]$ is defined by the composition:

$$T_{y_*} := td_* \circ \mathrm{MHC}_y, \tag{4}$$

with

$$td_* : K_0(\mathrm{Coh}(Z)) \to H^{BM}_{ev}(Z; \mathbb{Q}), \tag{5}$$

the Baum–Fulton–MacPherson Todd class transformation [BFM] (linearly extended over $\mathbb{Z}[y^{\pm 1}]$). This transformation T_{y_*} has good functorial properties, e.g., it commutes with proper push-down. The above mentioned characteristic classes $T_{y_*}(Z)$ and $IT_{y_*}(Z)$ are then obtained by evaluating the transformation T_{y_*} on the (class of the) constant Hodge complex $\mathbb{Q}^H_Z$,

$$T_{y_*}(Z) := T_{y_*}([\mathbb{Q}^H_Z]), \tag{6}$$

and, respectively, for pure-dimensional Z, on (a shift of) the intersection Hodge sheaf IC^H_Z:

$$IT_{y_*}(Z) := T_{y_*}([IC'^H_Z]), \tag{7}$$

with $IC'^H_Z := IC^H_Z[-\dim(Z)]$. We point out that both classes $T_{y_*}(Z)$ and $IT_{y_*}(Z)$ are extensions to the singular setting of the un-normalized cohomology Hirzebruch class $T_y{}^*(-)$ appearing in the generalized Hirzebruch–Riemann–Roch theorem [H], which in Hirzebruch's philosophy corresponds to the un-normalized or non-characteristic power series $Q_y(\alpha) = \frac{\alpha(1+ye^{-\alpha})}{1-e^{-\alpha}} \in \mathbb{Q}[y][[\alpha]]$. In fact the associated normalized or characteristic power series (which we need in §4) is $\widehat{Q}_y(\alpha) := \frac{Q_y(\alpha(1+y))}{1+y} = \frac{\alpha(1+y)}{1-e^{-\alpha(1+y)}} - \alpha y$, which defines the normalized cohomology Hirzebruch class $\widehat{T}^*_y(-)$. If we specialize the parameter y of $\widehat{T}^*_y(-)$ to the three distinguished values $y = -1, 0$ and 1, we recover the cohomology Chern, Todd, and L-class, respectively.

Moreover, as shown in [BSY, Theorem 3.1] and [Sch2, Example 5.2], the homology Hirzebruch classes $T_{y_*}(Z)$ and $IT_{y_*}(Z)$ contain only non-negative powers of y, so one is allowed to specialize the parameter y to the above three distinguished values $y = -1, 0$ and 1, in particular to $y = 0$.

Over a point space $Z = \{pt\}$, the transformation T_{y_*} (as well as its normalization defined in §4 below) reduces to the χ_y-polynomial ring homomorphism

$$\chi_y : K_0(\mathrm{mHs}^p) \to \mathbb{Z}[y, y^{-1}], \tag{8}$$

which is defined on the Grothendieck group $K_0(\mathrm{mHs}^p)$ of (graded) polarizable mixed Hodge structures by:

$$\chi_y([H]) := \sum_p \dim_{\mathbb{C}} Gr_F^p(H \otimes \mathbb{C}) \cdot (-y)^p, \tag{9}$$

with $F^\bullet$ the Hodge filtration on $H \in \mathrm{mHs}^p$. So, if Z is a compact variety, by pushing down to a point the classes $T_{y_*}(Z)$ and $IT_{y_*}(Z)$ (or their normalized counterparts from §4), one gets that the degrees of their zero-dimensional components are the corresponding Hodge polynomials $\chi_y(Z)$ and $I\chi_y(Z)$, respectively, defined in terms of dimensions of the graded parts of the Hodge filtration on the (intersection) cohomology of Z.

1.1. **Statement of results.** Let X be a complex quasi-projective variety, with n-th symmetric product $X^{(n)}$ and projection $\pi_n : X^n \to X^{(n)}$. For a complex of mixed Hodge modules $\mathcal{M} \in D^b\mathrm{MHM}(X)$, we let the *$n$-th symmetric power* of $\mathcal{M}$ be defined by:

$$\mathcal{M}^{(n)} := (\pi_{n*}\mathcal{M}^{\boxtimes n})^{\Sigma_n} \in D^b\mathrm{MHM}(X^{(n)}), \tag{10}$$

where $\mathcal{M}^{\boxtimes n} \in D^b\mathrm{MHM}(X^n)$ is the n-th exterior product of $\mathcal{M}$ with the Σ_n-action defined as in [MSS], and $(-)^{\Sigma_n}$ is the projector on the Σ_n-invariant sub-object. The action of Σ_n on $\mathcal{M}^{\boxtimes n}$ is, by construction, compatible with the natural action on the underlying $\mathbb{Q}$-complexes (see [MSS] for details). In what follows, we regard the exterior product $\mathcal{M}^{\boxtimes n}$ as an object in the category $D^{b,\Sigma_n}\mathrm{MHM}(X^n)$ of weakly Σ_n-equivariant complexes of mixed Hodge modules on X^n (compare with [CMSS1, Appendix A]). As special cases of (10), it was shown in [MSS] that for $\mathcal{M} = \mathbb{Q}_X^H$ the constant Hodge sheaf on X, one obtains:

$$\left(\mathbb{Q}_X^H\right)^{(n)} = \mathbb{Q}_{X^{(n)}}^H. \tag{11}$$

Also, for X pure-dimensional and $\mathcal{M} = IC'^H_X := IC^H_X[-\dim X]$ the (shifted) intersection Hodge sheaf on X, one has:

$$\left(IC'^H_X\right)^{(n)} = IC'^H_{X^{(n)}}. \tag{12}$$

The main result of this paper is the following generating series formula for the Hirzebruch classes of the symmetric powers $\mathcal{M}^{(n)} \in D^b\mathrm{MHM}(X^{(n)})$ of a fixed complex of mixed Hodge modules on the variety X:

Theorem 1.1. *Let X be a complex quasi-projective variety and $\mathcal{M} \in D^bMHM(X)$. Then the following identity holds in $\sum_n H_{ev}^{BM}(X^{(n)}; \mathbb{Q}[y^{\pm 1}]) \cdot t^n$:*

$$\sum_{n \geq 0} T_{(-y)_*}(\mathcal{M}^{(n)}) \cdot t^n = \exp\left(\sum_{r \geq 1} \Psi_r \left(d_*^r T_{(-y^r)_*}(\mathcal{M})\right) \cdot \frac{t^r}{r}\right), \tag{13}$$

where

(a) *$d^r : X \to X^{(r)}$ is the composition of the diagonal embedding $i_r : X \simeq \Delta_r(X) \hookrightarrow X^r$ with the projection $\pi_r : X^r \to X^{(r)}$.*

(b) Ψ_r *is the* r-th homological Adams operation*, which on* $H^{BM}_{2k}(X^{(r)};\mathbb{Q})$ *(*$k\in\mathbb{Z}$*) is defined by multiplication by* $\frac{1}{r^k}$ *(and is then linearly extended over the corresponding coefficient ring).*

(c) *The multiplication on the right-hand side of (13) is with respect to the* Pontrjagin product *induced by*

$$X^{(m)}\times X^{(n)}\to X^{(m+n)},\quad m,n\in\mathbb{N},$$

which in turn comes from the product $X^m\times X^n=X^{m+n}$*, with* $\Sigma_m\times\Sigma_n\subset\Sigma_{m+n}$ *acting on each factor. (Note that this Pontrjagin product is associative, commutative, and with unit* $1_{pt}\in H_0(pt)$*, so that the exponential series on the right-hand side of formula (13) makes sense (compare* [Mo]*).)*

The proof of Theorem 1.1 makes use of the equivariant Hirzebruch classes of [CMSS1], combined with the Lefschetz–Riemann–Roch theorem [BFQ, Mo], which in the context of symmetric products is related to the singular Adams–Riemann–Roch transformation for coherent sheaves (e.g., see [FL, Mo, N]).

If X is a projective variety, by pushing down to a point the result of Theorem 1.1, we recover the generating series formula for the Hodge polynomials $\chi_y(X^{(n)},\mathcal{M}^{(n)})$ (cf. [MSa, MSS]), namely:

$$\sum_{n\geq 0}\chi_{-y}(X^{(n)},\mathcal{M}^{(n)})\cdot t^n=\exp\left(\sum_{r\geq 1}\chi_{-y^r}(X,\mathcal{M})\cdot\frac{t^r}{r}\right).\tag{14}$$

Indeed, over a point space, the map d^r is the identity, and the r-th Adams operation Ψ_r also becomes the identity transformation.

If we let $\mathcal{M}$ be the constant Hodge sheaf $\mathbb{Q}^H_X$ or the shifted intersection chain sheaf IC'^H_X, respectively, we obtain by (11) and (12) the following special cases of formula (13), as announced in [MSb]:

Corollary 1.2. *For any complex quasi-projective variety* X *the following identity holds in* $\sum_{n\geq 0}H^{BM}_{ev}(X^{(n)};\mathbb{Q}[y])\cdot t^n$*:*

$$\sum_{n\geq 0}T_{(-y)_*}(X^{(n)})\cdot t^n=\exp\left(\sum_{r\geq 1}\Psi_r\left(d^r_*T_{(-y^r)_*}(X)\right)\cdot\frac{t^r}{r}\right),\tag{15}$$

and, if X *is pure-dimensional, then the identity*

$$\sum_{n\geq 0}IT_{(-y)_*}(X^{(n)})\cdot t^n=\exp\left(\sum_{r\geq 1}\Psi_r\left(d^r_*IT_{(-y^r)_*}(X)\right)\cdot\frac{t^r}{r}\right),\tag{16}$$

holds in $\sum_{n\geq 0}H^{BM}_{ev}(X^{(n)};\mathbb{Q}[y])\cdot t^n$*.*

If X is smooth and projective, the formulae of Corollary 1.2 specialize to Moonen's generating series formula for his generalized Todd classes $\tau_y(X^{(n)})$ (cf. [Mo, p.172]). Indeed, as shown in [CMSS1], Moonen's generalized Todd class $\tau_y(Y/G)$, which he could only define for a projective orbifold Y/G (with G a finite group of algebraic automorphisms of the projective manifold Y), coincides in this context with the Brasselet–Schürmann–Yokura (un-normalized) Hirzebruch class $T_{y*}(Y/G)$ considered in this paper.

We conclude this introduction with a discussion on important special cases of the formulae (15) and (16) of Corollary 1.2.

If $y = 0$, the formulae for the corresponding classes $T_{0*}(-)$ and $IT_{0*}(-)$ should be compared with Moonen's generating series formula for the Baum–Fulton–MacPherson Todd classes $td_*(X^{(n)})$ of symmetric products of a projective variety (see [Mo, p.162–164]). However, while these three classes satisfy the same generating series formula, they do not coincide in general, except in very special cases, e.g., if X is smooth so that the symmetric products $X^{(n)}$ have only rational (hence Du Bois) singularities (see [BSY, Example 3.2]). If X is smooth and projective, by taking the degrees of the zero-dimensional components in either (15) or (16), we recover Moonen's generating series formula for the arithmetic genus of symmetric products of a projective manifold (cf. [Mo, Corollary 2.7, p.161]):

$$\sum_{n\geq 0} \chi_a(X^{(n)})t^n = \exp\left(\sum_{r\geq 1} \chi_a(X)\cdot \frac{t^r}{r}\right) = (1-t)^{-\chi_a(X)}. \tag{17}$$

Let us now consider the case $y = -1$, and assume that X is projective and pure-dimensional. Then by taking the degree of the zero-dimensional components in (16), we recover the generating series formula for the Goresky–MacPherson intersection cohomology signature $\sigma(X^{(n)})$ of the symmetric products of X [1], i.e.

$$\sum_{n\geq 0} \sigma(X^{(n)})\cdot t^n = \frac{(1+t)^{\frac{\sigma(X)-\chi^{IH}(X)}{2}}}{(1-t)^{\frac{\sigma(X)+\chi^{IH}(X)}{2}}}, \tag{18}$$

with $\chi^{IH}(X) := \chi([IH^*(X;\mathbb{Q})])$ the intersection cohomology Euler characteristic of X, see [MSa, MSS]. If, moreover, X is smooth, formula (18) was proved by Zagier [Za]. With regard to characteristic classes, both formulae (15) and (16) specialize for X smooth and projective to the generating series for Moonen's class $\tau_{1*}(X^{(n)})$ of symmetric products of X. This differs from the Thom–Milnor homology L-class $L_*(X^{(n)})$ by a renormalization, defined by multiplying in each even degree by a suitable power of 2. More precisely, for any projective G-manifold Y, with G a finite group of algebraic automorphisms of Y, one has (cf. [Mo, Corollary 2.10, p.171]):

$$\Psi_2 T_{1*}(Y/G) = \Psi_2 \tau_{1*}(Y/G) = L_*(Y/G),$$

[1] Here we use Saito's Hodge index theorem, which asserts that if Z is a pure-dimensional complex projective variety, its Goresky–MacPherson (intersection cohomology) signature $\sigma(Z)$ is obtained from the intersection homology Hodge numbers of Z by the formula $\sigma(Z) = I\chi_1(Z)$, e.g., see [MSS] for an abstract Hodge index theorem.

with Ψ_2 the second homological Adams operation (as defined in Theorem 1.1). A formula for the Thom–Milnor L-classes of symmetric products was originally obtained by Zagier [Za] in the manifold context, and then re-proved by Moonen in [Mo] in the complex projective case.

If $y = 1$ and X is projective, by taking degrees in formula (15) we recover's MacDonald's generating series formula for the Euler characteristics of symmetric products [Mac]:

$$\sum_{n\geq 0} \chi(X^{(n)})t^n = \exp\left(\sum_{r\geq 1} \chi(X)\cdot \frac{t^r}{r}\right) = (1-t)^{-\chi(X)}. \tag{19}$$

Similarly, by taking degrees in formula (16), we obtain the generating series formula for the intersection cohomology Euler characteristic $\chi^{IH}(X^{(n)})$ of the symmetric products of X, see [MSa, MSS]. Finally, after a suitable re-normalization (as explained in §4), formula (15) specializes for the value $y = 1$ of the parameter to Ohmoto's generating series formula [O] for the rationalized MacPherson–Chern classes $c_*(X^{(n)})$ of the symmetric products of X (see §4 for details):

$$\sum_{n\geq 0} c_*(X^{(n)})\cdot t^n = \exp\left(\sum_{r\geq 1} d^r_* c_*(X)\cdot \frac{t^r}{r}\right). \tag{20}$$

By similar arguments (as explained in §4), we get from Theorem 1.1 the following generating series formula for the rationalized MacPherson–Chern classes of symmetric products of a constructible sheaf complex $\mathcal{F}$ underlying a complex of mixed Hodge modules $\mathcal{M}$:

$$\sum_{n\geq 0} c_*(\mathcal{F}^{(n)})\cdot t^n = \exp\left(\sum_{r\geq 1} d^r_* c_*(\mathcal{F})\cdot \frac{t^r}{r}\right), \tag{21}$$

which, for $\mathcal{F} = \mathbb{Q}_X$ the constant sheaf gives back formula (20).

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2. Proof of Theorem 1.1

An essential ingredient in the proof of Theorem 1.1 is the Atiyah–Singer class transformation (cf. [CMSS1])

$$T_{y*}(-;g) : K_0(\mathrm{MHM}^G(Z)) \to H^{BM}_{ev}(Z^g)\otimes \mathbb{C}[y^{\pm 1}],$$

which is defined by combining Saito's theory with the Lefschetz–Riemann–Roch transformation

$$td_*(-;g) : K_0(\mathrm{Coh}^G(Z)) \to H^{BM}_{ev}(Z^g;\mathbb{C})$$

of Baum–Fulton–Quart [BFQ] and Moonen [Mo]. These transformations are defined for any complex quasi-projective variety Z acted upon by a finite group G of algebraic automorphisms. Here $K_0(\mathrm{MHM}^G(Z))$ denotes the Grothendieck group of equivariant mixed Hodge modules, which is identified with a suitable Grothendieck group of "weakly" equivariant complexes of mixed Hodge modules (see [CMSS1, Appendix A]). Also, $K_0(\mathrm{Coh}^G(Z))$ denotes the Grothendieck group of G-equivariant algebraic coherent sheaves on Z. More details on the construction of the Atiyah–Singer class transformation $T_{y_*}(-;g)$ will be given in §3, as needed.

Let $\sigma \in \Sigma_n$ have cycle partition $\lambda = (k_1, k_2, \cdots)$, i.e., k_r is the number of length r cycles in σ and $n = \sum_r r \cdot k_r$. Let

$$\pi^\sigma : (X^n)^\sigma \to X^{(n)}$$

denote the composition of the inclusion of the fixed point set $(X^n)^\sigma \hookrightarrow X^n$ followed by the projection $\pi_n : X^n \to X^{(n)}$. For a cycle A of σ, we let $|A|$ denote its length. Then

$$(X^n)^\sigma \simeq \prod_{\text{A= } cycle\ in\ \sigma} (X^{|A|})^A \simeq \prod_r ((X^r)^{\sigma_r})^{k_r} \simeq \prod_r \Delta_r(X)^{k_r} \simeq X^{k_1+k_2+\cdots} \ ,$$

where σ_r denotes a cycle of length r, and $\Delta_r(X)$ is the diagonal in X^r. Also, $(X^{|A|})^A \simeq X$, diagonally embedded in $X^{|A|}$. Here the inclusion $X^{|A|} \hookrightarrow X^n$ is given by $X_{j_1} \times X_{j_2} \times \cdots$, for $A = (j_1, j_2, \cdots)$ and with X_j on the j-th place in X^n. Then the projection $\pi^\sigma : (X^n)^\sigma \to X^{(n)}$ is the product (over cycles A of σ) of projections

$$\pi^A : X \to X^{(|A|)}$$

defined by the composition

$$\pi^A : X \simeq \Delta_{|A|}(X) \hookrightarrow X^{|A|} \to X^{(|A|)},$$

with

$$\prod_A X^{(|A|)} \to X^{(n)}$$

induced by the Pontrjagin product. In the notations of Theorem 1.1, this amounts to saying that π^σ is the product of projections

$$d^r : X \simeq \Delta_r(X) \overset{i_r}{\hookrightarrow} X^r \overset{\pi_r}{\to} X^{(r)},$$

where each r-cycle contributes a copy of d^r.

Theorem 1.1 is a consequence of the following sequence of reductions (compare with [Mo] for a similar argument):

Lemma 2.1. *For $\mathcal{M} \in D^bMHM(X)$ and every $n \geq 0$, we have:*

$$T_{y_*}(\mathcal{M}^{(n)}) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \pi_*^\sigma T_{y_*}(\mathcal{M}^{\boxtimes n}; \sigma). \tag{22}$$

Proof. This follows directly from [CMSS1, Theorem 5.4], by regarding the exterior product $\mathcal{M}^{\boxtimes n}$ with its Σ_n-action (as defined in [MSS]) as a weakly equivariant complex, i.e., as an element in $D^{b,\Sigma_n}\mathrm{MHM}(X^n)$. □

Lemma 2.2. *If $\sigma \in \Sigma_n$ has cycle-type $(k_1, k_2, \cdots)$, then:*

$$T_{y_*}(\mathcal{M}^{\boxtimes n}; \sigma) = \prod_r \left(T_{y_*}(\mathcal{M}^{\boxtimes r}; \sigma_r)\right)^{k_r}. \tag{23}$$

Therefore,

$$\pi_*^\sigma T_{y_*}(\mathcal{M}^{\boxtimes n}; \sigma) = \prod_r \left(d_*^r T_{y_*}(\mathcal{M}^{\boxtimes r}; \sigma_r)\right)^{k_r} \tag{24}$$

Proof. This is a consequence of the multiplicativity property of the Atiyah–Singer class transformation, see [CMSS1, Corollary 4.2]. □

Lemma 2.3. *The following identification holds in $H_{ev}^{BM}(X) \otimes \mathbb{Q}[y^{\pm 1}] \subset H_{ev}^{BM}(X) \otimes \mathbb{C}[y^{\pm 1}]$:*

$$T_{(-y)_*}(\mathcal{M}^{\boxtimes r}; \sigma_r) = \Psi_r T_{(-y^r)_*}(\mathcal{M}), \tag{25}$$

with Ψ_r the r-th homological Adams operation, which is defined on $H_{2k}^{BM}(X; \mathbb{Q})$ ($k \in \mathbb{Z}$) by multiplication by $\frac{1}{r^k}$ and is then linearly extended over $\mathbb{Q}[y^{\pm 1}]$.

The proof of Lemma 2.3 is given in §3.

We now have all the ingredients for proving Theorem 1.1.

Proof. For a given *partition* $\Pi = (k_1, k_2, \cdots, k_n)$ of n, i.e., $n = \sum_r k_r \cdot r$, denote by N_Π the number of elements $\sigma \in \Sigma_n$ of cycle-type Π. Then it's easy to see that

$$N_\Pi = \frac{n!}{k_1!k_2! \cdots 1^{k_1} 2^{k_2} \cdots}.$$

Formula (13) follows now from the following sequence of identities:

$$\begin{aligned}
\sum_n T_{(-y)_*}(\mathcal{M}^{(n)}) \cdot t^n &\overset{(22)}{=} \sum_n t^n \cdot \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \pi_*^\sigma T_{(-y)_*}(\mathcal{M}^{\boxtimes n}; \sigma) \\
&\overset{(24)}{=} \sum_n \frac{t^n}{n!} \cdot \sum_{\Pi = (k_1, k_2, \cdots k_n)} N_\Pi \prod_{r=1}^n \left(d_*^r T_{(-y)_*}(\mathcal{M}^{\boxtimes r}; \sigma_r)\right)^{k_r} \\
&= \sum_n \sum_{\Pi = (k_1, k_2, \cdots k_n)} \frac{t^{k_1 \cdot 1 + k_2 \cdot 2 + \cdots}}{k_1!k_2! \cdots 1^{k_1} 2^{k_2} \cdots} \prod_{r=1}^n \left(d_*^r T_{(-y)_*}(\mathcal{M}^{\boxtimes r}; \sigma_r)\right)^{k_r}
\end{aligned}$$

$$\begin{aligned}
&= \sum_n \sum_{\Pi=(k_1,k_2,\cdots k_n)} \prod_{r=1}^{n} \frac{t^{k_r\cdot r}}{k_r! r^{k_r}} \cdot \left(d^r_* T_{(-y)_*}(\mathcal{M}^{\boxtimes r};\sigma_r)\right)^{k_r} \\
&= \prod_{r=1}^{\infty}\left(\sum_{k_r=0}^{\infty} \frac{t^{k_r\cdot r}}{k_r! r^{k_r}} \cdot \left(d^r_* T_{(-y)_*}(\mathcal{M}^{\boxtimes r};\sigma_r)\right)^{k_r}\right) \\
&= \prod_{r=1}^{\infty}\left(\sum_{k_r=0}^{\infty} \frac{1}{k_r!} \cdot \left(d^r_* T_{(-y)_*}(\mathcal{M}^{\boxtimes r};\sigma_r)\cdot \frac{t^r}{r}\right)^{k_r}\right) \\
&= \prod_{r=1}^{\infty} \exp\left(d^r_* T_{(-y)_*}(\mathcal{M}^{\boxtimes r};\sigma_r)\cdot \frac{t^r}{r}\right) \\
&= \exp\left(\sum_{r=1}^{\infty} d^r_* T_{(-y)_*}(\mathcal{M}^{\boxtimes r};\sigma_r)\cdot \frac{t^r}{r}\right) \\
&\overset{(25)}{=} \exp\left(\sum_{r=1}^{\infty} \Psi_r\left(d^r_* T_{(-y^r)_*}(\mathcal{M})\right)\cdot \frac{t^r}{r}\right),
\end{aligned}$$

where in the last equality we also use the functoriality with respect to proper push-down of the homological Adams transformation Ψ_r.

□

3. Proof of Lemma 2.3

The aim of this section is to supply a proof of the technical Lemma 2.3. We begin by recalling in §3.1 the construction of the Atiyah–Singer class transformation from [CMSS1]. In §3.2, we specialize to the case of symmetric products and study in §3.2.1 how Saito's functors $gr^F_* DR$ behave with respect to exterior powers (with the induced graded anti-symmetric action). Before finishing the proof of Lemma 2.3, we indicate in §3.2.2 how the Lefschetz–Riemann–Roch and Adams–Riemann-Roch transformations are related in the context of symmetric products.

3.1. **The Atiyah–Singer class transformation.** Let Z be a (possibly singular) quasi-projective variety acted upon by a finite group G of algebraic automorphisms. The Atiyah–Singer class transformation

$$T_{y*}(-;g) : K_0(\mathrm{MHM}^G(Z)) \to H^{BM}_{ev}(Z^g) \otimes \mathbb{C}[y^{\pm 1}]$$

is constructed in [CMSS1] in two stages. First, by using Saito's theory of algebraic mixed Hodge modules [Sa2], we construct an equivariant version of the motivic Chern class transformation of [BSY](see also [Sch2, Yo]), i.e., the *equivariant motivic Chern class transformation*:

$$\mathrm{MHC}^G_y : K_0(\mathrm{MHM}^G(Z)) \to K_0(\mathrm{Coh}^G(Z)) \otimes \mathbb{Z}[y^{\pm 1}], \tag{26}$$

for $K_0(\mathrm{Coh}^G(Z))$ the Grothendieck group of G-equivariant algebraic coherent sheaves on Z. Secondly, we employ the Lefschetz–Riemann–Roch transformation of Baum–Fulton–Quart [BFQ] and Moonen [Mo]:

$$td_*(-;g) : K_0(\mathrm{Coh}^G(Z)) \to H^{BM}_{ev}(Z^g;\mathbb{C}) \tag{27}$$

to obtain (localized) homology classes on the fixed-point set Z^g.

In order to define the equivariant motivic Chern class transformation MHC^G_y, we work in the category $D^{b,G}\mathrm{MHM}(Z)$ of G-equivariant objects in the derived category $D^b\mathrm{MHM}(Z)$ of algebraic mixed Hodge modules on Z, and similarly for $D^{b,G}_{\mathrm{coh}}(Z)$, the category of G-equivariant objects in the derived category $D^b_{\mathrm{coh}}(Z)$ of bounded complexes of $\mathcal{O}_Z$-sheaves with coherent cohomology. Let us recall that in both these cases, a G-equivariant element $\mathcal{M}$ is just an element in the underlying additive category (e.g., $D^b\mathrm{MHM}(Z)$), with a G-action given by isomorphisms

$$\psi_g : \mathcal{M} \to g_*\mathcal{M} \quad (g \in G),$$

such that $\psi_{id} = id$ and $\psi_{gh} = g_*(\psi_h) \circ \psi_g$ for all $g, h \in G$ (see [MSa, Appendix A]). These "weak equivariant derived categories" $D^{b,G}(-)$ are not triangulated in general. Nevertheless, one can define a suitable Grothendieck group, by using "equivariant distinguished triangles" in the underlying derived category $D^b(-)$, and get isomorphisms (cf. [CMSS1, Lemma 6.7]):

$$K_0(D^{b,G}\mathrm{MHM}(Z)) = K_0(\mathrm{MHM}^G(Z)) \quad \text{and} \quad K_0(D^{b,G}_{\mathrm{coh}}(Z)) = K_0(\mathrm{Coh}^G(Z)).$$

The equivariant motivic Chern class transformation MHC^G_y is defined by noting that Saito's natural transformations of triangulated categories (cf. [Sa2])

$$\mathrm{gr}^F_p DR : D^b\mathrm{MHM}(Z) \to D^b_{\mathrm{coh}}(Z)$$

commute with the push-forward g_* induced by each $g \in G$, thus inducing equivariant transformations (cf. [CMSS1, Example 6.6])

$$\mathrm{gr}^F_p DR^G : D^{b,G}\mathrm{MHM}(Z) \to D^{b,G}_{\mathrm{coh}}(Z).$$

Note that for a fixed $\mathcal{M} \in D^{b,G}\mathrm{MHM}(Z)$, one has that $\mathrm{gr}^F_p DR^G(\mathcal{M}) = 0$ for all but finitely many $p \in \mathbb{Z}$. This yields the following definition (cf. [CMSS1]):

Definition 3.1. *The G-equivariant motivic Chern class transformation*

$$MHC^G_y : K_0(MHM^G(Z)) \to K_0(D^{b,G}_{\mathrm{coh}}(Z)) \otimes \mathbb{Z}[y^{\pm 1}] = K_0(\mathrm{Coh}^G(Z)) \otimes \mathbb{Z}[y^{\pm 1}]$$

is defined by:

$$MHC^G_y([\mathcal{M}]) := \sum_p \left[\mathrm{gr}^F_{-p} DR^G(\mathcal{M})\right] \cdot (-y)^p = \sum_{i,p} (-1)^i \left[\mathcal{H}^i(\mathrm{gr}^F_{-p} DR^G(\mathcal{M}))\right] \cdot (-y)^p. \tag{28}$$

The Atiyah–Singer class transformation is defined by the composition

$$T_{y*}(-;g) := td_*(-;g) \circ MHC^G_y, \tag{29}$$

with

$$td_*(-;g) : K_0(Coh^G(Z)) \to H^{BM}_{ev}(Z^g;\mathbb{C}) \tag{30}$$

the Lefschetz–Riemann–Roch transformation (extended linearly over $\mathbb{Z}[y^{\pm 1}]$*).*

3.2. **The case of symmetric products.** In this section we develop the prerequisites needed in the proof of Lemma 2.3.

3.2.1. *Multiplicativity and equivariance of* $gr^F_* DR$.

Lemma 3.2. *Let* X *be a complex quasi-projective variety and fix* $\mathcal{M} \in D^bMHM(X)$*. Then there is a* Σ_r*-equivariant isomorphism of bounded graded objects in* $D^b_{\mathrm{coh}}(X^r)$*:*

$$gr^F_* DR(\mathcal{M}^{\boxtimes r}) \simeq \left(gr^F_* DR(\mathcal{M})\right)^{\boxtimes r}, \tag{31}$$

where the left-hand side underlies the weakly equivariant complex $gr^F_* DR^{\Sigma_r}(\mathcal{M}^{\boxtimes r})$*, and the* Σ_r*-action on the right-hand side is the usual action on exterior products of graded complexes.*

Proof. Since X is quasi-projective, we can assume X is embedded in a smooth complex algebraic variety M. We have $D^b\mathrm{MHM}(X) \simeq D^b\mathrm{MHM}_X(M)$, by using the identification of the category $\mathrm{MHM}(X)$ of mixed Hodge modules on X with the category $\mathrm{MHM}_X(M)$ of mixed Hodge modules on M supported on X ([Sa2, §4]). In this case, Saito's functor $gr^F_* DR$ is given as the graded transformation associated to a filtered de Rham functor

$$DR : D^b\mathrm{MHM}_X(M) \to D^b_{\mathrm{coh},X}F(\mathcal{O}_M, \mathrm{Diff})$$

taking values in Saito's category of bounded filtered differential complexes on M whose graded pieces have coherent cohomology sheaves of $\mathcal{O}_X$-modules ([Sa1, §2.2]). Moreover, this filtered de Rham functor is induced from a corresponding functor of complexes

$$DR : C^b\mathrm{MHM}(M) \to C^bF(\mathcal{D}_M) \to C^bF(\mathcal{O}_M, \mathrm{Diff}),$$

associating to a complex of mixed Hodge modules on M the filtered de Rham complex of the underlying complex of filtered right $\mathcal{D}_M$-modules.

By [MSS, Remark 1.6], for $\mathcal{M} \in C^b\mathrm{MHM}(M)$ there is a canonical map

$$\mathrm{can} : DR(\mathcal{M}^{\boxtimes r}) \to DR(\mathcal{M})^{\boxtimes r}$$

commuting with the corresponding Σ_r-actions as defined in [MSS]. This induces a Σ_r-equivariant map

$$\mathrm{gr(can)} : gr^F_* DR(\mathcal{M}^{\boxtimes r}) \to \left(gr^F_* DR(\mathcal{M})\right)^{\boxtimes r}$$

of the associated graded complexes. Moreover, gr(can) is a (graded) quasi-isomorphism, as can be checked locally using a suitable "locally free" resolution as in [Sa1, Lemma 2.1.17]

For $\mathcal{M} \in C^b\mathrm{MHM}_X(M)$, one has $gr^F_p DR(\mathcal{M}) \in D^b_{\mathrm{coh}}(X)$ for all p, with $gr^F_p DR(\mathcal{M}) \simeq 0$ for all but finitely many $p \in \mathbb{Z}$.

Finally, by the multiple Künneth formula for push-forwards of mixed Hodge modules ([MSS, §1.11]), the induced Σ_r-equivariant isomorphism (31) does not depend on the choice of the embedding.

□

3.2.2. *Lefschetz–Riemann–Roch vs. Adams–Riemann–Roch.* The following result is a generalization of a similar fact proved by Moonen in the case of (the class of) the structure sheaf $\mathcal{O}_X$ (see [Mo, Satz 2.4, p.162]).

Lemma 3.3. *Let σ_r be an r-cycle. Then for any $\mathcal{G} \in D^b_{\mathrm{coh}}(X)$, the following identity holds in $H^{BM}_{ev}(X;\mathbb{Q})$:*

$$td_*(\left[\mathcal{G}^{\boxtimes r}\right]; \sigma_r) = \Psi_r td_*([\mathcal{G}]). \tag{32}$$

Here $\mathcal{G}^{\boxtimes r} \in D^{b,\Sigma_r}_{\mathrm{coh}}(X^r)$ is considered with the induced action of the symmetric group Σ_r, and $X \simeq (X^r)^{\sigma_r}$.

Moonen's proof for the special case $\mathcal{G} = \mathcal{O}_X$ uses an embedding $i : X \hookrightarrow M$ into a smooth complex algebraic variety M, together with a bounded locally free resolution $\mathcal{F}$ of $i_*\mathcal{G}$. Then $i^r : X^r \to M^r$ is a Σ_r-equivariant embedding, with $\mathcal{F}^{\boxtimes r}$ a Σ_r-equivariant locally free resolution of $(i_*\mathcal{G})^{\boxtimes r} \simeq i^r_*(\mathcal{G}^{\boxtimes r})$. For the calculation of $td_*(\left[\mathcal{G}^{\boxtimes r}\right]; \sigma_r)$, one takes suitable traces of the induced σ_r-action on the restriction of $\mathcal{F}^{\boxtimes r}$ to $M \simeq (M^r)^{\sigma_r}$. Such traces are well-defined since σ_r acts trivially on the fixed point set M.

Starting with $\mathcal{G} \in D^b_{\mathrm{coh}}(X)$ instead of $\mathcal{O}_X$, Moonen's proof applies therefore mutatis mutandis to this more general context.

This leads to the following important consequence:

Proposition 3.4. *With the above notations, the following identity holds:*

$$td_*([gr^F_p DR^{\Sigma_r}(\mathcal{M}^{\boxtimes r})]; \sigma_r) = \begin{cases} \Psi_r td_*([gr^F_q DR(\mathcal{M})]) \ , & \text{if } p = q \cdot r \ , \\ 0 \ , & \text{if } p \not\equiv 0 \ mod \ r. \end{cases} \tag{33}$$

Proof. By taking the degree p part in (31), we have that:

$$gr^F_p DR^{\Sigma_r}(\mathcal{M}^{\boxtimes r}) = \bigoplus_{\sum_{j=1}^r q_j = p} gr^F_{q_1} DR(\mathcal{M}) \boxtimes \cdots \boxtimes gr^F_{q_r} DR(\mathcal{M}), \tag{34}$$

where the action of the r-cycle σ_r on the right-hand side is the (graded anti-symmetric) action by cyclic permutations of the factors in the multiple exterior product of complexes (as explained e.g., in [MSS]).

Fix a multi-index $(q_1, \cdots, q_r) \in \mathbb{Z}^r$, with $\sum_{j=1}^r q_j = p$. If $q_1 = \cdots = q_r = q$ (with $p = q \cdot r$), we get by Lemma 3.3 that

$$td_*([gr^F_q DR(\mathcal{M})^{\boxtimes r}]; \sigma_r) = \Psi_r td_*([gr^F_q DR(\mathcal{M})]). \tag{35}$$

Otherwise, the orbit of $(q_1, \cdots, q_r)$ under the permutation action of σ_r on $\mathbb{Z}^r$ has length r. This implies:

$$td_*\left(\left[\bigoplus_{j=1}^r \mathcal{G}_{\sigma_r^j(q_1)} \boxtimes \cdots \boxtimes \mathcal{G}_{\sigma_r^j(q_r)}\right]; \sigma_r\right) = 0 \ ,$$

for $\mathcal{G}_q := gr_q^F DR(\mathcal{M})$, $q \in \mathbb{Z}$. This can be seen as follows: choose an embedding $i : X \hookrightarrow M$ into a smooth complex algebraic variety M, together with a bounded locally free resolution $\mathcal{F}_q$ of $i_*\mathcal{G}_q$ ($q \in \mathbb{Z}$). Then

$$\bigoplus_{j=1}^{r} \mathcal{F}_{\sigma_r^j(q_1)} \boxtimes \cdots \boxtimes \mathcal{F}_{\sigma_r^j(q_r)}$$

is a σ_r-equivariant locally free resolution of

$$i_*^r \left(\bigoplus_{j=1}^{r} \mathcal{G}_{\sigma_r^j(q_1)} \boxtimes \cdots \boxtimes \mathcal{G}_{\sigma_r^j(q_r)} \right) .$$

Let $\Delta_r : M \to M^r$ denote the diagonal embedding, with $\Delta_r(M) \simeq (M^r)^{\sigma_r}$. Then

$$\Delta_r^* \left(\bigoplus_{j=1}^{r} \mathcal{F}_{\sigma_r^j(q_1)} \boxtimes \cdots \boxtimes \mathcal{F}_{\sigma_r^j(q_r)} \right)$$

is a complex, whose components are direct sums of terms of the form

$$\left(\mathcal{F}_{q_1}^{k_1} \otimes \cdots \otimes \mathcal{F}_{q_r}^{k_r} \right) \otimes \left(\oplus_{j=1}^{r} \mathcal{O}_M \right) ,$$

for $\mathcal{F}_q^k$ the k-th degree component of the complex $\mathcal{F}_q$, and with σ_r acting (up to suitable signs) by cyclic permutation of order r on the summands in $\oplus_{j=1}^{r} \mathcal{O}_M$. So the corresponding trace is zero.

Together with the additivity of $td_*(-;\sigma_r)$, this yields (33).

□

3.3. **Proof of Lemma 2.3.** We now have all the ingredients for proving Lemma 2.3.

Proof.

$$\begin{aligned}
T_{(-y)_*}(\mathcal{M}^{\boxtimes r};\sigma_r) &:= td_*(-;\sigma_r) \circ \mathrm{MHC}_{-y}^{\Sigma_r}(\mathcal{M}^{\boxtimes r}) \\
&:= td_*(-;\sigma_r) \left(\sum_p [gr_{-p}^F DR^{\Sigma_r}(\mathcal{M}^{\boxtimes r})] \cdot y^p \right) \\
&= \sum_p td_*([gr_{-p}^F DR^{\Sigma_r}(\mathcal{M}^{\boxtimes r})];\sigma_r) \cdot y^p \\
&\overset{(33)}{=} \sum_q \Psi_r td_*([gr_{-q}^F DR(\mathcal{M})]) \cdot (y^r)^q \\
&= \Psi_r td_* \left(\sum_q [gr_{-q}^F DR(\mathcal{M})] \cdot (y^r)^q \right) \\
&= \Psi_r \left(td_* \circ \mathrm{MHC}_{-y^r}(\mathcal{M}) \right) \\
&= \Psi_r T_{(-y^r)_*}(\mathcal{M}).
\end{aligned}$$

□

4. Comparison with Ohmoto's Chern class formula

In this section we show how Ohmoto's generating series formula (20) for the MacPherson–Chern classes of symmetric products can be derived as a special case of a suitable renormalization of our formula (15). We begin with a general discussion on normalized Hirzebruch classes.

The power series $Q_y(\alpha) = \frac{\alpha(1+ye^{-\alpha})}{1-e^{-\alpha}} \in \mathbb{Q}[y][[\alpha]]$ mentioned in the introduction is not normalized, as its zero-degree part is $1+y$, instead of 1. So one can consider the normalized power series

$$\widehat{Q}_y(\alpha) := \frac{Q_y\left(\alpha(1+y)\right)}{1+y} = \frac{\alpha(1+y)}{1-e^{-\alpha(1+y)}} - \alpha y \tag{36}$$

which defines the normalized cohomology Hirzebruch class $\widehat{T}_y^*(-)$. This specializes to the cohomology Chern, Todd, and L-class, for $y = -1, 0$ and 1, respectively.

In the singular context, the corresponding normalized homology Hirzebruch class transformation $\widehat{T}_{y*}$ is obtained from the transformation T_{y_*} defined in (4) by a simple renormalization procedure (e.g., see [BSY]). More precisely, for a complex algebraic variety Z we let $\widehat{T}_{y*}$ be defined by the composition

$$\widehat{T}_{y*} : K_0(\mathrm{MHM}(Z)) \overset{T_{y_*}}{\to} H_{ev}^{BM}(Z) \otimes \mathbb{Q}[y^{\pm 1}] \overset{\Psi_{(1+y)}}{\to} H_{ev}^{BM}(Z) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}], \tag{37}$$

with the normalization functor $\Psi_{(1+y)}$ given in degree $2k$ by multiplication by $(1+y)^{-k}$. The corresponding (normalized) characteristic classes associated to the (classes of) complexes $\mathbb{Q}_Z^H$ and IC'^H_Z will be denoted here by $\widehat{T}_{y*}(Z)$ and $\widehat{IT}_{y*}(Z)$, respectively. As shown in [Sch2, Proposition 5.21], the transformation $\widehat{T}_{y*}$ of (37) takes in fact values in $H_{ev}^{BM}(Z) \otimes \mathbb{Q}[y^{\pm 1}]$, so one is allowed to specialize the parameter y of the transformation to the values $y = \pm 1$.

It follows from [BSY] that $\widehat{T}_{y*}(Z) \in H_{ev}^{BM}(Z) \otimes \mathbb{Q}[y]$. Moreover, by loc. cit., if $y = -1$ one gets that

$$\widehat{T}_{-1*}(Z) = c_*(Z) \otimes \mathbb{Q} \tag{38}$$

is the rationalized homology Chern class of MacPherson [M]. Also, for a variety Z with at most "Du Bois singularities" (e.g., rational singularities), we have by [BSY, Example 3.2] that

$$\widehat{T}_{0*}(Z) = td_*(Z), \tag{39}$$

the Baum–Fulton–MacPherson homology Todd class [BFM]. And it is only conjectured that if Z is a compact algebraic variety, then

$$\widehat{IT}_{1*}(Z) = L_*(Z) \tag{40}$$

is the Goresky–MacPherson L-class of Z (cf. [BSY, Remark 5.4]). This conjecture is known to hold in some special cases, e.g., if Z has a small resolution (cf. [Sch2, §5.1]), or if $Z = Y/G$ is a global projective orbifold (cf. [CMSS1, Corollary 1.2]), or if Z is a compact complex algebraic variety with only isolated singularities, which is a rational homology manifold that can be realized as a global hypersurface in a complex algebraic manifold (cf. [CMSS2, §4]).

For simplicity, the main results of this note are formulated only in terms of the unnormalized Hirzebruch class transformation of (4). However, for the purpose of comparing our formula (15) with Ohmoto's generating series formula (20) for the MacPherson-Chern classes of symmetric products of a quasi-projective variety [O], we need to say a few words about the normalized version of our formula (15).

By applying the normalization functor $\Psi_{(1-y)}$ (note that due to our indexing conventions, y is replaced here by $-y$) to the left-hand side of (15), we get the generating series $\sum_{n\geq 0} \widehat{T}_{(-y)_*}(X^{(n)})$. Applying the same procedure to the right-hand side of (15), we first note that the normalization functor $\Psi_{(1-y)}$ commutes with push-forward for proper maps, as well as with exterior products, therefore $\Psi_{(1-y)}$ also commutes with the Pontrjagin product and the exponential; finally, it also commutes with the homological Adams operation Ψ_r of Theorem 1.1. But $\Psi_{(1-y)}T_{(-y^r)_*}(X)$ is not in general equal to $\widehat{T}_{-t*}(X)|_{t=y^r}$. Only in the case $y = 1$ we get the following:

Lemma 4.1. *With the above notations, the following identification holds:*

$$\lim_{y\to 1} \Psi_{(1-y)}\Psi_r T_{(-y^r)_*}(X) = \widehat{T}_{-1*}(X) = c_*(X) \otimes \mathbb{Q}. \tag{41}$$

Before giving the proof, we need to recall from [BSY] that, if Z is a complex algebraic variety, then the class of the constant Hodge sheaf $[\mathbb{Q}^H_Z]$ is in the image of the natural group homomorphism

$$\chi_{\mathrm{Hdg}} : K_0(var/Z) \to K_0(\mathrm{MHM}(Z)) \ , \ [f : Y \to Z] \mapsto [f_!\mathbb{Q}^H_Y] \ , \tag{42}$$

defined on the relative Grothendieck group $K_0(var/Z)$ of complex algebraic varieties over Z. Indeed, $\chi_{\mathrm{Hdg}}([id_Z]) = [\mathbb{Q}^H_Z]$. Therefore, the corresponding homology Hirzebruch class $T_{y_*}(Z)$ can be regarded as the image of the distinguished element $[id_Z] \in K_0(var/Z)$ under the natural motivic Hirzebruch transformation (cf. [BSY])

$$T_{y_*} : K_0(var/Z) \to H^{BM}_{ev}(X; \mathbb{Q}[y]) \tag{43}$$

defined by pre-composing (4) with the group homomorphism χ_{Hdg}. Similarly, one can consider the normalized motivic Hirzebruch transformation $\widehat{T}_{y*} := \Psi_{(1+y)} \circ T_{y_*}$, which maps $[id_Z]$ to the corresponding normalized Hirzebruch class $\widehat{T}_{y*}(Z) := \widehat{T}_{y*}([id_Z])$.

We next recall that the MacPherson–Chern class of an algebraic variety Z is defined by $c_*(Z) := c_*(1_Z)$, with $c_* : F(Z) \to H^{BM}_{ev}(Z)$ the Chern class transformation of MacPherson [M] defined on the group $F(Z)$ of complex algebraically constructible functions.

Lemma 4.1 follows by applying the following identity of transformations to the distinguished element $[id_X] \in K_0(var/X)$.

Lemma 4.2. *With the above notations, the following identification of transformations holds:*

$$\lim_{y\to 1} \Psi_{(1-y)}\Psi_r T_{(-y^r)_*}(-) = \widehat{T}_{-1*}(-) : K_0(var/X) \to H^{BM}_{ev}(X; \mathbb{Q}) \ . \tag{44}$$

Proof. Since both sides of (44) are defined by functorial group homomorphisms, this identity can be checked on generators. So, by functoriality for proper push-downs, it suffices to check it in the case when X is smooth. In this case, we can perform our calculations dually, in cohomology. Formula (44) follows now from a simple manipulation with power series, the main steps of which are sketched below.

The un-normalized cohomology Hirzebruch class $T^*_y(X)$ is defined by the power series $Q_y(\alpha) = \frac{\alpha(1+ye^{-\alpha})}{1-e^{-\alpha}}$, hence $T^*_{(-y^r)}(X)$ corresponds to the power series $Q_{(-y^r)}(\alpha)$. By applying the cohomological r-th Adams operation to $Q_{(-y^r)}(\alpha)$, we get the power series $f(\alpha) := \frac{\alpha(1-y^r e^{-r\alpha})}{1-e^{-r\alpha}}$. The cohomological version of the normalization coming from the left-hand side of formula (15) amounts to replacing $f(\alpha)$ with the power series $\frac{f(\alpha(1-y))}{1-y}$. The limit of the latter, as $y \to 1$, yields by l'Hôpital's rule:

$$\lim_{y\to 1} \frac{\alpha\left(1-y^r e^{-r\alpha(1-y)}\right)}{1-e^{-r\alpha(1-y)}} = \lim_{y\to 1} \frac{\alpha\left(-ry^{r-1}-r\alpha y^r\right)e^{-r\alpha(1-y)}}{-r\alpha e^{-r\alpha(1-y)}} = \frac{-r-r\alpha}{-r} = 1+\alpha$$

i.e. the power series defining the total Chern class in cohomology. This finishes the proof of (44).

□

Therefore, by specializing to $y = 1$ in our formula (15), we recover as a corollary Ohmoto's Chern class formula [O]:

Corollary 4.3. *For any quasi-projective complex algebraic variety X, the following formula holds in $\sum_{n\geq 0} H^{BM}_{ev}(X^{(n)};\mathbb{Q})\cdot t^n$:*

$$\sum_{n\geq 0} c_*(X^{(n)})\cdot t^n = \exp\left(\sum_{r\geq 1} d^r_* c_*(X)\cdot \frac{t^r}{r}\right), \tag{45}$$

with $c_*(-)$ denoting the rationalized homology Chern classes of MacPherson [M].

The above arguments can be extended to obtain a generating series formula for the rationalized MacPherson–Chern classes of symmetric products of a constructible sheaf complex $\mathcal{F}$ underlying a complex of mixed Hodge modules $\mathcal{M}$. For this, we use the commutativity of the following diagram (see [Sch2, Proposition 5.21]):

$$\begin{tikzcd} K_0(\mathrm{MHM}(X)) \arrow[r, "rat"] \arrow[d, "\widehat{T}_{-1*}"'] & K_0(D^b_c(X)) \arrow[d, "\chi_{\mathrm{stalk}}"] \\ H^{BM}_{ev}(X;\mathbb{Q}) & F(X) \arrow[l, "c_*\otimes\mathbb{Q}"] \end{tikzcd} \tag{46}$$

Here, $rat : D^b\mathrm{MHM}(X) \to D^b_c(X)$ is the forgetful functor associating to a complex of mixed Hodge modules the underlying constructible sheaf complex, and χ_{stalk} is defined by taking the Euler characteristics of the stalk complexes. Then we have:

Corollary 4.4. *For any quasi-projective complex algebraic variety X and $\mathcal{F} = rat(\mathcal{M})$ the underlying constructible sheaf complex of a complex of mixed Hodge modules $\mathcal{M} \in D^bMHM(X)$, the following formula holds in $\sum_{n\geq 0} H^{BM}_{ev}(X^{(n)};\mathbb{Q})\cdot t^n$:*

$$\sum_{n\geq 0} c_*(\mathcal{F}^{(n)})\cdot t^n = \exp\left(\sum_{r\geq 1} d^r_* c_*(\mathcal{F})\cdot \frac{t^r}{r}\right)\ , \tag{47}$$

with $c_*(\mathcal{F}) := c_*(\chi_{\mathrm{stalk}}(\mathcal{F}))$.

The proof is exactly the same as above, based on the fact that the functor rat commutes with symmetric products (see [MSS]), together with the identification of transformations:

$$\lim_{y\to 1} \Psi_{(1-y)}\Psi_r T_{(-y^r)_*}(-) = \widehat{T}_{-1*}(-) : K_0(\mathrm{MHM}(X)) \to H^{BM}_{ev}(X;\mathbb{Q}) \tag{48}$$

which follows from combining (44) with the proof of [Sch2, Proposition 5.21].

Finally, by using the localized Chern class transformation

$$c_*(-;g) : K_0(D^{b,G}_c(Z;\mathbb{C})) \to H^{BM}_{ev}(X^g;\mathbb{C})$$

of [Sch1][Ex.1.3.2], with $D^{b,G}_c(Z;\mathbb{C})$ the category of G-equivariant objects in the derived category of constructible sheaf complexes of $\mathbb{C}$-vector spaces, one can formally adapt our proof of Theorem 1.1 to give a direct proof of formula (47) for any constructible sheaf complex $\mathcal{F} \in D^b_c(X;\mathbb{C})$ on a quasi-projective algebraic variety X. Details of this will be discussed elsewhere.

References

[BFM] P. Baum, W. Fulton and R. MacPherson, *Riemann–Roch for singular varieties*, Inst. Hautes Études Sci. Publ. Math. **45** (1975), 101–145.

[BFQ] P. Baum, W. Fulton and G. Quart, *Lefschetz–Riemann–Roch for singular varieties*, Acta Math. **143** (1979), no. 3-4, 193–211.

[BL] L. Borisov and A. Libgober, *Elliptic genera of singular varieties, orbifold elliptic genus and chiral de Rham complex*, in Mirror symmetry, IV (Montreal, QC, 2000), 325-342, AMS/IP Stud. Adv. Math., 33, Amer. Math. Soc., Providence, RI, 2002.

[BSY] J. P. Brasselet, J. Schürmann and S. Yokura, *Hirzebruch classes and motivic Chern classes of singular spaces*, Journal of Topology and Analysis, **2** (2010), No. 1, 1–55.

[CMS] S. E. Cappell, L. Maxim and J. L. Shaneson, *Hodge genera of algebraic varieties, I.*, Comm. Pure Appl. Math. **61** (2008), No. 3, 422–449.

[CMSS1] S. E. Cappell, L. Maxim, J. Schürmann and J. L. Shaneson, *Equivariant characteristic classes of complex algebraic varieties*, arXiv:1004.1844.

[CMSS2] S. E. Cappell, L. Maxim, J. Schürmann and J. L. Shaneson, *Characteristic classes of complex hypersurfaces*, arXiv:0908.3240, Advances in Mathematics (in print), doi:10.1016/j.aim.2010.05.007

[Che] J. Cheah, *On the cohomology of Hilbert schemes of points*, J. Algebraic Geom. **5** (1996), No. 3, 479–511.

[FL] W. Fulton and S. Lang, *Riemann–Roch algebra.* Grundlehren der Mathematischen Wissenschaften, 277. Springer-Verlag, New York, 1985. x+203 pp.

[H] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer, 1966.

[M] R. MacPherson, *Chern classes for singular algebraic varieties*, Ann. of Math. (2) **100** (1974), 423–432.

[Mo] B. Moonen, *Das Lefschetz–Riemann–Roch Theorem für singuläre Varietäten*, Bonner Mathematische Schriften **106** (1978), viii+223 pp.

[Mac] I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, Proc. Cambridge Philos. Soc. **58**, 1962, 563–568.

[MSa] L. Maxim, J. Schürmann, *Twisted genera of symmetric products*, arXiv:0906.1264.

[MSb] L. Maxim, J. Schürmann, *Hirzebruch invariants of symmetric products*, to appear in the Proceedings of Lib60ber.

[MSS] L. Maxim, M. Saito, J. Schürmann, *Symmetric products of mixed Hodge modules*, preprint.

[N] M. Nori, *The Hirzebruch-Riemann-Roch theorem*, Michigan Math. J. 48 (2000), 473-482.

[O] T. Ohmoto, *Generating functions for orbifold Chern classes I: symmetric products*, Math. Proc. Cambridge Philos. Soc. **144** (2008), No. 2, 423–438.

[Sa1] M. Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849-995.

[Sa2] M. Saito, *Mixed Hodge Modules*, Publ. Res. Inst. Math. Sci. **26** (1990), No. 2, 221–333.

[Sch1] J. Schürmann, *A general construction of partial Grothendieck transformations*, arXiv:math/0209299

[Sch2] J. Schürmann, *Characteristic classes of mixed Hodge modules*, to appear in "Topology of Stratified Spaces", MSRI Publications Vol. **58**, Cambridge University Press (2010) , arXiv:0907.0584.

[SY] J. Schürmann and S. Yokura, *A survey of characteristic classes of singular spaces*, in "Singularity Theory" (Denis Cheniot et al, ed.), Dedicated to Jean-Paul Brasselet on his 60th birthday, Proceedings of the 2005 Marseille Singularity School and Conference, World Scientific (2007), 865–952,

[Yau] D. Yau, *Lambda-Rings*, World Sci. Publ., 2010.

[Yo] S. Yokura, *Motivic characteristic classes*, to appear in "Topology of Stratified Spaces", MSRI Publications Vol. **58**, Cambridge University Press (2010), arXiv:0812.4584

[Za] D. Zagier, *Equivariant Pontrjagin classes and applications to orbit spaces. Applications of the G-signature theorem to transformation groups, symmetric products and number theory*, Lecture Notes in Mathematics, Vol. 290. Springer-Verlag, Berlin-New York, 1972.

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