Symmetries of Surfaces

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§ 1 Introduction (1.1) Let Σ_g denote a compact orientable surface of genus g. Without explicit mention we shall only consider the effective and orientation-preserving actions of finite groups G on Σ_g. For brevity we shall simply say that G acts on Σ_g or that G is a symmetry group of Σ_g. The cases $g=0$ and 1 are well-known. For the latter case cf. for example [3], ch. 19. So henceforth we shall also assume $g \geq 2$. In this case it is well-known that each Σ_g admits only finitely many symmetry groups. Some experimentation shows however that their enumeration runs into difficult number-theoretic and finite-group-theoretic problems. In this note we investigate the problem in the reverse direction, namely we fix a finite group G and attempt an enumeration of those g's for which G is a symmetry group of Σ_g. It turns out that the 2-Sylow subgroup G_2 plays a special role. According to a certain structural property of G_2, cf. (2.2), G's are divided into two types I and II. If p^{n_p} is the order of a p-Sylow subgroup G_p of G and p^{e_p} is the exponent of G_p then the integer $n_p - e_p$ is called the cyclic p-deficiency of G.

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(1.2) Main Theorem Let G be a finite group. There exists an integer N depending only on the cyclic p-deficiencies of G with the following property.

1) If G acts on Σ_g then $g\equiv 1\ (N)$, (resp. $(2N)$), according as G is of type I (resp. II).

2) Conversely, if G is of type I, (resp. II), and $g\equiv 1\ (N)$, (resp. $(2N)$), then G acts on Σ_g except for finitely many exceptional values of g.

For the definition of N see (2.1). The exceptions mentioned in 2) depend on the solvability in non-negative integers of a certain linear diophantine equation. This equation depends in an essential way on the structure of the group.

(1.3) The method of proof of (1.2) combined with some results from the finite group theory provides some precise information on surface-symmetries. The sophisticated results from the finite group theory which we mention follow easily from the classification of finite simple groups. But partly they were in fact some of the initial steps in this classification program.

(1.3.1) G is a symmetry group of almost all (i.e. all but finitely many) Σ_g iff G_p is cyclic for p odd and G_2 is cyclic, dihedral, generalized quaternion or semi-dihedral.

Curiously, the family of groups mentioned is this statement includes the finite subgroups of SO(3) and SU(2) , the finite groups acting freely on complexes homotopic to spheres, the semi-direct products of two finite cyclic groups of co-prime orders etc. The only perfect groups among these are $SL_2(\mathbb{F}_q)$ and $PSL_2(\mathbb{F}_q)$, q=a prime ≥ 5 .

(1.3.2) The only simple symmetry groups of $\sum_g$, g even, come from

a) three infinite families : i) $PSL_2(\mathbb{F}_q)$, q odd ≥ 5 ,
ii) $PSL_3(\mathbb{F}_q)$, $q \equiv -1(4)$, and iii) $PSU_3(\mathbb{F}_q)$, $q \equiv 1(4)$,
and
b) two special groups : i) A_7 , and ii) M_{11}.

To get an overall understanding of the simple symmetry groups of $\sum_g$ it is useful to consider the subsets $S(r) = \{g \mid g \not\equiv 1 \ (2^r)\}$. We have $S(1) \subseteq S(2) \subseteq \ldots$.. and the result mentioned above refers to $g \in S(1)$. For each fixed r the simple symmetry groups of $\sum_g$, $g \in S(r)$, again come from
i) finitely many infinite families of Lie type where the congruences on q are dictated by the fact that the cyclic 2-deficiency is bounded by r , and ii) finitely many alternating, sporadic and other classical groups.

(1.4) Besides the structural properties of p-groups acting on $\sum_g$ which follow from (1.2) we also derive bounds on their order by a Hurwitz's-type analysis, cf. (3.5). The bounds for p-groups are substantially better than Hurwitz's "84 (g-1)" which applies to all symmetry groups of Σ_g. This considerably narrows down a list of possible p-groups acting on Σ_g, and in certain cases actually leads to their classification. (For example the p-groups acting on Σ_2 are $\mathbb{Z}_3, \mathbb{Z}_5$ and subgroups of SD_{16}; those on Σ_4 are $\mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_9, \mathbb{Z}_3 \times \mathbb{Z}_3$ and subgroups of SD_{32}). It seems possible to carry out the classification of simple groups or p-groups acting on Σ_g for small values of g.

(1.5) There are some purely finite-groups-theoretic consequences of this method. So far, they have been rather elementary but their "geometric" proofs are intriguing. For example

(1.5.1) A finite perfect group is of type I.

This result appears to be new. D.Gorenstein pointed out to me a purely group-theoretic proof of (1.5.1) using transfer. It appears that certain arguments in finite group theory can be interpreted geometrically in terms of realizability of a branch-data for a finite-group-action on a surface.

(1.6) A historical note: The first results on surface-symmetries $(g \geq 2)$ are due to Schwarz, Klein, von Dyck and Hurwitz. Perhaps the most celebrated among these is the Hurwitz's "84(g-1) theorem" cf. [6]. Partially more precisely Wiman [19], 20] showed that a cyclic symmetry group of Σ_g has order at most $4g+2$ and the bound is attained for each g. Accola [1] and Maclachlan [13] independently showed that $8g+2$ is the minimum of the max $|G|$ where G is a symmetry group of $\sum_g$, and for infinitely many g this is actually the max $|G|$ also. The Hurwitz's bounds are attained for infinitely many g, but the genera for which they are attained are very thinly distributed cf. [11], [8]. There is a vast literature on surface-symmetries expressed in the language of fuchsian groups, cf. e.g. [4], [8], [11], [12], [17] and the references there. The ubiquitous PSL_2's abundantly make their appearance in this literature. Nielsen cf. [15] obtained a topological classification of cyclic symmetry groups. This work is partially extended in [5], [18]. The algebraic geometers have studied surface-symmetries in the context of automorphism groups of function-fields in one variable cf. e.g. [9], [10], [19], [20]. Surface-symmetries also account for the singularities of the moduli spaces of Riemann surfaces, [7], [16].

(1.7) Acknowledgement I would like to thank Professors Gorenstein and Lyons for sharing their insights in finite simple groups.

§ 2 Congruences satisfied by the genera

(2.1) Let G be a finite group of order d. Let δ (resp. Δ) denote the set of all primes dividing d (resp. prime-powers q such that G contains an element of order q). Let G_p denote a Sylow p-subgroup of G. The maximum of the orders of the elements of G_p is called the *p-exponent* of G. Throughout we write $|G_p| = p^{n_p}$ and the p-exponent $= p^{e_p}$. The number $n_p - e_p$ is called the *cyclic p-deficiency* of G. Set

(2.1.1) $$f_p = \begin{cases} n_p - e_p, & \text{if } p \text{ is odd, or } p = 2 \ \& \ n_2 = e_2 \\ n_2 - e_2 - 1, & \text{if } p = 2 \text{ and } n_2 > e_2 \end{cases}$$

and

(2.1.2) $$N = \prod_{p \in \delta} p^{f_p}$$

(2.2) A finite 2-group G is said to be of *type I* if it is either cyclic or else the elements of order $< 2^{e_2}$ do not form a subgroup of index 2. Otherwise G is said to be of *type II.* In other words G is of type II if it is not cyclic and there exists a homomorphism $\varphi : G \longrightarrow \mathbb{Z}_2 = \{0,1\}$ such that $\varphi^{-1}(1)$ consists of elements of order 2^{e_2} and $\varphi^{-1}(0)$ consists of elements of order $< 2^{e_2}$. More generally a finite group G will be said to be of type I (resp. II) if G_2 is of type I (resp. II).

(2.3) *Theorem* Let G be a finite group. Using the notation and terminology introduced above we let S denote the set of all integers ≥ 2 and $\equiv 1(N)$ (resp. $(2N)$) if G is of type I (resp. II). Then

a) G acts on $\Sigma_g, g \geq 2 \Rightarrow g \in S$

b) For all but finitely many $g \in S$, G acts on Σ_g.

(2.4) The proof of (2.3) is divided into several parts. First note that if G acts on Σ_g than $G\backslash\Sigma_g \approx \Sigma_h$ for some h. Let $p:\Sigma_g \longrightarrow \Sigma_h$ denote the canonical projection and call

$$(2.4.1) \qquad S = \{x \in \Sigma_g \mid G_x \neq \{1\}\}$$

the <u>singular set</u> of the G-action. The set $B = p(S)$ is called the <u>branch set</u> of the G-action. For $x \in B$ and any $y \in p^{-1}(x)$ the integer $n_x = |G_y|$ is called the <u>branching-index</u> at x. The well-known Riemann-Hurwitz relation reads

$$(2.4.2) \qquad 2 - 2g = d\{2 - 2h - \sum_{x \in B} (1 - \frac{1}{n_x})\} .$$

(2.5) First consider the case when G is a p-group. Write n, e, f for n_p, e_p, f_p respectively. If G acts on Σ_g and there are a_j branch points of branching index p^j then (2.4.2) takes the form

$$(2.5.1) \qquad 2 - 2g = p^n(2 - 2h) - \sum_{j=1}^{e} (p^j - 1)p^{n-j}a_j$$

This shows that if p is odd then p^{n-e} divides $g - 1$ and if $p = 2$ and $n > e$ then 2^{n-e-1} divides $g - 1$. By the definitions of f and N it follows that $g \equiv 1\,(N)$. In the general case when G is not necessarily a p-group the same result follows from the existence of Sylow p-subgroups and the Chinese remainder theorem.

(2.6) Now suppose that G is a 2-group of type II. We again write n,e,f for n_2,e_2,f_2 respectively. By (2.5) we know that $g \equiv 1(2^f)$. Our claim now is that in fact $g \equiv 1(2^{1+f})$. Suppose this is not the case i.e. $g \not\equiv 1(2^{1+f})$. Then (2.5.1) shows that a_e is odd. By the well-known description of finite-group actions on surfaces going back to Hurwitz cf. [3] p. 420 it follows that a necessary and sufficient condition for the existence of such a G-action is the existence of generators u_i, v_i $1 \leq i \leq h$ and x_{jk}, $1 \leq j \leq e$, $1 \leq k \leq a_j$ of G such that x_{jk} has order 2^j and

(2.6.2) $$\prod_{i=1}^{h} (u_i, v_i) \prod_{j=k}^{e} \prod_{k=1}^{a_j} x_{jk} = 1.$$

By hypothesis there exists a homomorphism $\varphi: G \longrightarrow \mathbb{Z}_2 = \{0,1\}$ such that $\varphi^{-1}(1)$ consists of elements of order 2^e whereas $\varphi^{-1}(0)$ consists of elements of order $< 2^e$. It is also clear that $\varphi^{-1}(0)$ contains all commutators. But now applying φ to both sides of (2.6.2) and noting again that a_e is odd, we clearly get a desired contradiction. This proves the part a) of (2.3).

(2.7) Now we proceed to prove the part b) of (2.3). For $n \in \Delta$ set $n' = d/n$. The formula (2.4.2) suggests considering the diophantine equation

$$(2.7.1) \qquad \sum_{n\in\Delta} (n-1)n'x_n + 2dy = 2(z-1) + 2d$$

to be solved for x_n, y and z in non-negative integers. The g.c.d. of the coefficients of the L.H.S. of (2.7.1) is N if $n_2 = e_2$ and $2N$ if $n_2 > e_2$. So for a solution of (2.7.1) we must have $z \equiv 1(N)$. It is an elementary fact from number theory that if $g \equiv 1(N)$ and is sufficently large then (2.7.1) with $z = g$ can be solved for x_n and y in non-negative integers, or more generally x_n and y can be chosen to be greater than some fixed values. It would suffice to show that if $z = g$, $x_n = a_n$, $y = h$ is such a solution of (2.7.1), and $g \in S$, then we can choose the elements u_i, v_i, $x_{n1}, x_{n2}, \ldots, x_{na_n}$, $1 \leq i \leq h$, $n \in \Delta$ in G so that (2.6.2) holds. The lower bound for the values of g and a_n needed for this purpose will be indicated in the course of the proof.

(2.8) First set $u_i = v_i = 1$. Next note that G certainly admits a set of generators whose orders are prime-powers. If $\{e_1, \ldots, e_s\}$ is such a system we make sure that (2.6.2) contains a segment of the form $e_1e_1^{-1}e_2e_2^{-1} \ldots e_se_s^{-1}$ ⊕ In this process for each $n \in \Delta$ we have used 0 or 2 slots out of the a_n allowable slots.

(2.9) Now suppose n is a power of an odd prime p. If a_n is even then we again insert segments of the form $u \cdot u^{-1}$ in (2.6.2). If a_n is odd, then we assume that $a_n \geq 5$. So after possibly using 2 slots for the procedure described in

⊕Note that the order of the factors in (2.6.2) is immaterial.

(2.8) at least 3 slots remain. Now note that an element of order p^r, p = an odd prime, can always be written as a product of two elements of order p^r. So the remaining slots can be filled in by a segment of elements of order n whose product is identity.

(2.10) Now let n be a power of 2. Let G_2 be a Sylow 2-subgroup and write $r = |G_2|$ and s = the 2-exponent of G. We need to consider 3 cases.

Case 1) $r = s$: Then G_2 is cyclic. The coefficient of x_r in (2.7.1) is odd, and it is the only odd coefficient. So a_r must be even. In G_2 any element of order $< r$ may be written as product of two generators of G_2. So if some a_t, $t = 2^j < r$ is odd we use two slots in a_r and one in a_t to insert in (2.6.2) a segment of three elements (one of order t and two of order r) whose product is identity. The remaining $a_t - 1$ slots may be filled by $u \cdot u^{-1}$-type segments. This process repeated for each $t = 2^j < r$ takes up only even number of slots in a_r. (So we need to ensure that a_r is sufficiently large for this purpose). The remaining of the a_r slots may be filled by $u \cdot u^{-1}$-type segments.

Case 2) $r > s$, G of type 2: Then by choice $g \equiv 1\ (2N)$ and so a_r is even hence the process described in case 1 may be repeated.

Case 3) $r > s$, G of type 1: If a_r is even, again we can proceed as above. So suppose a_r is odd. This occurs precisely when $g \equiv 1(N)$ but $g \not\equiv 1(2N)$. Let P be the subgroup of G_2 generated by elements of order $<r$. Clearly P is normal and G_2/P is an elementary abelian 2-group. If $P = G_2$ then an element of order r can be expressed as a product of elements of order $< r$. So we can use one slot in a_r and certain others in a_t, $t = 2^j < r$ to be filled by a segment of elements (one of order r and the others of order $< r$) whose product is identity. If $P \neq G_2$ then it has index at least 4. So it is possible to express an element of order r as a product of two elements of order r. Thus we can use 3 slots in a_r to be filled by elements of order r whose product is identity. In any case there are even number of slots remaining in a_r. So we can proceed further if necessary as in case 1).

This finishes the proof of the theorem (2.3).

(2.11) Remark: For a precise enumeration of g such that G acts on Σ_g we shall need to solve the equation analogous to (2.7.1) where Δ is replaced by the set of all orders of elements of G.

§ 3 Applications

(3.1) Proposition Let G be a symmetry p-group of $\sum_g$, and $p \nmid (g-1)$.

a) If p is odd then G is cyclic.

b) If $p = 2$ then G is cyclic, dihedral, generalized quaternion or semi-dihedral.

(Recall that by definition, a dihedral group

$$D_{2^{n+1}} = \langle x,y \mid x^{2^n} = e = y^2,\ yxy^{-1} = x^{-1} \rangle \ ,\ n \geq 1 \ ,$$

a generalized quaternion group

$$Q_{2^{n+1}} = \langle x,y \mid x^{2^n} = e,\ x^{2^{n-1}} = y^2,\ yxy^{-1} = x^{-1} \rangle \ ,\ n \geq 2 \ ,$$

a semi-dihedral group

$$SD_{2^{n+1}} = \langle x,y \mid x^{2^n} = e = y^2,\ yxy^{-1} = x^{2^{n-1}-1} \rangle,\ n \geq 3 \ ,$$

and a twisted dihedral group

$$S_{*}D_{2^{n+1}} = \langle x,y \mid x^{2^n} = e = y^2 \ ,\ yxy^{-1} = x^{2^{n-1}+1} \rangle \ ,\ n \geq 3 \ .)$$

Proof The hypotheses imply that in the notation of (2.1) $f_p = 0$. So

G has cyclic deficiency 0 if p is odd and ≤ 1 if $p=2$. So G_p is cyclic for p odd. The 2-groups with cyclic deficiency ≤ 1 are well-known. Besides those stated in the proposition they are $\mathbb{Z}_{2^n} \oplus \mathbb{Z}_2$, $n \geq 2$ and $S_* D_{2^{n+1}}$, $n \geq 3$. A calculation for both of these types shows that they are of type II, and so by (2.3) they cannot act on $\sum_g$, if g is even.

q.e.d.

(3.2) <u>Proposition</u> Let G be a p-group acting on $\sum_g$ and p^r the highest power of p dividing $g-1$. Then the cyclic deficiency of G is $\leq r$ if p is odd, and $\leq r+1$ if $p=2$. In particular if G is an elementary abelian p-group of rank s then $s \leq r+1$ if p is odd, and $\leq r+2$ if $p=2$.

<u>Proof</u> The first assertion follows immediately from (2.3). If G is an elementary abelian p-group of rank s then its cyclic deficiency is $s-1$. Hence the second assertion.

q.e.d.

(3.3) <u>Proposition</u> Let G be a cyclic p-group acting on $\sum_g$ then $|G| \leq \frac{2p}{p-1} g$.

<u>Proof</u> Let ${}_G\backslash\sum_g \approx \sum_h$ with a_j branch points of branching index p^j. Then by the Riemann-Hurwitz formula

$$2-g = |G| \{(2-2h) - \sum_j a_j (1 - \frac{1}{p^j})\} .$$

If $h \geq 2$ then $2-2g \leq |G|\ (2-2h) \leq -2|G|$ or $|G| \leq g-1$. If $h=1$ there are at least two branch points and we have $|G| \leq \frac{p}{p-1}(g-1)$. Now suppose $h=0$. Then there are at least 3 branch points and at least two branching indices equal $|G|$. So

$$2-2g \leq |G|\ \{2-2(1-\frac{1}{|G|}) - (1-\frac{1}{p})\} = 2-|G|\ \frac{p-1}{p}$$

or $|G| \leq \frac{2p}{p-1}\,g$.

q.e.d.

(3.4) <u>Proposition</u> Let G be a non-cyclic p-group acting on $\sum_g$. Let $k \geq 2$ be the minimum number of generators of G . Then

$$|G| \leq \begin{cases} 16\ (g-1) & \text{if } k=2\ ,\ p=2, \\ 9\ (g-1) & \text{if } k=2\ ,\ p=3, \\ 8\ (g-1) & \text{if } k=3\ ,\ p=2, \\ \dfrac{2p\ (g-1)}{(k-1)p-(k+1)} & \text{otherwise}. \end{cases}$$

<u>Proof</u> Let $G\backslash\sum_g \approx \sum_h$. If $k \leq 2h-1$ then $h \geq 2$ and by the Riemann-Hurwitz formula we get

$$2-2g \leq |G|\ \{2-2h\} \leq |G|\{1-k\} \quad \text{i.e.} \quad |G| \leq \frac{2}{k-1}\,(g-1)\ .$$

So suppose $k \geq 2h$. We now use the fact that a minimal set of generators of a p-group maps injectively onto a minimal set of generators of its commutator quotient. So we see that there must be at least $k-2h+1$ branch points. So

$$2-2g \le |G| \{2-2h - (k-2h+1) (1-\frac{1}{p})\} , \text{ or}$$

$$(3.4.1) \quad |G| \le \frac{2p(g-1)}{(k-1)p-(k-2h+1)} \le \frac{2p(g-1)}{(k-1)p-(k+1)}$$

where the inequalities are valid as long as the denominators are positive. The denominator of the last term is already positive for all pairs (p,k) except $(p,k) = (3.2), (2.3)$ and (2.2). We analyze these exceptional cases separately.

case i), $P=3$, $k=2$: If $h \ge 1$ the denominators of the middle term of (3.4.1) is positive. This gives an estimate $|G| \le 3(g-1)$. Now suppose $h=0$. Then there are at least 3 branch points. If there are ≥ 4 branch points we get an estimate

$$2-2g \le |G| \{2-4(1-\frac{1}{3})\} \quad \text{or} \quad |G| \le 3(g-1) .$$

If there are 3 branch points then the branching indices $\{3,3,9\}$ give the best upper estimate

$$2-2g \le |G| \{2-2(1-\frac{1}{3})-(1-\frac{1}{9})\} , \text{ or } \quad |G| \le 9(g-1)$$

case ii), $p=2$, $k=3$: If $h \ge 1$ the middle estimate in (3.4.1) gives $|G| \le 2(g-1)$. If $h=0$ then there are at least 4 branch points, and the branching indices $\{2,2,2,4\}$ give the best upper estimate $|G| \le 8(g-1)$.

case iii), $p=2$, $k=2$: The analysis as in the above cases shows that the branching indices $\{2,4,8\}$ give the best upper estimate $|G| \leq 16(g-1)$.

q.e.d.

(3.5) Corollary If G is a p-group acting on Σ_g then

$$|G| \leq \begin{cases} 16\,(g-1) & \text{if } p=2, \\ 9\,(g-1) & \text{if } p=3, \\ \frac{2p}{p-3}\,(g-1) & \text{if } p\geq 5. \end{cases}$$

Moreover for each prime p these bounds are attained for infinitely many values of g , and also not attained for infinitely many values of g .

Proof Comparing (3.3) and (3.4) we see that for $p=2$ and 3 bounds for the 2-generator groups are higher than those for k-generator groups, $k \neq 2$. For $p\geq 5$ one sees that the bound $\frac{2p}{p-3}(g-1)$ for a 2-generator group is higher than that for a k-generator group, $k\geq 3$, and moreover $\frac{2p}{p-1}\,g \leq \frac{2p}{p-3}(g-1)$ as long as $p\leq 2g+1$. On the other hand (3.3) applied to $G=\mathbb{Z}_p$ does imply $p\leq 2g+1$. So $\frac{2p}{p-3}(g-1)$ is the estimate for $|G|$ for $p\geq 5$. The proof of (3.3), (3.4) also shows that the bounds are attained precisely when $G\backslash\Sigma_g \approx \Sigma_0$, with three branch points with branching indices (2,4,8) for $p=2$, (3,3,9) for $p=3$, and (p,p,p) for $p\geq 5$. Interpreted in terms of fuchsian groups this means that the bounds are attained for G's which are quotients of the Schwarz triangle groups $\Delta_{2,4,8}$ for $p=2$, $\Delta_{3,3,9}$ for $p=3$ and $\Delta_{p,p,p}$ for $p\geq 5$ with torsion-free kernel. One easily sees that there are infinitely many such possibilities. Also there are infinitely many genera not admitting such actions since existence of such actions implies non-trivial congruences on g. (In fact the genera admitting such actions have natural density zero in the set of positive integers.)

q.e.d.

(3.6) Remark The p-groups with cyclic deficiency ≤ 2 are known, cf. [3], [14]. So if $g-1$ is not divisible by 4 or by p^3, p odd, then using (3.1)-(3.5) one has a small list of p-groups using which one can actually enumerate the p-groups acting on $\sum_g$. Similar remarks apply to the actions of finite simple groups on $\sum_g$, cf. (1.3), and (3.7) below. (The topological equivalence classes of the actions can be determined by Hurwitz's procedure, cf. (2.6). Each such topological equivalence class defines a certain subvariety of the Teichmüller space of Riemann surfaces of genus g. The topology of this subvariety can be determined, in principle, from the branching data of the action.)

(3.7) Remark The proof of the assertion in (1.3.2) in the introduction follows from (3.1) and [2] which contains a classification of finite simple groups whose 2-Sylow subgroups contain an elementary abelian 2-group of rank at most 2. For the latter remarks we have to appeal to the full classifiaction of finite simple groups. For instance if $g \not\equiv 1$ (4) then the only simple symmetry groups of Σ_g , besides the ones listed in (1.3.2) , are the first Janko group J_1 , the Ree groups ${}^2G_2(q)$, $q=3^{2n+1}$, $n \geq 2$, and $PSL_2(\mathbb{F}_8)$. These results may be further extended in the following direction. Let G be a finite perfect group acting on Σ_g , g even, and $O(G)$ = the maximal normal subgroup of odd order in G . Then $G/O(G)$ is isomorphic to one of the groups listed at the beginning of (1.3.2) or else it is isomorphic to $SL_2(\mathbb{F}_q)$ q odd ≥ 5 or A^*_7 = the 2-fold central perfect extension of A_7 .

Notice that if ${}_G\backslash\Sigma_g$ has genus 0 and the branching indices are pairwise co-prime then G is necessarily perfect. The above remarks indicate the very special nature of such actions.

(3.8) Remark The proof of (1.3.1) in the introduction follows from (3.1), (3.7) and a simple check that in the list given in (1.3.2) only $PSL_2(\mathbb{F}_q)$ q=a prime ≥ 5 has cyclic p-Sylow subgroups for p odd, and $SL_2(\mathbb{F}_q)$ q=a prime ≥ 5 is its only perfect extension with the same property.

Now we mention two purely group-theoretic consequences, which are presented mainly for the novelty of their "geometric" proofs.

(3.9) Proposition Let G be a p-group and $H \leq G$. Then the cyclic deficiency of H is less than or equal to that of G. In particular the rank of an elementary abelian subgroup is $\le 1+$ {the deficiency of G}.

<u>Proof</u> Suppose p is odd. By (2.3) we may choose g such that G acts on Σ_g and the highest power of p dividing $g-1$ equals the cyclic deficiency of G. But then H acts on Σ_g and the result is clear from (2.3). A slight modification of this argument for $p=2$ is left to the reader. The last assertion is clear.

q.e.d.

(3.10) <u>Proposition</u> A finite perfect group is of type I.

<u>Proof</u> Let G be a finite perfect group of order d and suppose if possible that it is of type II. We use the notations of § 2. For $z = g \equiv 1(N)$ but $\not\equiv 1(2N)$, and g sufficiently large we can find a solution of (2.7.1) in non-negative integers for x_n, y and h, and we may assume that h is sufficiently large. According to (2.3) G cannot act on Σ_g. On the other hand in a perfect group the equation (2.6.2) clearly has a solution for large h so if h is sufficiently large then G would act on Σ_g. This contradiction shows that G must be of type I.

q.e.d.

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