Kuiper's theorem for Hilbert modules: the general case

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Abstract

Let $\operatorname{End} l_2(A)$ denote the algebra of all bounded A–operators in Hilbert module $l_2(A)$ and $\operatorname{End}^* l_2(A)$ denote the C*–algebra of operators admitting an adjoint. Through $\mathrm{GL}(A)$ and $\mathrm{GL}^*(A)$ we denote the correspondent groups of invertible elements. In the present paper we prove the contractibility of $\mathrm{GL}(A)$ and $\mathrm{GL}^*(A)$ for arbitrary C*–algebra A.

Let $\operatorname{End} l_2(A)$ denote the algebra of all bounded A–operators in Hilbert module $l_2(A)$ and $\operatorname{End}^* l_2(A)$ denote the C*–algebra of operators admitting an adjoint. Through $\mathrm{GL}(A)$ and $\mathrm{GL}^*(A)$ we denote the correspondent groups of invertible elements. The question on the contractibility of these linear groups is very important in K–theory for construction of classifying spaces and was the subject of a number of papers. In [6, 3, 7] the contractibility of $\mathrm{GL}^*(A)$ for unital A was proved. The author used these results for constructing the classifying spaces for $K^{p,q}(X;A)$ in [8]. In [9] the author obtained another proof of this fact as well as a proof of the contractibility of $\mathrm{GL}(A)$ for unital A. In [1] was proved the contractibility of $\mathrm{GL}^*(A)$ for A with strictly positive element.

In the present paper we prove the contractibility of $\mathrm{GL}(A)$ and $\mathrm{GL}^*(A)$ for arbitrary C*–algebra A.

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1 The first step of the homotopy

Definition 1.1. The set of the invertible elements of End $^*l_2(A)$ (correspondently End $l_2(A)$) we call *the general linear group* $\mathrm{GL}\,(A)$ (correspondently *the full general linear group* $\mathrm{GL}^*(A)$).

Remark 1.2. The groups $\mathrm{GL}\,(A)$ and $\mathrm{GL}^*(A)$ are open sets in Banach spaces $\mathrm{End}\, l_2(A)$ and $\mathrm{End}\,^*l_2(A)$ correspondently, hence, by the Milnor's theorem [5] they have homotopy type of CW–complexes. So, by the Whitehead theorem for the proof of the contractibility of $\mathrm{GL}\,(A)$ and $\mathrm{GL}^*(A)$ it is sufficient to prove the following. Let $f_0 : S \to \mathrm{GL}\,(A)$ be a continuous mapping of a sphere of arbitrary dimension, then f is homotopic to $f_1 : S \to 1 \in \mathrm{GL}\,(A)$ (similarly for $\mathrm{GL}^*(A)$).

So, let $f_0 : S \to \mathrm{GL}\,(A)$ be a continuous mapping. Any operator from $\mathrm{End}\, l_2(A)$ is represented by a matrix with entries from $\mathrm{LM}(A) \subset W = W^*(A)$ the universal envelopping von Neumann algebra. Then the following mapping (inclusion)

$$\mathrm{End}\, l_2(A) \subset \mathrm{End}\, l_2(W), \qquad \mathrm{End}\,^*l_2(A) \subset \mathrm{End}\,^*l_2(W).$$

arises. Let us denote the images of $\mathrm{GL}\,(A)$ and $\mathrm{GL}^*(A)$ under this mapping through $\mathrm{GL}_A(W)$ and $\mathrm{GL}^*_A(W)$.

Let us denote through p_M the projection on the free W–module of finite type L_M, generated by $e_1, \ldots, e_M$,

$$L_M = \mathrm{span}_W\langle e_1, \ldots, e_M\rangle$$

along $L_M^\perp$, and through q_j the projection on the free 1-generated W–module W_j, generated by e_j.

Lemma 1.3 *Let K be a norm-compact set of operators from* $\mathrm{End}\, l_2(A)$. *Then for any n and any $\varepsilon > 0$ there exists $k = k(\varepsilon, n)$ such that*

$$\|(1 - p_k)G|_{L_n}\| = \|(1 - p_k)Gp_n\| < \varepsilon \qquad \forall G \in K.$$

Proof. Let $G_1, \ldots, G_N$ be a finite $\varepsilon/2$–net for K. For each G_i there exists such $k(i)$ that

$$\|(1 - p_{k(i)})G_i e_s\| < \frac{\varepsilon}{2n} \qquad (s = 1, \ldots, n).$$

If $x \in L_n$, $\|x\| \leq 1$, then $x = \sum_{s=1}^n e_s\alpha_s$, $\|\alpha_s| < 1$ and for each $i = 1, \ldots, N$

$$\|(1 - p_{k(i)})G_i p_n x\| = \|(1 - p_{k(i)})G_i p_n \sum_s e_s\alpha_s\| \leq$$

$$\leq \sum_{s=1}^n \|(1 - p_{k(i)})G_i e_s\| \cdot \|\alpha_s\| \leq n \cdot \frac{\varepsilon}{2n} = \varepsilon/2.$$

If we define $k = \max\limits_{i=1,\ldots,N} k(i)$, then for every $G \in K$ there exists G_{i_0} with $\|G - G_{i_0}\| < \varepsilon$ and

$$\|(1 - p_k)Gp_n\| \leq \|(1 - p_k)(G - G_{i_0})p_n\| +$$

$$+\|(1-p_k)G_{i_0}p_n\| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \qquad \square$$

Let $\varepsilon > 0$ be so small that ε-neighborhood of K is contained in GL.

We would like to define a sequence of homotopies in such a way, that as a result the image of sphere will consist of operators mapping some subsequence of basis vectors $\{a_i^!\}_{i=1}^{\infty} \subset \{e_j\}_{j=1}^{\infty}$ to another subsequence $\{a_i\}_{i=1}^{\infty} \subset \{e_j\}_{j=1}^{\infty}$. These sequences will not depend on the choice of $s \in S$. We will define homotopies in $\mathrm{GL}(W)$, but we will verify that in fact they are in $\mathrm{GL}_A(W)$.

While reasoning we will define sequences of strictly increasing entire numbers $k(i)$, $k'(i)$, $k''(i)$, $i \in \mathbf{N}^+$.

Lemma 1.4 *There exists a homotopy $f_0 \sim f_{2/3}$ and a decomposition in W-Hilbert sum*

$$l_2(W) = E_1 \oplus E_2 \oplus \ldots,$$

$E_j = \mathrm{span}_W\langle e_{k(j)}, \ldots, e_{k(j+1)-1}\rangle$, *restricted to satisfy for every $F \in f_{2/3}(S)$ the following conditions:*

$$F(e_{k(j)}) \in L_{k'(j)}, \quad e_{k'(j)+1} \in F(E_1 \oplus \ldots \oplus E_j), \quad k(j) < k'(j) + 1 < k(j+1) - 1.$$

In addition the homotopy is in $\mathrm{GL}_A(W)$.

Proof. Let $k(1) = 1$. Let us choose $k'(1) > k(1)$ in such a way that (see Lemma 1.3)

$$\|(1 - p_{k'(1)})F(e_{k(1)})\| < \frac{1}{2} \cdot \frac{\varepsilon}{2}$$

for $\forall F \in K$. Let us define $F' \in \mathrm{GL}$ by

$$F' = \begin{cases} p_{k'(1)}F & \text{on } \mathrm{span}_W(e_1), \\ F & \text{on } \mathrm{span}_W(e_1)^{\perp}. \end{cases}$$

Let $p(F')_j = F'p_j(F')^{-1}$ be the projection on $F'(L_j)$ along $F'(L_j^{\perp})$. Let us define

$$y_j := (F')^{-1}p(F')_{k''(1)}e_j = p_{k''(1)}(F')^{-1}e_j,$$

$(1 \le j \le k'(1) + 1)$, while $k''(1) > k'(1) + 1$ is chosen in such a way, that y_j are the free generators,

$$\|\sum_{j=1}^{k'(1)+1} y_j\alpha_j\| \le 1 \quad \Rightarrow \|\alpha_j\| \le 2, \quad j = 1, \ldots, k'(1) + 1,$$

and

$$\|(1 - p(F')_{k''(1)})|_{L_{k'(1)+1}}\| < \frac{1}{2} \cdot \frac{\varepsilon}{2} \cdot \frac{1}{2(k'(1)+1)\|F'\|},$$

$\forall F \in K$ (Lemma 1.3). Let us define $(F^{(1)}) \in \mathrm{GL}$ as

$$\begin{cases} F^{(1)}(y_j) = e_j, & (j = 1, \ldots, k'(1) + 1) \\ F^{(1)}x = F'x, & x \in (F')^{-1}(L_{k'(1)+1}^{\perp}) \end{cases}$$

and $k(2) = k''(1) + 2$.

Let now $F^{(j)}$, $k(j)$, $k'(j)$, $k''(j)$ be already define for all $j \le m$. Let $k(m+1) = k''(m) + 2$. Let us choose $k'(m+1) > k(m+1)$ in such a way, that for every $F \in K$

$$\|(1 - p_{k'(m+1)})F^{(m)} p_{k(m+1)}\| < \frac{1}{2} \cdot \frac{\varepsilon}{2^{m+1}}. \tag{1}$$

(Lemma 1.3). Let us define $F_i^{(m+1)'} \in \mathrm{GL}$ by

$$F^{(m+1)'} = \begin{cases} p_{k'(m+1)} F^{(m)} & \text{on } L_{k(m+1)}, \\ F^{(m)} & \text{on } (L_{k(m+1)})^{\perp}. \end{cases} \tag{2}$$

Let us note, that since

$$F^{(m)}(L_{k(m)}) \subset L_{k'(m)} \subset L_{k'(m+1)},$$

then from formula (2) we get:

$$F^{(m+1)'}|_{L_{k(m)}} = F^{(m)}|_{L_{k(m)}}. \tag{3}$$

Let $p(F^{(m+1)'})_j = F^{(m+1)'} p_j (F^{(m+1)'})^{-1}$ be the projection on $F^{(m+1)'}(L_j)$ along $F^{(m+1)'}(L_j^{\perp})$. Let us define

$$y_j := (F^{(m+1)'})^{-1} p(F^{(m+1)'})_{k''(m+1)} e_j = p_{k''(m+1)} (F^{(m+1)'})^{-1} e_j,$$

$(1 \le j \le k'(m+1)+1)$, while $k''(m+1) > k'(m+1)+1$ are chosen in such a way that y_j are free generators,

$$\| \sum_{j=1}^{k'(m+1)+1} y_j \alpha_j \| \le 1 \quad \Rightarrow \|\alpha_j\| \le 2, \quad j = 1, \dots, k'(m+1)+1,$$

and

$$\|(1 - p(F^{(m+1)'})_{k''(m+1)})|_{L_{k'(m+1)+1}}\| < \frac{1}{2} \cdot \frac{\varepsilon}{2^{m+1}} \cdot \frac{1}{2(k'(m+1)+1)\|F^{(m+1)'}\|}, \tag{4}$$

$\forall F \in K$ (Lemma 1.3). Let us define $(F_i^{(m+1)}) \in \mathrm{GL}$ by

$$\begin{cases} F^{(m+1)}(y_j) = e_j, \quad (j = 1, \dots, k'(m+1)+1) \\ F^{(m+1)} x = F^{(m+1)'} x, \quad x \in (F^{(m+1)'})^{-1}(L^{\perp}_{k'(m+1)+1}) \end{cases} \tag{5}$$

$\forall F \in K$, or, equivalently,

$$\begin{cases} (F^{(m+1)})^{-1}|_{L_{k'(m+1)+1}} = p_{k''(m+1)} (F^{(m+1)'})^{-1} \\ (F^{(m+1)})^{-1}|_{L^{\perp}_{k'(m+1)+1}} = (F^{(m+1)'})^{-1} \end{cases} \tag{6}$$

If $\beta \leq k(m+1)$, then by construction

$$F^{(m+1)\prime}(e_\beta) \in F^{(m+1)\prime}(L_{k''(m+1)}) \cap L_{k'(m+1)+1}, \quad F^{(m+1)\prime}(e_\beta) = \sum_{j=1}^{k'(m+1)+1} e_j \alpha_j,$$

$$e_\beta = \sum_{j=1}^{k'(m+1)+1} (F^{(m+1)\prime})^{-1} e_j \alpha_j, \qquad e_\beta = p_{k''(m+1)} e_\beta = \sum_{j=1}^{k'(m+1)+1} y_j \alpha_j,$$

hence $F^{(m+1)}(e_\beta)$ is defined by the first line of (5) and

$$F^{(m+1)}(e_\beta) = \sum_{j=1}^{k'(m+1)+1} e_j \alpha_j = F^{(m+1)\prime}(e_\beta),$$

so the changes "do not touch the changes on the previous step". (In general changes are on

$$(F^{(m+1)\prime})^{-1}(p(F^{(m+1)\prime})_{k''(m+1)} L_{k'(m+1)+1}) =$$
$$= p_{k''(m+1)}(F^{(m+1)\prime})^{-1} L_{k'(m+1)+1} \subset L_{k''(m+1)}. \quad)$$

Due to (1 – 5) and the choice of ε there exists the limit

$$F'' = (\lim_{m\to\infty} F^{(m)}) \in \mathrm{GL} \qquad \forall F \in K$$

and the induced linear homotopy $f_{1/3} \sim f_{2/3}$ also lies in GL. By (3) and above reasoning on the changes at the second part of each step the desired conditions are fulfilled. Indeed, by (2) $F''(e_{k(j)}) \in L_{k'(j)}$, and by (5)

$$F''^{-1}(e_{k'(m+1)+1}) \subset L_{k''(m+1)},$$

hence

$$e_{k'(m+1)+1} \subset F''(L_{k''(m+1)}).$$

Since the projections on the basis modules W_j and their sums in $l_2(W)$ are from $\mathrm{End}\, l_2(A)$, then the homotopy also is in $\mathrm{GL}_A(W)$. □

2 The second step of the homotopy

Let

$$C_0 = \max\{\max_{F\in K_{2/3}} \|F\|, \max_{F\in K_{2/3}} \|F^{-1}\|\}.$$

Now for each

$$F \in K_{2/3} = f_{2/3}(S), \quad \varphi \in [0, \pi/2], \quad i \in \mathbf{N}^+$$

we will define operators

$$J_i(F, \varphi) : l_2(W) \to l_2(W).$$

Each of modules $\mathrm{span}_W(Fe_{k(i)})$ and $\mathrm{span}_W e_{k'(i)+1}$ is isomorphic to W, so by [2] has a W-orthogonal complement in $l_2(W)$.

Lemma 2.1

$$R_i^0 \oplus R_i^1 := \{\mathrm{span}_W(Fe_{k(i)}) \oplus \mathrm{span}_W e_{k'(i)+1}\}\oplus$$
$$\oplus\{F\,(\mathrm{span}_W e_{k(i)}^{\perp}) \cap (\mathrm{span}_W e_{k'(i)+1})^{\perp}\} = l_2(W).$$

Proof. Let $w \in l_2(W)$ be an arbitrary element,

$$w = v + u, \qquad u \in \mathrm{span}_W e_{k'(i)+1}, \qquad v \in (\mathrm{span}_W e_{k'(i)+1})^{\perp}.$$

Then we can decompose

$$v = v_1 + v_2, \qquad v_1 \in \mathrm{span}_W(Fe_{k(i)}), \qquad v_2 \in F\,((\mathrm{span}_W e_{k(i)})^{\perp}).$$

Hence $v_1 \in (\mathrm{span}_W e_{k'(i)+1})^{\perp}$, since by the construction $Fe_{k(i)} \perp e_{k'(i)+1}$, and

$$v_2 = v - v_1 \in (\mathrm{span}_W e_{k'(i)+1})^{\perp}.$$

Hence, $w = u + v_1 + v_2$ is the desired decomposition. □

Corollary 2.2 (from the proof) *We have the following partition of the identity*

$$1 = (q_{k'(i)+1}) + (Fq_{k(i)}F^{-1})(1 - q_{k'(i)+1}) + (1 - Fq_{k(i)}F^{-1})(1 - q_{k'(i)+1}),$$

into three complementary projections.

Lemma 2.3 *Let $i > j$, then*

$$Fe_{k(j)} \in R_i^1, \qquad e_{k'(j)+1} \in R_i^1.$$

Proof. By the construction $Fe_{k(j)} \perp e_{k'(i)+1}$ and $Fe_{k(j)} \in F(\mathrm{span}_W e_{k(i)}^{\perp})$, since $e_{k(j)} \perp e_{k(i)}$. Also $e_{k'(j)+1} \perp e_{k'(i)+1}$, and by the construction

$$e_{k'(i)+1} \in F(E_1 \oplus \ldots \oplus E_j) \subset F((e_{k(j+1)})^{\perp}).\square$$

Let us define $J_i(F, \varphi)$ by

$$\begin{cases} J_i(F,\varphi)(Fe_{k(i)}) &= \cos\varphi\, Fe_{k(i)} + \sin\varphi\, e_{k'(i)+1}, \\ J_i(F,\varphi)(e_{k'(i)+1}) &= -\sin\varphi\, Fe_{k(i)} + \cos\varphi\, e_{k'(i)+1}, \\ J_i(F,\varphi)(x) &= x, \qquad \text{if} \quad x \in R_i^1. \end{cases}$$

Lemma 2.4

$$J_i(F,\varphi)(x) = \{\cos\varphi\, Fe_{k(i)} + \sin\varphi\, e_{k'(i)+1}\}(F^{-1}(1 - q_{k'(i)+1})x)^{k(i)} +$$
$$+\{-\sin\varphi\, Fe_{k(i)} + \cos\varphi\, e_{k'(i)+1}\}(x)^{k'(i)+1} +$$
$$+(1 - Fq_{k(i)}F^{-1})(1 - q_{k'(i)+1})x,$$

where y^j is the j-th coordinate in the standard basis $\{e_i\}$.

Proof. Let us verify the coincidense for the following three types of elements

$$x = e_{k'(i)+1}, \quad x = Fe_{k(i)}, \quad x \in R_i^1.$$

Let $x = e_{k'(i)+1}$, then the first and the third lines in our expression vanish. The second line is equal to

$$\{-\sin\varphi\, Fe_{k(i)} + \cos\varphi\, e_{k'(i)+1}\}\cdot 1,$$

and this case is done.

Let now $x = Fe_{k(i)}$. Then $(1 - q_{k'(i)+1})(x) = x$, since by construction $Fe_{k(i)} \perp e_{k'(i)+1}$, and the first line is equal to

$$\{\cos\varphi\, Fe_{k(i)} + \sin\varphi\, e_{k'(i)+1}\}(F^{-1}(1 - q_{k'(i)+1})x)^{k(i)} =$$
$$= \{\cos\varphi\, Fe_{k(i)} + \sin\varphi\, e_{k'(i)+1}\}(F^{-1}Fe_{k(i)})^{k(i)} =$$
$$= \cos\varphi\, Fe_{k(i)} + \sin\varphi\, e_{k'(i)+1}.$$

the second line is equal to 0 by the same argument. The third line is equal to

$$(1 - Fq_{k(i)}F^{-1})Fe_{k(i)} = Fe_{k(i)} - Fe_{k(i)} = 0$$

and this case is also done.

Let $x \in R_i^1$, then

$$x = (1 - Fq_{k(i)}F^{-1})(1 - q_{k'(i)+1})(y)$$

and

$$q_{k(i)}F^{-1}(1 - q_{k'(i)+1})(1 - Fq_{k(i)}F^{-1})(1 - q_{k'(i)+1})(y) =$$
$$= (q_{k(i)}F^{-1} - q_{k(i)}F^{-1}q_{k'(i)+1})(1 - Fq_{k(i)}F^{-1} - q_{k'(i)+1} + Fq_{k(i)}F^{-1}q_{k'(i)+1})(y) =$$
$$= (q_{k(i)}F^{-1} - q_{k(i)}F^{-1}q_{k'(i)+1} -$$
$$- q_{k(i)}F^{-1}Fq_{k(i)}F^{-1} + q_{k(i)}F^{-1}q_{k'(i)+1}Fq_{k(i)}F^{-1} -$$
$$- q_{k(i)}F^{-1}q_{k'(i)+1} + q_{k(i)}F^{-1}q_{k'(i)+1}q_{k'(i)+1} +$$
$$+ q_{k(i)}F^{-1}Fq_{k(i)}F^{-1}q_{k'(i)+1} - q_{k(i)}F^{-1}q_{k'(i)+1}Fq_{k(i)}F^{-1}q_{k'(i)+1})(y) =$$
$$= (q_{k(i)}F^{-1} - q_{k(i)}F^{-1}q_{k'(i)+1} - q_{k(i)}F^{-1} + q_{k(i)}F^{-1}\cdot 0 \cdot F^{-1} -$$
$$- q_{k(i)}F^{-1}q_{k'(i)+1} + q_{k(i)}F^{-1}q_{k'(i)+1} +$$
$$+ q_{k(i)}F^{-1}q_{k'(i)+1} - q_{k(i)}F^{-1}\cdot 0 \cdot F^{-1}q_{k'(i)+1})(y) = 0,$$

the first line vanishes. The third line evidently gives x. Since

$$q_{k'(i)+1}(1 - Fq_{k(i)}F^{-1})(1 - q_{k'(i)+1})(y) =$$
$$= (q_{k'(i)+1} - q_{k'(i)+1}Fq_{k(i)}F^{-1})(1 - q_{k'(i)+1})(y) =$$
$$= (q_{k'(i)+1} - 0 \cdot F^{-1})(1 - q_{k'(i)+1})(y) = 0,$$

then the second line vanishes and this complete the proof. □

From the representation in the previous lemma is evident, that $J_i(F,\varphi)$ are in the image of $\operatorname{End} l_2(A)$. Norm of the operator for every i, $F \in K_{2/3}$, φ does not exceed

$$(C_0+1)C_0 + (C_0+1) + (1+C_0^2) = C.$$

The operator $J_i(F,\varphi)$ has the following matrix

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to decomposition

$$\operatorname{span}_W(Fe_{k(i)}) \oplus \operatorname{span}_W e_{k'(i)+1} \oplus R_i^1,$$

while $J_i(F,\varphi)^{-1}$ -

$$\begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular his norm is also not more then C.

Let us note that for every coordinate of sum of arbitrary vectors from $l_2(A)$

$$a_1 + \ldots + a_s = (a_1^1 + \ldots + a_s^1, a_1^2 + \ldots + a_s^2, \ldots)$$

we can write (here $a_j^i = b_j$)

$$(b_1+\ldots+b_s)^*(b_1+\ldots+b_s) \le (b_1+\ldots+b_s)^*(b_1+\ldots+b_s) + \sum_{i\neq j}(b_i-b_j)^*(b_i-b_j) =$$

$$= s(b_1^*b_1 + \ldots + b_s^*b_s),$$

hence

$$\langle a_1+\ldots+a_s, a_1+\ldots+a_s\rangle \le s(\langle a_1,a_1\rangle + \ldots + \langle a_s,a_s\rangle). \qquad (7)$$

Let us define a family $J(F,\varphi) : l_2(W) \to l_2(W)$, by

$$J(F,\varphi)|_{F(E_1\oplus\ldots\oplus E_s)} = J_s(F,\varphi)J_{s-1}(F,\varphi)\ldots J_1(F,\varphi).$$

By Lemma 2.3 this is a well-defined operator on a dense set and

$$J(F,\varphi)(Fe_{k(i)}) = J_i(F,\varphi)(Fe_{k(i)}),$$

$$J(F,\varphi)(e_{k'(i)+1}) = J_i(F,\varphi)(e_{k'(i)+1}).$$

Let us show that these operators are bounded and invertible. Let

$$y = (y_1+x_1)\ldots+(y_s+x_s)+z, \qquad y_i+x_i \in R_i^0,$$

$$y_i = Fe_{k(i)}\alpha_i = Fq_{k(i)}F^{-1}(1 - q_{k'(i)+1})y,$$
$$x_i = e_{k'(i)+1}\beta_i = q_{k'(i)+1}(y),$$

then

$$J(F,\varphi)y = \sum_{i=1}^{s} J_i(F,\varphi)(y_i + x_i) + z =$$

$$= \sum_{i=1}^{s}\Big[\{\cos\varphi\, Fe_{k(i)} + \sin\varphi\, e_{k'(i)+1}\}\alpha_i + \{-\sin\varphi\, Fe_{k(i)} + \cos\varphi\, e_{k'(i)+1}\}\beta_i\Big] + z.$$

With (7) this gives

$$\langle J(F,\varphi)y, J(F,\varphi)y\rangle \le 5\Big(\cos^2\varphi\langle\sum_{i=1}^{s} Fe_{k(i)}\alpha_i, \sum_{i=1}^{s} Fe_{k(i)}\alpha_i\rangle +$$

$$+\sin^2\varphi\langle\sum_{i=1}^{s} Fe_{k(i)}\alpha_i, \sum_{i=1}^{s} Fe_{k(i)}\alpha_i\rangle + \sin^2\varphi\sum_{i=1}^{s}\alpha_i^*\alpha_i + \cos^2\varphi\sum_{i=1}^{s}\beta_i^*\beta_i + \langle z,z\rangle\Big) \le$$

$$\le 5\Big((\|F\|^2+1)(\sum_{i=1}^{s}\alpha_i^*\alpha_i + \sum_{i=1}^{s}\beta_i^*\beta_i) + \langle z,z\rangle\Big).$$

Let us note, that

$$\sum_{i=1}^{s}\beta_i^*\beta_i = \sum_{i=1}^{s}\langle x_i, x_i\rangle = \sum_{i=1}^{s}\langle q_{k'(i)+1}y, q_{k'(i)+1}y\rangle \le \langle y,y\rangle.$$

Also,

$$\sum_{i=1}^{s}\alpha_i^*\alpha_i = \langle F^{-1}F\sum_{i=1}^{s} e_{k(i)}\alpha_i, \sum_{i=1}^{s} F^{-1}Fe_{k(i)}\alpha_i\rangle \le C_0^2\langle\sum_{i=1}^{s} y_i, \sum_{i=1}^{s} y_i\rangle =$$

$$= C_0^2\langle\sum_{i=1}^{s} Fq_{k(i)}F^{-1}(1-q_{k'(i)+1})y, \sum_{i=1}^{s} Fq_{k(i)}F^{-1}(1-q_{k'(i)+1})y\rangle \le$$

$$\le 2C_0^2\Big(\langle\sum_{i=1}^{s} Fq_{k(i)}F^{-1}y, \sum_{i=1}^{s} Fq_{k(i)}F^{-1}y\rangle +$$

$$+\langle\sum_{i=1}^{s} Fq_{k(i)}F^{-1}q_{k'(i)+1}y, \sum_{i=1}^{s} Fq_{k(i)}F^{-1}q_{k'(i)+1}y\rangle\Big) \le$$

$$\le 2C_0^2\Big(\|F\|^2\langle\sum_{i=1}^{s} q_{k(i)}F^{-1}y, \sum_{i=1}^{s} q_{k(i)}F^{-1}y\rangle +$$

$$+\|F\|^2\langle\sum_{i=1}^{s} q_{k(i)}F^{-1}q_{k'(i)+1}y, \sum_{i=1}^{s} q_{k(i)}F^{-1}q_{k'(i)+1}y\rangle\Big) \le$$

$$\le 2C_0^2\Big(C_0^2\langle F^{-1}y, F^{-1}y\rangle + C_0^2\langle\sum_{i=1}^{s} F^{-1}q_{k'(i)+1}y, \sum_{i=1}^{s} F^{-1}q_{k'(i)+1}y\rangle\Big) \le$$

$$\leq 2C_0^2\Big(C_0^4\langle y,y\rangle + C_0^4\langle\sum_{i=1}^{s} q_{k'(i)+1}y, \sum_{i=1}^{s} q_{k'(i)+1}y\rangle\Big) \leq$$

$$\leq 2C_0^2\Big(C_0^4\langle y,y\rangle + C_0^4\langle y,y\rangle\Big) = 4C_0^6\langle y,y\rangle,$$

hence

$$\langle z,z\rangle \leq 3\Big(\langle y,y\rangle + \langle\sum_{i=1}^{s} y_i, \sum_{i=1}^{s} y_i\rangle + \langle\sum_{i=1}^{s} x_i, \sum_{i=1}^{s} x_i\rangle\Big) \leq 3(1+1+4C_0^4).$$

We get an estimation, which does not depend on s, hence $J(F,\varphi)$ is a bounded operator as well as $J(F,\varphi)^{-1} = J(F,-\varphi)$.

Lemma 2.5 *The family of operators $J(F,\varphi)$ is continuous in*

$$(F,\varphi) \in K_{2/3} \times [0, \frac{\pi}{2}].$$

Proof. Let $y \in E_1 \oplus \ldots \oplus E_s$.

$$J(F,\varphi)Fy - J(F',\varphi)F'y =$$

$$= \sum_{i=1}^{s}\Big[\{\cos\varphi\, Fe_{k(i)} + \sin\varphi\, e_{k'(i)+1}\}\alpha_i + \{-\sin\varphi\, Fe_{k(i)} + \cos\varphi\, e_{k'(i)+1}\}\beta_i\Big] + z -$$

$$- \sum_{i=1}^{s}\Big[\{\cos\varphi\, F'e_{k(i)} + \sin\varphi\, e_{k'(i)+1}\}\alpha_i' + \{-\sin\varphi\, F'e_{k(i)} + \cos\varphi\, e_{k'(i)+1}\}\beta_i'\Big] - z',$$

where

$$Fy = (y_1+x_1) + \ldots + (x_s+y_s) + z, \qquad F'y = (y_1'+x_1') + \ldots + (x_s'+y_s') + z',$$

$$y_i = Fe_{k(i)}\alpha_i = (Fq_{k(i)}F^{-1})(1-q_{k'(i)+1})Fy,$$

$$x_i = e_{k'(i)+1}\beta_i = q_{k'(i)+1}Fy,$$

$$y_i' = F'e_{k(i)}\alpha_i' = (F'q_{k(i)}F'^{-1})(1-q_{k'(i)+1})F'y,$$

$$x_i' = e_{k'(i)+1}\beta_i' = q_{k'(i)+1}F'y,$$

then

$$\langle J(F,\varphi)Fy - J(F',\varphi)F'y, J(F,\varphi)Fy - J(F',\varphi)F'y\rangle \leq$$

$$\leq 7\Big[\cos^2\varphi\langle\sum_{i=1}^{s} Fe_{k(i)}(\alpha_i-\alpha_i'), \sum_{i=1}^{s} Fe_{k(i)}(\alpha_i-\alpha_i')\rangle +$$

$$+ \cos^2\varphi\langle\sum_{i=1}^{s}(F-F')e_{k(i)}\alpha_i', \sum_{i=1}^{s}(F-F')e_{k(i)}\alpha_i'\rangle +$$

$$+ \sin^2\varphi\langle\sum_{i=1}^{s} Fe_{k(i)}(\beta_i-\beta_i'), \sum_{i=1}^{s} Fe_{k(i)}(\beta_i-\beta_i')\rangle +$$

$$+\sin^2\varphi\langle\sum_{i=1}^{s}(F-F')e_{k(i)}\beta_i',\sum_{i=1}^{s}(F-F')e_{k(i)}\beta_i'\rangle+$$

$$+\sin^2\varphi\langle\sum_{i=1}^{s}e_{k'(i)+1}(\alpha_i-\alpha_i'),\sum_{i=1}^{s}e_{k'(i)+1}(\alpha_i-\alpha_i')\rangle+$$

$$+\cos^2\varphi\langle\sum_{i=1}^{s}e_{k'(i)+1}(\beta_i-\beta_i'),\sum_{i=1}^{s}e_{k'(i)+1}(\beta_i-\beta_i')\rangle+\langle z-z',z-z'\rangle\Big]\leq$$

$$\leq 7\Big[\|F-F'\|^2(\sum_{i=1}^{s}\alpha_i'^*\alpha_i'+\sum_{i=1}^{s}\beta_i'^*\beta_i')+$$

$$+(C_0^2+1)\Big(\sum_{i=1}^{s}(\alpha_i-\alpha_i')^*(\alpha_i-\alpha_i')+\sum_{i=1}^{s}(\beta_i-\beta_i')^*(\beta_i-\beta_i')\Big)+\langle z-z',z-z'\rangle\Big].$$

Since

$$\alpha_i=(F^{-1}(1-q_{k'(i)+1})Fy)^{k(i)},$$
$$\alpha_i'=(F'^{-1}(1-q_{k'(i)+1})F'y)^{k(i)},$$

we get

$$\sum_{i=1}^{s}(\alpha_i-\alpha_i')^*(\alpha_i-\alpha_i')=$$

$$=\sum_{i=1}^{s}\langle q_{k(i)}(F^{-1}(1-q_{k'(i)+1})F-F'^{-1}(1-q_{k'(i)+1})F')y,$$

$$q_{k(i)}(F^{-1}(1-q_{k'(i)+1})F-F'^{-1}(1-q_{k'(i)+1})F')y\rangle\leq$$

$$\leq 4\Big[\sum_{i=1}^{s}\langle q_{k(i)}(F^{-1}-F'^{-1})Fy,q_{k(i)}(F^{-1}-F'^{-1})Fy\rangle+$$

$$+\sum_{i=1}^{s}\langle q_{k(i)}(F^{-1}-F'^{-1})q_{k'(i)+1}Fy,q_{k(i)}(F^{-1}-F'^{-1})q_{k'(i)+1}Fy\rangle+$$

$$+\sum_{i=1}^{s}\langle q_{k(i)}F'^{-1}(F-F')Fy,q_{k(i)}F'^{-1}(F-F')Fy\rangle+$$

$$+\sum_{i=1}^{s}\langle q_{k(i)}F'^{-1}q_{k'(i)+1}(F-F')Fy,q_{k(i)}F'^{-1}q_{k'(i)+1}(F-F')Fy\rangle\Big]\leq$$

$$\leq 16\|F^{-1}-F'^{-1}\|^2C_0^2\langle y,y\rangle. \tag{8}$$

Since

$$\beta_i=(Fy)^{k'(i)+1},\qquad \beta_i'=(F'y)^{k'(i)+1},$$

we get

$$\sum_{i=1}^{s}(\beta_i-\beta_i')^*(\beta_i-\beta_i')=\sum_{i=1}^{s}\langle q_{k'(i)+1}(F-F')y,q_{k'(i)+1}(F-F')y\rangle\leq$$

$$\leq \|F - F'\|^2 \langle y, y \rangle.$$

Hence

$$\langle z - z', z - z' \rangle = \langle F(y) - \sum_{i=1}^{s} y_i - \sum_{i=1}^{s} x_i - F'(y) + \sum_{i=1}^{s} y'_i + \sum_{i=1}^{s} x'_i,$$

$$F(y) - \sum_{i=1}^{s} y_i - \sum_{i=1}^{s} x_i - F'(y) + \sum_{i=1}^{s} y'_i + \sum_{i=1}^{s} x'_i \rangle \leq$$

$$\leq 3\Big(\langle (F - F')y, (F - F')y \rangle + \langle \sum_{i=1}^{s} (y_i - y'_i), \sum_{i=1}^{s} (y_i - y'_i) \rangle + \langle \sum_{i=1}^{s} (x_i - x'_i), \sum_{i=1}^{s} (x_i - x'_i) \rangle \Big) \leq$$

$$\leq 3\Big(\|F - F'\|^2 \langle y, y \rangle + \langle \sum_{i=1}^{s} \Big[(F q_{k(i)} F^{-1})(1 - q_{k'(i)+1}) F y - (F' q_{k(i)} F'^{-1})(1 - q_{k'(i)+1}) F' y \Big],$$

$$\sum_{i=1}^{s} \Big[(F q_{k(i)} F^{-1})(1 - q_{k'(i)+1}) F y - (F' q_{k(i)} F'^{-1})(1 - q_{k'(i)+1}) F' y \Big] \rangle$$

$$+ \langle \sum_{i=1}^{s} q_{k'(i)+1} (F - F') y, \sum_{i=1}^{s} q_{k'(i)+1} (F - F') y \rangle \Big) \leq$$

$$\leq 3\Big(\|F - F'\|^2 \langle y, y \rangle + \langle \sum_{i=1}^{s} \Big[(F - F') q_{k(i)} F^{-1} F y + (F - F') q_{k(i)} F^{-1} q_{k'(i)+1} F y +$$

$$+ F' q_{k(i)} (F^{-1} - F'^{-1}) F y + F' q_{k(i)} (F^{-1} - F'^{-1}) q_{k'(i)+1} F y +$$

$$+ F' q_{k(i)} F'^{-1} (F - F') y + F' q_{k(i)} F'^{-1} q_{k'(i)+1} (F - F') y \Big],$$

$$\sum_{i=1}^{s} \Big[(F - F') q_{k(i)} F^{-1} F y + (F - F') q_{k(i)} F^{-1} q_{k'(i)+1} F y +$$

$$+ F' q_{k(i)} (F^{-1} - F'^{-1}) F y + F' q_{k(i)} (F^{-1} - F'^{-1}) q_{k'(i)+1} F y +$$

$$+ F' q_{k(i)} F'^{-1} (F - F') y + F' q_{k(i)} F'^{-1} q_{k'(i)+1} (F - F') y \Big] \rangle +$$

$$+ \|F - F'\| \langle y, y \rangle \leq$$

$$\leq 3\Big(2\|F - F'\|^2 \langle y, y \rangle + 6\Big\{ \langle \sum_{i=1}^{s} (F - F') q_{k(i)} y, \sum_{i=1}^{s} (F - F') q_{k(i)} y \rangle +$$

$$+ \langle \sum_{i=1}^{s} (F - F') q_{k(i)} F^{-1} q_{k'(i)+1} F y, \sum_{i=1}^{s} (F - F') q_{k(i)} F^{-1} q_{k'(i)+1} F y \rangle +$$

$$+ \langle \sum_{i=1}^{s} F' q_{k(i)} (F^{-1} - F'^{-1}) F y \sum_{i=1}^{s} F' q_{k(i)} (F^{-1} - F'^{-1}) F y \rangle +$$

$$+ \langle \sum_{i=1}^{s} F' q_{k(i)} (F^{-1} - F'^{-1}) q_{k'(i)+1} F y, \sum_{i=1}^{s} F' q_{k(i)} (F^{-1} - F'^{-1}) q_{k'(i)+1} F y \rangle +$$

$$+\langle \sum_{i=1}^{s} F' q_{k(i)} F'^{-1}(F-F')y, \sum_{i=1}^{s} F' q_{k(i)} F'^{-1}(F-F')y\rangle +$$

$$+\langle \sum_{i=1}^{s} F' q_{k(i)} F'^{-1} q_{k'(i)+1}(F-F')y, \sum_{i=1}^{s} F' q_{k(i)} F'^{-1} q_{k'(i)+1}(F-F')y\rangle\Big) \leq$$

$$\leq 3\Big[2\|F-F'\|^2\langle y,y\rangle + 6\Big\{\|F-F'\|^2\langle y,y\rangle + 5\|F-F'\|^2 C_0^4\langle y,y\rangle\Big\}\Big], \qquad (9)$$

(the last as (8)).

So,

$$\|J(F,\varphi)-J(F',\varphi')\| \leq C_0\|J(F,\varphi)F - J(F',\varphi')F\| \leq$$

$$\leq C_0\|J(F,\varphi)F - J(F',\varphi)F' + (J(F',\varphi)-J(F',\varphi'))F' + J(F',\varphi')(F'-F)\| \leq$$

$$\leq \varepsilon + \|J(F',\varphi)F' - J(F',\varphi')F'\|,$$

where $\varepsilon \to 0$ while $F' \to F$. Let us estimate the last norm again on $y \in E_1 \oplus \dots E_s$, where (in the previous notation)

$$F'y = (y_1 + x_1) + \dots + (x_s + y_s) + z.$$

Then

$$J(F',\varphi)F'y - J(F',\dot\varphi)F'y = \sum_{i=1}^{s}\Big[\{(\cos\varphi - \cos\varphi')Fe_{k(i)} + (\sin\varphi - \sin\varphi')e_{k'(i)+1}\}\alpha_i +$$

$$+\{-(\sin\varphi - \sin\varphi')Fe_{k(i)} + (\cos\varphi - \cos\varphi')e_{k'(i)+1}\}\beta_i\Big],$$

and the estimation from the proof of the boundeness gives now continuity. □

Let us denote $e_{k(i)} = a_i$, $e_{k'(i)+1} = a_i^1$, $e_{k(i+1)-1} = a_i^0$.

Lemma 2.6 *There exists a homotopy $f_{2/3} \sim f_1$, such that for $f \in f_1(S)$ we have*

$$Fa_i = a_i^1,$$

and which lies $\mathrm{GL}_A(W)$.

Proof. By the previous lemma it is sufficient to define the homotopy by

$$f_t(s) = J(f_{2/3}(s), \varphi)\, f_{2/3}(s),$$

where $\varphi = (4t-3)(\pi/2)$. □

3 The contractibility of GL (A)

Let us define $K_i(F,\varphi)$, being W–unitary automorphisms of E_i when $F\in f_1(S), 0\le\varphi\le\pi$. We define for $0\le\varphi\le\pi/2$

$$\begin{cases} K_i(F,\varphi)(a_i^1) &= \cos\varphi\, a_i^1+\sin\varphi\, a_i^0,\\ K_i(F,\varphi)(a_i^0) &= -\sin\varphi\, a_i^1+\cos\varphi\, a_i^0,\\ K_i(F,\varphi)(x) &= x,\ \text{if } x\perp \operatorname{span}_W(a_i^1,a_i^0),\end{cases}$$

and for $\pi/2\le\varphi\le\pi$ –

$$\begin{cases} K_i(F,\varphi)K_i^{-1}(F,\pi/2)(a_i^0) &= \cos(\varphi-(\pi/2))\, a_i^0+\sin(\varphi-(\pi/2))\, a_i,\\ K_i(F,\varphi)K_i^{-1}(F,\pi/2)(a_i) &= -\sin(\varphi-(\pi/2))\, a_i^0+\cos(\varphi-(\pi/2))\, a_i,\\ K_i(F,\varphi)K_i^{-1}(F,\pi/2)(x) &= x,\ \text{if } x\perp \operatorname{span}_W(a_i,a_i^0).\end{cases}$$

We have $K_i(F,\pi)(Fa_i)=a_i$.

Lemma 3.1 *The homotopy $K_i(F,\varphi)$ a continuous function of $F\in f_1(S)$ and φ uniformly with respect to i.*

Proof. Since $K_i(F,\varphi)$ is W–unitary, then

$$\|K_i(F',\varphi')-K_i(F,\varphi)\|\le$$
$$\le\|K_i(F',\varphi')-K_i(F',\varphi)\|+\|K_i(F',\varphi')-K_i(F,\varphi)\|\le$$
$$\le\|K_i(F',\varphi')K_i^{-1}(F',\varphi)-1\|+\|K_i(F',\varphi)K_i^{-1}(F,\varphi)-1\|. \tag{10}$$

Let us consider $\varphi,\varphi'\in[0,\pi/2]$ and $\varphi,\varphi'\in[\pi/2,\pi]$, separately. then it is clear, that the first summand can be estimated by the norm of the operator $G: W\bigoplus W\to W\bigoplus W$ with matrix

$$\begin{pmatrix} \cos(\varphi-\varphi')-1 & \sin(\varphi-\varphi')\\ -\sin(\varphi-\varphi') & \cos(\varphi-\varphi')-1\end{pmatrix}.$$

Let $\|\alpha_1e_1+\alpha_2e_2\|=1$, then $\|\alpha_1\|\le1$, $\|\alpha_2\|\le1$ and

$$\|G(\alpha_1e_1+\alpha_2e_2)\|\le\|\alpha_1\|\,\|Ge_1\|+\|\alpha_2\|\,\|Ge_2\|\le$$
$$\le\{(\cos^2(\varphi-\varphi')-2\cos(\varphi-\varphi')+1)+\sin^2(\varphi-\varphi')\}^{1/2}+$$
$$+\{\sin^2(\varphi-\varphi')+(\cos^2(\varphi-\varphi')-2\cos(\varphi-\varphi')+1)\}^{1/2}=$$
$$=2\sqrt{2}\{1-\cos(\varphi-\varphi')\}^{1/2}=4\sin\left|\frac{\varphi-\varphi'}{2}\right|\le4\left|\frac{\varphi-\varphi'}{2}\right|=2|\varphi-\varphi'|.$$

The second summand in (10) is constant while $\pi/2\le\varphi\le\pi$. Hence, let us consider $0\le\varphi\le\pi/2$, but since the choice of a_i, a_i^0, a_i^1 does not depend on F, then the second summand vanishes. □

Since we have got a uniform estimation, then from it follows

Lemma 3.2 *The family of A-unitary homomorphisms $K(F,\varphi)$ of Hilbert module $l_2(W)$, defined (when $F \in f_1(S)$, $0 \le \varphi \le \pi$) by the formula*

$$K(F,\varphi)|_{E_i} = K_i(F,\varphi),$$

is continuous in F and φ, and

$$K(F,0) = 1, \qquad K(F,\pi)(Fa_i) = a_i. \qquad \square \tag{11}$$

Let us define a homotopy $f_1 \sim f_2$ by

$$f_t(s) = K(f_1(s), \pi(t-1))\, f_1(s), \qquad 1 \le t \le 2.$$

When $t = 1$:

$$K(f_1(s), 0) = 1, \qquad f_t(s) = f_1(s),$$

when $t = 2$:

$$f_2(s) = K(f_1(s), \pi)\, f_1(s),$$

thus by (11) we get the following statement.

Lemma 3.3 *The mapping f_1 is homotopic in $\mathrm{GL}_A(W)$ to such f_2, that*

$$f_2(s)a_i = a_i. \qquad \square$$

Now we reason as in [4]. We can work now only with operators $l_2(A) \to l_2(A)$, but for the convinience of notation we will stay in $\mathrm{GL}_A(W)$.

Lemma 3.4 *Let $H' \cong l_2(W)$ be generated by W-basis $\{a_i\}$, $H_1 = (H')^\perp \cong l_2(W)$, then $f_2 \sim f_3$, where*

$$f_3(s)|_{H'} = \mathrm{Id}_{H'}, \qquad f_3(s)(H_1) = H_1.$$

Proof. With the respect to the decomposition $l_2(W) = H' \oplus H_1$ let us define the homotopy by the formula

$$f_t(s) = \begin{pmatrix} 1 & \beta(3-t) \\ 0 & \gamma \end{pmatrix}.$$

Let

$$\begin{pmatrix} \varphi & \psi \\ \chi & \xi \end{pmatrix}$$

be the inverse of

$$\begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix},$$

so

$$\begin{pmatrix} \varphi & \psi \\ \chi & \xi \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} \varphi & \varphi\beta + \psi\gamma \\ \chi & \chi\beta + \xi\gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \varphi & \psi \\ \chi & \xi \end{pmatrix} = \begin{pmatrix} \varphi + \beta\chi & \psi + \beta\xi \\ \gamma\chi & \gamma\xi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

hence

$$\varphi = 1, \qquad \chi = 0, \qquad \gamma\xi = \xi\gamma = 1,$$
$$\beta + \psi\gamma = 0, \qquad \psi + \beta\xi = 0,$$

and

$$\begin{pmatrix} 1 & \psi(3-t) \\ 0 & \xi \end{pmatrix} \begin{pmatrix} 1 & \beta(3-t) \\ 0 & \gamma \end{pmatrix} =$$
$$= \begin{pmatrix} 1 & \beta(3-t) + (3-t)\psi\gamma \\ 0 & \xi\gamma \end{pmatrix} = \begin{pmatrix} 1 & (3-t)\cdot 0 \\ 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & \beta(3-t) \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & \psi(3-t) \\ 0 & \xi \end{pmatrix} =$$
$$= \begin{pmatrix} 1 & \psi(3-t) + \beta\xi(3-t) \\ 0 & \gamma\xi \end{pmatrix} = \begin{pmatrix} 1 & (3-t)\cdot 0 \\ 0 & 1 \end{pmatrix}.$$

so the homotopy is in GL. □

Lemma 3.5 *The subset* $V \subset \mathrm{GL}$, *defined by*

$$V = \{g \in \mathrm{GL} \mid g|_{H'} = \mathrm{Id}_{H'},\ g(H_1) = H_1\},$$

is contractible inside itself to $1 \in \mathrm{GL}$.

Proof is just the same as in [4]. □

Theorem 3.6 *The space* $\mathrm{GL}(A)$ *is contractible.*

Proof. We have shown, that $f = f_0 \sim f_4 : S \to 1 \in \mathrm{GL}$, where f_t for $0 \le t \le 3$ is defined above, $f_t(s) = \eta_{t-3} f_3(s)$ for $3 \le t \le 4$, if η_r (for $0 \le r \le 1$) is the contraction from the previous lemma. All the homotopies are in $\mathrm{GL}_A(W)$. In accordance with the remark from the beginning of the paper, it is sufficient to complete the proof. □

4 The contractibility of $\mathrm{GL}^*(A)$ and $\mathrm{GL}(A)$ in the case $\mathrm{LM}(A) = \mathrm{M}(A)$

Let $B = \mathrm{LM}(A) = \mathrm{M}(A)$, then any operator $F \in \mathrm{End}\, l_2(A)$ is defined by the matrix F^i_j, $F^i_j \in B$. Moreover, if $x = (b_1, b_2, \dots) \in l_2(B)$, and we define

$$\hat{F}x = \left(\dots, \sum_{i=1}^{\infty} F^i_j b_i, \dots\right)$$

(i.e. the operator with the same matrix), then

$$\|\hat{F}x\|^2_{l_2(B)} = \|\langle \hat{F}x, \hat{F}x\rangle\|_B =$$

$$= \left\| \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} F^i_j b_i \right)^* \left(\sum_{i=1}^{\infty} F^i_j b_i \right) \right\|_B = \sup_{\substack{a\in A \\ \|a\|=1}} \left\| a^* \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} F^i_j b_i \right)^* \left(\sum_{i=1}^{\infty} F^i_j b_i \right) a \right\|_A =$$

$$= \sup_{\substack{a\in A \\ \|a\|=1}} \left\| \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} F^i_j b_i a \right)^* \left(\sum_{i=1}^{\infty} F^i_j b_i a \right) \right\|_A = \sup_{\substack{a\in A \\ \|a\|=1}} \|F(xa)\|^2 \le$$

$$\le \sup_{\substack{a\in A \\ \|a\|=1}} \|F\|^2 \|xa\|^2 = \|F\|^2\|x\|^2.$$

Quite similarly

$$\sup_{\|x\|=1} \|\hat{F}x\|^2_{l_2(B)} = \sup_{\substack{\|x\|=1 \\ \|a\|=1}} \|F(xa)\|^2 = \|F\|^2.$$

Hence, the correspondence

$$F \mapsto \hat{F}, \qquad \mathrm{End}^{(*)} l_2(A) \to \mathrm{End}^{(*)} l_2(B),$$

is a continuous isometric inclusion (and $*$–homomorphism in the case of adjointable operators). Here $(*)$ denotes that it is possible to put on this place both algebras.

Conversely, let $G \in \mathrm{End}^{(*)} l_2(B)$, then this operator is defined by matrix $\|G^i_j\|$, $G^i_j \in B$ (since B is a unital algebra). Let us define for $a = (a_1, a_2, \ldots) \in l_2(A)$

$$\check{G}a = \left(\ldots, \sum_{i=1}^{\infty} G^i_j a_i, \ldots \right).$$

Then we have the commutative diagram

$$\begin{array}{ccc} l_2(A) & \hookrightarrow & l_2(B) \\ \downarrow \check{G} & & \downarrow G \\ l_2(A) & \hookrightarrow & l_2(B) \end{array}$$

and to prove the continuity of $\check{G}$ and of the correspondence $G \mapsto \check{G}$ it is sufficient to prove, that the horiszontal inclusions are the isometries, i.e.

$$\| \sum_{i=1}^{\infty} a_i^* a_i \|_A = \| \sum_{i=1}^{\infty} a_i^* a_i \|_B,$$

which follows from the fact, that $A \hookrightarrow B$ is an isometry:

$$\|a\|_A = \sup_{\substack{x\in A \\ \|x\|=1}} \|ax\|_A = \|a\|_B.$$

Theorem 4.1 *If* $\mathrm{LM}(A) = \mathrm{M}(A)$, *then* $\mathrm{GL}^*(A)$ *and* $\mathrm{GL}\,(A)$ *are contractible.*

Proof. It is evident, that

$$\check{\hat{F}} = F, \qquad \hat{\check{G}} = G,$$

and we can identify $\mathrm{End}^{(*)} l_2(A)$ and $\mathrm{End}^{(*)} l_2(B)$, as well as $\mathrm{GL}^{(*)}(A)$ and $\mathrm{GL}^{(*)}(B)$. Since B is unital, then the statement follows from [3, 6, 9]. □

5 Proof of the contractibility of $\mathrm{GL}^*(A)$ in general case

Theorem 5.1 *The group* $\mathrm{GL}^*(A)$ *is contractible.*

Proof.
The first way. Matrices of operators from End $^*l_2(A)$ have the entries from the unital C*–algebra M(A). The argument from the previous section gives the isometry between $\mathrm{GL}^*(A)$ and $\mathrm{GL}^*(\mathrm{M}(A))$. Applying [9], [1] or [7] we finish the proof.
The second way. In this case we can choose $k(i)$ and $k'(i)+1$ in such a way, that (after small homotopy) the system $\{Fe_{k(i)}, e_{k'(i)+1}\}$ would be orthonormal (c.f. [9]). Then $J(F,\varphi)$ can be defined as $J(F,\varphi)|_{E_i} = J_i(F,\varphi)|_{E_i} : E_i \to E_i$, and to prove that it is adjointable it is sufficient to prove that J_i is such an operator and the norms of J_i^* are bounded uniform in i.

$$\langle Fe_{k(i)}\langle e_{k(i)}, F^{-1}(1-q_{k'(i)+1})x\rangle, y\rangle = \langle e_{k(i)}\langle e_{k(i)}, F^{-1}(1-q_{k'(i)+1})x\rangle, F^*y\rangle =$$
$$= \langle x, (1-q_{k'(i)+1})((F^*)^{-1}e_{k(i)})\langle e_{k(i)}, F^*y\rangle\rangle,$$
$$\langle e_{k'(i)+1}\langle e_{k(i)}, F^{-1}(1-q_{k'(i)+1})x\rangle, y\rangle = \langle F^{-1}(1-q_{k'(i)+1})x, e_{k'(i)+1}\langle e_{k'(i)+1}, y\rangle\rangle =$$
$$= \langle x, (1-q_{k'(i)+1})(F^*)^{-1}e_{k'(i)+1}\langle e_{k'(i)+1}, y\rangle\rangle,$$
$$\langle Fe_{k(i)}\langle e_{k'(i)+1}, x\rangle, y\rangle = \langle e_{k(i)}\langle e_{k'(i)+1}, x\rangle, F^*y\rangle = \langle x, e_{k'(i)+1}\langle e_{k(i)}, F^*y\rangle\rangle,$$
$$\langle e_{k'(i)+1}\langle e_{k'(i)+1}, x\rangle, y\rangle = \langle x, e_{k'(i)+1}\langle e_{k'(i)+1}, y\rangle\rangle,$$
$$(Fq_{k(i)}F^{-1})^* = (F^*)^{-1}q_{k(i)}F^*, \qquad (q_{k'(i)+1})^* = q_{k'(i)+1},$$

hence J_i is adjointable and the norms of J_i^* are uniformly bounded. The operators from the other steps of homotopy are adjointable in this case in a trivial way. □

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