Stable Harmonic Maps from Pinched Manifolds

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Abstract

In this paper, it is proved that for $n \geq 3$ there exists a constant $\delta(n)$ with $1/4 \leq \delta(n) < 1$ such that if M is a simply connected Riemannian manifold of dimension n with $\delta(n)$–pinched curvatures then for every Riemannian manifold N every stable harmonic map $\phi : M \longrightarrow N$ is constant. The proof is completely different from that of the author's previous paper and here the pinching constants are easy to compute by elementary functions.

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Stable Harmonic Maps from Pinched Manifolds

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1. Introduction

A harmonic map is a critical point of the energy functional and a harmonic map is said to be stable if for any deformation vector field, its second variation is always non–negative.

As well known, when the source or the target manifold is the Euclidean sphere $S^n (n \geq 3)$, every stable harmonic map must be constant ([4], [8]). A natural question is "Does the above fact hold too for a simply connected δ–pinched Riemannian manifold ?". Here by a δ–pinched Riemannian manifold we mean a Riemannian manifold whose sectional curvatures are between the interval $(\delta K, K]$ with constants $K > 0$ and $1 \geq \delta > 0$.

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For the case that the target manifold is a simply connected δ–pinched Riemannian manifold, Howard in 1985 proved that Let $n \geq 3$. There is a number $\delta(n)$ with $1/4 \leq \delta(n) < 1$ such that if M^n is a simply connected Riemannian manifold with $\delta(n)$–pinched curvatures then for every compact Riemannian manifold N every stable harmonic map $\phi : N \longrightarrow M^n$ is constant on [3]. Recently, Okayasu obtains a dimension–independent pinching constant. He proves in [5] that Let M^n be a compact simply connected 0.83–pinched Riemannian manifold $(n \geq 3)$: Then for every compact Riemannian manifold N , any stable harmonic map $\phi : N \longrightarrow M^n$ is constant.

There is no result for the case that the source manifold is a simply connected δ–pinched Riemannian manifold up to now. Recently, the author in a previous paper [7] gives an affirmative answer to it with dimension–depending pinching constants. But there the pinching constants are difficult to compute. The aim of the present paper is to give a new proof of the above answer in a completely different way from which one can practically compute those pinching constants. We shall prove the following

<u>Main Theorem</u>. Let $n \geq 3$. There is a number $\delta(n)$ with $1/4 \leq \delta(n) < 1$ such that if M^n is a simply connected Riemannian manifold with $\delta(n)$–pinched curvatures then for any Riemannian manifold N every stable harmonic map $\phi : M^n \longrightarrow N$ is constant.

Some values of $\delta(n)$ are given in the following table.

n	3	4	5	6	7	8	9	10	11	12
$\delta(n)$	0.94	0.95	0.95	0.96	0.96	0.97	0.97	0.97	0.97	0.98

2. Preliminaries

From now on, we always assume that M is a compact simply connected δ–pinched Riemannian manifold of dimension n.

As in [2], we normalize the δ–pinched metric of M by multiplication with $(1+\delta)/2$. Put $E = TM \oplus \epsilon(M)$, where TM is the tangent bundle of M and $\epsilon(M)$ is a trivial line bundle on M with a metric. Thus E naturally becomes a Euclidean vector bundle on M. Let e be a section of length one in $\epsilon(M)$. We define a metric connection ∇'' on E as follows:

$$\nabla''_X Y = \nabla_X Y - \langle X,Y\rangle \cdot e, \tag{1}$$

$$\nabla''_X e = X, \tag{2}$$

where X and Y are any vector fields on M, $\langle\ ,\ \rangle$ and ∇ are the Riemannian metric and connection of M, respectively. As shown in [2], the curvature R'' of ∇'' satisfies the following relations:

$$R''(X,Y)Z = R(X,Y)Z - \langle Y,Z\rangle X + \langle X,Z\rangle Y, \tag{3}$$

$$R''(X,Y)e = 0 \tag{4}$$

where X,Y,Z are any vector fields on M and $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ is the curvature operator of ∇.

Under the assumption on M, we can obtain a flat metric connection ∇' close to ∇'' exactly as in [2]. To measure the closeness, we define

$$||\nabla'-\nabla''|| := \text{Max}\left\{||\nabla'_X Y-\nabla''_X Y||;\ X \in TM,\ ||X||=1,\ Y \in E,\ ||Y||=1\right\}.$$

Note that our $||\nabla'-\nabla''||$ is half of $||\nabla'-\nabla''||$ in [1]. Set

$$k_1(\delta) = \tfrac{4}{3}(1-\delta)\delta^{-1}[1+(\delta^{1/2}\sin\tfrac{1}{2}\pi\delta^{-1/2})^{-1}]\ , \tag{5}$$

$$k_2(\delta) = [(1+\delta)/2]^{-1}\cdot k_1(\delta)\ , \tag{6}$$

$$k_3(\delta) = k_2(\delta)\cdot\left\{1+[1-\tfrac{1}{24}\pi^2(k_1(\delta))^2]^{-2}\right\}^{1/2}. \tag{7}$$

By [1, 4.13], we have

$$||\nabla'-\nabla''|| \le \tfrac{1}{2}k_3(\delta)\ . \tag{8}$$

Now let N be any Riemannian manifold of dimension m and $\phi : M \longrightarrow N$ any harmonic map from M into N. Choose local fields of orthonormal farmes $\{e_i\}$ and $\{e'_\alpha\}$ in M and N, respectively. We shall make the following convention on the ranges of indices: $1 \le i,j,k,\ldots,\le n$; $1 \le \alpha,\beta,\gamma,\ldots,\le m$, and use the summation convention. Let $\phi_* : TM \longrightarrow TN$ be the tangential map of ϕ. We also can consider ϕ_* as a $\phi^{-1}TN$ valued 1–form $d\phi$, i.e., $d\phi(X) = \phi_* X$, for $X \in TM$. The induced bundle $\phi^{-1}TN \longrightarrow M$ possesses the induced Riemannian connection as follows

$$\tilde{\nabla}_X(S\circ\phi) = (\nabla_{\phi_* X}S)\circ\phi\ , \tag{9}$$

where $X \in TM$, S is any section of $\phi^{-1}TN$, and ∇ is the Riemannian connection of N .

Set $\phi_* e_i = a_{\alpha i} e'_\alpha$ and $e(\phi) = \sum_{\alpha,i} a^2_{\alpha i}$. Then the energy of ϕ is $E(\phi) = \frac{1}{2}\int_M e(\phi)*1_M$, and the tension field of ϕ is $\tau = \sum_{\alpha,i} a_{\alpha ii} e'_{\alpha'}$, where $a_{\alpha ij}$ is the covariant derivative of $a_{\alpha i}$. For a harmonic map ϕ, $\tau = 0$, i.e., $\sum_i a_{\alpha ii} = 0$.

For any section of E , say V , we denote by V^T and V^e the TM–component and the $\epsilon(M)$–component of V , respectively. If we take $\phi_* V^T$ as the deformation vector field, the second variation formula of the energy can be reduced to the following form as shown in [6]:

$$I(\phi_* V^T, \phi_* V^T) = \int_M \langle d\phi(\nabla_{e_i}\nabla_{e_i} V^T) - 2\nabla_{e_i}(d\phi(\nabla_{e_i} V^T)) - \phi_*(Ric^M(V^T)), \phi_* V^T \rangle_N *1 , \tag{10}$$

where Ric^M is the Ricci curvature operator of M , $Ric^M(e_i) = R_{ij} e_j$.

For any fixed point $p \in M$, choosing $\{e_i\}$ such that $\nabla_{e_i} e_j |_p = 0$, we make the following calculations

$$\nabla_{e_i} V^T = (\nabla''_{e_i} V^T)^T = (\nabla''_{e_i}(V - \langle V,e\rangle e))^T$$

$$= (\nabla''_{e_i} V)^T - \langle V,e\rangle e_i , \tag{11}$$

$$\nabla_{e_i}\nabla_{e_i} V^T = \nabla_{e_i}(\nabla''_{e_i} V)^T - (\nabla_{e_i}\langle V,e\rangle)e_i - \langle V,e\rangle \nabla_{e_i} e_i$$

$$= (\nabla''_{e_i}(\nabla''_{e_i} V)^T)^T - \langle \nabla''_{e_i} V, e\rangle e_i - \langle V, \nabla''_{e_i} e\rangle e_i$$

$$= [\nabla''_{e_i}(\nabla''_{e_i}V - <\nabla''_{e_i}V,e>e)]^T - <\nabla''_{e_i}V,e>e_i - <V,e_i>e_i$$

$$= <\nabla''_{e_i}\nabla''_{e_i}V,e_j>e_j - 2<\nabla''_{e_i}V,e>e_i - <V,e_i>e_i \,. \tag{12}$$

Noting $d\phi(e_i) = a_{\alpha i}e'_\alpha$ and the harmonicity $a_{\alpha ii} = 0$, we have

$$d\phi(\nabla_{e_i}\nabla_{e_i}V^T) = <\nabla''_{e_i}\nabla''_{e_i}V,e_j>a_{\alpha j}e'_\alpha - 2<\nabla''_{e_i}V,e>a_{\alpha i}e'_\alpha - <V,e_i>a_{\alpha i}e'_\alpha \,, \tag{13}$$

$$-2\,\tilde{\nabla}_{e_i}(d\phi(\nabla_{e_i}V^T)) = -2\,\tilde{\nabla}_{e_i}(<\nabla''_{e_i}V,e_j>a_{\alpha j}e'_\alpha - <V,e>a_{\alpha i}e'_\alpha)$$

$$= -2<\nabla''_{e_i}\nabla''_{e_i}V,e_j>a_{\alpha j}e'_\alpha + 2<\nabla''_{e_j}V,e>a_{\alpha j}e'_\alpha$$

$$-2<\nabla''_{e_i}V,e_j>a_{\alpha ji}e'_\alpha + 2<\nabla''_{e_i}V,e>a_{\alpha i}e'_\alpha + 2<V,e_i>a_{\alpha i}e'_\alpha \,. \tag{14}$$

Thus, the second variation formula reduces to

$$I(\phi_*V^T,\phi_*V^T) = \int_M Q{*}1 \,, \tag{15}$$

where

$$Q = -<\nabla''_{e_i}\nabla''_{e_i}V,e_j><V,e_k>a_{\alpha j}a_{\alpha k} + 2<\nabla''_{e_i}V,e><V,e_k>a_{\alpha j}a_{\alpha k}$$

$$+ <V,e_i><V,e_k>a_{\alpha i}a_{\alpha k} - 2<\nabla''_{e_i}V,e_j><V,e_k>a_{\alpha ji}a_{\alpha k} - <V,e_j><V,e_k>R_{ij}a_{\alpha i}a_{\alpha k} \, . \tag{16}$$

3. Proof of the main theorem

We now define $\mathscr{V} = \{V \in \Gamma(E) \mid \nabla' V = 0\}$, where $\Gamma(E)$ denotes the vector space consisting of all smooth sections of E . Then $\mathscr{V}$ is isomorphic to $\mathbb{R}^{n+1}$ and has a natural inner product and $I(\phi_* V^T, \phi_* V^T)$ is a quadratic form on $\mathscr{V}$. We compute the trace of $I(\phi_* V^T, \phi_* V^T)$ over $\mathscr{V}$ and show for a appropriate chosen δ depending on n the rsult is negative if ϕ is not a constant harmonic map.

Let $\{V^r, r=1,\ldots,n+1\}$ be an orthonormal basis of $\mathscr{V}$. We get

$$\mathrm{tr}\, I(\phi_* V^T, \phi_* V^T) = \int_M \mathrm{tr} Q^* 1 \, , \tag{17}$$

and

$$\mathrm{tr} Q = -<\nabla''_{e_i}\nabla''_{e_i}V^r,e_j><V^r,e_k>a_{\alpha j}a_{\alpha k} + 2<\nabla''_{e_i}V^r,e><V^r,e_k>a_{\alpha k}a_{\alpha j}$$

$$+ <V^r,e_i><V^r,e_k>a_{\alpha i}a_{\alpha k} - 2<\nabla''_{e_i}V^r,e_j><V^r,e_k>a_{\alpha ji}a_{\alpha k} - <V^r,e_j><V^r,e_k>R_{ij}a_{\alpha i}a_{\alpha k} \, . \tag{18}$$

Lemma 1. It holds that

$$<\nabla''_{e_i} V^r, V^s> = -<\nabla''_{e_i} V^s, V^r> , \tag{19}$$

and

$$<\nabla''_{e_\ell} \nabla''_{e_i} V^r, V^s> + <\nabla''_{e_\ell} \nabla''_{e_i} V^s, V^r> + <\nabla''_{e_\ell} V^r, \nabla''_{e_i} V^s> + <\nabla''_{e_\ell} V^s, \nabla''_{e_i} V^r> = 0 . \tag{20}$$

$(r, s = 1,\dots,n + 1)$

Proof. Since $\{V^r\}$ is orthonromal to each other, we have $<V^r,V^s> = \delta_{rs}$. Differentiating it, we get

$$0 = \nabla_{e_i} <V^r,V^s> = <\nabla''_{e_i} V^r, V^s> + <\nabla''_{e_i} V^s, V^r> . \tag{21}$$

It follows that (19) holds. Differentiating (21), we get (20).

q.e.d.

In the following, we transform the bad term $-2<\nabla''_{e_i} V^r, e_j><V^r, e_k> a_{\alpha ji} a_{\alpha k}$ into a form in which the quantities can be estimated.

Noting $a_{\alpha ij} = a_{\alpha ji}$, we have

$$-2<\nabla''_{e_i} V^r, e_j><V^r, e_k> a_{\alpha ji} a_{\alpha k}$$

$$= -2\,\nabla_{e_j}\{<\nabla''_{e_i}V^r, e_j><V^r, e_k>a_{\alpha i}a_{\alpha k}\} + 2<\nabla''_{e_j}\nabla''_{e_i}V^r, e_j><V^r, e_k>a_{\alpha i}a_{\alpha k}$$

$$+2<\nabla''_{e_i}V^r, e_j><\nabla''_{e_j}V^r, e_k>a_{\alpha i}a_{\alpha k} - 2n<\nabla''_{e_i}V^r, e><V^r, e_k>a_{\alpha i}a_{\alpha k}$$

$$-2<\nabla''_{e_i}V^r, e_j><V^r, e>a_{\alpha i}a_{\alpha j} + 2<\nabla''_{e_i}V^r, e_j><V^r, e_k>a_{\alpha i}a_{\alpha kj}\,. \tag{22}$$

In the computation, since the computation is pointwisely done, we can omit the terms in which $\nabla_{e_i}e_j$ appears.

By using the Ricci identity

$$\nabla''_{e_j}\nabla''_{e_i}V^r = \nabla''_{e_i}\nabla''_{e_j}V^r + R''(e_j, e_i)V^r$$

and $a_{\alpha ii} = 0$, we have

$$2<\nabla''_{e_j}\nabla''_{e_i}V^r, e_j><V^r, e_k>a_{\alpha i}a_{\alpha k}$$

$$= 2<\nabla''_{e_i}\nabla''_{e_j}V^r, e_j><V^r, e_k>a_{\alpha i}a_{\alpha k} + 2<R''(e_j, e_i)V^r, e_j><V^r, e_k>a_{\alpha i}a_{\alpha k}$$

$$= 2\nabla''_{e_i}\{<\nabla''_{e_j}V^r, e_j><V^r, e_k>a_{\alpha i}a_{\alpha k}\} + 2<\nabla''_{e_i}V^r, e><V^r, e_k>a_{\alpha i}a_{\alpha k}$$

$$-2<\nabla''_{e_j}V^r, e_j><\nabla''_{e_i}V^r, e_k>a_{\alpha i}a_{\alpha k} + 2<\nabla''_{e_j}V^r, e_j><V^r, e>e(\phi)$$

$$-2<\nabla''_{e_j}V^r,e_j><V^r,e_k>e(\phi),_k + 2<R''(e_j,e_i)V^r,e_j><V^r,e_k>a_{\alpha i}a_{\alpha k}\,. \tag{23}$$

Now we compute

$$-<\nabla''_{e_j}V^r,e_j><V^r,e_k>e(\phi),_k$$

$$= -\nabla_{e_k}\{<\nabla''_{e_j}V^r,e_j><V^r,e_k>e(\phi)\} + <\nabla''_{e_k}\nabla''_{e_j}V^r,e_j><V^r,e_k>e(\phi)$$

$$-<\nabla''_{e_j}V^r,e><V^r,e_j>e(\phi) + <\nabla''_{e_j}V^r,e_j><\nabla''_{e_k}V^r,e_k>e(\phi) - n<\nabla''_{e_j}V^r,e_j><V^r,e>e(\phi)\,. \tag{24}$$

From (22)~(24) and using Stokes formula, we have

$$\int_M -2<\nabla''_{e_i}V^r,e_j><V^r,e_k>a_{\alpha ji}a_{\alpha k}{}^*1$$

$$= \int_M\{2(1-n)<\nabla''_{e_i}V^r,e><V^r,e_k>a_{\alpha i}a_{\alpha k} - 2<\nabla''_{e_j}V^r,e_j><\nabla''_{e_i}V^r,e_k>a_{\alpha i}a_{\alpha k}$$

$$+ (2-n)<\nabla''_{e_j}V^r,e_j><V^r,e>e(\phi)+<\nabla''_{e_k}\nabla''_{e_j}V^r,e_j><V^r,e_k>e(\phi)-<\nabla''_{e_j}V^r,e><V^r,e_j>e(\phi)$$

$$+ <\nabla''_{e_j}V^r,e_j><\nabla''_{e_k}V^r,e_k>e(\phi) + 2<\nabla''_{e_i}V^r,e_j><\nabla''_{e_j}V^r,e_k>a_{\alpha i}a_{\alpha k}$$

$$-2<\nabla''_{e_i}V^r,e_j><V^r,e>a_{\alpha i}a_{\alpha j} + 2\nabla''_{e_i}V^r,e_j><V^r,e_k>a_{\alpha i}a_{\alpha kj}$$

$$= \frac{1}{2}(<\nabla''_{e_i}V^k, e_j> + <\nabla''_{e_i}V^j, e_k>)a_{\alpha i}a_{\alpha jk}$$

$$= 0 .$$

So (28) follows.

Noting (3) and $R_{ij} = <R(e_k,e_i)e_j,e_k>$, we get (29).

q.e.d.

Concerning the second derivatives of V^r , we have

<u>Lemma 3</u>. It holds at each point p that

$$-<\nabla''_{e_i}\nabla''_{e_i}V^r, e_j><V^r, e_k>a_{\alpha j}a_{\alpha k} = \nabla''_{e_i}V^j, \nabla''_{e_i}V^k>a_{\alpha j}a_{\alpha k} , \tag{30}$$

$$<\nabla''_{e_k}\nabla''_{e_j}V^r, e_j><V^r, e_k> = -\frac{1}{2}R + \frac{1}{2}n(n-1) - \frac{1}{2}<\nabla''_{e_i}V^j, \nabla''_{e_j}V^i> - \frac{1}{2}<\nabla''_{e_j}V^j, \nabla''_{e_i}V^i> , \tag{31}$$

where R is the scalar curvature of M .

Proof. Choose an orthonormal basis $\{V^r\}$ as in Lemma 2. Letting $\ell = i$, $r = k$ and $s = j$ and then multiplying $a_{\alpha j}a_{\alpha k}$ and summing over the indices, we get

$$2<\nabla''_{e_i}\nabla''_{e_i}V^k, e_j>a_{\alpha j}a_{\alpha k} + 2<\nabla''_{e_i}V^k, \nabla''_{e_i}V^j>a_{\alpha j}a_{\alpha k} = 0 . \tag{32}$$

Thus

$$+ 2\langle R''(e_j,e_i)V^r,e_j\rangle\langle V^r,e_k\rangle a_{\alpha i}a_{\alpha k}\} * 1 . \tag{25}$$

Since the trace of Q is independent of the choice of an orthonormal basis for each fibre of E and the computation is pointwisely done, at each point $p \in M$ we can choose an orthonormal basis $\{V^1,\dots,V^n,V^{n+1}\}$ such that $V^i = e_i$, $i = 1,\dots,n$, and $V^{n+1} = e$ at the point p . Thus we have

Lemma 2. It holds at each point p that

$$\langle V^r,e_i\rangle\langle V^r,e_k\rangle a_{\alpha i}a_{\alpha k} = e(\phi) , \tag{26}$$

$$\langle V^r,e_j\rangle\langle V^r,e_k\rangle R_{ij}a_{\alpha i}a_{\alpha k} = R_{ij}a_{\alpha i}a_{\alpha j} , \tag{27}$$

$$\langle \nabla''_{e_i} V^r,e_j\rangle\langle V^r,e_k\rangle a_{\alpha i}a_{\alpha kj} = 0 , \tag{28}$$

$$\langle R''(e_j,e_i)V^r,e_j\rangle\langle V^r,e_k\rangle a_{\alpha i}a_{\alpha k} = R_{ik}a_{\alpha i}a_{\alpha k} - (n-1)e(\phi) . \tag{29}$$

Proof. Choose orthornormal basis $\{V^r\}$ as above. (26) and (27) are obvious. Letting $r = k$ and $s = j$ in (19), at the point p , we get $\langle \nabla''_{e_i} V^k,e_j\rangle = -\langle \nabla''_{e_i} V^j,e_k\rangle$. Thus we get

$$\langle \nabla''_{e_i} V^r,e_j\rangle\langle V^r,e_k\rangle a_{\alpha i}a_{\alpha kj} = \langle \nabla''_{e_i} V^k,e_j\rangle a_{\alpha i}a_{\alpha kj}$$

$$= \frac{1}{2}\left(\langle \nabla''_{e_i} V^k,e_j\rangle a_{\alpha i}a_{\alpha kj} + \langle \nabla''_{e_i} V^j,e_k\rangle a_{\alpha i}a_{\alpha jk}\right)$$

$$-<\nabla''_{e_i}\nabla''_{e_i}V^r,e_j><V^r,e_k>a_{\alpha j}a_{\alpha k} = -<\nabla''_{e_i}\nabla''_{e_i}V^k,e_j>a_{\alpha j}a_{\alpha k}$$

$$= <\nabla''_{e_i}V^k,\nabla''_{e_i}V^j>a_{\alpha j}a_{\alpha k}\ .$$

It follows that (30) holds. In a similar way, letting $\ell = r = k$, $i = s = j$ and summing over the indices, from the Ricci identity we get

$$2<\nabla''_{e_k}\nabla''_{e_j}V^k,e_j> = -\tfrac{1}{2}R + \tfrac{1}{2}n(n-1) - \tfrac{1}{2}<\nabla''_{e_i}V^j,\nabla''_{e_j}V^i> - \tfrac{1}{2}<\nabla''_{e_i}V^i,\nabla''_{e_j}V^j> \tag{33}$$

and (31) follows from (33).

q.e.d.

<u>Lemma 4</u>. It holds at each point p that

$$<\nabla''_{e_j}V^r,e><V^r,e_k> = -<\nabla''_{e_j}V^r,e_k><V^r,e>\ . \tag{34}$$

Proof. Since $\{e_1,\ldots,e_n,e\}$ is a local orthonormal basis, we have $<e_k,e> = 0$ and $<e_j,e_k> = \delta_{jk}$, i.e., $<V^r,e_k><V^r,e> = 0$ and $<V^r,e_j><V^r,e_k> = \delta_{jk}$. Differentiating the first equality, we get

$$0 = \nabla_{e_j}(<V^r,e_k><V^r,e>)$$

$$= <\nabla''_{e_j}V^r,e_k><V^r,e> - \delta_{jk}<V^r,e><V^r,e> + <\nabla''_{e_j}V^r,e><V^r,e_k>$$

$$+ <V^r,e_j><V^r,e_k>$$

$$= <\nabla''_{e_j} V^r,e_k><V^r,e> + <\nabla''_{e_j} V^r,e><V^r,e_k> .$$

q.e.d.

Now, we integrate (18). By means of Lemma 2,3,4 and (25) we get

$$\mathrm{tr}\, I(\phi_* V^T,\phi_* V^T) = \int_M \mathrm{tr} Q^*1$$

$$= \int_M \{<\nabla''_{e_i} V^j,\nabla''_{e_i} V^k> a_{\alpha j} a_{\alpha k} + 2(3-n)<\nabla''_{e_j} V^r,e_k> a_{\alpha k} a_{\alpha j}$$

$$-2<\nabla''_{e_j} V^r,e_j><\nabla''_{e_i} V^r,e_k> a_{\alpha i} a_{\alpha k} + (3-n)<\nabla''_{e_j} V^r,e_j><V^r,e>e(\phi)$$

$$-\tfrac{1}{2}<\nabla''_{e_i} V^j,\nabla''_{e_j} V^i>e(\phi) - \tfrac{1}{2}<\nabla''_{e_i} V^i,\nabla''_{e_j} V^j>e(\phi)$$

$$+ <\nabla''_{e_j} V^r,e_j><\nabla''_{e_k} V^r,e_k>e(\phi) + 2<\nabla''_{e_i} V^r,e_j><\nabla''_{e_j} V^r,e_k> a_{\alpha i} a_{\alpha k}$$

$$+ \tfrac{1}{2}n(n-1)e(\phi) - \tfrac{1}{2}R\, e(\phi) + R_{ij} a_{\alpha i} a_{\alpha j} - (2n-3)e(\phi)\}^*1 . \qquad (35)$$

Noting (8), we have the following estimations:

$$|<\nabla''_{e_i} V^j,\nabla''_{e_i} V^k> a_{\alpha j} a_{\alpha k}| \le \tfrac{1}{4}n\, k_3^2 e(\phi) ,$$

$$|<\nabla''_{e_j} V^r, e><V^r, e_k> a_{\alpha k} a_{\alpha j}| \le \frac{1}{2} k_3 e(\phi) ,$$

$$|<\nabla''_{e_j} V^r, e_j><\nabla''_{e_i} V^r, e_k> a_{\alpha i} a_{\alpha k}| \le \frac{1}{4} n\, k_3^2 e(\phi) ,$$

$$|<\nabla''_{e_j} V^r, e_j><V^r, e>| \le \frac{1}{2} n\, k_3 ,$$

$$|<\nabla''_{e_i} V^j, \nabla''_{e_j} V^i>| \le \frac{1}{4} n^2 k_3^2 ,$$

$$|<\nabla''_{e_j} V^r, e_j><\nabla''_{e_k} V^r, e_k>| \le \frac{1}{4} n^2 k_3^2 ,$$

$$|<\nabla''_{e_i} V^r, e_j><\nabla''_{e_j} V^r, e_k>| \le \frac{1}{4} n(n+1) k_3^2 . \tag{36}$$

Noting that we have normalized the δ–pinched metric of M , we have

$$R \ge \frac{2\delta}{1+\delta} n(n-1) , \tag{37}$$

and

$$R_{ij} a_{\alpha i} a_{\alpha j} \le \frac{2}{1+\delta}(n-1) . \tag{38}$$

From (36)~(38), we get the estimation:

$$\mathrm{tr}\, I(\phi_* V^T, \phi_* V^T) \le \int_M e(\phi) \cdot \left\{ \frac{7n^2+10n}{8} \cdot k_3^2 + \frac{n^2-n-6}{2} k_3 \right.$$

$$+ \frac{n^2-n+2-(n^2+3n-6)\delta}{2(1+\delta)} \Big] * 1 \qquad (39)$$

We observe that the RHS of (39) is a continous functions of δ and for any fixed $n \geq 3$ its value at $\delta = 1$ is

$$-(n-2)\int_M e(\phi)^*1 < 0 ,$$

because $k_3(1) = 0$. Thus we can take

$$\delta(n) = \inf\{\tfrac{1}{4} < \delta < 1 \text{ s.t. the RHS of (39) is negative}\} .$$

Now the main theorem is proved.

Remark 1. Since the values of $\delta(n)$ here are actually greater than 0.83. Okayasu's result holds for these $\delta(n)$–pinched Riemannian manifolds too.

Remark 2. Unfortunately, $\lim_{n\to\infty} \delta(n) = 1$, and the pinching constant depends on the dimension of the manifold. It seems that there should exist a dimension–independent pinching constant.

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