MODULI VARIETIES OF REAL AND QUATERNIONIC VECTOR BUNDLES OVER A CURVE

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Abstract. We examine the moduli problem for real and quaternionic vector bundles over a curve, and we give a gauge-theoretic construction of moduli varieties for such bundles. These moduli varieties are irreducible subsets of real points inside a complex projective variety. We relate our point of view to previous work by Biswas, Huisman and Hurtubise ([BHH10]), and we use this to study the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action $[\mathcal{E}] \mapsto [\overline{\sigma^*\mathcal{E}}]$ on moduli varieties of semistable holomorphic bundles over a complex curve with given real structure σ. We show in particular a Harnack-type theorem, bounding the number of connected components of the fixed-point set of that action by $2^g + 1$, where g is the genus of the curve. Moreover, any two such connected components are homeomorphic.

1. Introduction

A *real structure* on a smooth complex projective curve M is, by definition, a $\mathbb{C}$-antilinear involution σ of M, and the fixed points of σ are called the *real points* of M. The pair (M, σ) is called a *real space* (see [Ati66]). For a given real structure, one may define two special kinds of vector bundles over M.

Definition 1.1 (Real and quaternionic vector bundles)**.** *A complex vector bundle $(E \to M)$ is called a* ***real bundle*** *(with respect to the real structure σ on M) if it is endowed with a map $\widetilde{\sigma} : E \to E$ such that:*

(1) *The diagram*

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{\sigma}} & E \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{\sigma} & M \end{array}$$

is a commutative diagram.

(2) *$\widetilde{\sigma}$ is $\mathbb{C}$-antilinear fibrewise.*

(3) *$\widetilde{\sigma}^2 = \mathrm{Id}_E$.*

The complex vector bundle $(E \to M)$ is called a ***quaternionic bundle*** *if it is endowed with a map $\widetilde{\sigma}$ satisfying the first two conditions above, and such that $\widetilde{\sigma}^2 = -\mathrm{Id}_E$. We call $\widetilde{\sigma}$ the real or the quaternionic structure of E, respectively.*

Quaternionic bundles are also called symplectic bundles ([Har78]). In this work, we address the problem of constructing moduli varieties for families of real and quaternionic bundles, respectively. While there are various ways of achieving such a construction, it is the purpose of this paper to throw a bridge between the work of Biswas, Huiswan and Hurtubise on vector bundles over a real algebraic curve in [BHH10], and the differential geometric approach followed by the present author in [Sch09], the novelty being that we now have a characterisation of quaternionic bundles similar to the one obtained in [Sch09] for real bundles only, as well as a complete analysis of the moduli problem for real and quaternionic bundles.

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2. Real structures on moduli varieties of holomorphic bundles

The starting point of our study is the remark that the real structure $\sigma : M \to M$ induces a real structure $[E] \mapsto [\overline{\sigma^* E}]$ of the set of isomorphism classes of complex vector bundles over M. The two main features of this map are as follows: it takes a complex vector bundle E to a complex vector bundle having same rank and degree, and it fixes both real and quaternionic bundles. The first property shows that σ induces a real structure on the moduli variety $\mathcal{M}od_M^{ss}(r,d)$ of semistable holomorphic vector bundles of rank r and degree d over M, for any choice of r and d. We denote $\mathcal{M}od_{(M,\sigma)}^{ss}(r,d)$ the real space thus defined. Somewhat as a converse to the second property, one has the following result.

Proposition 2.1 ([BHH10], Proposition 3.1)**.** *Let $(\mathcal{E} \to M)$ be a stable holomorphic bundle such that $\overline{\sigma^*\mathcal{E}}$ is isomorphic to $\mathcal{E}$. Then $\mathcal{E}$ is either real or quaternionic.*

Proof. We reproduce the proof of [BHH10]. A homomorphism of vector bundles $\overline{\sigma^*\mathcal{E}} \xrightarrow{\simeq} \mathcal{E}$ covering Id_M is the same as a $\mathbb{C}$-antilinear map $\widetilde{\sigma} : \mathcal{E} \to \mathcal{E}$ covering σ. As $\sigma^2 = Id_M$, the map $\widetilde{\sigma}^2$ is a $\mathbb{C}$-linear map covering Id_M. As $\mathcal{E}$ is stable, this implies that $\widetilde{\sigma}^2 = \lambda \in \mathbb{C}^*$. Replacing φ with $\frac{1}{\sqrt{|\lambda|}}\varphi$ if necessary, we may assume that $\widetilde{\sigma}^2 \in S^1$. Then $\lambda\widetilde{\sigma} = (\widetilde{\sigma}^2)\widetilde{\sigma} = \widetilde{\sigma}(\widetilde{\sigma}^2) = \widetilde{\sigma}(\lambda\cdot) = \overline{\lambda}\widetilde{\sigma}$, so $\lambda = \overline{\lambda}$. As a consequence, $\widetilde{\sigma}^2 = \pm 1$, making $\mathcal{E}$ real or quaternionic. If $\widetilde{\sigma}'$ is another $\mathbb{C}$-antilinear map covering σ, then, as $\mathcal{E}$ is stable, $\widetilde{\sigma}' \circ \widetilde{\sigma} = \nu \in \mathbb{C}^*$, so $(\widetilde{\sigma}')^2 = |\nu|^2(\widetilde{\sigma})^2$. Therefore, $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ are either both real or both quaternionic. □

In particular, the real structure of the moduli variety is too coarse to tell the real bundles apart from the quaternionic ones (both types are real points of $[\mathcal{E}] \mapsto [\overline{\sigma^*\mathcal{E}}]$). It is desirable to remedy this and construct distinct moduli varieties for real and quaternionic bundles. In this paper, we show that differential geometry provides such a construction, along with additional topological information on the set of real points of $\mathcal{M}od_{(M,\sigma)}^{ss}(r,d)$, such as the number of connected components. Of course, it may happen that, for certain values of r, d, and the genus g of M, there exist only real bundles, or only quaternionic ones (see section 4 of [BHH10], and theorem 5.5 in the present paper), but this is not always the case (for instance, if $\sigma : M \to M$ has no fixed points, $g = 3$, $r = 1$, and $d = 2$, there are both real and quaternionic bundles over M -note that the moduli variety is a smooth projective variety in that case). If σ, g, r, and d are so chosen that there exists only real or only quaternionic bundles over (M,σ), our methods still give good understanding of the set of real points of $\mathcal{M}od_{(M,\sigma)}^{ss}(r,d)$ in terms of the topological invariants of real and quaternionic bundles.

3. Unitary connections on a fixed Hermitian bundle

The differential geometric construction of the moduli variety $\mathcal{M}od_M^{ss}(r,d)$ of semistable holomorphic bundles of rank r and degree d is due to Atiyah-Bott ([AB83]) and Donaldson ([Don83]). It gives a presentation of the moduli space $\mathcal{M}od_M^{ss}(r,d)$ as a symplectic quotient, obtained by reduction from the infinite-dimensional Kähler manifold $\mathcal{A}(E,h)$, the affine space of all unitary connections on a fixed Hermitian bundle of rank r and degree d. In this approach, the smooth projective curve M is seen as compact Kähler manifold of complex dimension one, and differential geometry enters the picture by means of the identification of the set of isomorphism classes of holomorphic vector bundles of rank r and degree d over M with $\mathcal{A}(E,h)/\mathcal{G}_{(E,h)}^{\mathbb{C}}$, where $\mathcal{G}_{(E,h)}^{\mathbb{C}}$ is the group of all complex linear automorphisms of E (the complex gauge group). As M is a compact oriented manifold of real dimension two, the Hodge star operator $*$ induces a complex structure on

$\mathcal{A}(E,h)$, and the two-form

$$\omega_A(a,b) = \int_M -\mathrm{tr}(a \wedge b),$$
$$A \in \mathcal{A}(E,h),\ a,b \in T_A\mathcal{A}(E,h) \simeq \Omega^1(M;\mathfrak{u}(E,h)),$$

is symplectic and compatible with the complex structure of $\mathcal{A}(E,h)$. The action of the gauge group $\mathcal{G}_{(E,h)}$ (the group of unitary automorphisms of (E,h)) on the space of conections is symplectic, and admits the curvature map

$$F: \begin{array}{ccc} \mathcal{A}(E,h) & \longrightarrow & \Omega^2(M;\mathfrak{u}(E,h)) \simeq \mathrm{Lie}(\mathcal{G}_{(E,h)}) \\ A & \longmapsto & F_A \end{array}$$

as a momentum map. By a theorem of Donaldson ([Don83]), a holomorphic vector bundle $\mathcal{E}$ of rank r and degree d over M is stable if and only if the $\mathcal{G}^{\mathbb{C}}_{(E,h)}$-orbit of unitary connections that it defines contains an irreducible connection A having curvature

$$F_A = * \begin{pmatrix} i2\pi\frac{d}{r} & & \\ & \ddots & \\ & & i2\pi\frac{d}{r} \end{pmatrix} = *t_{i2\pi\frac{d}{r}},$$

and such a connection is then unique up to a unitary automorphism of (E,h). A connection with constant central curvature as above is called a Yang-Mills connection (it minimises the Yang-Mills functional, defined on the space of all unitary connections, see [AB83]). Since, by a theorem of Seshadri ([Ses67]), two semistable holomorphic bundles of rank r and degree d, $\mathcal{E}$ and $\mathcal{E}'$, say, define the same point in the moduli variety if and only if the graded bundles $\mathrm{gr}(\mathcal{E})$ and $\mathrm{gr}(\mathcal{E}')$ (associated to arbitrary Jordan-Hölder filtrations of $\mathcal{E}$ and $\mathcal{E}'$, respectively) are isomorphic (see [VLP85]), it is a corollary of Donaldson's theorem that

$$\begin{array}{rcl} \mathcal{M}od^{ss}_M(r,d) & = & F^{-1}(\{*t_{i2\pi\frac{d}{r}}\})/\mathcal{G}_{(E,h)} \\ & = & \mathcal{A}(E,h)/\!\!/_{*t_{i2\pi\frac{d}{r}}}\mathcal{G}(E,h). \end{array}$$

The remainder of this paper is devoted to showing that, on $\mathcal{A}(E,h)$, one may define a finite number of distinct real structures, each one of which fixes a particular family of connections, namely those connections which define real or quaternionic bundles of a given topological type. As a consequence, we obtain connected moduli varieties for real and quaternionic bundles of each topological type. These embed into $\mathcal{M}od^{ss}_{(M,\sigma)}(r,d)$ and the closed subvarieties (defined over the field of real numbers) thus obtained are precisely the connected components of the set of real points of $[\mathcal{E}] \mapsto [\overline{\sigma^*\mathcal{E}}]$.

4. Real unitary connections

The construction recalled in the previous section is independent of the choice of the bundle E and the metric h. In the present section, we analyse the case where (E,h) is additionally endowed with a real Hermitian structure (a $\mathbb{C}$-antilinear isometry $\widetilde{\sigma}: E \to E$ which covers $\sigma: M \to M$ and squares to the identity). In particular, there is, for such a bundle, a canonical choice of an isomorphism $\varphi: \overline{\sigma^*E} \xrightarrow{\simeq} E$. The choice of a real structure $\widetilde{\sigma}$ on (E,h) also determines so-called *real invariants*, which classify $(E,h,\widetilde{\sigma})$ up to isomorphism of smooth (or topological) real Hermitian bundles (an isomorphism of vector bundles over M that commutes to the given real structures on the bundles). In the remainder of this paper, we denote M^σ the fixed-point set of $\sigma: M \to M$.

Proposition 4.1 ([BHH10], Propositions 4.1 and 4.2)**.** *One has:*

(1) *if $M^\sigma = \emptyset$, then real Hermitian bundles are topologically classified by their rank and degree. It is necessary and sufficient for a real Hermitian bundle of rank r and degree d to exist that d should be even.*
(2) *if $M^\sigma \neq \emptyset$, then $(E^{\widetilde{\sigma}} \to M^\sigma)$ is a real vector bundle in the ordinary sense, over the disjoint union $M^\sigma = \mathcal{C}_1 \sqcup \ldots \sqcup \mathcal{C}_k$ of at most $(g+1)$ circles. Denoting $w^{(j)} := w_1(E_j^{\widetilde{\sigma}}) \in H^1(S^1;\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ the first Stiefel-Whiteny class of $E^{\widetilde{\sigma}}$ restricted to $\mathcal{C}_j$, real Hermitian bundles over M are topologically classified by their rank, their degree and the sequence $(w^{(1)},\ldots,w^{(k)})$. It is necessary and sufficient for a real Hermitian bundle with given invariants r, d and $(w^{(1)},\ldots,w^{(k)})$ to exist that*
$$w^{(1)} + \cdots + w^{(k)} \equiv d \pmod 2.$$

The choice of a real structure $\widetilde{\sigma}$ on (E,h) induces real structures on the complex vector space of smooth global sections of E,

$$\begin{array}{ccl} \Omega^0(M;E) & \longrightarrow & \Omega^0(M;E) \\ s & \longmapsto & \widetilde{\sigma}\circ s\circ\sigma : (x\in M)\mapsto(\widetilde{\sigma}(s(\sigma(x)))\in E_x), \end{array}$$

and on the space of E-valued 1-forms on M,

$$\begin{array}{ccl} \Omega^1(M;E) & \longrightarrow & \Omega^1(M;E) \\ \beta & \longmapsto & \widetilde{\sigma}\circ s\circ\sigma : (v\in T_xM)\mapsto(\widetilde{\sigma}(\beta_{\sigma(x)}(T_x\sigma.v))\in E_x). \end{array}$$

Definition 4.2 (Real unitary connections)**.** *A unitary connection*
$$d_A : \Omega^0(M;E) \longrightarrow \Omega^1(M;E)$$
*is called **real** if it commutes to the real structures of $\Omega^0(M;E)$ and $\Omega^1(M;E)$:*
$$d_A(\widetilde{\sigma}\circ s\circ\sigma) = \widetilde{\sigma}\circ(d_As)\circ\sigma \text{ for all } s\in\Omega^0(M;E).$$

Thus, if d_A is a real unitary connection, the real structure of $\Omega^0(M;E)$ leaves $\ker d_A$ invariant, so (E,d_A) is a *real holomorphic* bundle. In [Sch09], we showed that, when (E,h) is real, the space $\mathcal{A}(E,h)$ of h-unitary connections on E also has a real structure. To define that real structure, we made use of the canonical isomorphism $\varphi : \overline{\sigma^*E} \to E$ determined by the real structure $\widetilde{\sigma}$ on E. If we choose a unitary frame
$$(g_{\tau\tau'} : U_\tau\cap U_{\tau'} \to \mathbf{U}_r)_{\tau,\tau'},$$
(a $\mathbf{U}_r$-valued one-cocyle representing E and subordinate to a covering $(U_\tau)_{\tau\in T}$ of M by trivialising open sets which satisfy $\sigma(U_\tau)=U_\tau$), then φ may be represented by a family $(\varphi_\tau : U_\tau \to \mathbf{U}_r)_{\tau\in T}$ such that
$$\varphi_\tau\overline{\sigma^*g_{\tau\tau'}}\varphi_\tau^{-1} = g_{\tau\tau'} \quad\text{and}\quad \overline{\sigma^*\varphi_\tau} = \varphi_\tau^{-1}.$$
One may then verify that if the family $(A_\tau\in\Omega^1(U_\tau;\mathfrak{u}(r)))_{\tau\in T}$ is a framed unitary connection on (E,h), then so is the family
$$\left(\varphi_\tau\overline{\sigma^*A_\tau}\varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1}\right)_{\tau\in T}.$$
As a matter of fact, this only uses the relation $\varphi_\tau\overline{\sigma^*g_{\tau\tau'}}\varphi_\tau^{-1} = g_{\tau\tau'}$. And the relation $\overline{\sigma^*\varphi_\tau} = \varphi_\tau^{-1}$ shows that the transformation of $\mathcal{A}(E,h)$ thus defined is an involution. We simply denote

$$\alpha_{\widetilde{\sigma}} : \begin{array}{ccc} \mathcal{A}(E,h) & \longrightarrow & \mathcal{A}(E,h) \\ A & \longmapsto & \overline{A} \end{array}$$

that real structure. This suggestive notation is justified by the following result.

Proposition 4.3 ([Sch09]). *The unitary connection* $d_A : \Omega^0(M;E) \to \Omega^1(M;E)$ *is real if, and only if,* $\overline{A} = A$.

Note that it is appropriate to call the involution $(A \mapsto \overline{A})$ of $\mathcal{A}(E,h)$ a real structure, for $\mathcal{A}(E,h)$ has a complex structure (induced on $T_A\mathcal{A}(E,h) \simeq \Omega^1(M;\mathfrak{u}(E,h))$ by the Hodge star of M), with respect to which $(A \mapsto \overline{A})$ is $\mathbb{C}$-antilinear. We now observe that there are involutions on $\Omega^2(M;\mathfrak{u}(E,h))$ and $\mathcal{G}_{(E,h)}$ which come from real structures defined respectively on $\Omega^2(M;\mathfrak{gl}(E))$ and $\mathcal{G}^{\mathbb{C}}_{(E,h)}$. The induced involutions are

$$\begin{array}{rcl} \Omega^2(M;\mathfrak{u}(E,h)) & \longrightarrow & \Omega^2(M;\mathfrak{u}(E,h)) \\ (R_\tau \in \Omega^2(U_\tau;\mathfrak{u}(r)))_{\tau\in T} & \longmapsto & (\varphi_\tau \overline{\sigma^* R_\tau} \varphi_\tau^{-1})_{\tau \in T} \end{array}$$

and

$$\begin{array}{rcl} \mathcal{G}_{(E,h)} & \longrightarrow & \mathcal{G}_{(E,h)} \\ (u_\tau \in \Omega^0(U_\tau;\mathbf{U}_r))_{\tau\in T} & \longmapsto & (\varphi_\tau \overline{\sigma^* u_\tau} \varphi_\tau^{-1})_{\tau \in T} \end{array}$$

The definition of both these involutions depends on the choice of the real structure $\widetilde{\sigma}$ on (E,h). When $M^\sigma \neq \emptyset$, the subgroup of $\mathcal{G}_{(E,h)}$ consisting of fixed points of the involution is the automorphism group of the bundle $(E^{\widetilde{\sigma}} \to M^\sigma)$, whose structural group is the orthogonal group $\mathbf{O}_r$ (the group of fixed points of the involution $u \mapsto \overline{u}$ of $\mathbf{U}_r$). We denote it $\mathcal{G}^{\widetilde{\sigma}}_{(E,h)}$ and call it the *real gauge group*. From now on, we simply denote $R \mapsto \overline{R}$ and $u \mapsto \overline{u}$ the above involutions of $\Omega^2(M;\mathfrak{u}(E,h))$ and $\mathcal{G}_{(E,h)}$.

Proposition 4.4 ([Sch09]). *One has the following compatibility relations:*

(1) *between the real structure of* $\mathcal{A}(E,h)$ *and the gauge action:*
$$\overline{u(A)} = \overline{u}(\overline{A}).$$
(2) *between the real structure of* $\mathcal{A}(E,h)$ *and the momentum map of the gauge action:*
$$F_{\overline{A}} = \overline{F_A}.$$

Moreover, $\alpha_{\widetilde{\sigma}}$ *is an antisymplectic involution of* $\mathcal{A}(E,h)$.

As $*t_{i2\pi\frac{d}{r}} \in \Omega^2(M;\mathfrak{u}(E,h))$ is a fixed point of the involution $R \mapsto \overline{R}$, the proposition shows that the real structure $A \mapsto \overline{A}$ on $\mathcal{A}(E,h)$ induces a real structure on the moduli variety
$$\mathcal{M}od^{ss}_{(M,\sigma)}(r,d) = F^{-1}(\{*t_{i2\pi\frac{d}{r}}\})/\mathcal{G}_{(E,h)}.$$
On the dense open subset of isomorphism classes of stable bundles, this real structure is simply $[\mathcal{E}] \mapsto [\overline{\sigma^*\mathcal{E}}]$, *regardless of the chosen real structure* $\widetilde{\sigma}$ *on* (E,h). In particular, quaternionic bundles are also real points for that real structure. This motivates what follows.

Instead of looking at the set of real points of the induced real structure on the moduli variety, we shall eventually look at the projection, on the moduli variety, of the set of real points for the real structure $\alpha_{\widetilde{\sigma}} : A \mapsto \overline{A}$ on $\mathcal{A}(E,h)$. To that end, we first form the so-called *real quotient*
$$\mathcal{L}^{ss}_{\widetilde{\sigma}}(r,d) := (F^{-1}(\{*t_{i2\pi\frac{d}{r}}\}))^{\alpha_{\widetilde{\sigma}}}/\mathcal{G}^{\widetilde{\sigma}}_{(E,h)}.$$
This is our would-be moduli variety of semistable real bundles of rank r and degree d having topological type that of $(E,h,\widetilde{\sigma})$. By definition, it consists of real gauge equivalence of holomorphic vector bundles of rank r and degree d which are both polystable and real, and we shall show that this is the correct space for a moduli variety of real bundles. For the next definition, we recall that the slope $\mu(\mathcal{E})$ of a

non-trivial bundle $\mathcal{E}$ is by definition the ratio of its degree and its rank, and that a subbundle $\mathcal{F}$ of $\mathcal{E}$ is called non-trivial if it is neither $\{0\}$ nor $\mathcal{E}$. Finally, a subbundle $\mathcal{F}$ of a real bundle $(\mathcal{E}, \widetilde{\sigma})$ is called real if it is invariant under $\widetilde{\sigma}$.

Definition 4.5 (Stability for real bundles)**.** *A real holomorphic bundle* $(\mathcal{E} \to M)$ *over a smooth projective curve is called:*

- ***stable****, if the underlying holomorphic bundle* $(\mathcal{E} \to M)$ *is stable, that is, if for any non-trivial subbundle* $\mathcal{F} \subset \mathcal{E}$*, the slope of* $\mathcal{F}$ *is strictly smaller than that of* $\mathcal{E}$*.*
- ***semistable****, if the underlying holomorphic bundle* $(\mathcal{E} \to M)$ *is semistable, that is, if for any non-trivial subbundle* $\mathcal{F} \subset \mathcal{E}$*, the slope of* $\mathcal{F}$ *is smaller than, or equal to, that of* $\mathcal{E}$*.*

Put plainly, a stable real bundle is a bundle which is both stable and real, and likewise for semistable real bundles. This is consistent with the definition in [HN75] and the fact that stability is a *geometric* notion. If one defines stability for real bundles by considering the slope condition for real subbundles only, the following phenomenon may occur. If $\mathcal{F}$ is a stable bundle of slope μ, then $\mathcal{F} \oplus \overline{\sigma^*\mathcal{F}}$ has a real structure given fibrewise by

$$\begin{array}{ccc} \mathcal{F}_x \oplus \overline{\mathcal{F}_{\sigma(x)}} & \longrightarrow & \mathcal{F}_{\sigma(x)} \oplus \overline{\mathcal{F}_x} \\ (v_1, v_2) & \longmapsto & (v_2, v_1). \end{array}$$

One may observe that $\mathcal{F} \oplus \overline{\sigma^*\mathcal{F}}$ interpretes nicely as the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbit of $\mathcal{F}$. As $\mathcal{F}$ is stable, any real subbundle $\mathcal{F}' \oplus \overline{\sigma^*\mathcal{F}'}$ of $\mathcal{F} \oplus \overline{\sigma^*\mathcal{F}}$ satisfies

$$\mu(\mathcal{F}' \oplus \overline{\sigma^*\mathcal{F}'}) = \mu(\mathcal{F}') < \mu(\mathcal{F}) = \mu(\mathcal{F} \oplus \overline{\sigma^*\mathcal{F}}),$$

but $\mathcal{F} \oplus \overline{\sigma^*\mathcal{F}}$ is not stable as a holomorphic bundle, for it is a direct sum. One may observe, however, that if the rank and the degree of a real holomorphic bundle $\mathcal{E}$ are coprime, then it is stable as a holomorphic bundle as soon as the slope condition is satisfied for all real subbundles. Indeed, assume this condition satisfied and let $\mathcal{F} \subset \mathcal{E}$ be a non-trivial, holomorphic subbundle of $\mathcal{E}$, not necesarily real. As $\mathcal{E}$ is real, $\overline{\sigma^*\mathcal{F}}$ may be considered a holomorphic subbundle of $\mathcal{E}$, having rank and degree equal to those of $\mathcal{F}$. Let us set $\mathcal{F}_1 := \mathcal{F} \cap \overline{\sigma^*\mathcal{F}}$ and $\mathcal{F}_2 := \mathcal{F} + \overline{\sigma^*\mathcal{F}}$, and denote $d_i := \deg(\mathcal{F}_i)$ and $r_i := \mathrm{rk}(\mathcal{F}_i)$. Both $\mathcal{F}_1$ and $\mathcal{F}_2$ are real subbundles of $\mathcal{E}$. Note that either $\mathcal{F}_1 \neq \{0\}$ or $\mathcal{F}_2 \neq \mathcal{E}$. Indeed, if $\mathcal{F}_1 = \{0\}$ and $\mathcal{F}_2 = \mathcal{E}$, then $\mathcal{E} = \mathcal{F} \oplus \overline{\sigma^*\mathcal{F}}$, contradicting the coprimality condition. Moreover,

$$r_1 + r_2 = 2\,\mathrm{rk}(\mathcal{F}) \quad \text{and} \quad d_1 + d_2 = 2\deg(\mathcal{F}),$$

so

$$\frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} = \frac{d_1 + d_2}{r_1 + r_2} < \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})},$$

for, always,

$$\frac{d_1}{r_1} < \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} \quad \text{or} \quad \frac{d_2}{r_2} < \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}.$$

If $\varphi : \mathcal{E} \to \mathcal{F}$ is a homomorphism of real bundles, then the real structure of $\mathcal{E}$ leaves $\ker\varphi$ invariant, and the real structure of $\mathcal{F}$ leaves $\mathrm{Im}\,\varphi$ invariant. This implies that the category of semistable real bundles with fixed slope μ is an Abelian category (a strict subcategory of the Abelian category of semistable holomorphic bundles of slope μ, compare [VLP85], Exposé 2). Moreover, it is stable by extensions.

Proposition 4.6. *The simple objects in the Abelian category of semistable real bundles of slope* μ *are the stable real bundles of slope* μ*.*

Corollary 4.7. *In particular, any semistable real bundle has a filtration whose successive quotients are holomorphic bundles of equal slope which are both stable and real. We call such a Jordan-Hölder filtration a* ***real Jordan-Hölder filtration****.*

Accordingly, we call a real bundle **polystable** if it is a direct sum of stable real bundles of equal slope, and this is the same as being both polystable and real. In categorical terms, polystable real bundles are the semisimple objects in the category of semistable real bundles of slope μ.

Lemma 4.8. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two stable real bundles of slope μ. Then $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic as real holomorphic bundles if, and only if, they are isomorphic as holomorphic bundles.*

Proof. It suffices to prove that, if $\mathcal{E}$ and $\mathcal{E}'$ are both stable and real, and isomorphic as holomorphic bundles, then they are isomorphic as real holomorphic bundles. To that end, take two irreducible Yang-Mills connections A and A' on E representing $\mathcal{E}$ and $\mathcal{E}'$, respectively. Since $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic and A and A' are irreducible Yang-Mills, there exists, by Donaldson's theorem, a gauge transformation $u \in \mathcal{G}_{(E,h)}$ such that $u(A) = A'$. As $\mathcal{E}$ and $\mathcal{E}'$ are real, one has $\overline{A} = A$ and $\overline{A'} = A'$, so

$$u(A) = A' = \overline{A'} = \overline{u(A)} = \overline{u}(\overline{A}) = \overline{u}(A).$$

The irreducibility of A then implies that $u^{-1}\,\overline{u} = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Put $v = e^{i\frac{\theta}{2}}u$. Then $v(A) = u(A) = A'$ and

$$\overline{v} = e^{-i\frac{\theta}{2}}\overline{u} = e^{-i\frac{\theta}{2}}e^{i\theta}u = e^{i\frac{\theta}{2}}u = v.$$

So v is a real gauge transformation taking A to A'. This gives an isomorphism of real holomorphic bundles between $\mathcal{E} = (E, d_A)$ and $\mathcal{E}' = (E, d_{A'})$. □

We are now in a position to show that $\mathcal{L}_{\widetilde{\sigma}}^{ss}(r,d)$ is the correct space for a moduli variety of semistable real bundles.

Theorem 4.9. *The map*

$$i_{\widetilde{\sigma}} : \mathcal{L}_{\widetilde{\sigma}}^{ss}(r,d) \longrightarrow \mathcal{M}od_{(M,\sigma)}^{ss}(r,d),$$

obtained by forgetting the real structure on a real holomorphic bundle, is a closed embedding -defined over $\mathbb{R}$- onto a single connected component of the set of real points of $\mathcal{M}od_{(M,\sigma)}^{ss}(r,d)$, for the real structure induced by $[\mathcal{E}] \mapsto [\overline{\sigma^*\mathcal{E}}]$. This connected component consists of moduli of semistable real holomorphic bundles of rank r and degree d with real topological invariants those determined by $\widetilde{\sigma}$.

Note that if we choose a different real structure $\widetilde{\sigma}'$ but with the *same* real invarinats as $\widetilde{\sigma}$, then $\widetilde{\sigma}' = \varphi\widetilde{\sigma}\varphi^{-1}$ for some $\varphi \in \mathcal{G}_{(E,h)}^{\widetilde{\sigma}}$, so the quotient $\mathcal{L}_{\widetilde{\sigma}}^{ss}(r,d) = (F^{-1}(\{*t_{i2\pi\frac{d}{r}}\}))^{\alpha_{\widetilde{\sigma}}}/\mathcal{G}_{(E,h)}^{\widetilde{\sigma}}$ is independent of such a choice. Also, the fact that $\alpha_{\widetilde{\sigma}}$ should be antisymplectic on $\mathcal{A}(E,h)$ implies that the embedding in the theorem is a Lagrangian embedding, meaning that $\mathcal{L}_{\widetilde{\sigma}}^{ss}(r,d)$ is sent onto a union of connected components of the fixed-point set of the involution of $\mathcal{M}od_{(M,\sigma)}^{ss}(r,d) = \mathcal{A}(E,h)/\!\!/_{*t_{i2\pi\frac{d}{r}}}\mathcal{G}(E,h)$ induced by $\alpha_{\widetilde{\sigma}}$; see [Sch09] for details).

Proof. Recall that, by Seshadri's theorem, two semistable holomorphic bundles define the same point in the moduli space $\mathcal{M}od_{(M,\sigma)}^{ss}(r,d)$ if, and only if, their associated graded bundles are isomorphic. By corollary 4.7, the points in the image of the map $i_{\widetilde{\sigma}}$ are semistable holomorphic bundles which admit a real Jordan-Hölder filtration. Since, by lemma 4.8, $i_{\widetilde{\sigma}}$ is injective when restricted to simple objects, the injectivity of $i_{\widetilde{\sigma}}$ follows, by induction on the length of the real Jordan-Hölder filtration.

Let $\mathcal{E}$ be a semistable holomorphic bundle admitting a real Jordan-Hölder filtration and with real topological invariants those determined by $\widetilde{\sigma}$. The associated graded bundle is both polystable and real so, by proposition 4.3, its holomorphic structure is defined by a real gauge orbit of real connections, that is, an element of

$$(F^{-1}(\{*t_{i2\pi\frac{d}{r}}\}))^{\alpha_{\widetilde{\sigma}}}/\mathcal{G}^{\widetilde{\sigma}}_{(E,h)}.$$

This shows that the image of $i_{\widetilde{\sigma}}$ is as stated in the theorem.
To conclude, we only need to show that $\mathcal{L}^{ss}_{\widetilde{\sigma}}(r,d)$ is connected. This is theorem 6.5 in [BHH10]. □

The theorem says that moduli varieties of semistable real bundles with a given topological type are given by real gauge equivalence classes of real Yang-Mills connections, which are the real points of a real structure on the space of Yang-Mills connections. The real moduli varieties themselves are closed connected subsets of real points in a complex projective variety with a real structure on it. Not all real points of $\mathcal{M}od^{ss}_{(M,\sigma)}(r,d)$ are moduli of real bundles, though, as we have already noted. And, when two real points do indeed correspond to real bundles, they only come come from topologically equivalent bundles if they lie in a same connected component of the set of real points of $\mathcal{M}od^{ss}_{(M,\sigma)}(r,d)$. As a more algebraic formulation of the theorem, the same type of result should hold true for a real structure, induced by $\sigma : M \to M$, on the moduli stack of semistable bundles of rank r and degree d over M.

Corollary 4.10. *Denote k the number of connected components of M^σ. Then, the number of connected components of the set of real points of $\mathcal{M}od^{ss}_{(M,\sigma)}(r,d)$ which are moduli varieties of real bundles is 2^{k-1} if $k>0$, and 1 if $k=0$. Moreover, any two such components are homeomorphic.*

Note that, by Harnack's theorem, one has $0 \leq k \leq g+1$, so the total number of moduli varieties of real bundles of rank r and degree d over a real curve (M,σ) of genus g is at most 2^g.

Proof. By the previous theorem, the connected components of $(\mathcal{M}od^{ss}_{(M,\sigma)}(r,d))^\sigma$ that we are interested in counting correspond bijectively to topological types of real Hermitian bundles $(E,h,\widetilde{\sigma})$ over (M,σ). If $k=0$, any two real holomorphic bundles having same rank and degree are topologically isomorphic, and there is only one connected component of moduli of semistable real bundles. If $k>0$, the complete real topological invariants of a real Hermitian bundle are $(w^{(1)},\dots,w^{(k)})$, where $w^{(j)}$ is the first Stiefel-Whitney class of $(E^{\widetilde{\sigma}} \to M^\sigma)$ restricted to the j^{th} connected component of M^σ. This is subject to the condition $w^{(1)}+\dots+w^{(k)} \equiv d \pmod 2$. Setting $e = d \bmod 2 \in \mathbb{Z}/2\mathbb{Z}$, we see that the equation $w^{(1)}+\dots+w^{(k)} = e$ has 2^{k-1} solutions in $\mathbb{Z}/2\mathbb{Z}$. Finally, if $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ are two real Hermitian structures with different real invariants on (E,h), then $\widetilde{\sigma}\widetilde{\sigma}'$ is a gauge automorphism of the bundle which conjugates $\alpha_{\widetilde{\sigma}}$ and $\alpha_{\widetilde{\sigma}'}$, as well as $\mathcal{G}^{\widetilde{\sigma}}_{(E,h)}$ and $\mathcal{G}^{\widetilde{\sigma}'}_{(E,h)}$ in $\mathcal{G}_{(E,h)}$, thus inducing a homeorphism between

$$\mathcal{L}^{ss}_{\widetilde{\sigma}}(r,d) = (F^{-1}(\{*t_{i2\pi\frac{d}{r}}\}))^{\alpha_{\widetilde{\sigma}}}/\mathcal{G}^{\widetilde{\sigma}}_{(E,h)}$$

and

$$\mathcal{L}^{ss}_{\widetilde{\sigma}'}(r,d) = (F^{-1}(\{*t_{i2\pi\frac{d}{r}}\}))^{\alpha_{\widetilde{\sigma}'}}/\mathcal{G}^{\widetilde{\sigma}'}_{(E,h)}.$$

□

5. Quaternionic unitary connections

We shall now treat in a similar fashion the case of quaternionic holomorphic bundles. This time, the Hermitian bundle (E, h) is endowed with a quaternionic Hermitian structure (a $\mathbb{C}$-antilinear isometry $\widetilde{\sigma} : E \to E$ which covers $\sigma : M \to M$ and squares to *minus* the identity). Just as for real bundles, this fixes a canonical isomorphism $\varphi : \overline{\sigma^* E} \xrightarrow{\simeq} E$. The topological (or smooth) classification of quaternionic Hermitian bundles is as follows.

Proposition 5.1 ([BHH10], Proposition 4.3)**.** *Denote g the genus of the smooth complex projective curve M. Quaternionic Hermitian bundles are topologically classified by their rank and degree. It is necessary and sufficient for a topological quaternionic bundle of rank r and degree d to exist over (M, σ), that*

$$d + r(g + 1) \equiv 0 \pmod 2.$$

Observe that, when $M^\sigma = \emptyset$, the complex rank of a quaternionic bundle is allowed to be odd, while when $M^\sigma \neq \emptyset$, it must be even, for the fibres of $E|_{M^\sigma} \to M^\sigma$ are left modules over the field of quaternions. The choice of a quaternionic structure on (E, h) also gives the complex vector spaces $\Omega^0(M; E)$ and $\Omega^1(M; E)$ quaternionic structures ($\mathbb{C}$-antilinear automorphisms which square to minus the identity; this turns these complex vector spaces into left modules over the field of quaternions). The quaternionic structure on $\Omega^k(M; E)$ is formally the same as the real case, with $\widetilde{\sigma}$ squaring to $-Id$ this time:

$$J^{(k)}_{\widetilde{\sigma}} : \begin{array}{ccc} \Omega^k(M; E) & \longrightarrow & \Omega^k(M; E) \\ \beta & \longmapsto & \widetilde{\sigma} \circ \beta \circ \sigma. \end{array}$$

Definition 5.2 (Quaternionic unitary connection)**.** *A unitary connection*

$$d_A : \Omega^0(M; E) \longrightarrow \Omega^1(M; E)$$

is called ***quaternionic*** *if it commutes to the quaternionic structures of $\Omega^0(M; E)$ and $\Omega^1(M; E)$:*

$$d_A(\widetilde{\sigma} \circ s \circ \sigma) = \widetilde{\sigma} \circ (d_A s) \circ \sigma \text{ for all } s \in \Omega^0(M; E).$$

Thus, if d_A is a quaternionic unitary connection on $(E, h, \widetilde{\sigma})$, the operator $J^{(0)}_{\widetilde{\sigma}}$ of $\Omega^0(M; E)$ leaves $\ker d_A$ invariant, so (E, d_A) is a *quaternionic holomorphic* bundle. We now set out to show that a *quaternionic* structure on (E, h) gives rise to a *real* structure on $\mathcal{A}(E, h)$, whose real points are exactly the quaternionic connections. This is intuitively likely because the compact symplectic group

$$\mathbf{Sp}_r = \{u \in \mathbf{U}_{2r} \mid u^t J u = J\},$$

where $J = \begin{pmatrix} 0 & \mathrm{I}_r \\ -\mathrm{I}_r & 0 \end{pmatrix}$, is the group of fixed points of the involution

$$\begin{array}{ccc} \mathbf{U}_{2r} & \longrightarrow & \mathbf{U}_{2r} \\ u & \longmapsto & J\overline{u}J^{-1}. \end{array}$$

As in the real case, we consider a covering $(U_\tau)_{\tau \in T}$ of M by open sets which trivialise E and satisfy $\sigma(U_\tau) = U_\tau$, and we represent (E, h) by $(g_{\tau\tau'} : U_\tau \cap U_{\tau'} \to \mathbf{U}_r)_{\tau,\tau'}$, and $\varphi : \overline{\sigma^* E} \xrightarrow{\simeq} E$ by $(\varphi_\tau : U_\tau \to \mathbf{U}_r)_{\tau \in T}$, in such a way that $\varphi_\tau \overline{\sigma^* g_{\tau\tau'}} \varphi_{\tau'}^{-1} = g_{\tau\tau'}$ and $\overline{\sigma^* \varphi_\tau} = -\varphi_\tau^{-1}$ (with a minus sign, this time). Then, if $(A_\tau \in \Omega^1(U_\tau; \mathfrak{u}(r)))_{\tau \in T}$ is a framed unitary connection, then so is

$$\left(\varphi_\tau \overline{\sigma^* A_\tau} \varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1}\right)_{\tau \in T}.$$

The fact that $\overline{\sigma^*\varphi_\tau} = -\varphi_\tau^{-1}$ implies that this is indeed an involution (the φ_τ appear in pairs, so the minus signs cancel out when applying the transformation twice, see the Appendix). We simply denote this involution

$$\alpha_{\widetilde{\sigma}} : \begin{array}{ccc} \mathcal{A}(E,h) & \longrightarrow & \mathcal{A}(E,h) \\ A & \longmapsto & J\overline{A}J^{-1}. \end{array}$$

Again, this suggestive notation is justified by the following result.

Proposition 5.3. *The unitary connection $d_A : \Omega^0(M;E) \to \Omega^1(M;E)$ is quaternionic if, and only if, $J\overline{A}J^{-1} = A$.*

Proof. By definition, the connection d_A is quaternionic if, and only if,

$$J^{(1)}_{\widetilde{\sigma}}(d_A s) = d_A(J^{(0)}_{\widetilde{\sigma}} s) \quad \text{for all } s \in \Omega^0(M;E),$$

which writes, locally,

$$\varphi_\tau\big(\overline{\sigma^*(ds_\tau + A_\tau s_\tau)}\big) = d\big(\varphi_\tau\overline{\sigma^* s_\tau}\big) + A_\tau\big(\varphi_\tau\overline{\sigma^* s_\tau}\big),$$

so

$$\begin{array}{rl} & \varphi_\tau\big(d(\overline{\sigma^* s_\tau})\big) + \big(\varphi_\tau\overline{\sigma^* A_\tau}\varphi_\tau^{-1}\big)\big(\varphi_\tau\overline{\sigma^* s_\tau}\big) \\ = & \big((d\varphi_\tau)\varphi_\tau^{-1}\big)\big(\varphi_\tau\overline{\sigma^* s_\tau}\big) + \varphi_\tau\big(d(\overline{\sigma^* s_\tau})\big) + A_\tau\big(\varphi_\tau\overline{\sigma^* s_\tau}\big), \end{array}$$

hence

$$\varphi_\tau\overline{\sigma^* A_\tau}\varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1} = A_\tau.$$

□

Observe that the proof is valid verbatim for real connections (as it does not use the fact that $\overline{\sigma^*\varphi_\tau} = -\varphi_\tau^{-1}$). Also similar to the case of real bundles is the fact that the involution $\alpha_{\widetilde{\sigma}} : A \to J\overline{A}J^{-1}$ anti-commutes to the complex structure of $\mathcal{A}(E,h)$, so it is indeed a real structure on $\mathcal{A}(E,h)$. Further, the vector space $\Omega^2(M;\mathfrak{u}(E,h))$ and the gauge group $\mathcal{G}_{(E,h)}$ also admit involutions, which are defined as in the case of a real structure on (E,h), by means of the canonical isomorphism $\varphi : \overline{\sigma^* E} \xrightarrow{\simeq} E$ determined by $\widetilde{\sigma}$. When $M^\sigma \neq \emptyset$, the group $\mathcal{G}^{\widetilde{\sigma}}_{(E,h)}$ of fixed points of the involution, that we now call the *quaternionic gauge group*, is the automorphism group of the bundle $(E|_{M^\sigma} \to M^\sigma)$, whose structural group is the symplectic group $\mathbf{Sp}_r$ (the group of fixed points of the involution $u \mapsto J\overline{u}J^{-1}$ of $\mathbf{U}_{2r}$).

Proposition 5.4. *One has the following compatibility relations:*

(1) *between the real structure of $\mathcal{A}(E,h)$ and the gauge action:*

$$J\big(\overline{u(A)}\big)J^{-1} = \big(J\overline{u}J^{-1}\big)\big(J\overline{A}J^{-1}\big).$$

(2) *between the real structure of $\mathcal{A}(E,h)$ and the momentum map of the gauge action:*

$$F_{J\overline{A}J^{-1}} = J\overline{F_A}J^{-1}.$$

Moreover, $\alpha_{\widetilde{\sigma}}$ is an antisymplectic involution of $\mathcal{A}(E,h)$.

Proof. We compute using local frames.

(1) One has

$$\begin{aligned}
& \big(J\overline{\big(u(A)\big)}J^{-1}\big)_\tau \\
= \;& \varphi_\tau\overline{\big(\sigma^*\big(u_\tau A_\tau u_\tau^{-1} - (du_\tau)u_\tau^{-1}\big)\big)}\varphi_\tau^{-1} - \big(d\varphi_\tau\big)\varphi_\tau^{-1} \\
= \;& \big(\varphi_\tau\overline{\sigma^* u_\tau}\varphi_\tau^{-1}\big)\big(\varphi_\tau\overline{\sigma^* A_\tau}\varphi_\tau^{-1}\big)\big(\varphi_\tau\overline{\sigma^* u_\tau}\varphi_\tau^{-1}\big) - \varphi_\tau\big(d(\overline{\sigma^* u_\tau})\overline{\sigma^* u_\tau}^{-1}\big)\varphi_\tau^{-1} \\
& -\big(d\varphi_\tau\big)\varphi_\tau^{-1} \\
= \;& \big(\varphi_\tau\overline{\sigma^* u_\tau}\varphi_\tau^{-1}\big)\big(\varphi_\tau\overline{\sigma^* A_\tau}\varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1}\big)\big(\varphi_\tau\overline{\sigma^* u_\tau}\varphi_\tau^{-1}\big) \\
& +\varphi_\tau\overline{\sigma^* u_\tau}\varphi_\tau^{-1}\big(d\varphi_\tau\big)\overline{\sigma^* u_\tau}^{-1}\varphi_\tau^{-1} - \varphi_\tau\big(d(\overline{\sigma^* u_\tau})\big)\overline{\sigma^* u_\tau}^{-1}\varphi_\tau^{-1} \\
& -\big(d\varphi_\tau\big)\varphi_\tau^{-1} \\
= \;& \big(\varphi_\tau\overline{\sigma^* u_\tau}\varphi_\tau^{-1}\big)\big(\varphi_\tau\overline{\sigma^* A_\tau}\varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1}\big)\big(\varphi_\tau\overline{\sigma^* u_\tau}\varphi_\tau^{-1}\big) \\
& -\big(d(\varphi_\tau\overline{\sigma^* u_\tau}\varphi_\tau^{-1})\big)\big(\varphi_\tau\overline{\sigma^* u_\tau}\varphi_\tau^{-1}\big)^{-1} \\
= \;& \big(J\overline{u}J^{-1}\big)_\tau\big(J\overline{A}J^{-1}\big)_\tau.
\end{aligned}$$

(2) One has

$$\begin{aligned}
& \big(F_{J\overline{A}J^{-1}}\big)_\tau \\
= \;& d\big(\varphi_\tau\overline{\sigma^* A_\tau}\varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1}\big) \\
& +\big(\varphi_\tau\overline{\sigma^* A_\tau}\varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1}\big) \wedge \big(\varphi_\tau\overline{\sigma^* A_\tau}\varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1}\big) \\
= \;& \big(d\varphi_\tau\big) \wedge \big(\overline{\sigma^* A_\tau}\varphi_\tau^{-1}\big) + \varphi_\tau\overline{\sigma^*\big(dA_\tau\big)}\varphi_\tau^{-1} \\
& -\big(\varphi_\tau\overline{\sigma^* A_\tau}\big) \wedge \big(d(\varphi_\tau^{-1})\big) + \big(d\varphi_\tau\big) \wedge \big(d(\varphi_\tau^{-1})\big) + \varphi_\tau\big(\overline{\sigma^* A_\tau} \wedge \overline{\sigma^* A_\tau}\big)\varphi_\tau^{-1} \\
& -\big(\varphi_\tau\overline{\sigma^* A_\tau}\varphi_\tau^{-1}\big) \wedge \big((d\varphi_\tau)\varphi_\tau^{-1}\big) - \big((d\varphi_\tau)\varphi_\tau^{-1}\big) \wedge \big(\varphi_\tau\overline{\sigma^* A_\tau}\varphi_\tau^{-1}\big) \\
& +\big((d\varphi_\tau)\varphi_\tau^{-1}\big) \wedge \big((d\varphi_\tau)\varphi_\tau^{-1}\big) \\
= \;& \varphi_\tau\overline{\sigma^*\big(dA_\tau + A_\tau \wedge A_\tau\big)}\varphi_\tau^{-1} + \big(d\varphi_\tau\big) \wedge \big(\overline{\sigma^* A_\tau}\varphi_\tau^{-1}\big) \\
& +\big(\varphi_\tau\overline{\sigma^* A_\tau}\varphi_\tau^{-1}\big) \wedge \big((d\varphi_\tau)\varphi_\tau^{-1}\big) - \big((d\varphi_\tau)\varphi_\tau^{-1}\big) \wedge \big((d\varphi_\tau)\varphi_\tau^{-1}\big) \\
& -\big(\varphi_\tau\overline{\sigma^* A_\tau}\varphi_\tau^{-1}\big) \wedge \big((d\varphi_\tau)\varphi_\tau^{-1}\big) - \big(d\varphi_\tau\big) \wedge \big(\overline{\sigma^* A_\tau}\varphi_\tau^{-1}\big) \\
& +\big((d\varphi_\tau)\varphi_\tau^{-1}\big) \wedge \big((d\varphi_\tau)\varphi_\tau^{-1}\big) \\
= \;& \varphi_\tau(F_A)_\tau\varphi_\tau^{-1} + 0 \\
= \;& \big(J\overline{F_A}J^{-1}\big)_\tau.
\end{aligned}$$

□

Consequently, $\alpha_{\widetilde{\sigma}}$ induces a real structure on $\mathcal{M}od^{ss}_{(M,\sigma)}(r,d)$, which again coincides with the one induced by $[\mathcal{E}] \mapsto [\overline{\sigma^*\mathcal{E}}]$. From this point, we can carry over the analysis of the moduli problem that we made in the real case over to the quaternionic case. The definition of stability for quaternionic bundles is, again, stability of the underlying holomorphic bundle. Note that a bundle of the form $\mathcal{F} \oplus \overline{\sigma^*\mathcal{F}}$ admits a quaternionic structure defined fibrewise by

$$\begin{array}{ccc}
\mathcal{F}_x \oplus \overline{\mathcal{F}_{\sigma(x)}} & \longrightarrow & \mathcal{F}_{\sigma(x)} \oplus \overline{\mathcal{F}_x} \\
(v_1, v_2) & \longmapsto & (-v_2, v_1).
\end{array}$$

If we form the quotient

$$\mathcal{L}^{ss}_{\widetilde{\sigma}}(r,d) = (F^{-1}(\{*t_{i2\pi\frac{d}{r}}\}))^{\alpha_{\widetilde{\sigma}}}/\mathcal{G}^{\widetilde{\sigma}}_{(E,h)},$$

consisting of quaternionic gauge equivalence classes of holomorphic bundles of rank r and degree d which are both polystable and quaternionic, we obtain, as in the real case, a closed (Lagrangian) embedding

$$i_{\widetilde{\sigma}} : \mathcal{L}^{ss}_{\widetilde{\sigma}}(r,d) \hookrightarrow \mathcal{M}od^{ss}_{(M,\sigma)}(r,d)$$

onto a single connected component of the set of real points of $\mathcal{M}od^{ss}_{(M,\sigma)}(r,d)$. This connected component consists of moduli of semistable quaternionic bundles of rank r and degree d. It is non-empty if, and only if, $d+r(g+1) \equiv 0 \pmod 2$. In that case, this connected component is homeomorphic to any given connected component of the set of real points of $\mathcal{M}od^{ss}_{(M,\sigma)}(r,d)$, such a homeomorphism being induced by the gauge transformation $\widetilde{\sigma}\widetilde{\sigma}'$ of (E,h), where $\widetilde{\sigma}$ is quaternionic and $\widetilde{\sigma}'$ is real. Assembling our results, we obtain the following theorem.

Theorem 5.5. *Let (M,σ) be a smooth complex projective curve with a real structure. Denote k the number of connected components of M^σ. There is a real structure $[\mathcal{E}] \mapsto [\overline{\sigma^*\mathcal{E}}]$ on the moduli space $\mathcal{M}od^{ss}_{(M,\sigma)}(r,d)$ of semistable holomorphic bundles of rank r and degree d and the number of non-empty connected components of the set of real points of $\mathcal{M}od^{ss}_{(M,\sigma)}(r,d)$ is:*

- 0 *if $k=0$, $d \equiv 1 \pmod 2$, and $r(g+1) \equiv 0 \pmod 2$. In that case, there are neither real nor quaternionic bundles over (M,σ) -note that this is so for purely topological reasons.*
- 1 *if $k=0$, $d \equiv 0 \pmod 2$, and $r(g+1) \equiv 1 \pmod 2$. In that case, there are only real bundles over (M,σ).*
- 1 *if $k=0$, $d \equiv 1 \pmod 2$, and $r(g+1) \equiv 1 \pmod 2$. In that case, there are only quaternionic bundles over (M,σ).*
- 2 *if $k=0$, $d \equiv 0 \pmod 2$, and $r(g+1) \equiv 0 \pmod 2$. In that case, there are both real and quaternionic bundles over (M,σ).*
- 2^{k-1} *if $k>0$ and $r \equiv 1 \pmod 2$. In that case, there are only real bundles over (M,σ).*
- 2^{k-1} *if $k>0$, $r \equiv 0 \pmod 2$ and $d \equiv 1 \pmod 2$. In that case, there are only real bundles over (M,σ).*
- $2^{k-1}+1$ *if $k>0$, $r \equiv 0 \pmod 2$ and $d \equiv 0 \pmod 2$. In that case, there are both real and quaternionic bundles over (M,σ).*

Any two connected components are homeomorphic and a given connected component is, in a natural way, a moduli variety for real or quaternionic holomorphic bundles of a given topological type.

These results are consistent with those obtained by Gross and Harris in the rank 1 case ([GH81]).

Corollary 5.6. *The number of connected components of the set of real points of $\mathcal{M}od^{ss}_{(M,\sigma)}(r,d)$ is at most 2^g+1.*

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Appendix

In the present appendix, we perform some verifications that are perhaps too elementary to appear in the body of the paper. We shall represent by $(g_{\tau\tau'} : U_\tau \cap U_{\tau'} \to \mathbf{U}_r)_{\tau,\tau'}$ a real or quaternionic bundle $(E, h, \widetilde{\sigma})$, and by $(\varphi_\tau : U_\tau \to \mathbf{U}_r)_{\tau\in T}$ the canonical isomorphism $\varphi : \overline{\sigma^* E} \stackrel{\simeq}{\to} E$ determined by $\widetilde{\sigma}$. The open covering $(U_\tau)_{\tau\in T}$ is assumed to satisfy $\sigma(U_\tau) = U_\tau$ (such a covering always exists, see [Ati66], page 374).

- If $(A_\tau : U_\tau \to \Omega^1(U_\tau; \mathfrak{u}(r)))_{\tau\in T}$ is a framed unitary connection on (E, h), then so is the family $(B_\tau := \varphi_\tau \overline{\sigma^* A_\tau} \varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1})_{\tau\in T}$. One needs to check that
$$B_\tau = g_{\tau\tau'} B_{\tau'} g_{\tau\tau'}^{-1} - (dg_{\tau\tau'}) g_{\tau\tau'}^{-1},$$
using that $A_\tau = g_{\tau\tau'} A_{\tau'} g_{\tau\tau'}^{-1} - (dg_{\tau\tau'}) g_{\tau\tau'}^{-1}$ and $\varphi_\tau \overline{\sigma^* g_{\tau\tau'}} \varphi_{\tau'}^{-1} = g_{\tau\tau'}$. Indeed,
$$\begin{aligned} & g_{\tau\tau'} B_{\tau'} g_{\tau\tau'}^{-1} - (dg_{\tau\tau'}) g_{\tau\tau'}^{-1} \\ = \ & g_{\tau\tau'} (\varphi_{\tau'} \overline{\sigma^* A_{\tau'}} \varphi_{\tau'}^{-1} - (d\varphi_{\tau'})\varphi_{\tau'}^{-1}) g_{\tau\tau'}^{-1} - \big(d(\varphi_\tau \overline{\sigma^* g_{\tau\tau'}} \varphi_{\tau'}^{-1})\big) \varphi_{\tau'} \overline{\sigma^* g_{\tau\tau'}}^{-1} \varphi_\tau^{-1} \\ = \ & \varphi_\tau \overline{\sigma^* (g_{\tau\tau'} A_{\tau'} g_{\tau\tau'}^{-1})} \varphi_\tau^{-1} - g_{\tau\tau'} (d\varphi_{\tau'}) \varphi_{\tau'}^{-1} g_{\tau\tau'}^{-1} - (d\varphi_\tau)\varphi_\tau^{-1} \\ & - \varphi_\tau \overline{\sigma^* \big((dg_{\tau\tau'}) g_{\tau\tau'}^{-1}\big)} \varphi_\tau^{-1} + \varphi_\tau \overline{\sigma^* g_{\tau\tau'}} \varphi_{\tau'}^{-1} (d\varphi_{\tau'}) \overline{\sigma^* g_{\tau\tau'}}^{-1} \varphi_\tau^{-1} \\ = \ & \varphi_\tau \overline{\sigma^* A_\tau} \varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1} \\ = \ & B_\tau. \end{aligned}$$

- The relation $\overline{\sigma^* \varphi_\tau} = \pm \varphi_\tau^{-1}$ implies that the map $(A_\tau)_{\tau\in T} \mapsto (B_\tau)_{\tau\in T}$ is an involution of $\mathcal{A}(E, h)$. Indeed,
$$\begin{aligned} & \varphi_\tau B_\tau \varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1} \\ = \ & \varphi_\tau \big(\overline{\sigma^* \varphi_\tau} A_\tau \overline{\sigma^* \varphi_\tau}^{-1} - \big(d(\overline{\sigma^* \varphi_\tau})\big) \overline{\sigma^* \varphi_\tau}^{-1}\big) \varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1} \\ = \ & (\pm 1) A_\tau (\pm 1) - (\pm 1) \varphi_\tau \big(d(\varphi_\tau^{-1})\big)(\pm 1) \varphi_\tau \varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1} \\ = \ & A_\tau + (d\varphi_\tau)\varphi_\tau^{-1} - (d\varphi_\tau)\varphi_\tau^{-1} \\ = \ & A_\tau. \end{aligned}$$

- If $(s_\tau : U_\tau \to \mathbb{C}^r)_{\tau\in T}$ is framed global section of E, then so is $(\varphi_\tau \overline{\sigma^* s_\tau})_{\tau\in T}$. One needs to check that $g_{\tau\tau'}(\varphi_{\tau'} \overline{\sigma^* s_{\tau'}}) = \varphi_\tau \overline{\sigma^* s_\tau}$, using that $g_{\tau\tau'} s_{\tau'} = s_\tau$ and $\varphi_\tau \overline{\sigma^* g_{\tau\tau'}} \varphi_{\tau'}^{-1} = g_{\tau\tau'}$. Indeed,
$$\begin{aligned} g_{\tau\tau'} \varphi_{\tau'} \overline{\sigma^* s_{\tau'}} &= g_{\tau\tau'} \varphi_{\tau'} \overline{\sigma^* g_{\tau\tau'}}^{-1} \overline{\sigma^* s_\tau} \\ &= \varphi_\tau \overline{\sigma^* s_\tau}. \end{aligned}$$

- If $(u_\tau : U_\tau \to \mathbf{U}_r)_{\tau \in T}$ is a framed gauge automorphism (a frame global section of $U(E,h)$), then so is $(\varphi_\tau \overline{\sigma^* u_\tau} \varphi_\tau^{-1})_{\tau \in T}$. As $(\mathrm{Int}\, g_{\tau\tau'})_{\tau,\tau'}$ is a 1-cocycle representing $U(E,h)$, one has $g_{\tau\tau'} u_{\tau'} g_{\tau\tau'}^{-1} = u_\tau$, and one needs to check that
$$g_{\tau\tau'}(\varphi_{\tau'} \overline{\sigma^* u_{\tau'}} \varphi_{\tau'}^{-1}) g_{\tau\tau'}^{-1} = \varphi_\tau \overline{\sigma^* u_\tau} \varphi_\tau^{-1}.$$
Indeed,
$$\begin{aligned} g_{\tau\tau'}(\varphi_{\tau'} \overline{\sigma^* u_{\tau'}} \varphi_{\tau'}^{-1}) g_{\tau\tau'}^{-1} &= (g_{\tau\tau'} \varphi_{\tau'} \overline{\sigma^* g_{\tau\tau'}}^{-1}) \overline{\sigma^* u_\tau} (\overline{\sigma^* g_{\tau\tau'}} \varphi_{\tau'}^{-1} g_{\tau\tau'}^{-1}) \\ &= \varphi_\tau \overline{\sigma^* u_\tau} \varphi_\tau^{-1}. \end{aligned}$$
The same proof shows that, if $(R_\tau \in \Omega^2(U_\tau; \mathfrak{u}(r)))_{\tau \in T}$ is a framed global section of $\Omega^2(M; \mathfrak{u}(E,h))$, then so is $(\varphi_\tau \overline{\sigma^* R_\tau} \varphi_\tau^{-1})_{\tau \in T}$.

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