VANISHING AND NON-VANISHING THEOREMS

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MPI/87-61

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At the C.I.R.M. conference in Luminy the first author gave a report on "Logarithmic DeRham complexes" and sketched the vanishing theorems as well as the applications included in [4] and [5]. The Kodaira-Nakano vanishing theorem can often be improved by regarding the "logarithmic version" of the vanishing theorem for invertible sheaves directly. Following this theme we discuss in this note some applications already indicated but not worked out in [4] and [5].

In particular, using remark 2.3.6 in [4], we prove A. Sommese's vanishing theorem for k-ample invertible sheaves $\mathcal{L}$, with an improvement on the bounds if k is larger than the Iitaka dimension $\kappa(\mathcal{L})$ (§2).

§1 contains some remarks concerning cohomology of local constant systems. We recall methods from [4] as far as they are needed in §2 and §3.

In §3 we just extend [5] to local constant systems of rank one without imposing conditions on the monodromy. This part was motivated by a talk by A.N. Varchenko at the International Conference on Topology at Baku (October 1987) on "Combinatoric and Topology of Configurations of Hyperplanes" where he used an explicite description by differential forms of a base of $H^n(\mathbb{R}^n, \bigcup_{i=1}^{N} A_i;\mathbb{C})$ for N hyperplanes A_i in general position. We reformulate the content of [5] in such a

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1 Supported by "Deutsche Forschungsgemeinschaft, Heisenberg Program"

way that the main result, the non-vanishing of cohomology classes given by certain differential forms, can be applied to constant coefficients as well.

Recently several authors studied vanishing theorems for logarithmic differential forms (for example D. Arapura [1] and K. Maehara [7]). Some of the results described here may overlap with some contained explicitly or implicitly in their papers.

Throughout this note we use the notations introduced in [4].

X will always denote a connected complex compact manifold of dimension n, bimeromorphically dominated by a Kähler manifold and $D = \sum_{i=1}^{s} D_i$ a normal crossing divisor on X. We write $U = X - D$ and $j : U \longrightarrow X$ for the inclusion.

§1 Local constant systems and logarithmic DeRham complexes

Definition 1.1. Let $g : Y \longrightarrow Z$ be a morphism of analytic varieties. We define $r(g) = \mathrm{Max}\{\dim \Gamma - \dim g(\Gamma) - \operatorname{codim} \Gamma;\ \Gamma$ closed subvariety of $Y\}$.

Of course we can write as well $r(g) = \mathrm{Max}\{\dim$ (generic fibre of $g|_\Gamma) - \operatorname{codim} \Gamma;\ \Gamma$ closed subvariety of $Y\}$
and moreover, there are only finitely many subvarieties Γ where the maximum is achieved. If b denotes the maximal fibre dimension of g and $\kappa = \dim Z$ one has $r(g) \leq \mathrm{Max}\{\dim Y - \kappa;\ b - 1\}$.

Let $\mathcal{V}$ be a closed local constant system on U.

Lemma 1.2. (see [4], 2.3.6). Assume that there exists a proper surjective morphism g from U to an affine variety W.
Then $H^k(U,\mathcal{V}) = 0$ for $k > n + r(g)$.

Proof. By [9], 2.3.1 the sheaves $R^q g_* \mathcal{V}$ are analytically constructible and $S_q = \mathrm{Support}\ (R^q g_* \mathcal{V})$ must be a Stein space. Since $2 \cdot (\dim g^{-1}(S_q) - \dim S_q) \geq q$ one has $H^p(W, R^q g_* \mathcal{V}) = 0$ for $p + q > n + r(g) \geq 2 \dim g^{-1}(S_q) - \dim S_q \geq q + \dim S_q$. By the Leray spectral sequence

$$E_2^{pq} = H^p(W, R^q g_* \mathcal{V}) \Rightarrow H^{p+q}(U,\mathcal{V})$$

one obtains 1.2.

Corollary 1.3. If in addition none of the monodromies of $\mathcal{V}$ around D_i has one as an eigenvalue then $H^k(U,\mathcal{V}) = 0$ for $k < n - r(g)$ as well.

Proof. The condition on the monodromy is equivalent to $Rj_*\mathcal{V} = j_!\mathcal{V}$ (see [4], 1.6). By Poincaré duality one has

$$\begin{aligned} H^k(U,\mathcal{V}) &= \mathbb{H}^k(X, Rj_*\mathcal{V}) = \mathbb{H}^k(X, j_!\mathcal{V}) = H_c^k(U,\mathcal{V}) = \\ &= H^{2n-k}(U, \mathrm{Hom}_{\mathbb{C}}(\mathcal{V},\mathbb{C})) \end{aligned}$$

and by 1.2 all the cohomology groups are zero for $2n - k > n + r(g)$.

1.4. From now on we fix a locally free $\mathcal{O}_X$-module $\mathcal{M}$ and a logarithmic holomorphic integrable connection

$$\nabla : \mathcal{M} \longrightarrow \Omega^1_X(\log D) \otimes \mathcal{M}$$

with $\mathcal{V} = \mathrm{Ker}(\nabla|_U)$.
By the Riemann-Hilbert correspondence of P. Deligne, [2], such a pair $(\mathcal{M},\nabla)$ exists. ∇ gives rise to a logarithmic DeRham complex $\Omega^\bullet_X(\log D) \otimes \mathcal{M}$, quasi isomorphic to $\mathcal{V}$ on U. Let $\Gamma_i \in \mathrm{End}(\mathcal{O}_{D_i} \otimes \mathcal{M})$ be the residue of ∇ along D_i, i.e. the endomorphism

$$\mathcal{M} \xrightarrow{\nabla} \Omega^1_X(\log D) \otimes \mathcal{M} \longrightarrow \mathcal{O}_{D_i} \otimes \mathcal{M}.$$

If none of the eigenvalues of Γ_i lies in $\mathbb{Z}_{>0}$, for $i = 1,\dots,s$, the complexes $\Omega^\bullet_X(\log D) \otimes \mathcal{M}$ and $Rj_*\mathcal{V}$ are quasi isomorphic (see [2]). By duality ([4], Appendix A, for example) $\Omega^\bullet_X(\log D) \otimes \mathcal{M}$ is quasi isomorphic to $j_!\mathcal{V}$ if none of the eigenvalues of the Γ_i lies in $\mathbb{Z}_{\leq 0}$.

More generally, let us assume that we can write $D = D^* + D^!$ such that none of the eigenvalues of Γ_i lies in $\mathbb{Z}_{>0}$, if $D_i \leq D^*$, and none in $\mathbb{Z}_{\leq 0}$, if $D_i \leq D^!$. Writing

$$\begin{tikzcd} U=X-D \arrow[r, "\upsilon'"'] \arrow[d, "\sigma'"] & X-D^* \arrow[d, "\sigma"] \\ X-D^! \arrow[r, "\upsilon"'] & X \end{tikzcd}$$

we have

<u>Lemma 1.5.</u> The three complexes $\Omega^\bullet_X(\log D) \otimes \mathcal{M}$, $R\sigma_*\upsilon'_!\mathcal{V}$ and $\upsilon_!R\sigma'_*\mathcal{V}$ are quasi isomorphic.

Proof. By [4], A.2 and 1.2,e, the Verdier duality exchanges the role of D^* and $D^!$. Therefore it is sufficient to prove that the first two complexes are quasi isomorphic. $R\sigma_* \upsilon_! \mathcal{V}$ is quasi isomorphic to $R\sigma_*(\Omega^{\cdot}_{X-D*}(\log D^!) \otimes \mathcal{M})$. To show that the map

$$\Omega^{\cdot}_X(\log D) \otimes \mathcal{M} \longrightarrow R\sigma_*(\Omega^{\cdot}_{X-D*}(\log D^!) \otimes \mathcal{M})$$

is a quasi isomorphism as well we may reduce the statement to poly-disks and then to rank one sheaves $\mathcal{M}$ (following the proof of II,3.13 in [2], as we did in [4], A.8). In this case one may assume $(\mathcal{M},\nabla)$ to be the product of rank one sheaves with connections obtained by pull-back from those living on disks. Since $R\sigma_*$ is compatible with this construction we are reduced to the one-dimensional case, where the statement is a consequence of the quotations made above.

1.6 The main lemma ([4], 2.2 and 2.3)

Assume that g is a proper surjective morphism from U to an affine variety W. Let $\mathcal{M}$ be a locally free $\mathcal{O}_X$-module and ∇ a holomorphic integrable connection of $\mathcal{M}$ with logarithmic poles along D. Assume that none of the eigenvalues of the residues Γ_i is an integer. If the spectral sequence

$$E_1^{pq}(\mathcal{M}) = H^q(X,\Omega^p_X(\log D) \otimes \mathcal{M}) \Rightarrow \mathbb{H}^{p+q}(X,\Omega^{\cdot}_X(\log D) \otimes \mathcal{M})$$

degenerates at E_1, then

$$H^q(X,\Omega^p_X(\log D) \otimes \mathcal{M}) = 0 \quad \text{for}$$

$p + q > n + r(g)$ and for $p + q < n - r(g)$.

Proof. The assumptions just imply $H^k(U,\mathrm{Ker}\,\nabla|_U) \cong \bigoplus_{p+q=k} E_1^{pq}(\mathcal{M})$. Then 1.2 and 1.3 finish the proof.

Remark 1.7. Under the monodromy assumptions of 1.6 it is sometimes useful to introduce additional divisors C and E such that $C + E + D$ has still normal crossings and to study the complex

$$\Omega^{\bullet}_X(\log(D + C + E)) \otimes \mathcal{M} \otimes \mathcal{O}_X(-E).$$

If one denotes the inclusions by

$$\begin{tikzcd} U-(C+E) \arrow[r, "\upsilon'"] \arrow[d, "\sigma'"] & U-C \arrow[r, "j'"] \arrow[d, "\sigma"] & X-C \arrow[d, "\eta"] \\ U-E \arrow[r, "\upsilon"'] & U \arrow[r, "j"'] & X \end{tikzcd}$$

the complex is quasi isomorphic to each of:

$$R(j \circ \sigma)_* \upsilon'_! \mathcal{V}', \quad (j \circ \upsilon)_! R\sigma'_* \mathcal{V}' \quad \text{or}$$
$$R\eta_*(j' \circ \upsilon')_! \mathcal{V}' \text{ where } \mathcal{V}' = \mathcal{V}|_{U-(C+E)}$$

(use 1.5).

Assume that the spectral sequence

$$H^q(X,\Omega^p_X(\log(D+E+C))\otimes \mathcal{M} \otimes \mathcal{O}_X(-E)) \Rightarrow \mathbb{H}^{p+q}(X,\Omega^{\bullet}_X(\log(D+E+C)) \otimes \mathcal{M} \otimes \mathcal{O}_X(-E))$$

degenerates at E_1. Then again, geometric properties of (X,D,E,C) imply the vanishing of some of the cohomology groups occuring as E_1-terms in the spectral sequence (see 2.1).

The following lemma is, for $E = \phi$, one example which will be needed in 2.6.

<u>Lemma 1.8.</u> Assume in addition that $g : U \longrightarrow W$ is smooth and that $C|_U$ has relative normal crossings. Then $H^q(X,\Omega^p_X(\log(D+C))\otimes\mathcal{M}) = 0$ for $p + q < \dim W$.

<u>Proof.</u> The pair (U,C) is locally topological trivial over W. Therefore - keeping the notations from (1.7) - the cohomology of $R(g \circ \sigma)_*\mathcal{V}'$ is locally constant. Then

$$H^k(U,R\sigma_*\mathcal{V}') = H^k_c(U,R\sigma_*\mathcal{V}') = \mathbb{H}^k_c(W,R(g \circ \sigma)_*\mathcal{V}') = 0 \quad \text{for} \quad k < \dim W.$$

§2 Vanishing theorems for k-ample invertible sheaves

2.1. The main example. Even if the statements obtained in [4] or §1 are more general, many applications follow from the example [4], 2.7:

Let $\mathcal{L}$ be an invertible sheaf such that $\mathcal{L}^N = \mathcal{O}_X(\sum_{i=1}^{s} \nu_i D_i)$ with $N > \nu_i \geq 0$. Then $\mathcal{L}^{-1}$ has a holomorphic integrable connection ∇ with logarithmic poles along D, whose residues Γ_i are the multiplication with ν_i/N and the spectral sequence $E_1^{pq}(\mathcal{L}^{-1})$ degenerates at E_1. In fact the complex $\Omega_X^\bullet(\log D) \otimes \mathcal{L}^{-1}$ is a direct summand of a complex $\pi_*\Omega_Y^\bullet(\log \pi^*D)$ where $\pi : Y \longrightarrow X$ is the desingularization of the cyclic cover obtained by taking the N-th root of the section with zero divisor $\sum_{i=1}^{s} \nu_i D_i$. The $E_1(\mathcal{L}^{-1})$-degeneration is implied by P. Deligne's theorem, that the logarithmic Hodge-DeRham spectral sequence degenerates at E_1. If X is algebraic, P. Deligne and L. Illusie gave recently a beautiful purely algebraic proof by reduction modulo p of this theorem ([3]). There one also finds a proof of the degeneration of the spectral sequence given by $\Omega_Y^\bullet(\log(\pi^*D)) \otimes \mathcal{O}_Y(-B)$ for any reduced subdivisor B of D. Interpreted in the same way we obtain the degeneration of

$$\begin{aligned} &H^q(X,\Omega_X^p(\log(D + E + C)) \otimes \mathcal{L}^{-1} \otimes \mathcal{O}_X(-E)) \Rightarrow \\ &\Rightarrow \mathbb{H}^{p+q}(X,\Omega_X^\bullet(\log(D + E + C)) \otimes \mathcal{L}^{-1} \otimes \mathcal{O}_X(-E)) \end{aligned}$$

for all reduced divisors C and E such that $D + C + E$ has normal crossings. If $0 < \nu_i < N$, for all i, and if $\mathcal{V}' = \mathrm{Ker}(\nabla|_{X-(D+C+E)})$ we obtain (notations as in 1.7):

$$\begin{aligned} &\mathbb{H}^k(X,R(j \circ \sigma)_*\nu_!\mathcal{V}') = \mathbb{H}^k(X,(j\circ\nu)_!R\sigma_*'\mathcal{V}') = \\ &\mathbb{H}^k(X,R\eta_*(j' \circ \nu')_!\mathcal{V}') \cong \bigoplus_{p+q=k} H^q(X,\Omega_X^p(\log(D + E + C)) \otimes \mathcal{L}^{-1} \otimes \mathcal{O}_X(-E)). \end{aligned}$$

Several vanishing theorems for differential forms (as well as for morphisms between cohomology groups, as in [4] §3) can be so obtained.

Some are stated and discussed in [1] and [7]. We return to the simple case where $C = E = \phi$:

<u>Corollary 2.2.</u> Assume there exists a proper surjective morphism g from $U = X - D$ to an affine variety W. Let $\mathcal{L}$ be an invertible sheaf on X and assume that $\mathcal{L}^N = \mathcal{O}_X(\sum_{i=1}^{s} \nu_i D_i)$ for $0 < \nu_i < N$. then $H^q(X, \Omega_X^p(\log D) \otimes \mathcal{L}^{-1}) = 0$ for $p + q > n + r(g)$ and for $p + q < n - r(g)$.

<u>Notations 2.3.</u> An invertible sheaf is <u>semiample</u> if some of its powers are generated by global sections. A. Sommese (see [8]) defined $\mathcal{L}$ to be <u>b-ample</u> if for some $N > 0$ $\mathcal{L}^N$ is generated by its global sections and if the corresponding morphism $\phi_{\mathcal{L}^N} : X \longrightarrow \mathbb{P}(H^0(X, \mathcal{L}^N))$ has at most b-dimensional fibres. We write for a semiample invertible sheaf $\mathcal{L}$ $r(\mathcal{L}) = r(\phi_{\mathcal{L}^N})$ where N is any positive number such that $\mathcal{L}^N$ is generated by its global sections. It is easy to see that $r(\mathcal{L})$ is well defined.

Using those notations we obtain an improvement of A. Sommese's vanishing theorem (see [8], Chapter III):

<u>Theorem 2.4</u>. Let $\mathcal{L}$ be a semiample invertible sheaf on x. Then $\mathbb{H}^q(X, \Omega_X^p \otimes \mathcal{L}^{-1}) = 0$ for $p + q < n - r(\mathcal{L})$. Especially, if $\mathcal{L}$ is b-ample of Iitaka dimension $\kappa(\mathcal{L})$, this holds for

$$p + q < \mathrm{Min}(\kappa(\mathcal{L}), n - b + 1).$$

<u>Proof.</u> If $\kappa(\mathcal{L}) = 0$, there is nothing to show. For $\kappa(\mathcal{L}) > 0$ we choose $N > 1$ such that $\mathcal{L}^N$ is generated by its sections and write $\phi = \phi_{\mathcal{L}^N} : X \longrightarrow Z = \phi_{\mathcal{L}^N}(X)$. Let D be the zero divisor of a general section of $\mathcal{L}^N$. D is non singular and $Z - \phi(D)$ affine. By 2.2 $H^q(X, \Omega_X^p(\log D) \otimes \mathcal{L}^{-1}) = 0$ for $p + q < n - r(\phi)$. For those p and q the exact sequence

$$0 \longrightarrow \Omega_X^p \longrightarrow \Omega_X^p(\log D) \longrightarrow \Omega_D^{p-1} \longrightarrow 0$$

gives rise to a surjection

$$H^{q-1}(D,\Omega_D^{p-1} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}) \longrightarrow H^q(X,\Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}).$$

Since $\kappa(\mathcal{L}|_D) = \kappa(\mathcal{L}) - 1$ and since $\mathcal{L}|_D$ is again semiample with $r(\mathcal{L}|_D) \leq r(\mathcal{L}) + 1$ the lefthand side is zero by induction on $\kappa(\mathcal{L})$ (In fact, since D is in general position we even have $r(\mathcal{L}|_D) \leq r(\mathcal{L})$).

<u>Remarks 2.5.</u> a) If $\phi_{\mathcal{L}^N}$ is equidimensional the bound for $p + q$ given in 2.4 is the same one as in A. Sommese's original theorem. If $\mathcal{L}$ is b-ample and $\kappa(\mathcal{L}) \geq n - b + 1$ then $n - r(\mathcal{L}) = n - b + 1$ if and only if the union of all b-dimensional fibres of $\phi_{\mathcal{L}^N}$ has codimension one. In this case the bound is just improved by one. On the other hand, C.P. Ramanujam gave an example (see [8], 3.23) of a threefold X and a 2-ample sheaf $\mathcal{L}$ of Iitaka dimension 3, such that $H^1(X,\Omega_X^1 \otimes \mathcal{L}^{-1}) \neq 0$. Therefore, as long as the "bad locus" consists of divisors one can not expect further improvements.

b) It should be possible to replace the assumption "b-ample of Iitaka dimension $\kappa(\mathcal{L})$" in 2.4 by some numerical condition. But anything we could imagine looked quite unnatural. However for applications it is often sufficient to use 2.2 for a suitable divisor D as illustrated in part ii) of the following lemma:

<u>Lemma 2.6.</u> Let $\mathcal{L}$ be an invertible sheaf on X and $C \subseteq X$ be a normal crossing divisor. Assume that one of the following assumptions holds:

i) $\mathcal{L}$ is semi-ample

ii) X is Moišezon and $\mathcal{L}$ is numerically good. Then there exists a bimeromorphic morphism $\tau : X' \longrightarrow X$ of compact complex manifolds and a normal crossing divisor D' on X' containing $(\tau^*C)_{red}$ such that

$$H^q(X',\Omega_{X'}^p(\log D') \otimes \tau^*\mathcal{L}^{-1}) = 0 \quad \text{for} \quad p + q < \kappa(\mathcal{L}).$$

Before sketching the proof (similar to [4], 2.11 and 2.12), let us recall the definition and some properties of numerically good invertible sheaves, both due to Y. Kawamata, [6].

Definition 2.7. An invertible sheaf $\mathcal{L}$ is called numerically good if it is numerically effective (i.e. if $\deg \mathcal{L}|_\Gamma \geq 0$ for all curves $\Gamma \subseteq X$) and if $\kappa(\mathcal{L}) = \mathrm{Min}\{k, c_1(\mathcal{L})^{k+1}$ numerically trivial$\}$.

Lemma 2.8. (see [6]) Let X be the Moišezon and $\mathcal{L}$ be numerically good. Then there are projective manifolds X' and Z, a birational morphism $\tau : X' \longrightarrow X$, a surjective morphism $g : X' \longrightarrow Z$ and an invertible numerically effective sheaf $\mathcal{N}$ on Z, such that $g^*\mathcal{N} = \tau^*\mathcal{L}^\alpha$ for some $\alpha > 0$ and $\dim Z = \kappa(\mathcal{L}) = \kappa(\mathcal{N})$.

It is easy to see that all numerically effective sheaves $\mathcal{L}$ with $\kappa(\mathcal{L}) \geq \dim X - 1$ are numerically good.

The proof of 2.6. Under either one of the assumptions made we can find $X', Z, \mathcal{N}, \tau, g$ and α as in 2.8. (For i) we take $X' = X$ and $g = \phi_{\mathcal{L}^N}$). Let $C' = (\tau^* C)_{red} = \sum_{j=1}^{k} C_j'$. We can find - blowing up X', if necessary - a divisor Γ on Z such that $B = g^*\Gamma$ as well as $C' + B$ are normal crossing divisors, such that $g|_{X'-B}$ is smooth and $C|_{X'-B}$ a relative normal crossing divisor. $\kappa(\mathcal{N})$ is maximal and for $\nu >> 0$ $\mathcal{N}^\nu \otimes \mathcal{O}_Z(-\Gamma)$ will contain an ample invertible sheaf. Replacing Γ by a larger divisor and blowing up X' a little bit more we may as well assume $\mathcal{N}^\nu \otimes \mathcal{O}_Z(-\Gamma)$ to be ample. $\mathcal{N}$ is numerically effective, which allows to enlarge ν until $N = \alpha \cdot \nu >$ Multiplicities of the components of B. This inequality remains true if we replace ν and Γ by the same multiple and we may assume that $\mathcal{N}^\nu \otimes \mathcal{O}_Z(-\Gamma)$ is very ample. Pulling back a general section we get a nonsingular divisor H on X' such that $D = H + B$ and $D' = H + B + C'$ are both normal crossing divisors. For $\mathcal{L}' = \tau^*\mathcal{L}$ we have $\mathcal{L}'^N = \mathcal{O}_{X'}(D)$ and the assumptions of 2.2 are satisfied. 1.8 allows to add the divisor C' to the boundary and we obtain the vanishing of $H^q(X', \Omega^p_{X'}(\log D') \otimes \mathcal{L}'^{-1})$ for $p + q < n - r(g|_{X'-D}) = \dim Z = \kappa(\mathcal{L})$.

§3 Cohomology classes represented by logarithmic differential forms

3.1. Let $\mathcal{M}$ be an invertible $\mathcal{O}_X$-module with a holomorphic integrable connection ∇ with logarithmic poles along D and $\mathcal{V} = \operatorname{Ker} \nabla|_U$. The residues Γ_i of ∇ along D_i are given by multiplication with constants γ_i and - as in 1.4 - we write $D = D^* + D^!$, where for

$D_i \leq D^*$ $\gamma_i \notin \mathbb{Z}_{>0}$ and for

$D_i \leq D^!$ $\gamma_i \in \mathbb{Z}_{\leq 0}$. As in 1.5 we fix one of those decompositions and write $U \xrightarrow{\upsilon'} X - D^* \xrightarrow{\sigma} X$. By 1.5 we have

$$\mathbb{H}^k(X, \Omega_X^{\bullet}(\log D) \otimes \mathcal{M}) = \mathbb{H}^k(X, R\sigma_* \upsilon_!' \mathcal{V})$$

which is (by definition)

$$H^k(X - D^*,\ D^! \cap (X - D^*); \mathcal{V}).$$

Theorem 3.2. For $(\mathcal{M}, \nabla)$ as above we assume that either X is Moišezon and $\mathcal{M}^{-1}$ is numerically good or that $\mathcal{M}^{-1}$ is semiample. Then for $\kappa = \kappa(\mathcal{M}^{-1})$ the morphism

$$H^0(X, (\Omega_X^k(\log D) \otimes \mathcal{M})_{cl}) \xrightarrow{\beta} \mathbb{H}^k(X, R\sigma_* \upsilon_!' \mathcal{V})$$

is injective.

Remark 3.3. a) For the sheaf $(\Omega_X^p(\log D) \otimes \mathcal{M})_{cl}$ of closed $\mathcal{M}$-valued p-forms we have

$$\begin{aligned} H^0(X, (\Omega_X^p(\log D) \otimes \mathcal{M})_{cl}) &= \\ &= \mathbb{H}^p(X, F^p(\Omega_X^{\bullet}(\log D) \otimes \mathcal{M})) \end{aligned}$$

where F^p denotes the Hodge filtration. β is given by the inclusion

$$F^p(\Omega_X^{\bullet}(\log D) \otimes \mathcal{M}) \longrightarrow \Omega_X^{\bullet}(\log D) \otimes \mathcal{M}.$$

b) By F. Bogomolov's vanishing theorem one knows that for $p < \kappa$ $H^0(X,\Omega^p_X(\log D) \otimes \mathcal{M}) = 0$. In fact, this can be obtained from 2.2 by using the arguments given in [4], 2.11.

<u>Proof of 3.2.</u> (see also [5])
By 2.6 there is a bimeromorphic morphism $\tau : X' \longrightarrow X$ and a normal crossing divisor D' containing $(\tau^*D)_{red}$ such that $H^q(X',\Omega^p_{X'}(\log D') \otimes \tau^*\mathcal{M}) = 0$ for $p + q < \kappa$. Then

$$\mathbb{H}^{\kappa-1}(X',\Omega^{\bullet\leq\kappa-1}_{X'}(\log D') \otimes \tau^*\mathcal{M}) = 0$$

and the morphism β' in the following diagram is injective:

$$\begin{tikzcd}
\mathbb{H}^{\kappa}(X,F^{\kappa}(\Omega^{\bullet}_X(\log D) \otimes \mathcal{M})) \arrow[r, "\beta"] \arrow[d, "\tau^*"] & \mathbb{H}^{\kappa}(X,\Omega^{\bullet}_X(\log D) \otimes \mathcal{M}) \arrow[d] \\
\mathbb{H}^{\kappa}(X',F^{\kappa}(\Omega^{\bullet}_{X'}(\log D') \otimes \tau^*\mathcal{M})) \arrow[r, "\beta'"] & \mathbb{H}^{\kappa}(X',\Omega^{\bullet}_{X'}(\log D') \otimes \tau^*\mathcal{M})
\end{tikzcd}$$

As remarked in 3.3 a) the groups on the left hand side are those of global closed forms and τ^* is injective as well.

<u>3.4.</u> $j_{m_*}(\mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_U) := \mathcal{M} \otimes \mathcal{O}_X(*D)$ is the regular meromorphic extension of $\mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_U$ to X, unique up to isomorphism ([2]). We call $\omega \in H^0(U,\Omega^p_U \otimes_{\mathbb{C}} \mathcal{V})$ <u>meromorphic along D</u> if ω lies in

$$H^0(X,j_{m_*}(\mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_U) \otimes_{\mathcal{O}_X} \Omega^p_X(\log D)).$$

The canonical extension $\mathcal{V}_{can}$ is an invertible subsheaf of $j_{m_*}(\mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_U)$ which is determined by the property that ∇ induces a connection on $\mathcal{V}_{can}$ with logarithmic poles along d such that the real part of the residues Γ_i lies in $[0,1[$ for all i. We say that <u>ω swallows D_j</u>, if the monodromy of $\mathcal{V}$ around D_j is one and if for some $\nu_i \in \mathbb{Z}$

$$\omega \in H^0(X,\Omega^p_X(\log D) \otimes \mathcal{V}_{can} \otimes \mathcal{O}_X(\sum_{i\neq j}\nu_i D_i - D_j)).$$

<u>Corollary 3.5.</u> Let $\omega \in H^0(U,\Omega_U^n \otimes_{\mathbb{C}} \mathscr{V})$ be meromorphic along D. Let $D^!$ be the union of all components of D which are swallowed by ω and $D^* = D - D^!$. Let Z be the closure of the zero divisor of ω on U. If $\mathscr{L} = \Omega_X^n(\log D) \otimes \mathcal{O}_X(-Z)$ is numerically effective and $\kappa(\mathscr{L}) = n$, then ω defines a non vanishing cohomology class in

$$H^n(X - D^*, D^! \cap (X - D^*); \mathscr{V})$$

<u>Proof</u>. Let $\mathscr{M}$ be the smallest extension of $\mathscr{V} \otimes {}_{\mathbb{C}}\mathcal{O}_U$ in $j_{m_*}\mathscr{V} \otimes {}_{\mathbb{C}}\mathcal{O}_U$ such that $\omega \in H^0(X,\Omega_X^n(\log D) \otimes \mathscr{M} \otimes \mathcal{O}_X(-Z))$. Then $\omega : \mathcal{O}_X \longrightarrow \Omega_X^n(\log D) \otimes \mathscr{M} \otimes \mathcal{O}_X(-Z)$ is an isomorphism and $\mathscr{M}^{-1} = \mathscr{L}$. Moreover $\mathscr{M} \subset \mathscr{V}_{can} \otimes \mathcal{O}_X(-D_j) \otimes \mathcal{O}_X(*(D-D_j))$ if and only if ω swallows D_j. Therefore the choice of $D^!$ and D^* satisfies the assumptions made in 3.1 and by 3.2 we have an injection

$$H^0(X,\Omega_X^n(\log D) \otimes \mathscr{M}) \longrightarrow \mathbb{H}^n(X, R\sigma_* \upsilon_! \mathscr{V}).$$

<u>3.6.</u> We write again $D = D^* + D^!$. The relative cohomology $H^n(X - D^*, D^! \cap (X-D^*); \mathbb{C})$ is given by the n-th hypercohomology of the complex $\Omega_X^{\cdot}(\log D) \otimes \mathcal{O}_X(-D^!) \otimes \mathcal{O}_X(*D^*)$. If $X - D^*$ is affine (otherwise we should replace the holomorphic forms by $\mathscr{C}^\infty$-forms) we can as well take the n-th cohomology of the complex of vector spaces

$$H^0(X - D^*, \Omega_{X-D^*}^{\cdot}(\log D^!) \otimes \mathcal{O}_X(-D^!)).$$

3.2 says that in this complex no non zero form out of

$$H^0(X,\Omega_X^n(\log D) \otimes \mathcal{O}_X(-D^! + \upsilon \cdot D^*)) \cong$$
$$= H^0(X,\Omega_X^n \otimes \mathcal{O}_X((\upsilon + 1) \cdot D^*))$$

is exact, provided $\mathcal{O}_X(D^! - \upsilon \cdot D^*)$ is numerically effective and of maximal Iitaka dimension.

We heard the following example (over $\mathbb{R}$) form A.N. Varchenko.

<u>Corollary 3.7.</u> Let $A_1,\dots,A_N$ be hyperplanes in $\mathbb{C}^n$ in general position, $n \leq N$, and let $Y_1,\dots,Y_n$ be the coordinate functions. Then a base of $H^n(\mathbb{C}^n, A_1 \cup \dots \cup A_N;\mathbb{C})$ is given by the differential forms $Y_1^{m_1}\cdot\ldots\cdot Y_n^{m_n} \cdot dY_1\wedge\ldots\wedge dY_n$ for $m_i \geq 0$ and $\sum_{i=1}^{n} m_i \leq N-n-1$.

<u>Proof.</u> On $\mathbb{P}^n$ we write $D^! = \bar{A}_1 \cup \dots \cup \bar{A}_N$ and D^* for the hyperplane at ∞. The differential forms given form a base of $H^0(\mathbb{P}^n,\Omega^n_{\mathbb{P}^n}\otimes \mathcal{O}(N\cdot D^*)) = H^0(\mathbb{P}^n,\Omega^n_{\mathbb{P}^n}(\log D))\otimes \mathcal{M})$ for $\mathcal{M} = \mathcal{O}_{\mathbb{P}^n}(-D^! + (N-1)\cdot D^*) = \mathcal{O}_{\mathbb{P}^n}(-1)$. Obviously, if we take for ∇ the usual differential on $\mathcal{O}_{\mathbb{P}^n}(-D^! + (N-1)\cdot D^*)$ the assumptions of 3.2 are all satisfied and we have an injection

$$H^0(\mathbb{P}^n,\Omega^n_{\mathbb{P}^n}(\log D)\otimes \mathcal{M}) \longrightarrow H^n(\mathbb{C}^n, A_1 \cup\ldots\cup A_n;\mathbb{C}) =$$
$$= \mathbb{H}^n(\mathbb{P}^n,\Omega^{\cdot}_{\mathbb{P}^n}(\log D)\otimes \mathcal{M}).$$

The cokernel is contained in $\mathbb{H}^n(\mathbb{P}^n,\Omega^{\cdot \leq n-1}_{\mathbb{P}^n}(\log D)\otimes \mathcal{M})$ and 3.7 follows from the presumably well known

<u>Lemma 3.8.</u> Let $D_0,\dots,D_N$ be hyperplanes in $\mathbb{P}^n$ in general position, $n \leq N$. Then $H^q(\mathbb{P}^n,\Omega^p_{\mathbb{P}^n}(\log D)\otimes \mathcal{O}_{\mathbb{P}^n}(-1)) = 0$ for $q > 0$.

<u>Proof.</u> $D_0,\dots,D_n$ form a complete coordinate system. If $N = n$ then

$$\Omega^1_{\mathbb{P}^n}(\log D) = \bigoplus^{n} \mathcal{O}_{\mathbb{P}^1}$$

and the cohomology group considered is $\bigoplus^{n} H^q(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(-1))$.

For $N > n$ we write $D' = \sum_{i=0}^{N-1} D_i$ and consider the long exact sequence

$$0 \longrightarrow \Omega^p_{\mathbb{P}^n}(\log D') \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \Omega^p_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow$$

$$\longrightarrow \Omega^{p-1}_{D_N}(\log(D_N \cap D')) \otimes \mathcal{O}_{D_N}(-1)) \longrightarrow 0$$

By induction on N the left hand side has no higher cohomology and by induction on n neither does the right hand side.

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