# Max-Planck-Institut für Mathematik
# Bonn

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by

Wadim Zudilin

Max-Planck-Institut für Mathematik
Preprint Series 2012 (42)

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Wadim Zudilin

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

School of Mathematical and Physical Sciences
The University of Newcastle
Callaghan, NSW 2308
Australia

MPIM 12-42

# Periodness of $L$-values

Wadim Zudilin*

School of Mathematical and Physical Sciences,
The University of Newcastle, Callaghan, NSW 2308, Australia

June 2012

**Abstract**

In our recent work with Rogers on resolving some Boyd's conjectures on two-variate Mahler measures, a new analytical machinery was introduced to write the values $L(E,2)$ of $L$-series of elliptic curves as periods in the sense of Kontsevich and Zagier. Here we outline, in slightly more general settings, the novelty of our method with Rogers, and provide two illustrative period evaluations of $L(E,2)$ and $L(E,3)$ for a conductor 32 elliptic curve $E$.

## 1 Introduction

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients [1]. Without much harm, the three appearances of the adjective "rational" can be replaced by "algebraic". The set of periods $\mathcal{P}$ is countable and admits a structure of ring.

The extended period ring $\widehat{\mathcal{P}} := \mathcal{P}[1/\pi] = \mathcal{P}[(2\pi i)^{-1}]$ (rather than the period ring $\mathcal{P}$ itself) contains many natural examples, like special $L$-values. For example, a general theorem [1] due to Beilinson and Deninger–Scholl states that the (non-critical) value of the $L$-series attached to a cusp form $f(\tau)$ of weight $k$ at a positive integer $m \ge k$ (cf. formula (2) below) belongs to $\widehat{\mathcal{P}}$. In spite of an effective nature of proof of the theorem, computing the $L$-values as periods remains a difficult problem even for particular examples; it is this phenomenon which we refer to as "perioddness". Most such computations are motivated by (conjectural) evaluations of the logarithmic Mahler measures of multi-variate polynomials.

With the purpose of establishing such evaluations in the two-variate case, Rogers and the present author [2] have developed a machinery of writing the $L$-values $L(f,2)$ as periods for cusp forms $f(\tau)$ of weight 2, the machinery which is different from that

*This work is supported by the Max Planck Institute for Mathematics (Bonn, Germany) and the Australian Research Council (grant DP110104419).

of Beilinson. In this note, we overview the method of [2, 3] on a particular example of $L(E,2)$ in Section 2, and then attempt in Section 3 to describe a general algorithm behind the method. Finally, in Section 4 we present an example of evaluating $L(E,3)$ as a period, a computation we failed to find in the existing literature. In the examples of Sections 2 and 4, $E$ stands for an elliptic curve of conductor 32. There are at least two reasons for choosing this conductor. First of all, this conductor is not discussed in our joint works [2, 3], and secondly, the involved modular parametrisations are sufficiently classical and remarkably simple.

Throughout the note we keep the notation $q=e^{2\pi i\tau}$ for $\tau$ from the upper half-plane $\operatorname{Re}\tau>0$, so that $|q|<1$. Our basic constructor of modular forms and functions is Dedekind's eta-function

$$\eta(\tau):=q^{1/24}\prod_{m=1}^{\infty}(1-q^m)=\sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24}$$

with its modular involution

$$\eta(-1/\tau)=\sqrt{-i\tau}\,\eta(\tau). \tag{1}$$

We also set $\eta_k:=\eta(k\tau)$ for short.

For functions of $\tau$ or $q=e^{2\pi i\tau}$ we use the differential operator

$$\delta:=\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau}=q\frac{\mathrm{d}}{\mathrm{d}q}$$

and denote by $\delta^{-1}$ the corresponding anti-derivative normalised by 0 at $\tau=i\infty$ (or $q=0$):

$$\delta^{-1}f=\int_0^q f\,\frac{\mathrm{d}q}{q}.$$

In particular, for a modular form $f(\tau)=\sum_{n=1}^{\infty}a_nq^n$, whose expansion vanishes at infinity, we have

$$L(f,m)=\frac{1}{(m-1)!}\int_0^1 f\log^{m-1}q\,\frac{\mathrm{d}q}{q}=\sum_{n=1}^{\infty}\frac{a_n}{n^m}=(\delta^{-m}f)|_{q=1} \tag{2}$$

whenever the latter sum has sense.

## 2 $L(E,2)$

For a conductor 32 elliptic curve $E$, the $L$-series is known to coincide with that for the cusp form $f(\tau):=\eta_4^2\eta_8^2$.

Note the (Lambert series) expansion

$$\begin{aligned}\frac{\eta_8^4}{\eta_4^2}&=\sum_{m\ge1}\left(\frac{-4}{m}\right)\frac{q^m}{1-q^{2m}}=\sum_{\substack{m,n\ge1\\ n\text{ odd}}}\left(\frac{-4}{m}\right)q^{mn}\\&=\sum_{m,n\ge1}a(m)b(n)q^{mn},\qquad\text{where}\quad a(m):=\left(\frac{-4}{m}\right),\quad b(n):=n\bmod 2,\end{aligned}$$

and $\left(\frac{-4}{m}\right)$ denotes the quadratic residue character modulo 4.

Then

$$
\begin{aligned}
f(it) &= \frac{\eta_8^4}{\eta_4^2}\frac{\eta_4^4}{\eta_8^2}\bigg|_{\tau=it} = \frac{\eta_8^4}{\eta_4^2}\bigg|_{\tau=it}\cdot\frac{1}{2t}\frac{\eta_8^4}{\eta_4^2}\bigg|_{\tau=i/(32t)} \\
&= \frac{1}{2t}\sum_{m_1,n_1\geq 1} a(m_1)b(n_1)e^{-2\pi m_1 n_1 t}\sum_{m_2,n_2\geq 1} b(m_2)a(n_2)e^{-2\pi m_2 n_2/(32t)}, \qquad (3)
\end{aligned}
$$

where $t>0$ and the modular involution (1) was used.

Now,

$$
\begin{aligned}
L(E,2) = L(f,2) &= \int_0^1 f\,\log q\,\frac{\mathrm{d}q}{q} = -4\pi^2\int_0^\infty f(it)t\,\mathrm{d}t \\
&= -2\pi^2\int_0^\infty \sum_{m_1,n_1,m_2,n_2\geq 1} a(m_1)b(n_1)b(m_2)a(n_2) \\
&\qquad\times\exp\left(-2\pi\left(m_1n_1t+\frac{m_2n_2}{32t}\right)\right)\mathrm{d}t \\
&= -2\pi^2\sum_{m_1,n_1,m_2,n_2\geq 1} a(m_1)b(n_1)b(m_2)a(n_2) \\
&\qquad\times\int_0^\infty \exp\left(-2\pi\left(m_1n_1t+\frac{m_2n_2}{32t}\right)\right)\mathrm{d}t.
\end{aligned}
$$

Here comes the crucial transformation of purely analytical origin: we make the change of variable $t = n_2u/n_1$. It does not change the form of the integrand but affects the differential, and we obtain

$$
\begin{aligned}
L(E,2) &= -2\pi^2\sum_{m_1,n_1,m_2,n_2\geq 1}\frac{a(m_1)b(n_1)b(m_2)a(n_2)n_2}{n_1} \\
&\qquad\times\int_0^\infty \exp\left(-2\pi\left(m_1n_2u+\frac{m_2n_1}{32u}\right)\right)\mathrm{d}u \\
&= -2\pi^2\int_0^\infty \sum_{m_1,n_2\geq 1} a(m_1)a(n_2)n_2e^{-2\pi m_1 n_2 u} \\
&\qquad\times\sum_{m_2,n_1\geq 1}\frac{b(m_2)b(n_1)}{n_1}e^{-2\pi m_2 n_1/(32u)}\mathrm{d}u.
\end{aligned}
$$

The first double series in the integrand corresponds to

$$
\sum_{m,n\geq 1} a(m)a(n)n\,q^{mn} = \sum_{m,n\geq 1}\left(\frac{-4}{mn}\right)n\,q^{mn} = \sum_{n\geq 1} n\left(\frac{-4}{n}\right)\frac{nq^n}{1+q^{2n}} = \frac{\eta_2^4\eta_8^4}{\eta_4^4},
$$

while the second one is

$$\sum_{m,n\ge1}\frac{b(m)b(n)}{n}\,q^{mn}=\sum_{m,n\ge1}\frac{q^{mn}}{n}-\frac{q^{(2m)n}}{n}-\frac{q^{m(2n)}}{2n}+\frac{q^{(2m)(2n)}}{2n}$$
$$=\frac{1}{2}\sum_{m,n\ge1}\frac{2q^{mn}-3q^{2mn}+q^{4mn}}{n}$$
$$=-\frac{1}{2}\log\prod_{m\ge1}\frac{(1-q^m)^2(1-q^{4m})}{(1-q^{2m})^3}=-\frac{1}{2}\log\frac{\eta_1^2\eta_4}{\eta_2^3},$$

hence

$$L(E,2)=\pi^2\int_0^\infty\frac{\eta_2^4\eta_8^4}{\eta_4^4}\bigg|_{\tau=iu}\cdot\log\frac{\eta_1^2\eta_4}{\eta_2^3}\bigg|_{\tau=i/(32u)}\mathrm{d}u.$$

Applying the involution (1) to the eta quotient under the logarithm sign we obtain

$$L(E,2)=\pi^2\int_0^\infty\frac{\eta_2^4\eta_8^4}{\eta_4^4}\log\frac{\sqrt{2}\eta_8\eta_{32}^2}{\eta_{16}^3}\bigg|_{\tau=iu}\mathrm{d}u.$$

Now comes the modular magic: choosing a particular modular function $x(\tau):=\eta_2^4\eta_8^2/\eta_4^6$, which ranges from 0 to 1 when $\tau$ ranges from $i\infty$ to 0, one can easily verify that

$$\frac{1}{2\pi i}\,\frac{x\,\mathrm{d}x}{2\sqrt{1-x^4}}=-\frac{\eta_2^4\eta_8^4}{\eta_4^4}\,\mathrm{d}\tau\quad\text{and}\quad\left(\frac{\sqrt{2}\eta_8\eta_{32}^2}{\eta_{16}^3}\right)^2=\frac{1-x}{1+x}.$$

Thus, we arrive at the following result.

**Theorem 1.** *For an elliptic curve $E$ of conductor* 32*,*

$$L(E,2)=\frac{\pi}{8}\int_0^1\frac{x}{\sqrt{1-x^4}}\log\frac{1+x}{1-x}\,\mathrm{d}x=0.9170506353\dots.$$

# 3 General $L$-values

To summarise our evaluation of $L(E,2)=L(f,2)$ in Section 2, we first split $f(\tau)$ into a product of two Eisenstein series of weight 1 and at the end we arrive at a product of two Eisenstein(-like) series $g_2(\tau)$ and $g_0(\tau)$ of weights 2 and 0, respectively, so that $L(f,2)=c\pi L(g_2g_0,1)$ for some rational $c$. The latter object is doomed to be a period as $g_0(\tau)$ is a logarithm of a modular function, while $2\pi i\,g_2(\tau)\,\mathrm{d}\tau$ is, up to a modular function multiple, the differential of a modular function, and finally any two modular functions are tied up by an algebraic relation over $\overline{\mathbb{Q}}$.

The method can be further formalised to more general settings, and it is this extension which we attempt to outline in this section.

For two *bounded* sequences $a(m)$, $b(n)$, we refer to an expression of the form

$$g_k(\tau)=a+\sum_{m,n\ge1}a(m)b(n)n^{k-1}q^{mn}\tag{4}$$

as to an Eisenstein-like series of weight $k$, especially in the case when $g_k(\tau)$ is a modular form of certain level, that is, when it transforms sufficiently 'nice' under $\tau \mapsto -1/(N\tau)$ for some positive integer $N$. This automatically happens when $g_k(\tau)$ is indeed an Eisenstein series (for example, when $a(m) = 1$ and $b(n)$ is a Dirichlet character modulo $N$ of designated parity, $b(-1) = (-1)^k$), in which case $\widehat{g}_k(\tau) := g_k(-1/(N\tau))(\sqrt{-N}\tau)^{-k}$ is again an Eisenstein series. It is worth mentioning that the above notion has perfect sense in case $k \le 0$ as well. Indeed, modular units, or weak modular forms of weight 0, that are the logarithms of modular functions are examples of Eisenstein-like series $g_0(\tau)$. Also, for $k \le 0$ examples are given by Eichler integrals, the $(1-k)$ th $\tau$-antiderivatives of holomorphic Eisenstein series of weight $2-k$, a consequence of the famous lemma of Hecke [5, Section 5].

Suppose we are interested in the $L$-value $L(f, k_0)$ of a cusp form $f(\tau)$ of weight $k = k_1 + k_2$ which can be represented as a product (in general, as a linear combination of several products) of two Eisenstein(-like) series $g_{k_1}(\tau)$ and $\widehat{g}_{k_2}(\tau)$, where the first one vanishes at infinity ($a = g_{k_1}(i\infty) = 0$ in (4)) and the second one vanishes at zero ($\widehat{g}_{k_2}(i0) = 0$). (The vanishing happens because the product is a cusp form!) In reality, we need the series $g_{k_2}(\tau) := \widehat{g}_{k_2}(-1/(N\tau))(\sqrt{-N}\tau)^{-k_2}$ to be Eisenstein-like:

$$g_{k_1}(\tau) = \sum_{m,n\ge1} a_1(m)b_1(n)n^{k_1-1}q^{mn} \quad \text{and} \quad g_{k_2}(\tau) = \sum_{m,n\ge1} a_2(m)b_2(n)n^{k_2-1}q^{mn}.$$

We have

$$\begin{aligned}
L(f, k_0) = L(g_{k_1}\widehat{g}_{k_2}, k_0) &= \frac{1}{(k_0-1)!}\int_0^1 g_{k_1}\widehat{g}_{k_2}\log^{k_0-1} q\,\frac{\mathrm{d}q}{q} \\
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!}\int_0^\infty g_{k_1}(it)\widehat{g}_{k_2}(it)t^{k_0-1}\,\mathrm{d}t \\
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}}\int_0^\infty g_{k_1}(it)g_{k_2}(i/(Nt))t^{k_0-k_2-1}\,\mathrm{d}t \\
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}}\int_0^\infty \sum_{m_1,n_1\ge1} a_1(m_1)b_1(n_1)n_1^{k_1-1}e^{-2\pi m_1n_1t} \\
&\qquad \times \sum_{m_2,n_2\ge1} a_2(m_2)b_2(n_2)n_2^{k_2-1}e^{-2\pi m_2n_2/(Nt)}t^{k_0-k_2-1}\mathrm{d}t \\
&= \frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}}\sum_{m_1,n_1,m_2,n_2\ge1} a_1(m_1)b_1(n_1)a_2(m_2)b_2(n_2)n_1^{k_1-1}n_2^{k_2-1} \\
&\qquad \times \int_0^\infty \exp\left(-2\pi\left(m_1n_1t + \frac{m_2n_2}{Nt}\right)\right)t^{k_0-k_2-1}\mathrm{d}t;
\end{aligned}$$

the interchange of integration and summation is legitimate because of the exponential decay of the integrand at the endpoints. After performing the change of variable

$t=n_2u/n_1$ and interchanging back summation and integration we obtain

$$\begin{aligned}
L(f,k_0)&=\frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}}\sum_{m_1,n_1,m_2,n_2\ge1}a_1(m_1)b_1(n_1)a_2(m_2)b_2(n_2)n_1^{k_1+k_2-k_0-1}n_2^{k_0-1}\\
&\qquad\times\int_0^\infty\exp\biggl(-2\pi\biggl(m_1n_2u+\frac{m_2n_1}{Nu}\biggr)\biggr)u^{k_0-k_2-1}\mathrm{d}u\\
&=\frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}}\int_0^\infty\sum_{m_1,n_2\ge1}a_1(m_1)b_2(n_2)n_2^{k_0-1}e^{-2\pi m_1n_2u}\\
&\qquad\times\sum_{m_2,n_1\ge1}a_2(m_2)b_1(n_1)n_1^{k_1+k_2-k_0-1}e^{-2\pi m_2n_1/(Nu)}u^{k_0-k_2-1}\mathrm{d}u\\
&=\frac{(-1)^{k_0-1}(2\pi)^{k_0}}{(k_0-1)!\,N^{k_2/2}}\int_0^\infty g_{k_0}(iu)g_{k_1+k_2-k_0}(i/(Nu))u^{k_0-k_2-1}\,\mathrm{d}u.
\end{aligned}$$

Assuming a modular transformation of the Eisenstein-like series $g_{k_1+k_2-k_0}(\tau)$ under $\tau\mapsto-1/(N\tau)$, we can realise the resulting integral as $c\pi^{k_0-k_1}L(g_{k_0}\widehat{g}_{k_1+k_2-k_0},k_1)$, where $c$ is algebraic (plus some extra terms when $g_{k_1+k_2-k_0}(\tau)$ is an Eichler integral). Alternatively, if $g_{k_0}(\tau)$ transforms under the involution, we perform the transformation and switch to the variable $v=1/(Nu)$ to arrive at $c\pi^{k_0-k_1}L(\widehat{g}_{k_0}g_{k_1+k_2-k_0},k_1)$. In both cases we obtain an identity which relates the starting $L$-value $L(f,k_0)$ to a different '$L$-value' of a modular-like object of the same weight.

The case $k_1=k_2=1$ and $k_0=2$, discussed in [2, 3] and in Section 2 above, allows one to reduce the $L$-values to periods. As we will see in Section 4, the perioddness can be achieved in a more general situation, based on the fact that Eichler integrals are related to solutions of inhomogeneous linear differential equations.

# 4 $L(E,3)$

To manipulate with $L(E,3)$ for a conductor 32 elliptic curve, we use again $L(E,3)=L(f,3)$ with $f(\tau):=\eta_4^2\eta_8^2$ and write the decomposition in (3) as

$$f(it)=\frac{1}{2t}\sum_{m_1,n_1\ge1}b(m_1)a(n_1)e^{-2\pi m_1n_1t}\sum_{m_2,n_2\ge1}b(m_2)a(n_2)e^{-2\pi m_2n_2/(32t)}.$$

Then

$$\begin{aligned}
L(E,3)=L(f,3)&=\frac12\int_0^1 f\,\log^2q\,\frac{\mathrm{d}q}{q}=4\pi^3\int_0^\infty f(it)t^2\,\mathrm{d}t\\
&=2\pi^3\int_0^\infty\sum_{m_1,n_1,m_2,n_2\ge1}b(m_1)a(n_1)b(m_2)a(n_2)\\
&\qquad\times\exp\biggl(-2\pi\biggl(m_1n_1t+\frac{m_2n_2}{32t}\biggr)\biggr)t\,\mathrm{d}t
\end{aligned}$$

$$= 2\pi^3 \sum_{m_1,n_1,m_2,n_2\ge 1} b(m_1)a(n_1)b(m_2)a(n_2) \times \int_0^\infty \exp\left(-2\pi\left(m_1n_1t + \frac{m_2n_2}{32t}\right)\right)t\,\mathrm{d}t$$

(here we perform the change of variable $t = n_2u/n_1$)

$$= 2\pi^3 \sum_{m_1,n_1,m_2,n_2\ge 1} \frac{b(m_1)a(n_1)b(m_2)a(n_2)n_2^2}{n_1^2} \times \int_0^\infty \exp\left(-2\pi\left(m_1n_2u + \frac{m_2n_1}{32u}\right)\right)u\,\mathrm{d}u$$

$$= 2\pi^3 \int_0^\infty \sum_{m_1,n_2\ge 1} b(m_1)a(n_2)n_2^2 e^{-2\pi m_1n_2u} \times \sum_{m_2,n_1\ge 1} \frac{b(m_2)a(n_1)}{n_1^2} e^{-2\pi m_2n_1/(32u)}\,u\,\mathrm{d}u.$$

Furthermore,

$$\sum_{m,n\ge 1} b(m)a(n)n^2q^{mn} = \sum_{\substack{m,n\ge 1\\ m \text{ odd}}} \left(\frac{-4}{n}\right)n^2q^{mn} = \frac{\eta_2^8\eta_8^4}{\eta_4^6},$$

$$\sum_{m,n\ge 1} b(m)a(n)m^2q^{mn} = \sum_{\substack{m,n\ge 1\\ m \text{ odd}}} \left(\frac{-4}{n}\right)m^2q^{mn} = \frac{\eta_4^{18}}{\eta_2^8\eta_8^4},$$

so that

$$r(\tau) := \sum_{m,n\ge 1} \frac{a(m)b(n)}{n^2}\,q^{mn} = \delta^{-2}\left(\frac{\eta_4^{18}}{\eta_2^8\eta_8^4}\right).$$

Continuing the previous computation,

$$L(E,3) = 2\pi^3 \int_0^\infty \frac{\eta_2^8\eta_8^4}{\eta_4^6}\bigg|_{\tau=iu} \cdot r(i/(32u))\,u\,\mathrm{d}u$$

(we apply the involution to the eta quotient)

$$= \frac{\pi^3}{8}\int_0^\infty \frac{\eta_4^4\eta_{16}^8}{\eta_8^6}\,r(\tau)\bigg|_{\tau=i/(32u)} \frac{\mathrm{d}u}{u^2}$$

(we change the variable $u = 1/(32v)$)

$$= 4\pi^3 \int_0^\infty \frac{\eta_4^4\eta_{16}^8}{\eta_8^6}\,r(\tau)\bigg|_{\tau=iv}\,\mathrm{d}v.$$

This is so far the end of the algorithm we have discussed in Section 3. In order to show that the resulting integral is a period we require to do one step more. As in Section 2 we make a modular parametrisation; this time we take the modular function $x(\tau) := 4\eta_2^4\eta_8^8/\eta_4^{12}$ which also ranges from 0 to 1 when $\tau$ goes from $i\infty$ to 0. Then

$$\delta x = \frac{4\eta_2^{12}\eta_8^8}{\eta_4^{16}}, \quad (1-x^2)^{1/4} = \frac{\eta_2^4\eta_8^2}{\eta_4^6}, \quad s(x) := \frac{(1-\sqrt{1-x^2})^2}{x(1-x^2)^{3/4}} = \frac{16\eta_4^{10}\eta_{16}^8}{\eta_2^8\eta_8^{10}}.$$

Furthermore, the substitution $z = x^2(\tau)$ into the hypergeometric function

$$F(z) := {}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix}\,\middle|\, z\right) = \frac{2}{\pi}\int_0^1 \frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-zy^2)}}$$

results in the modular form

$$\varphi(\tau) := F(x^2) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \left(\frac{x}{4}\right)^{2n} = \frac{\eta_4^{10}}{\eta_2^4\eta_8^4}$$

of weight 1. Because $F(z)$ (along with $F(1-z)$) satisfy the hypergeometric differential equation

$$z(1-z)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + (1-2z)\frac{\mathrm{d}F}{\mathrm{d}z} - \frac{1}{4}F = 0,$$

it is not hard to write down the corresponding linear second order differential operator

$$\mathcal{L} := x(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (1-3x^2)\frac{\mathrm{d}}{\mathrm{d}x} - x$$

(in terms of $x$) such that $\mathcal{L}\varphi = 0$.

With this notation in mind, we obtain

$$\begin{aligned} L(E,3) &= \pi^3 \int_0^\infty \frac{\eta_4^{10}\eta_{16}^8}{\eta_2^8\eta_8^{10}}\,\varphi(\tau)\,r(\tau)\delta x\bigg|_{\tau=iv}\,\mathrm{d}v \\ &= \frac{\pi^3}{16}\int_0^\infty s(x(\tau))\,\varphi(\tau)\,r(\tau)\delta x\bigg|_{\tau=iv}\,\mathrm{d}v, \end{aligned}$$

and at this point we make an observation that the function $h(\tau) := 4\varphi(\tau)r(\tau)$ solves the inhomogeneous differential equation

$$\mathcal{L}h = \frac{1}{1-x^2} \qquad \left(\text{which is nothing but [4, 6]} \quad \frac{\delta^2 r}{\delta x\cdot\varphi} = \frac{\eta_4^{24}}{4\eta_2^{16}\eta_8^8}\right),$$

so that it can be written as an integral using the method of variation of constants:

$$\begin{aligned} h &= \frac{\pi}{2}\left(F(x^2)\int\frac{F(1-x^2)}{1-x^2}\,\mathrm{d}x - F(1-x^2)\int\frac{F(x^2)}{1-x^2}\,\mathrm{d}x\right) \\ &= \frac{\pi x}{2}\int_0^1 \frac{F(x^2)F(1-x^2w^2) - F(1-x^2)F(x^2w^2)}{1-x^2w^2}\,\mathrm{d}w \\ &= x + \frac{5}{9}x^3 + \frac{89}{225}x^5 + \frac{381}{1225}x^7 + \frac{25609}{99225}x^9 + \frac{106405}{480249}x^{11} + \cdots. \end{aligned}$$

This implies that

$$L(E,3)=\frac{\pi^2}{128}\int_0^1 s(x)\,h(x)\,\mathrm{d}x,$$

and we have the following period expression.

**Theorem 2.** *For an elliptic curve $E$ of conductor 32,*

$$\begin{aligned}L(E,3)=\frac{\pi}{64}&\int_0^1\frac{(1-\sqrt{1-x^2})^2}{(1-x^2)^{3/4}}\,\mathrm{d}x\int_0^1\frac{\mathrm{d}w}{1-x^2w^2}\\&\times\biggl(\int_0^1\frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-x^2y^2)}}\cdot\int_0^1\frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-(1-x^2w^2)y^2)}}\\&\quad-\int_0^1\frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-(1-x^2)y^2)}}\cdot\int_0^1\frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-x^2w^2y^2)}}\biggr)\\&=0.9826801478\dots.\end{aligned}$$

**Acknowledgements.** I am thankful to Anton Mellit, Mat Rogers, Evgeny Shinder, Masha Vlasenko and James Wan for fruitful conversations on the subject, and to Don Zagier for his encouragement to isolate the $L$-series transformation part from [2, 3].

The final part of this work was done during my visit in the Max Planck Institute for Mathematics (Bonn). I would like to thank the staff of the institute for hospitality and enjoyable working conditions.

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