Moduli of Half Conformally Flat Structures

by

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ABSTRACT The moduli of half conformally flat structures on a 4–manifold is studied. The moduli is equipped with a real analytic variety structure and a canonical L^2–metric structure.

This moduli with the L^2–metric turns out in a K3 surface case to be isometric through the period map with a domain in the Grassmannian $SO(3,19)/_{SO(3)\times SO(19)}$. The moduli of zero scalar curvature type, the subspace of the full moduli is also investigated.

1. We would like to study in this article the moduli of "conformal structures" on a given 4–manifold. Here the moduli of conformal structures or more precisely the moduli of half conformally flat structures means the set of all half conformally flat structures $[g]$ on a 4–manifold M modulo the action of the gauge group $\mathrm{Diff}(M)$, the diffeomorphism group of M.

There is a significant notion in conformal geometry, the conformal flatness.

A Riemannian n–manifold (M,g) is called conformally flat if (M,g) has at every point a locally defined conformal map into a Euclidean flat space $\mathbb{R}^n$. When $n \geq 4$ this is equivalent to the vanishing of the Weyl conformal tensor W.

In four dimensional case one has a notion "half conformal flatness", in other words, the vanishing of a half part of W, W^+ or W^-.

Let (M,g) be an oriented Riemannian 4–manifold. Then a 2–form $\alpha \in \Omega^2$ splits with respect to the star operator $*$ into the self–dual part $\alpha^+ = (\alpha + *\alpha)/2$ and the anti–self–dual part $\alpha^- = (\alpha - *\alpha)/2$, $\alpha = \alpha^+ + \alpha^-$.

The Weyl conformal tensor W viewed as an $\mathrm{End}(TM)$–valued 2–form decomposes into $W = W^+ + W^-$ and we say (M,g) is self–dual or anti–self–dual (or simply half conformally flat) if $W^- = 0$ or $W^+ = 0$.

Obviously a conformally flat 4–manifold is self–dual and anti–self–dual.

Examples of conformally flat manifolds which are well known are manifolds of constant curvature and Riemann surfaces. These manifolds are divided into spaces of positive, negative or zero curvature.

Similarly the sign of the scalar curvature divides the set of all Riemannian 4–manifolds up to conformal change into three classes (see § 2 for the details and [5], [44]) so that a half conformally flat structure $[g]$ is called type positive, zero or negative according to the sign of the scalar curvature.

We denote by $\mathscr{C}_M$ the set of smooth conformal structures on a given compact connected oriented 4–manifold M and define an action $\mathscr{W} : \mathscr{C}_M \longrightarrow \mathbb{R}$;

$\mathscr{W}(\gamma) = \frac{1}{2}\int_M |W(g)|_g^2\, dV_g = \frac{1}{2}\int_M \mathrm{Tr}\, W(g) \wedge * W(g)$ for $W = W(g)$, the Weyl conformal tensor of a representative g of γ.

The topological identity $\tau(M) = \frac{1}{12\pi^2}\int (|W^+|^2 - |W^-|^2) dV_g$ then indicates the absolute inequality; $\mathscr{W}(\gamma) \geq 6\pi^2 |\tau(M)|$ for the Hirzebruch signature of M, $\tau(M)$, and " = " holds if and only if γ or g is self–dual (necessarily $\tau(M) \geq 0$) or anti–self–dual ($\tau(M) \leq 0$).

The moduli $\mathscr{M} = \mathscr{M}_M$ of anti–self–dual conformal structures on M is defined as all equivalence classes of anti–self–dual conformal structures. Here γ, $\gamma_1 \in \mathscr{C}_M$ are equivalent if $g_1 = \varphi^* g$ for a diffeomorphism φ of M and for some representatives g and g_1 of γ and γ_1, respectively and we write $\gamma_1 = \varphi^* \gamma$.

DEFINITION 1 The moduli of anti–self–dual conformal structures $\mathscr{M}_M$ is defined as the quotient

$$\mathscr{M}_M = \{\gamma = [g] \in \mathscr{C}_M ; W(g)^+ = 0\}_{/\mathrm{Diff}^+(M)} ,$$

modulo the group of orientation preserving diffeomorphisms of M, $\mathrm{Diff}^+(M)$.

To simplify the argument we deal mainly with anti–self–dual case, since reversing the orientation transfers an anti–self–dual conformal structure into self–dual conformal structure.

Another type of definition of the moduli is

$$\mathscr{M}'_M = \{[g] \in \mathscr{C}_M ; W(g)^+ = 0\}_{/\mathrm{Diff}^0(M)} ,$$

where $\mathrm{Diff}^0(M)$ denotes the group of diffeomorphisms homotope to the identity id_M.

Then $\tilde{\mathscr{M}}_M \longrightarrow \mathscr{M}_M$ is a fibration whose fibre is the "mapping class group".

The moduli $\tilde{\mathscr{M}}_M$ corresponds to the Teichmüller moduli of Riemann surfaces and its analysis is much easier to deal with than that of $\mathscr{M}_M$.

Works for moduli of some special geometric structures, for instance the moduli of Einstein metrics on 4–manifolds are recently done by several geometers ([31], [1], [42]) and our investigation of the moduli of half conformally flat structures seems to be an approach along the similar lines.

However, there are other moduli spaces which share common feature with our moduli from conformal geometric viewpoint, the moduli of Riemann surfaces and the moduli of Yang–Mills instantons ([7], [16]).

Being guided by established theories of these moduli spaces one can develop the study of our moduli.

Like the Yang–Mills instanton case our moduli has a "quantum number",the Hirzebruch signature corresponding to the instanton number. It admits also an elliptic complex describing the local data.

We have few examples of manifold for which the moduli is completely known.

For S^4 $\mathscr{M}$ consists of a single point, the standard conformally flat structure ([35]).

The complex projective plane $\mathbb{C}P^2$ has the Fubini–Study metric as an isolated point in $\mathscr{M}$ ([27], [43]).

The conformally flat case is another example whose moduli is somewhat known. In fact each conformally flat structure has by making use of the developing map a holonomy correspondence $\pi_1(M) \longrightarrow SO(5,1)$, the conformal group of S^4 with the standard metric, so that the moduli of conformally flat structures is mapped into the representation space $\mathscr{R}(\pi_1(M);SO(5,1))$, the space of conjugacy classes of representations $\pi_1(M) \longrightarrow SO(5,1)$.

A product 4–manifold $\Sigma_k \times \mathbb{C}P^1$ with metrics of opposite constant curvatures is a nontrivial example of conformally flat 4–manifold. Here Σ_k denotes a genus $k(> 1)$

compact Riemann surface.

By counting the dimensions the moduli of conformally flat structures on $\Sigma_k \times \mathbb{C}P^1$ is naturally embedded in $\mathscr{R}(\pi_1(\Sigma_k);SO(5,1))$, since $\dim \mathscr{R} = 30(k-1)$ is the minus sign of the index (1.1).

As in the Yang–Mills instanton case $\mathscr{M}$ is in general described locally as a conformal group quotient of a real analytic subvariety in a finite dimensional vector space, the first cohomology group H^1 of the elliptic complex:
$C^\infty(TM) \longrightarrow C^\infty(\mathrm{Hom}(\Omega^+,\Omega^-)) \longrightarrow C^\infty(S_0(\Omega^+))$ (see § 3, (ii) for the precise definition).

This complex has the index

$$\dim H^0 - \dim H^1 + \dim H^2 = \frac{1}{2}(29\tau(M) + 15\chi(M)) \tag{1.1}$$

from the Atiyah–Singer index theorem ($\chi(M)$ is the Euler characteristic of M).

The 0–th cohomology group $H^0 = \mathrm{Ker}\, L$ at $\gamma \in \mathscr{C}_M$ is the Lie algebra of the conformal group $C^0(\gamma) = \{\varphi \in \mathrm{Diff}^0(M);\ \varphi^*\gamma = \gamma\}$.

By applying a slice theorem (Theorem 3.3, § 3) and the Kuranishi map (Theorem 3.4, § 3) one has indeed

<u>THEOREM 2</u> For any $\bar{\gamma} \in \mathscr{M}_M$ there exists a neighborhood $U_{\bar{\gamma}}$ represented by the group quotient of the zero's of a map $\Phi : H^1 \longrightarrow H^2$;

$$U_{\bar{\gamma}} = \mathrm{Zero}(\Phi : H^1 \longrightarrow H^2)/_{C^0(\gamma)}$$

By virtue of the formulation of $\mathscr{C}_M$ given in § 3 the tangent space $T_\gamma \mathscr{C}_M$ is identified with $C^\infty(\mathrm{Hom}(\Omega^+,\Omega^-))$. A positive definite inner product on it is defined as

$$\|A\|^2 = \int_M (-\mathrm{Tr}AA^*)(x)dV_g(x)\ , \quad A \in C^\infty(\mathrm{Hom}(\Omega^+,\Omega^-)) \tag{1.2}$$

in terms of a "canonical" volume form dV_g, where A^* is the adjoint of $A : \Omega^- \longrightarrow \Omega^+$.

The notion "canonical" requires dV_g to satisfy the conformal invariance and the naturality with respect to diffeomorphisms, from which the inner product $\|A\|^2$ is $\mathrm{Diff}^+(M)$–invariant.

By using a basis of $H^+ = \{$self–dual harmonic 2–forms$\}$, for instance, which is orthonormal with respect to the cup product on $H^2(M;\mathbb{Z})$ one can exhibit such a canonical volume form (see § 3 v) for the details).

Thus this L^2–inner product is able to descend to the quotient $\mathscr{C}_{M/\mathrm{Diff}^0(M)}$.

By restricting this inner product we have

<u>THEOREM 3</u> The moduli of anti–self–dual conformal structures is endowed with a Riemannian metric even if a point $\overline{\gamma}$ has a quotient singularity.

We would like to state several consequences and applications of our theorems.

The first one is a local Torelli–type theorem on a "periodic map".

There is a natural map, the period map, $p : \mathscr{C}_M \longrightarrow G^+_{b^+}(H^2) = \{$positive b^+–planes in $H^2(M;\mathbb{R}) \cong \mathbb{R}^{b^2}\}$, where $H^2(M;\mathbb{R})$ is equipped with the cup product of type (b^+,b^-) ([17, Appendix]). At a tangential level this is

$$p_* : C^\infty(\mathrm{Hom}(\Omega^+,\Omega^-)) \longrightarrow \mathrm{Hom}(H^+,H^-) \tag{1.3}$$

for the spaces $H^\pm$ of self–dual (anti–self–dual) harmonic 2–forms.

<u>THEOREM 4</u> For a K3 surface M the p_* restricted to the tangent space of $\mathscr{M}_M$ at

any Ricci flat metric becomes an isometry with respect to the L^2–metric and the invariant metric on $\mathrm{Hom}(H^+,H^-)$, so that the component of $\tilde{\mathcal{M}}_M$ containing a Ricci flat metric (and hence a type zero anti–self–dual conformal structure) is isometric onto some open domain in the symmetric space $SO(3,19)/_{SO(3)\times SO(19)}$.

This theorem is already shown in terms of polarized Ricci flat Kähler metrics ([31], [8] and see also for the brief survey [2]). However it will be verified from our formulation of $\tilde{\mathcal{M}}_M$ in § 5 (Proposition 5.2).

As a consequence of this theorem there is no type negative anti–self–dual conformal structure on a K3 surface M, close to any Ricci flat metric.

The moduli $\tilde{\mathcal{M}}_M$ is divided into disjoint three parts $\tilde{\mathcal{M}}_M = \tilde{\mathcal{M}}_M{}^{(+)} \sqcup \tilde{\mathcal{M}}_M{}^{(0)} \sqcup \tilde{\mathcal{M}}_M{}^{(-)}$ according to the sign of constant scalar curvature.

The presence of each piece implies a geometric restriction. In fact, if $\tilde{\mathcal{M}}_M{}^{(+)}$ is not empty, then the quadratic form on $H^2(M;\mathbb{Z})$ is negative or zero so that from the definite intersection form theorem M is homeomorphic to $\overline{\mathbb{C}P^2} \# \ldots \# \overline{\mathbb{C}P^2}$ (b^2–times) provided M is simply connected.

On the other hand, if $\tilde{\mathcal{M}}_M{}^{(0)} \neq \phi$ and $H^+ \neq 0$, then M must be a Kähler surface with an extremal Kähler metric in the sense of Calabi [13] (see § 2, and [28] for the classification of candidates of those M's of nonempty $\mathcal{M}^{(0)}$).

So we obtain a map from $\tilde{\mathcal{M}}_M{}^{(0)}$ into the moduli of complex structures on M, $\tilde{\mathcal{J}}_M$.

For a ruled surface, a typical anti–self–dual 4–manifold of which $\tilde{\mathcal{M}}_M{}^{(0)} \neq \phi$ we are able to present $\tilde{\mathcal{M}}_M{}^{(0)}$ as in the representation space $\mathcal{R}(\pi_1(M); SL(2,\mathbb{R})\times PU(2))$, $PU(2) = SU(2)/_{Z_2}$ whose dimension coincides with the dimension of $\tilde{\mathcal{J}}_M$ (see Theorem 5.1).

This phenomenon can be explained by the investigation of a fibred space structure of $\mathscr{N}_M{}^{(0)}$ over $\mathscr{T}_M$ in a more general setting, namely for M of $b^+ = 1$ (i.e. $p_g = 0$).

Infinitesimally for a fixed pair (J,g) , a complex structure and a zero scalar curvature Kähler metric traceless symmetric 2–tensors which are J–invariant give rise to the "vertical" direction of $\mathscr{N}_M{}^{(0)} \longrightarrow \mathscr{T}_M$ and the "horizontal" direction is just a traceless symmetric 2–tensor induced from a complex structure deformation.

We discuss this observation in § 5 and as a result from Theorem 5.3 and Propositions 5.4, 5.5, 5.6 we can assert at least that for M of $p_g = 0$ $j : \mathscr{N}_M{}^{(0)} \longrightarrow \mathscr{T}_M$ has a fibre space structure over the image $j(\mathscr{N}_M{}^{(0)})$, some number of connected components of $\mathscr{T}_M$. Its vertical tangent space must be in terms of the first cohomology description the linear subspace of $\{$J–invariant $h \in T_{\overline{\gamma}} \mathscr{N} \subset H^1_g\}$ annihilated by the Ricci form of g . For the precise statement see Proposition 5.6. This linear subspace is thought to be the exact space describing the "vertical" tangent direction of the j .

The importance of half conformally flat 4–manifolds is that they are equipped with twistor spaces. It is an interesting question how our moduli relates with the moduli of complex structures on the twistor space, while we only remark on it in § 5.

However, more interesting is an investigation of the ends of the moduli of half conformally flat structures. The action $\mathscr{W}(\gamma) = 6\pi^2 |\tau(M)|$ for $\overline{\gamma} \in \mathscr{N}_M$ so that a bubble off phenomenon may occur at points where the Weyl conformal tensor concentrates. A Uhlenbeck's type theorem is expected as in the Yang–Mills instanton case.

The essential difference from the Yang–Mills instanton case is that by bubbling off a half conformally flat 4–manifold may separate into some half conformally flat 4–orbifolds $M_1,\dots,M_k$ such that $M = M_1 \# \dots \# M_k$ (see also [2] , [42]). So possibility of bubble off is detected by the quadratic form on $H^2(M;\mathbb{Z})$ ([23]). Here the connected sum is considered as generalized one being attached along homology 3–sphere. At any rate the one point blown up of $\mathbb{C}^2$ with anti–self–dual Kähler metric whose conformal compactification is $\overline{\mathbb{C}P^2}$ with the Fubini–Study metric ([30], [40]) and the Eguchi–Hanson metric on an ALE 4–manifold must play roles as "1–instantons" in the compactification of the moduli.

We discuss in § 2 the scalar curvature type and the connected sum operation. In § 3 we review briefly the fundamental properties of the Weyl conformal tensor and study the moduli of half conformally flat structures to show the main theorems (the real analytic subvariety theorem and the L^2–metric theorem).

We specify our argument in § 4 to the moduli of Ricci flat metrics of unit volume, identified with the moduli of type zero conformal structures when the Hitchin's bound $\chi + \frac{3}{2}\tau = 0$ is satisfied, and exhibit the detailed proof for the local Torelli–type theorem. § 5 is devoted to the investigation of the moduli $\tilde{\mathscr{M}}_M^{(0)}$ in terms of complex structures. Indeed we derive the "horizontal" direction theorem from the $\frac{2}{3}$ complex–homomorphism between the Kodaira–Spencer deformation complex and the half conformally flat deformation complex (Theorem 5.2) and obtain also the "vertical" direction theorems, as Propositions 5.4, 5.5, 5.6.

We summarize in Appendix some formulae needed in deriving the linearization of the Weyl conformal tensor.

For general references of (half) conformally flat manifolds we refer to [37], [4], [8], [15], [28], [39].

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2. Scalar curvature type

i) Before discussing the moduli of half conformally flat structures we begin with scalar curvature type.

As is shown as Yamabe problem solved by Aubin, Schoen, a compact connected oriented Riemannian 4–manifold (M,g) admits a constant scalar curvature metric, conformally equivalent to g ([5], [44]).

A conformal change $g' = f^2 g$, $f \in C^\infty(M)$, > 0, has the scalar curvature ρ' obeying the equation

$$\rho' f^3 = 6\,\Delta f + \rho f \tag{2.1}$$

for the Laplacian $\Delta = \Delta_g$ and the scalar curvature ρ of g.

From (2.1) one has the following proposition from which the value of constant scalar curvature is unique up to volume normalized conformal change provided the value is nonpositive.

PROPOSITION 2.1 Let g, g' be two conformally equivalent metrics of same volume. If they have constant scalar curvature ≤ 0, then $g' = g$.

PROOF We assume $\int_M dV_g = 1$. The metric $g' = f^2 g$ is a conformal change. So $\int_M f^4 dv_g = 1$. The proposition is obvious if $\rho = \rho' = 0$. So assume $\rho = \rho' < 0$. At a point $x \in M$ where f has the maximal value $\Delta f = -g^{ij}\partial_i\partial_j f \geq 0$ so that from the equality $(-\rho)f(1-f^2) = 6\Delta f$ $1-f^2(x) \geq 0$ and hence $1 \geq f$ on M. So $f \equiv 1$ because $\int f^4 dv_g = 1$. The case $\rho' \leq \rho < 0$ is similarly proved. Q.E.D.

Now we divide $\mathscr{C}_M$, the set of conformal structures, into three parts $\mathscr{C}_M^{(+)}$,

$\mathscr{C}_M^{(0)}$, $\mathscr{C}_M^{(-)}$ according to the sign of the constant scalar curvature and decompose $\mathscr{M}_M$ as $\mathscr{M}_M = \mathscr{M}_M^{(+)} \amalg \mathscr{M}_M^{(0)} \amalg \mathscr{M}_M^{(-)}$.

To every $\gamma \in \mathscr{C}_M \setminus \mathscr{C}_M^{(+)}$ we choose a representative g of unit volume and assign the value of constant scalar curvature of a conformal change of g within the volume normalized conformal class. So we get a map, Diff(M)–invariant $\rho : \mathscr{C}_M \setminus \mathscr{C}_M^{(+)} \longrightarrow \mathbb{R}$ which descends to a "smooth" function on $\mathscr{C}_M \setminus \mathscr{C}_M^{(+)} / \mathrm{Diff}^0(M)$ in certain Sobolev norm.

ii) Non negative type

The following are known with respect to half conformally flat 4–manifolds of nonnegative type.

<u>THEOREM 2.2</u> ([14], [9], [15], [28]). Let (M,g) be a connected 4–manifold endowed with a complex Kähler structure. (i) If (M,g) is compact and self–dual, then (M,g) is a complex space form, i.e., $\mathbb{C}P^2$ with a Fubini–Study metric, $\mathbb{C}^2/_\Lambda$ with a flat standard metric, $D^2/_\Gamma$ with a standard Kähler metric, or a compact quotient of $D^1 \times \mathbb{C}P^1$ with opposite curvature metrics (here D^1, D^2 are the unit balls). (ii) (M,g) is anti–self–dual if and only if the scalar curvature $\rho = 0$.

<u>THEOREM 2.3</u> ([39]). Let (M,g) be a compact connected oriented anti–self–dual 4–manifold of type positive or zero. If M admits a harmonic self–dual 2–form $\theta \not\equiv 0$ i.e., $b^+(M) > 0$, then (M,g) carries a complex structure for which g is a Kähler metric of type zero and the normalized $|\theta|^{-1}\theta$ is the Kähler form.

It follows from Theorem 2.3 that (i) if $\mathscr{M}_M^{(+)} \neq \phi$, then $b^+(M) = 0$, namely the intersection form of $H_2(M;\mathbb{Z})$ is negative definite or zero so that for such M of $\pi_1 = 1$, M is homeomorphic to the connected sum of $b^2(M)$ copies of $\overline{\mathbb{C}P^2}$, $\mathbb{C}P^2$ with reversed orientation, due to Donaldson's theorem [16] and (ii) if $\mathscr{M}_M^{(0)} \neq \phi$ and $b^+(M) > 0$, then $\mathscr{M}_M^{(+)} = \phi$ and M carries a complex structure with a Kähler metric of zero scalar curvature.

It is concluded moreover from Theorems 2.2, 2.3 that (i) type positive self–dual compact Kähler surface is only $\mathbb{C}P^2$ with a Fubini–Study metric, (ii) type negative self–dual compact Kähler surface is only a complex space form of negative constant holomorphic curvature, (iii) a Kähler metric is anti–self–dual if and only if it is type zero and (iv) compact conformally flat Kähler surfaces are only a Kähler flat torus T^4 and a compact quotient $(D^1 \times \mathbb{C}P^1)/_\Gamma$.

The last 4–manifold is in the algebraic geometric terminology a complex ruled surface M_k, a holomorphic $\mathbb{C}P^1$ bundle over a Riemann surface Σ_k of genus $k(> 1)$.

We remark against this 4–dimensional special feature that every conformally flat Kähler manifold of complex dimension ≥ 3 is flat ([46]).

A Hopf surface, diffeomorphic to $S^1 \times S^3$, is an example of compact conformally flat 4–manifold ([11], [38]). Its scalar curvature type is positive.

iii) Connected sum

A fundamental operation in conformal geometry is taking the connected sum. The class of conformally flat manifolds is closed under the connected sum operation ([36]). The subclass, a class of type positive conformally flat manifolds is also closed under this operation ([45]).

For half conformally flat case the connected sum operation must be specifically important since the "quantum number" τ behaves additively, $\tau(M \# N) = \tau(M) + \tau(N)$ and it is reasonably expected that the operation # works on half conformally flat 4–manifolds with "one instanton" $\mathbb{C}P^2$ with a Fubini–Study metric. Actually the connected sum of n copies of $\mathbb{C}P^2$ for arbitrary n is endowed with a self–dual conformal structure ([43], [21], [18], [41]).

iv) Negative type case

Type positive manifolds are well investigated because of Lichnerowicz–Hitchin $\hat{A}$–vanishing theorem for spin structure.

However, type negative 4–manifolds seem so far to be less known.

THEOREM 2.4 Let $M = N_1 \# N_2$ be a connected sum of compact connected oriented conformally flat 4–manifolds. If N_i, $i = 1,2$ is a flat torus T^4 or a ruled surface M_k, $k > 1$, with a conformally flat structure, then M admits a conformally flat structure and moreover any conformally flat structure on M must be negative.

PROOF From Kulkarni's theorem [36] M admits a conformally flat structure. Let [g] be any conformally flat structure on M. Assume its type is nonnegative. Since $b^+(M) = b^+(N_1) + b^+(N_2) > 0$, (M,g) must be Kähler from Theorem 2.3 so that M is T^4 ($b^2 = 6, \chi = 0$) or M_k ($b^2 = 2, \chi = 4(1-k)$). On the other hand $b^2(M) = b^2(N_1) + b^2(N_2)$, $\chi(M) = \chi(N_1) + \chi(N_2) - 2$. So the topological type of M differs from T^4 and M_k.

REMARK The class of type negative conformally flat 4–manifold is closed under the connected sum operation, as pointed out by Lafontaine ([38]).

3. Moduli of anti–self–dual conformal structures

i) Let M be a compact connected oriented 4–manifold. We fix for a technical reason a volume form dv on M.

For a smooth metric g on M we denote by $[g]$ the conformal structure represented by g.

We note first that any conformal structure γ has the unique representative metric g whose volume form $dv_g = \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$ coincides with dv. We call this g the normalized representative of γ.

Since two metrics g, g' on M are related as $g'(X,Y) = g(h(X),Y)$ for a positive definite symmetric tensor h, we regard conformal structures as smooth sections of a fibre bundle $V \longrightarrow M$ whose fibre at $x \in M$ is $S_+(T_x^*M)/_{\mathbb{R}^+}$. Here $S_+(T_x^*M)$ is the cone of positive definite symmetric bilinear forms on T_xM and $\mathbb{R}^+$ operates by scalar multiplication; $\mathscr{C}_M \cong C^\infty(M;V)$.

This is the standard description of conformal structures, valid for arbitrary dimension.

We have another formulaton of $\mathscr{C}_M$ from the four dimensionality.

The star operator $* : \Omega^2 \longrightarrow \Omega^2$ depending on a conformal structure and the orientation of M gives the splitting $\Omega_x^2 = \Omega_x^+ \oplus \Omega_x^-$, $x \in M$, into $\pm$ eigenspaces Ω_x^+, Ω_x^- with $\Omega_x^+ \wedge \Omega_x^- = 0$ so that the wedge product $\cdot\wedge\cdot : \Omega^2 \longrightarrow \Omega^4 = \mathbb{R}dv$ is positive on Ω^+ and negative on Ω^-, respectively.

Conversely a choice of an appropriate 3–dimensional subspace U in Ω_x^2 on which $\cdot\wedge\cdot$ is positive determines uniquely a conformal structure γ at $x \in M$ so that U and the subspace $U^\perp$ annihilated by U give the splitting $\Omega_x^2 = \Omega_x^+ \oplus \Omega_x^-$, $\Omega_x^+ = U$, $\Omega_x^- = U^\perp$.

So given a 4–manifold M fixing a conformal structure means equivalently a choice of an appropriate rank 3 subbundle Ω' of $\Omega^2 \longrightarrow M$ (see [21], [17, Appendix] for this formulation). Thus, once we fix a conformal structure γ with splitting $\Omega^2 = \Omega_\gamma^+ \oplus \Omega_\gamma^-$, we can identify $\mathscr{C}_M$ with an open set in $C^\infty(\mathrm{Hom}(\Omega_\gamma^+, \Omega_\gamma^-))$;

$$\mathscr{C}_M \cong \{A \in C^\infty(\mathrm{Hom}(\Omega^+_\gamma, \Omega^-_\gamma))\ ;\ \eta \wedge \eta + A\eta \wedge A\eta > 0\ ,\ \eta \in \Omega^+\}\ .$$

<u>REMARK</u> These two identifications are very natural because we have an SO(4)–isomorphism between $\Omega^+ \otimes \Omega^-$ and $S_0(T^*M) = \{$traceless symmetric 2–tensors$\}$:

$$\begin{aligned} \Omega^+ \otimes \Omega^-(\cong \mathrm{Hom}(\Omega^+,\Omega^-)) &\longrightarrow S_0(T^*M) \\ (\eta^+,\eta^-) &\longmapsto h = (h_{ij})\ , \end{aligned} \tag{3.1}$$

$h_{ij} = g^{kl}\ \eta^+_{ik}\ \eta^-_{lj}$ (Lemma 4.6 [10]) and $h \in S_0(T^*M)$ induces a homomorphism $A = A_h : \Omega^+ \longrightarrow \Omega^-$, $A\eta^+ = (A\eta^+_{ij})$;

$$A\eta^+_{ij} = h_i^k \eta^+_{kj} + h_j^k \eta^+_{ik}\ ,\ \ \eta^+ \in \Omega^+ \tag{3.2}$$

giving the inverse.

We adopt the Einstein summation convention throughout this article unless any confusion occurs.

ii) Elliptic complex

Our next investigation is to derive the linearization of W^+, the self–dual part of W.

The tensor W is composed of the Riemannian curvature tensor R, the Ricci tensor Ric and the scalar curvature ρ.

R is regarded as a self–adjoint operator: $\Omega^2 \longrightarrow \Omega^2$;

$$R(e_i \wedge e_j) = \frac{1}{2} R_{ijkl}\, e_k \wedge e_l$$

for an orthonormal basis $\{e_i\}$ of 1–forms so that

$$R = \begin{bmatrix} R^{++} & R^{+-} \\ R^{-+} & R^{--} \end{bmatrix}$$

with respect to the splitting $\Omega^2 = \Omega^+ \oplus \Omega^-$. Each of R^{++} , R^{--} has $W^{\pm} \in C^{\infty}(S_0(\Omega^{\pm}))$ as the traceless component and actually

$$R^{++} = W^+ + \frac{1}{12}\rho$$

$$R^{--} = W^- + \frac{1}{12}\rho \;,$$

where $S_0(\Omega^+)$ denotes the traceless symmetric product of Ω^+ ([4], [24]).

Raising indices of W we consider W^+ , W^- as sections of $\Omega^+ \otimes so(3)^+$, $\Omega^- \otimes so(3)^-$, respectively and then as End(TM)–2–forms

$$W^+ = \frac{1}{2}(W + *W)\,,\; W^- = \frac{1}{2}(W - *W) \;.$$

Here $so(3)^{\pm}$ is the Lie algebra of skew adjoint endomorphisms of T_xM caused by the operation of $\Omega^{\pm}_x$.

We denote by $D = D_g : C^{\infty}(S_0(T^*M)) \longrightarrow C^{\infty}(S_0(\Omega^+))$ the directional derivative of W^+ at $\gamma = [g]$, $D_\gamma(h) = (\delta W^+)(h)$ for $h \in T_\gamma \mathscr{C}_M$. The tangent space $T_\gamma \mathscr{C}_M$ is here identified through the first identification of $\mathscr{C}_M$ with $C^{\infty}(S_0(T^*M))$, the space of traceless symmetric 2–tensors, since as we note in i) we can choose for any $\gamma \in \mathscr{C}_M$ the normalized representative g uniquely.

PROPOSITION 3.1 Let $\gamma = [g]$ be an anti–self–dual conformal structure. Then the directional derivative D is a second order differential operator represented as

$$D(h) = (\delta W_g(h))^+ \quad , \tag{3.3}$$

that is, $D(h)$ is the self–dual part of the directional derivative of the full Weyl conformal tensor W.

PROOF The proof involves only calculation. By the definition of W^+, $(\delta W_g^+)(h) = (\delta W_g(h))^+ + \frac{1}{2}(\delta *_g(h))(W(g))$.

Since for any metric g_1, the star operator satisfies $*_{g_1} = (h_1^*) \circ *_g \circ (h_1^{-1})^*$ for $h_1 \in C^\infty(\mathrm{End}(TM))$ given by $g_1(X,Y) = g(h_1(X),Y)$, the derivative $(\delta *_g)(h)$ is $(\delta *_g)(h) = h \circ *_g - *_g \circ h$ where h is considered as acting on Ω^2 as derivation. So $D_g(h) = (\delta W_g(h))^+ + \frac{1}{2}h(*_g W(g)) - \frac{1}{2}*_g(hW(g))$ reduces to $(\delta W_g(h))^+$, because $W(g) \in C^\infty(S_0(\Omega^-))$. Q.E.D.

The action of diffeomorphisms of M on $\mathscr{C}_M$ yields the Lie derivative operation on the tangent space $T_\gamma \mathscr{C}_M$ by choosing a representative g within γ as

$$\begin{aligned} L = L_g : C^\infty(TM) &\longrightarrow C^\infty(S_0(T^*M)) \\ X &\longmapsto L(X) \ , \end{aligned} \tag{3.4}$$

$$L(X)_{ij} = \nabla_i X_j + \nabla_j X_i - \frac{1}{2}(\nabla_i X^i) g_{ij} \ .$$

We derive then a complex at any anti–self–dual $\gamma = [g]$;

$$C^{\infty}(TM) \xrightarrow{L_g} C^{\infty}(S_0(T^*M)) \xrightarrow{D_g} C^{\infty}(S_0(\Omega^+)) \tag{3.5}$$

PROPOSITION 3.2 ([21]). This complex is elliptic.

This complex has the index $\frac{1}{2}(29\tau(M) + 15\chi(M))$. See also [20].

iii) Slice theorem

To get the real analytic variety structure theorem for $\tilde{\mathscr{A}}_M$ we discuss a slice theorem and then a Kuranishi map theorem, even though these theorems are quite common for the Yang–Mills instanton case ([22]).

Consider the L^2–adjoint of L_g , $[g] \in \mathscr{C}_M$, denoted by L^* with respect to Sobolev spaces

$$L^* : L^p_{k+1}(\mathrm{Hom}(\Omega^+,\Omega^-)) \longrightarrow L^p_k(TM) \ .$$

The kernel of L^* , an L^2–orthogonal to Im L , gives a slice in $L^p_{k+1}(\mathrm{Hom}(\Omega^+,\Omega^-))$.

We use here the second identification, since $\mathscr{C}_M$ is considered as an open subset of an affine space.

Consider the composite map

$$\Psi : \mathrm{Diff}^0(M) \times \mathscr{C}_M(\subset L^p_{k+1}(\mathrm{Hom}(\Omega^+,\Omega^-))$$

$$\longrightarrow \mathscr{C}_M \xrightarrow{L^*} C^{\infty}(TM) \ , \tag{3.6}$$

where $\mathrm{Diff}^0(M)$ is completed by the L^p_k Sobolev norm and the first map is the diffeomorphism pull back.

To obtain the slice theorem we follow § 3, [22] and [19].

The partial derivative at $(\mathrm{id}_M,0)$ $\delta_1\Psi = L^*L$ is self–adjoint and elliptic. So the restriction of Ψ

$$\tilde{\Psi} : \exp((\mathrm{Ker}\ L)^{\perp}) \times \mathscr{C}_M \longrightarrow (\mathrm{Ker}\ L)^{\perp} \tag{3.7}$$

has at $(\mathrm{id}_M,0)$ the invertible partial derivative. By the implicit function theorem $\tilde{\Psi}(\varphi,A) = 0$ has then a unique solution $\varphi = \varphi(A)$ for A in a neighborhood U of γ so that one has a map $f : A \longmapsto \tilde{A} = \varphi(A)^*(A)$, $L^*(\tilde{A}) = 0$.

Since the conformal group $C^0(\gamma) = \{\varphi \in \mathrm{Diff}^0(M); \varphi^*\gamma = \gamma\}$ acts on $\mathscr{C}_M$ as isometries and leaves Ker L invariant, each $\psi \in C^0(\gamma)$ maps $(\mathrm{Ker}\ L)^{\perp}$ into itself and hence on the group level $\exp(\mathrm{Ker}\ L)^{\perp}$ into itself.

Thus for $\tilde{A} = f(A)$ ψ acts as $\psi^*(\tilde{A}) = (\psi^{-1}\varphi(A)\psi)^*\psi^*(A)$ and $L^*(\psi^*(\tilde{A})) = 0$. So $f(\psi^*(A)) = \psi^* f(A)$.

We can now follow the argument given by Ebin in the case of space of Riemannian metrics and we have the following theorem for a sufficiently small ball in $\mathrm{Ker}\ L^*$.

<u>THEOREM</u> 3.3 For any $\gamma \in \mathscr{C}_M$ there exists a slice $\mathscr{S}$ in $\mathscr{C}_M$, $\gamma \in \mathscr{S}$, a ball in $\mathrm{Ker}\ L^*$ such that

(i) any $\psi \in C^0(\gamma)$ fixes $\mathscr{S}$ invariantly

(ii) if $\varphi \in \mathrm{Diff}^0(M)$, $\varphi^*(\mathscr{S}) \cap \mathscr{S} \neq \phi$, then $\varphi \in C^0(\gamma)$,

(iii) there exists a local section $\chi : \mathrm{Diff}^0(M)/_{C^0(\gamma)} \longrightarrow \mathrm{Diff}^0(M)$ defined on a neighborhood U of the origin such that the map $F : (u,A) \longmapsto (\chi(u))^*(A)$; $U \times \mathscr{S} \longrightarrow \mathscr{C}_M$ is a homeomorphism onto a neighborhood of γ , diffeomorphic off fixed points of $C^0(\gamma)$.

From this theorem for any $\overline{\gamma} \in \mathscr{C}_M/\mathrm{Diff}^0(M)$ there exists a neighborhood of $\overline{\gamma}$, homeomorphic to $\mathscr{S}/C^0(\gamma)$ and diffeomorphic off quotient singular points.

iv) Kuranishi map

Let γ be an anti–self–dual conformal structure on a 4–manifold M .

Consider the anti–self–dual equation in the slice $\mathscr{S} = \mathscr{S}_\gamma$

$$\begin{aligned} & W^+(\gamma + A) = 0 \ , \\ & L^*(A) = 0 \ . \end{aligned} \tag{3.8}$$

The second equation is a gauge fixing equation. $W^+(\gamma+A)$ is the self–dual Weyl conformal tensor of a conformal structure $\gamma + A$ close to γ .

Choose metrics g , g_1 , for instance, the volume normalized representatives of γ , $\gamma + A$, respectively. Since $*_{g_1} = (h_1{}^*)^{-1} \circ *_g \circ (h_1)^*$ for $h = h_A \in C^\infty(\mathrm{End}(TM))$, $g(X,Y) = g_1(h(X),Y)$, the first equation is replaced by $(h_A(W(\gamma + A)))^{+_g} = 0$. So we rewrite (3.8) as

$$\begin{aligned} & (h_A(W(\gamma + A)))^{+_g} = 0 \ , \\ & L^*(A) = 0 \ . \end{aligned} \tag{3.8$'$}$$

Define a map

$$W^+ : \mathscr{A} \longrightarrow L^p_k(S_0(\Omega^+))$$
$$A \longmapsto (h_A(W(\gamma + A))^{+_g} \tag{3.9}$$

and expand $W^+(A)$ as

$$W^+(A) = W^+(\gamma) + D_g(A) + R(A)$$

with a remainder term $R(A) = R_g(A)$. So for anti–self–dual γ we have

$$W^+(A) = D_g(A) + R(A) \tag{3.10}$$

As a routine business for solving the equation $W^+(A) = 0$ we introduce a map $\Theta = \Theta_g$, the Kuranishi map, from a small ball in $L^p_{k+1}(\mathrm{Hom}(\Omega^+,\Omega^-))$ into $L^p_{k+1}(\mathrm{Hom}(\Omega^+,\Omega^-))$;

$$\Theta : A \longmapsto A + D^* G(R(A)) \tag{3.11}$$

for D^* , the adjoint of D and G , the Green operator of DD^* on $L^p_k(S_0(\Omega^+))$.

As was discussed in the deformations of complex structures ([34]) we can show the following, since Θ is locally invertible and $C^0(\gamma)$–equivariant.

<u>THEOREM</u> 3.4 (i) There exists for small $\epsilon > 0$ a $C^0(\gamma)$–equivariant map Φ from an ϵ–ball H^1_ϵ of $H^1_\gamma \cong \mathrm{Ker}\, L^* \cap \mathrm{Ker}\, D$ to $H^2_\gamma \cong \mathrm{Ker}\, D\, D^* : A \longmapsto \pi R(\Theta^{-1}(A))$ such that anti–self–dual conformal structures in the slice $\mathscr{A}_\gamma$ are described as

$\mathrm{Zero}(\Phi) = \{A \in H^1_\epsilon;\ \Phi(A) = 0\}$ and (ii) each gauge equivalence class $\bar{\gamma} \in \mathscr{M}_M$ has a neighborhood, homeomorphic to the quotient $\mathrm{Zero}(\Phi)/_{C^0(\gamma)}$, diffeomorphic off singular points. Here π is the projection of $C^\infty(S_0(\Omega^+))$ onto H^2_γ.

v) L^2–metric

As a first step towards for defining a Riemannian metric on the moduli $\mathscr{M}_M$ we define a $\mathrm{Diff}^+(M)$–gauge invariant L^2–metric on $\mathscr{C}_M$.

Throughout this section as in iv) we keep the identification $\mathscr{C}_M \subset C^\infty(\mathrm{Hom}(\Omega^+,\Omega^-))$.

For $A \in C^\infty(\mathrm{Hom}(\Omega^+,\Omega^-))$ define the adjoint $A^* : \Omega^-_\gamma \longrightarrow \Omega^+_\gamma$ with respect to the volume form dv_g (g is a representative of γ), in other words

$$\eta^+ \wedge A^* \eta^- = A\eta^+ \wedge \eta^- , \quad \eta^\pm \in \Omega^\pm . \tag{3.12}$$

Then the trace $-\mathrm{Tr}\, AA^*$ is a scalar function on M, positive definite and depends only on γ.

In fact, choose at a point orthonormal basis $\{\eta^+_i\}$, $\{\eta^-_i\}$ of $\Omega^\pm{}_\gamma$, i.e., $\pm\, \eta^\pm{}_i \wedge \eta^\pm{}_j = \delta_{ij} dv_g$, $i = 1,2,3$ and set $A\eta^+_i = A_i{}^j \eta^-_j$. Then $A^* \eta^-_i = A^*{}_i{}^j \eta^+_j$ has $A^*{}_i{}^j = -A_j{}^i$ and hence $-\mathrm{Tr}\, AA^* = A_i{}^j A_j{}^i$ is positive definite.

From this definition the trace is obviously independent of the choice of g.

A diffeomorphism φ acts on $C^\infty(\mathrm{Hom}(\Omega^+,\Omega^-))$:
$A \in C^\infty(\mathrm{Hom}(\Omega^+_\gamma,\Omega^-_\gamma)) \longmapsto A^\varphi \in C^\infty(\mathrm{Hom}(\Omega^+_{\gamma_1},\Omega^-_{\gamma_1}))$, $\gamma_1 = \varphi^* \gamma$ by the following diagram

$$\begin{array}{ccc}
\Omega^+_{g,\varphi(x)} & \xrightarrow{\quad \varphi_x^* \quad} & \Omega^+_{\varphi^* g, x} \\
{\scriptstyle A_{\varphi(x)}}\big\downarrow & & \big\downarrow{\scriptstyle (A^\varphi)_x} \\
\Omega^-_{g,\varphi(x)} & \xrightarrow{\quad \varphi_x^* \quad} & \Omega^-_{\varphi^* g, x}
\end{array}$$

where $x \in M$ and g is a representative of γ. So $(A^\varphi)_x = \varphi_x^* \circ A_{\varphi(x)} \circ (\varphi_x^*)^{-1}$ and $(A^\varphi)_x^* = \varphi_x^* \circ (A^*)_{\varphi(x)} \circ (\varphi_x^*)^{-1}$. Then the pointwise inner product satisfies

$$(-\mathrm{Tr}\, A^\varphi A^{\varphi *})(x) = (-\mathrm{Tr}\, AA^*)(\varphi(x)) \ . \tag{3.13}$$

To define an L^2–inner product on $\mathscr{C}_M$, invariant under $\mathrm{Diff}^+(M)$–action we need from (3.13) a "canonical" volume form $g \longmapsto dV_g$ satisfying the conformal invariance, $dV_{fg}(x) = dV_g(x)$, $f \in C^\infty(M)$, > 0, and the naturality, $dV_{\varphi^* g}(x) = (\varphi^* dV_g)(x)$.

Assume the existence of the canonical volume form. We then obtain an L^2–inner product on $C^\infty(\mathrm{Hom}(\Omega^+_\gamma, \Omega^-_\gamma))$ as

$$\|A\|^2 = \int_M (-\mathrm{Tr}\, AA^*)(x) dV_g(x) \ , \quad A \in C^\infty(\mathrm{Hom}(\Omega^+, \Omega^-)) \ , \tag{3.14}$$

integrated in terms of the canonical volume form.

So the remaining problem is to verify the existence of such a volume form.

To investigate it we notice that the quadratic form induced from the cup product: $H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \longrightarrow H^4(M;\mathbb{Z}) \cong \mathbb{Z}$ gives a nondegenerate symmetric form on $H^2(M;\mathbb{R})$ of type (b^+, b^-), identified with the wedge product on the de Rham cohomologies: $H^2(M;\mathbb{R}) \times H^2(M;\mathbb{R}) \longrightarrow H^4(M;\mathbb{R}) = \mathbb{R}[dv]$; $([\theta],[\omega]) \longmapsto [\theta \wedge \omega]$ (dv is a volume form of unit volume).

For any metric g $H_g^{\pm} = \{$(anti–) self–dual harmonic 2–forms$\}$ are

$b^{\pm}$–dimensional subspaces of $H^2(M;\mathbb{R})$, respectively.

To simplify the argument we assume $b^+ > 0$ (when $b^+ = 0$, $b^- > 0$ is assumed so that anyway $b^2 > 0$ is primarily assumed).

We choose an orthonormal basis $\{\psi_i^+ = \psi_{g,i}^+\}$, $1 \leq i \leq b^+$, of H_g^+. The orthonormality is measured by the cup product; $[\psi_i^+] \wedge [\psi_j^+] = \delta_{ij}[dv]$.

Define

$$dV_g(x) = \sum_{i=1}^{b^+} \|\psi_i^+\|_g^2(x)dv_g(x)\,,\quad x \in M\,, \tag{3.15}$$

where $\|\cdot\|_g$ is the norm measured by g.

This does not depend on choices of orthonormal basis. This is conformally invariant since for each i $\|\psi_i\|_g^2\, dv_g = \psi_i \wedge *\psi_i = \psi_i \wedge \psi_i$.

The canonical volume form (3.15) depends smoothly on the metric g, since $b^+ = \dim H_g^+$ is a topological invariant (see for example Theorem 4.5, p. 178 [34]).

The naturality of $dV_g(x)$ is indicated as follows. Any $\varphi \in \mathrm{Diff}^+(M)$ induces a quadratic form isometry $\varphi^*: H^2(M:\mathbb{Z}) \longrightarrow H^2(M:\mathbb{Z})$ so that $\{\varphi^*\psi_i^+\}$ gives rise to an orthonormal basis of $H^+_{\varphi^* g}$ and hence

$$dV_{\varphi^* g}(x) = \sum_i \|\varphi^*\psi_i^+\|^2_{\varphi^* g}(x)\, dv_{\varphi^* g}(x)$$

$$= \sum_i \|\psi_i^+\|_g^2(\varphi(x))(\varphi^* dv_g)(x)$$

Thus one has

PROPOSITION 3.5 The inner product (3.15) is positive definite and $\mathrm{Diff}^+(M)$–invariant.

Theorem 1, § 1 is concluded from Theorems 3.3, 3.4 and 3.5.

REMARK dV_g is the Riemannian volume form dv_g multiplied by a nonnegative weight function which has in general discrete zero from the result of [3]. However in some special case dV_g coincides with dv_g up to a constant scalar factor. Indeed this is the case if each of ψ_j^+ has constant norm.

We also remark that through the identification (3.1) $-\mathrm{Tr}\, AA^*$ is just $4\, \mathrm{Tr}\, hh$, $A = A_h$ from (3.2).

4. K3 surfaces

Recall the following formula for a compact connected oriented Riemannian 4–manifold (M,g)

$$\chi(M) + \frac{3}{2}\,\tau(M) = \frac{1}{4\pi^2}\int |W^+|^2 + \frac{1}{48\pi^2}\int \rho^2 - 3|\mathrm{Ric}|^2 \tag{4.1}$$

(see p. 72 [24]). So as an easy observation from (4.1)

<u>PROPOSITION</u> 4.1 Let M be as before a compact connected oriented 4–manifold. If M satisfies $2\chi(M) + 3\tau(M) = 0$ (this is the case for a complex torus, a quotient of a complex torus, a K3 surface, an Enriques surface and the quotient of an Enriques surface by an antiholomorphic involution [26]). Then any anti–self–dual Riemannian metric g is of zero scalar curvature if and if g is Ricci flat.

The moduli $\mathscr{M}_M^{(0)}$ of type zero anti–self–dual conformal structures on M of $2\chi + 3\tau = 0$ is then identified with the moduli of Ricci flat metrics of unit volume.

Now let M be a K3 surface, a simply connected compact complex surface with the trivial canonical bundle K_M.

The topological invariants are $\chi = 24$, $b^2 = 22$, $(b^+, b^-) = (3,19)$ so $\tau = -16$ and $2\chi + 3\tau = 0$.

The moduli $\mathscr{M}_M^{(0)}$ is well investigated in terms of the periodic map. Actually the quadratic form q_M on $H^2(M;\mathbb{Z})$ has type $(3,19)$ and the Grassmannian $G_3^+ = SO(3,19)/_{SO(3)\times SO(19)}$ of oriented positive definite 3–planes in $H^2(M;\mathbb{R})$ gives the Ricci flat Kähler metrics on M provided we ignore the action of $\mathrm{Aut}(H^2(M;\mathbb{Z});q_M)$; $p : \mathscr{E} \longrightarrow G_3^+$. Here $\mathscr{E}$ denotes the moduli of Ricci flat metrics of unit volume.

Then $\mathscr{E}$ admits a structure of 57 dimensional symmetric space with an invariant metric. This means that the space $\mathrm{Hom}(H^+,H^-) = H^- \otimes (H^+)^*$ gives the tangent space $T_g \mathscr{E}$ and the invariant metric is $-\mathrm{Tr}\, XX^t$, $X \in \mathrm{Hom}(H^+,H^-)$ from the standard argument of symmetric spaces.

On the other hand the index of the complex (3.5) is -52 and $\dim H^0 = 0$ and moreover from Corollary A.5 in Appendix $\dim H^2 = 5$. The virtual dimension of our moduli at each $\bar{\gamma}$ represented by a Ricci flat metric g is then at most 57.

The following proposition asserts as exhibited in Theorem 4, § 1 that $\mathscr{M}_M^{(0)}$ has actually 57 dimension and the connected component of $\mathscr{M}_M$ containing $\mathscr{M}_M^{(0)}$ is itself $\mathscr{M}_M^{(0)}$ and is isometric to $p(\mathscr{E})$ in G_3^+. As an easy observation there is no type negative anti–self–dual conformal structure nearby $\mathscr{M}_M^{(0)}$.

<u>PROPOSITION 4.2</u> Let g be a Ricci flat metric on a K3 surface M. Let $\phi_a^+ \in H^+$, $a = 1,2,3$ and $\phi_b^- \in H^-$, $b = 1,\dots,19$ be harmonic 2–forms being orthonormal basis of H^+, H^-, respectively. Then $\phi_b^- \otimes \phi_a^+ \in H^- \otimes H^+$, $1 \le a \le 3$, $1 \le b \le 19$ form through the identification $H^+ \cong (H^+)^*$ an orthonormal basis of the tangent space $T_{\bar{\gamma}} \mathscr{M}_M$, $\gamma = [g]$ with respect to the L^2–metric.

<u>PROOF</u> First we remark that the metric g is Kähler from Theorem 2.3 and each ϕ_a^+ is covariantly constant so that $dV_g = 3\, dv_g$ and then the L^2–inner product (3.14) is just the ordinary inner product $\|h\|^2 = \int_M \mathrm{Tr}\, hh\, dv_g$ of $C^\infty(S_0(T^*M))$ through the identification (3.1).

Let $h \in C^\infty(S_0(T^*M))$ be given through the map (3.1) by $\phi_b^- \otimes \phi_a^+$. Then $h = (h_{ij})$ is $h_{ij} = g^{k\ell} \phi^-_{ik} \phi^+_{\ell j} = \phi^-_{ik} \phi^{+k}{}_j$.

We verify $h \in \mathrm{Ker}\, L^* \cap \mathrm{Ker}\, D$ at g.

Since $d^* \phi^- = 0$ and $\nabla \phi^+ = 0$, $L^*(h)$ is from (3.4)

$L^*(h) = -2g^{\ell i}\nabla^j h_{j\ell} = -2g^{\ell i}(\nabla^j \phi^-_{jk})\phi^{+k}{}_\ell = 0$, so $h \in \operatorname{Ker} L^*$.

To show $Dh = 0$ we make use of the anti–self–duality of ϕ^- and apply (3.3) and (A.1), Appendix. Apply $\phi^{+k}{}_\ell$ to $\nabla_i\phi^-_{jk} + \nabla_j\phi^-_{ki} + \nabla_k\phi^-_{ij} = 0$. Then we have

$$\nabla_i h_{j\ell} - \nabla_j h_{i\ell} + \nabla_s \phi^-_{ij} \cdot \phi^{+s}{}_\ell = 0 \tag{4.2}$$

and hence

$$\nabla_k\nabla_i h_{j\ell} - \nabla_k\nabla_j h_{i\ell} + (\nabla_k\nabla_s\phi^-_{ij})\phi^{+s}{}_\ell = 0 \ , \tag{4.3}$$

or interchange k and ℓ

$$\nabla_\ell\nabla_i h_{jk} - \nabla_\ell\nabla_j h_{ik} + (\nabla_\ell\nabla_s\phi^-_{ij})\phi^{+s}{}_k = 0 \ . \tag{4.4}$$

So the tensor $U \in C^\infty(\Omega^2 \otimes \Omega^2)$ defined in (A.2) is

$$2\, U_{ijk\ell} = (\nabla_k\nabla_s\phi^-_{ij})\phi^{+s}{}_\ell - (\nabla_\ell\nabla_s\phi^-_{ij})\phi^{+s}{}_k \, . \tag{4.5}$$

$D(h)$ is the $S0(\Omega^+)$–component of U since g is Ricci flat.

Without loss of generality we can assume $\phi^+_1 = \omega$, the Kähler form and ϕ^+_2 , ϕ^+_3 are the real and imaginary parts of a covariantly constant holomorphic 2–form, respectively.

We use the complex coordinate indices.

For $\phi^+ = \omega$, $\phi^{+i}{}_j = \sqrt{-1}\ \delta^i_j$, $\phi^{+\bar{i}}{}_{\bar{j}} = -\sqrt{-1}\ \delta^i_j$, $i,j = 1,2$ and others are zero. Then $U_{ijk\ell} = 0$ for $k,\ell \in \{1,2\}$ and $i,j \in \{1,2,\bar{1},\bar{2}\}$ since $[\nabla_k,\nabla_\ell] = 0$ and also $U_{ijk\ell} = 0$ for $i,j \in \{1,2\}$ and $k,\ell \in \{1,2,\bar{1},\bar{2}\}$ since ϕ^- is a (1,1)–form.

Similarly $U_{ijk\ell} = 0$ for all indices running over $\bar{1}$, $\bar{2}$. Therefore the components of

U in $\Omega^+ \otimes \Omega^+$ remain are only the $\omega \otimes \omega$–component. But it is

$g^{\bar{j}i} g^{\bar{\ell}k} U_{i\bar{j}k\bar{\ell}} = g^{\bar{j}i} g^{\bar{\ell}k} (\nabla_k \nabla_{\bar{\ell}} + \nabla_{\bar{\ell}} \nabla_k) \psi_{i\bar{j}}$ which vanishes from the fact that ψ^- is primitive.

The similar argument works for other ψ_2^+, ψ_3^+ so that $\psi_b^- \otimes \psi_a^+ \in \operatorname{Ker} L^* \cap \operatorname{Ker} D$ for any a, b.

That $\psi_b^- \otimes \psi_a^+$, $1 \le a \le 3$, $1 \le b \le 19$ enjoy an L^2–orthonormal basis of $T_{\bar{\gamma}} \mathscr{N}_M$ follows from the definition of the L^2–inner product (3.14) and the remark mentioned at the beginning of the proof. Q.E.D.

5. Half conformal flatness and complex structures

i) Moduli on ruled surface

Since any ruled surface $M = M_k$ has $\tau = 0$, every anti–self–dual structure is conformally flat. Also $\mathscr{M}_M^{(+)} = \phi$ because $b^+ = 1$. The moduli $\mathscr{M}_M = \mathscr{M}_M^{(0)} \perp\!\!\perp \mathscr{M}_M^{(-)}$, the moduli of "conformally flat" structures on M_k is considered to lie inside the representation space $\mathscr{R}(\pi_1(M_k)\,;\ SO(5,1))$, as explained in § 1.

Now we are interested in $\mathscr{M}_M^{(0)}$, the moduli of type zero conformally flat structures on M_k.

Let $\overline{\gamma} \in \mathscr{M}_M^{(0)}$. Then one has from Theorem 2.3 a representative g of γ, a Kähler metric of zero scalar curvature. From Theorem 2.2 (M_k, g) is then covered by the Kähler product $D^1 \times \mathbb{C}P^1$; $(M_k, g) = D^1 \times \mathbb{C}P^1/_\Gamma$ for a discrete subgroup Γ of $\mathrm{Aut}(D^1 \times \mathbb{C}P^1) = SL(2,\mathbb{R}) \times PU(2)$ acting freely and properly discontinuously. Since every $a \in PU(2)$ has a fixed point, Γ is the graph of a homomorphism $\phi : \Gamma_1 \subset SL(2,\mathbb{R}) \longrightarrow PU(2) = \mathrm{Aut}(\mathbb{C}P^1)$, where Γ_1 is a subgroup isomorphic to $\pi_1(\Sigma_k)$ acting on D^1 freely and properly discontinuously.

It follows then that every type zero conformally flat structure $\overline{\gamma} \in \mathscr{M}_M^{(0)}$ one–to–one corresponds to an appropriate conjugacy class of representation $\pi_1(\Sigma_k) \longrightarrow SL(2,\mathbb{R}) \times PU(2)$. More precisely, $\mathscr{M}_M^{(0)}$ is exactly the set of all conjugacy classes $[\phi]$, $\phi : \pi_1(\Sigma_k) \longrightarrow SL(2,\mathbb{R}) \times PU(2)$ satisfying that ϕ is the composite of $\phi_1 : \pi_1(\Sigma_k) \longrightarrow SL(2,\mathbb{R})$ and $\phi_2 : \mathrm{Im}(\phi_1) \subset SL(2,\mathbb{R}) \longrightarrow PU(2)$ and ϕ_1 acts on the disk D^1 freely and properly discontinuously.

Since the homomorphism ϕ_2 induces a $PU(2)$ flat connection on a complex vector bundle over the Riemann surface $\Sigma_k = D^1/_{\mathrm{Im}\ \phi_1}$; $D^1 \times_{\phi_2} \mathbb{C}^2 \longrightarrow \Sigma_k$, the following fibration structure theorem is available.

THEOREM 5.1 The moduli $\mathscr{M}_M^{(0)}$ on a ruled surface $M = M_k$, $k > 1$ has a structure of fibration $\mathscr{M}_M^{(0)} \longrightarrow \mathscr{M}_{\Sigma_k}$, the Teichmüller moduli of Riemann surfaces, whose fibre over a Riemann surface represented by $[\phi_1]$, $\phi_1 : \pi_1(\Sigma_k) \longrightarrow SL(2,\mathbb{R})$ is the moduli of $PU(2)$ flat connections on the complex smooth vector bundle induced by $\phi_2 : \mathrm{Im}\ \phi_1 \longrightarrow PU(2)$.

From this theorem it is expected that the fibration yields a Riemannian submersion with respect to the L^2–metric and the Weil–Petersson metric on $\mathscr{M}_{\Sigma_k}$ such that the L^2–metric restricted to each fibre is the metric introduced in [29].

Since $SL(2,\mathbb{R})\times PU(2)$ is immersed in $SO(5,1)$ as a proper subgroup, $\mathscr{R}(\pi_1(\Sigma_k)$; $SL(2,\mathbb{R}) \times PU(2))$ and hence $\mathscr{M}_M^{(0)}$ are immersed in $\mathscr{R}(\pi_1(M_k); SO(5.1))$ .Therefore

COROLLARY 5.2 Any ruled surface admits type negative anti–self–dual structures around a type zero anti–self–dual structure. Namely, if $\mathscr{M}_M^{(0)} \neq \phi$, then $\mathscr{M}_M^{(-)}$ is also not empty.

REMARK There exists a ruled surface admitting no type zero anti–self–dual conformal structure ([12]).

ii) Moduli of complex structures

Let M be a compact complex surface. We investigate how the moduli of complex structures $\mathscr{J}_M$ of M affects our moduli.

The Kodaira–Spencer complex for complex structure deformations has the index $\frac{1}{6}(7\ c_1^2(M) - 5\ c_2(M)) = \frac{1}{6}(21\ \tau(M) + 9\ \chi(M))$ ([33]).

This index is for $M = M_k$, a ruled surface, $6(1-k)$, so that $H^1(M_k,TM)$ has the virtual complex dimension $6(k-1)$. This dimension coincides from Theorem 5.1 with the "complex dimension" of $\mathscr{M}_M^{(0)}$.

This phenomenon is fortunately not accidental.

Let M be now a compact complex surface of $p_g = 0$ (or equivalently $b^+(M) = 1$). Then from Theorem 2.3 every type zero anti–self–dual structure $\overline{\gamma} \in \mathscr{M}^{(0)}$ yields the unique complex structure J_γ (up to diffeomorphisms) so that one has a map

$$j : \mathscr{M}_M^{(0)} \longrightarrow \mathscr{I}_M = \{\text{complex structures}\}/_{\mathrm{Diff}^0(M)} \, ; \; \overline{\gamma} \longmapsto [J_\gamma] \; .$$

Relative to a fixed complex structure there are two possibilities of conformal structure deformations. One is a deformation fixing a complex structure and varying a metric and another is a deformation varying a complex structure.

We postpone investigating the first possibility.

Consider the second one from which we derive information on $\mathscr{M}_M^{(0)}$ being affected by $\mathscr{I}_M$.

Let J be a complex structure on M and g a zero scalar curvature J–Kähler metric.

Consider a deformation of complex structures J(t) of J . The infinitesimal deformation $I = \dot{J}(0) = I^i_j \frac{\partial}{\partial x^i} \otimes dx^j$ then satisfies

$$IJ + JI = 0 \, , \;\; \delta N_J(I) = 0 \tag{5.1}$$

Here N_J is the Nijenhuis tensor and $\delta N_J(I) = \frac{d}{dt} N_{J(t)} \Big|_{t=0}$.

From the first equation $I \in C^\infty(\mathrm{End}(TM))$ is regarded as a section of $\Omega^{0,1} \otimes \mathbf{T}M$, $I = I^i_{\bar{j}} \frac{\partial}{\partial z^i} \otimes d\bar{z}^j$ for a complex coordinate (z^1,z^2) and the second equation means $\overline{\partial} I = 0$ for $\overline{\partial} = \overline{\partial}_J : C^\infty(\Omega^{0,1} \otimes \mathbf{T}M) \longrightarrow C^\infty(\Omega^{0,2} \otimes \mathbf{T}M)$, where $\Omega^{0,p}$ is the (0,p)–form bundle and $\mathbf{T}M = T^{1,0}M$ is the holomorphic tangent bundle.

Trivial deformations $L_X J$, $X \in C^\infty(TM)$ are obviously sections of (5.1). Since $\bar\partial Z = \frac{1}{2\sqrt{-1}}(L_X J)^{(1,1)}$ for $X = Z + \bar Z$, $Z \in C^\infty(TM)$ ([8]), we obtain the Kodaira–Spencer complex

$$C^\infty(TM) \xrightarrow{\bar\partial} C^\infty(\Omega^{0,1}\otimes TM) \xrightarrow{\bar\partial} C^\infty(\Omega^{0,2}\otimes TM) \quad . \tag{5.2}$$

<u>THEOREM</u> 5.3 Let (M,J,g) be a compact complex surface with an anti–self–dual Kähler metric. Then there exist homomorphisms between complexes (3.5) and (5.2);

$$\begin{tikzcd} C^\infty(TM) \arrow[r, "\bar\partial"] & C^\infty(\Omega^{0,1}\otimes TM) \arrow[r, "\bar\partial"] \arrow[d, "\beta^1"] & C^\infty(\Omega^{0,2}\otimes TM) \arrow[d, "\beta^2"] \\ C^\infty(TM) \arrow[r] & C^\infty(S_0(T^*M)) \arrow[r] & C^\infty(S_0(\Omega^+)) \end{tikzcd} \tag{5.3}$$

For the proof of Theorem 5.3 we need to define homomorphisms β^1, β^2 .

$\beta^1(I)$, $I \in C^\infty(\Omega^{0,1}\otimes TM)$ is defined as $\beta^1(I) = h_I$, the traceless symmetric tensor; $h_I(X,Y) = g(IX,Y) + g(X,IY)$.

Note that $\mathrm{Ker}\,\beta^1$ consists of those $I's$ being g–skew symmetric.

To define β^2 we introduce an operator ϑ: $C^\infty(\Omega^{0,2}\otimes TM) \longrightarrow C^\infty(\Omega^{0,2}\otimes\Omega^{0,2})$ as the composite of the operators

$$\begin{gathered} C^\infty(\Omega^{0,2}\otimes TM) \xrightarrow{\#} C^\infty(\Lambda^2(TM)\otimes\Omega^{0,1}) \xrightarrow{\bar\partial} \\ C^\infty(\Lambda^2(TM)\otimes\Omega^{0,2}) \xrightarrow{b} C^\infty(\Omega^{0,2}\otimes\Omega^{0,2}) \quad . \end{gathered} \tag{5.4}$$

β^2 is the traceless symmetric part of the real form of ϑ . Here $\#$ is the operator raising and lowering indices so that for $a = \frac{1}{2}\, a^i_{\bar{j}\bar{k}} \frac{\partial}{\partial z^i} \otimes dz^{\bar{j}} \wedge dz^{\bar{k}} \in C^\infty(\Omega^{0,2} \otimes TM)$

$$\vartheta a = (\nabla_{\bar{i}}\, a_{\bar{j}\,\bar{k}\,\bar{\ell}} - \nabla_{\bar{j}}\, a_{\bar{i}\,\bar{k}\,\bar{\ell}})(dz^{\bar{i}} \wedge dz^{\bar{j}}) \otimes (dz^{\bar{k}} \wedge dz^{\bar{\ell}}) \;,$$
$$a_{\bar{i}\,\bar{j}\,\bar{k}} = g_{\ell \bar{i}}\, a^{\ell}_{\;\bar{j}\,\bar{k}} \;. \tag{5.5}$$

<u>PROOF</u> of Theorem 5.3 What to show is that $\beta^2(\bar{\partial}(I)) = D(h_I)$ for any $I \in C^\infty(\Omega^{0,1} \otimes TM)$.

We calculate $\vartheta(\bar{\partial}(I))$ from (5.5) as

$$\vartheta(\bar{\partial}(I)) = (\nabla_{\bar{i}}\nabla_{\bar{k}} I_{\bar{j}\,\bar{\ell}} - \nabla_{\bar{i}}\nabla_{\bar{\ell}} I_{\bar{j}\,\bar{k}} - \nabla_{\bar{j}}\nabla_{\bar{k}} I_{\bar{i}\,\bar{\ell}} + \nabla_{\bar{j}}\nabla_{\bar{\ell}} I_{\bar{i}\,\bar{k}}) \times$$
$$(dz^{\bar{i}} \wedge dz^{\bar{j}}) \otimes (dz^{\bar{k}} \wedge dz^{\bar{\ell}}) \;. \tag{5.6}$$

On the other hand from Proposition 3.1 and the formula (A.1) in Appendix $D(h)$ is

$$D(h) = U(h)^+ + V(h)^+ \tag{5.7}$$

where $U(h)^+$, $V(h)^+$ are the $SO(\Omega^+)$–components of $U = U(h)$, $V = V(h)$, respectively;

$$U_{ijk\ell} = \tfrac{1}{2}\,(\nabla_k \nabla_j h_{i\ell} - \nabla_\ell \nabla_j h_{ik} - \nabla_k \nabla_i h_{j\ell} + \nabla_\ell \nabla_i h_{jk}) \;,$$
$$V_{ijk\ell} = -\tfrac{1}{4}\,(R_{jk} h_{i\ell} - R_{j\ell} h_{ik} - R_{ik} h_{j\ell} + R_{\ell i} h_{jk}) \;. \tag{5.8}$$

Since $\mathrm{Ric} \in \Omega^{1,1}$ for the Kähler metric g and the self–dual 2–form bundle is $\Omega^+ = \mathbb{R}\omega \oplus \Omega_0^+$, $\Omega_0^+ = (\Omega^{2,0} \oplus \Omega^{0,2})_{\mathbb{R}}$ for the Kähler form ω, V^+ vanishes and also the $S_0(\Omega_0^+)$–component of U agrees with that of (5.6). Moreover the $\Omega^+ \otimes \omega$–component of U is zero since $(h_I)_{i\bar{j}} = 0$. Q.E.D.

We would like to define a homomorphism between 1st cohomology groups. But this is not automatically defined because of the lack of 0–th homomorphism. However for any $I \in H_J^1 = \mathrm{Ker}\,\bar{\partial} \cap \mathrm{Ker}\,\bar{\partial}^*$, $\beta^1(I)$ belongs to $\mathrm{Ker}\, D$. We add a compensating term $-LGL^*(\beta^1(I))$ such that $\beta^1(I) - LGL^*(\beta^1(I))$ is in $H_\gamma^1 = \mathrm{Ker}\, L^* \cap \mathrm{Ker}\, D$. So we derive a homomorphism

$$\beta_*^1 : H_J^1 \longrightarrow H_\gamma^1 . \tag{5.9}$$

Here G is the Green operator of $L^* L$.

<u>PROPOSITION</u> 5.4 For the infinitesimal deformation $\beta^1(I) \in \mathrm{Ker}\, D$ caused by $I \in H_J^1$ the scalar curvature derivative $\delta\rho$ vanishes.

<u>PROOF</u> For the scalar curvature derivation $\delta\rho$ is $\delta\rho(h) = g^{ij}(\delta R)_{ij} - h^{ij}R_{ij}$ and from (A.6), Appendix $(\delta R)_{ij} = \frac{1}{2}(\nabla_\ell \nabla_i h^\ell_j + \nabla_\ell \nabla_j h^\ell_i - \nabla_\ell \nabla^\ell h_{ij})$. Since the metric g is Kähler and $h = \beta^1(I)$ is type (0.2), in complex coordinates $\delta\rho(h)$ is $g^{\bar{j}i}(\delta R)_{i\bar{j}}$. But $(\delta R)_{j\bar{\ell}} = \frac{1}{2}\{\nabla_{\bar{\ell}}(\nabla_{\bar{i}} h^{\bar{i}}{}_j) + \nabla_j(\nabla_i h^i{}_{\bar{\ell}})\}$ vanishes because $I = I^i{}_{\bar{j}} \in \mathrm{Ker}\,\bar{\partial}^*$ and $h_{\bar{i}\bar{j}}$ is given by $h_{\bar{i}\bar{j}} = I_{\bar{i}\bar{j}} + I_{\bar{j}\bar{i}}$. Q.E.D.

From this proposition the deformation of metrics g_t induced from any complex structure deformation in the above way keeps the scalar curvature constant zero.

We remark that the geometric genus p_g gives rise to an obstruction of the injectivity of the $\beta^1 : H^1_J \longrightarrow \operatorname{Ker} D$. Indeed we can show

PROPOSITION 5.5 The kernel of β^1 restricted to H^1_J is isomorphic with $H^0(M,\mathcal{O}(K_M))$.

PROOF Let $I \in H^1_J$ be in the kernel. Then this means that $(X,Y) \longmapsto g(IX,Y)$ is a skew symmetric tensor $I_{\bar i \bar j} dz^{\bar i} \wedge dz^{\bar j} \in C^\infty(\Omega^{0,2})$. On the other hand $\bar\partial I = 0$, i.e. $\nabla_{\bar k} I_{\bar i \bar j} = \nabla_{\bar j} I_{\bar i \bar k}$ so it follows that $\nabla_{\bar i} I_{\bar j \bar k} = 0$ for all i, j, k . Applying the Ricci identity

$$g^{\bar i h}\nabla_{\bar i}\nabla_h I_{\bar j \bar k} = g^{\bar i h}\nabla_h \nabla_{\bar i} I_{\bar j \bar k} + \rho I_{\bar j \bar k} = 0$$

we see $I_{\bar i \bar j} dz^{\bar i} \wedge dz^{\bar j}$ is covariantly constant. So the kernel $\{I \in H^1_J, \beta^1(I) = 0\}$ is isomorphic through the complex conjugation with the space of covariantly constant (2,0)–forms. This space is from Lemma 3.1, [28] exactly $H^0(M;\mathcal{O}(K_M))$.

Q.E.D.

Now we consider the inverse image of the map $j : \tilde{\mathcal{M}}_M \longrightarrow \tilde{\mathcal{J}}_M$.

We let (J,g) be a complex structure and an anti–self–dual J–Kähler metric.

Let g_t be a deformation of anti–self–dual J–Kähler metric. Then g_t satisfies $S_t \wedge \omega_t = 0$ for the Ricci form $S_t = \sqrt{-1}\, R^t_{i \bar j}\, dz^i \wedge dz^{\bar j}$. Let $f_t g_t$, $f_t \in C^\infty(M)$, > 0 be a conformal change so that its volume form agrees with dv_g . Then $h = \frac{d}{dt} f_t g_t \Big|_{t=0} \in C^\infty(S_0(T^*M))$ satisfies $D(h) = 0$. We differentiate $*_t \omega_t = \omega_t$ (ω_t : the Kähler form of g_t) and have $h(*\omega) - *h(\omega) = \dot\omega - *\dot\omega$. From (3.2) $h(\omega) \in \Omega^-$ so the anti–self–dual part $(\dot{\omega})^-$ is $(\dot{\omega})^- = h(\omega)$. From $S_t \wedge \omega_t = 0$ we have $\dot{S} \wedge \omega + S \wedge \dot{\omega} = 0$. But $\dot{S}$ is an exact fomr so by integration $\int S \wedge (\dot{\omega})^- = \int S \wedge h(\omega) = 0$. Thus we have

PROPOSITION 5.6 Let g_t, $g_0 = g$ be an anti–self–dual J–Kähler metric deformation. Let $h \in C^\infty(S_0(T^*M))$ be an infinitesimal deformation of g_t. Then the anti–self–dual harmonic part of $h(\omega)$ is annihilated by $[S]$ as a cohomology element, where ω, S are the Kähler form and the Ricci form of g.

For any anti–self–dual J–Kähler metric g on a ruled surface M_k, the Ricci form $S \neq 0$ spans H_g^- $(b^- = 1)$ so that from Proposition 5.6 there is no anti–self–dual J–Kähler metric deformations g_t, $g_0 = g$. This means that the map $j : \mathscr{M}_M^{(0)} \longrightarrow \mathscr{J}_M$ is an immersed map and moreover from Proposition 5.5 the injective homomorphism $\beta^1 : H_J^1 \longrightarrow \mathrm{Ker}\, D_g$ must give the inverse of j_*.

We expect that these arguments explain a fibred space structure of $\mathscr{M}_M^{(0)}$ when M is one of other complex surfaces of $p_g = 0$ admitting an anti–self–dual Kähler metric, for instance, a Ricci flat Enriques surface.

iii) Remark for twistor spaces

By the twistor correspondence any anti–self–dual conformal structure γ on a 4–manifold induces a complex structure J_γ on the unit sphere bundle $U(\Omega^+)$ over M, called the twistor space $Z = Z_M = (U(\Omega^+), J_\gamma)$ ([27]).

This correspondence induces a canonical map from $\mathscr{M}_M$ to the moduli $\mathscr{I}_Z$ of complex structure on Z. This map is an embedding since there is a twistorial characterization of complex 3–manifold (Th. 13, 69 [8]).

Correspondingly to this we have a homomorphism between the complex (3.5) and the

Kodaira–Spencer complex of Z (see (3.3) in [21]);

$$\begin{array}{ccccccc} C^{\infty}(TM) & \xrightarrow{L} & C^{\infty}(\mathrm{Hom}(\Omega^{+},\Omega^{-})) & \xrightarrow{D} & C^{\infty}(S_0(\Omega^{+})) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ C^{\infty}(TZ) & \xrightarrow{\bar\partial} & C^{\infty}(\Omega^{0,1}\otimes TZ) & \xrightarrow{\bar\partial} & C^{\infty}(\Omega^{0,2}\otimes TZ) & \xrightarrow{\bar\partial} & \end{array} \tag{5.10}$$

which induces the injective homomorphism of 1st cohomology groups, the "tangent spaces" of $\tilde{\mathscr{M}}_M$ and $\tilde{\mathscr{T}}_M$.

A conformally flat structure corresponds to a holonomy homomorphism $\pi_1(M) \longrightarrow SO(5.1)$. As was pointed out in p. 439, [4] the natural homomorphism $SO(5.1) \longrightarrow SO(6,\mathbb{C}) \longrightarrow PSL(4,\mathbb{C})$ then defines on the twistor space $U(\Omega^{+})$ a projective flat complex structure ([25], [32]).

The twistor space of a conformally flat 4–manifold is in fact represented locally as a neighborhood in $\mathbb{C}P^3$ containing a complex line.

Our investigation of the moduli of conformally flat 4–manifolds yields examples of family of projectively flat complex 3–manifolds.

A projectively flat complex compact 3–manifold Z_M satisfies for Chern numbers $16\, c_3(Z) = \frac{8}{3}\, c_1 c_2(Z) = c_1{}^3(Z) = 32\chi(M)$ (p. 135, [32] and [27]).

Appendix

In this appendix we will show

PROPOSITION A.1 Let g be an anti–self–dual conformal structure. Then the linear map: $C^\infty(S_0(T^*M)) \longrightarrow C^\infty(S_0(\Omega^+))$; $h \longmapsto (\delta W_g(h))^+$ is written as

$$(\delta W_g(h))^+ = U(h)^+ + V(h)^+ \ , \tag{A.1}$$

where U^+, V^+ are the $S_0(\Omega^+)$–components of $U,V \in C^\infty(\Omega^2 \otimes \Omega^2)$ defined by

$$U_{ijk\ell} = \frac{1}{2}(\nabla_k\nabla_j h_{i\ell} - \nabla_\ell\nabla_j h_{ik} - \nabla_k\nabla_i h_{j\ell} + \nabla_\ell\nabla_i h_{jk}) \ , \tag{A.2}$$

$$V_{ijk\ell} = -\frac{1}{4}(R_{kj}h_{i\ell} - R_{\ell j}h_{ik} - R_{ki}h_{j\ell} + R_{\ell i}h_{jk}) \ , \tag{A.3}$$

for $h = (h_{ij}) \in C^\infty(S_0(T^*M))$.

The proof needs a straightforward calculation. For two metrics g, $\tilde{g}$ we calculate the difference of the Christoffel symbols as

$$\{{}_j{}^i{}_k\}^\sim - \{{}_j{}^i{}_k\} = \frac{1}{2}\tilde{g}^{i\ell}(g_{\ell s}\nabla_j h^s{}_k + g_{js}\nabla_k h^s{}_\ell - g_{is}\nabla_\ell h^s{}_k) \tag{A.4}$$

for $h = (h^j{}_k) \in C^\infty(\mathrm{End}(TM))$, $g(hX,Y) = \tilde{g}(X,Y)$.

From this one has

$$\delta\{{}_j{}^i{}_k\}(h) = \frac{1}{2}(\nabla_j h^i{}_k + \nabla_k h^i{}_j - \nabla^i h_{jk}) \ , \tag{A.5}$$

$$h_{ij} = \frac{d}{dt}g_{ij}(0) \ , \ h^i{}_j = g^{ik}h_{kj} \ .$$

Applying the chain rule, one gets

$$\delta\, R_g(h)^i{}_{jk\ell} = \nabla_k(\delta\{{}_j{}^i{}_\ell\}(h)) - \nabla_\ell(\delta\{{}_j{}^i{}_k\}(h)) \ ,$$

and then from (A.5)

$$\begin{aligned}\delta\, R_g(h)^i{}_{jk\ell} &= \tfrac{1}{2}(\nabla_k\nabla_\ell h^i{}_j - \nabla_\ell\nabla_k h^i{}_j) \\ &\quad + \tfrac{1}{2}(\nabla_k\nabla_j h^i{}_\ell - \nabla_\ell\nabla_j h^i{}_k) \\ &\quad - \tfrac{1}{2}(\nabla_k\nabla^i h_{j\ell} - \nabla_\ell\nabla^i h_{jk}) \ .\end{aligned} \tag{A.6}$$

Hence

$$\begin{aligned}(\delta\, R_g(h))_{ijk\ell} &= \tfrac{1}{2}(\nabla_k\nabla_j h_{i\ell} - \nabla_\ell\nabla_j h_{ik} - \nabla_k\nabla_i h_{j\ell} + \nabla_\ell\nabla_i h_{jk}) \\ &\quad + \tfrac{1}{2}(h^t{}_i R_{tjk\ell} + h^t_j R_{itk\ell}) \ .\end{aligned} \tag{A.7}$$

The Weyl conformal tensor W has three parts

$$W = R + R' + R'' \ ,$$

$$R'_{ijk\ell} = -\tfrac{1}{2}(g_{ik}R_{j\ell} - g_{i\ell}R_{jk} + R_{ik}g_{j\ell} - R_{i\ell}g_{jk}) \ , \tag{A.8}$$

$$R''_{ijk\ell} = \tfrac{1}{6}\rho(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}) \ . \tag{A.9}$$

By calculating $\delta R'$ and $\delta R''$ we derive the following formula valid for any metric and any $h \in C^\infty(S^2(T^*M))$.

FORMULA A.2

$$\delta W_g(h)_{ijk\ell} = h^t{}_i W_{tjk\ell} + h^t{}_j W_{itk\ell} + U_{ijk\ell}$$

$$-\tfrac{1}{2}(h^t{}_i R_{tjk\ell} + h^t{}_j R_{itk\ell})$$

$$-\tfrac{1}{2}(g_{ik}\delta R^t{}_\ell g_{jt} - g_{i\ell}\delta R^t{}_k g_{jt} + g_{it}\delta R^t{}_k g_{j\ell} - g_{it}\delta R^t{}_\ell g_{jk})$$

$$+\tfrac{1}{6}(\delta\rho)(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}) \,. \tag{A.10}$$

Now assume that g is anti–self–dual and h is traceless. Then the $S_0(\Omega^+)$–component $(\delta W_g(h))^+$ is

$$(\delta W_g(h))^+ = U^+ + V^+ \ ,$$

where V^+ is the $S_0(\Omega^+)$–component of the third term V of (A.10), since the first term and the last two terms of (A.10) vanish when we take the $S_0(\Omega^+)$–component. Here we characterize the traceless symmetric product $S_0(\Omega^+)$ as

LEMMA A.3 The traceless symmetric product $S_0(\Omega^+_x)$ of Ω^+_x at x is $\{Z = (Z_{ijk\ell})$; Z is Ricci flat curvature like tensor satisfying the first Bianchi identity; $Z_{ijk\ell} = -Z_{jik\ell} = -Z_{ij\ell k} = Z_{k\ell ij}$, $g^{ik}Z_{ijk\ell} = 0$, $Z_{ijk\ell} + Z_{ik\ell j} + Z_{i\ell jk} = 0\}$.

We substitute $R = W - R' - R''$ into V as

$$V_{ijk\ell} = -\tfrac{1}{2}(h^t{}_i W_{tjk\ell} + h^t{}_j W_{itk\ell})$$

$$+\tfrac{1}{2}(h^t{}_i R'{}_{tjk\ell} + h^t{}_j R'{}_{itk\ell})$$

$$+\tfrac{1}{2}(h^t{}_i R''{}_{tjk\ell} + h^t{}_j R''{}_{itk\ell})$$

and take its $S_0(\Omega^+)$–component. Then

$$V^+{}_{ijk\ell} = -\tfrac{1}{4}(R_{kj}h_{i\ell} - R_{\ell j}h_{ik} - R_{ki}h_{j\ell} + R_{\ell i}h_{jk})^+$$

from which the proposition follows.

REMARK If g is anti–self–dual and Einstein, then $V^+ = 0$, namely $(\delta W_g(h))^+ = U^+$.

We would like to obtain a formula for the adjoint D^* of $D : C^\infty(S_0(\Omega^+)) \longrightarrow C^\infty(S_0(T^*M))$.

PROPOSITION A.4 For an anti–self–dual conformal structure $\gamma = [g]$ the adjoint D^* has the form

$$(D^*Z)_{ij} = \nabla^k\nabla^\ell Z_{ik\ell j} + \nabla^\ell\nabla^k Z_{ik\ell j} + R^{k\ell}Z_{ik\ell j} \tag{A.11}$$

PROOF D^* is defined by

$$\int_M (h, D^* Z) dv_g = \int_M (Dh, Z) dv_g \ ,$$

of which the right hand side we calculate as $(Dh,Z) = (U^+,Z) + (V^+,Z)$ from Proposition A.1.

Here

$$(U^+,Z) = (\nabla_k \nabla_j h_{i\ell} - \nabla_k \nabla_i h_{j\ell}) Z^{ijk\ell}$$

and

$$(V^+,Z) = h_{ij} R_{k\ell} Z^{ik\ell j} \ .$$

Then the formula (A.11) is derived from the integration

$$\int (U^+,Z) dV_g = \int h_{ij} (\nabla_k \nabla_\ell Z^{ik\ell j} + \nabla_\ell \nabla_k Z^{ik\ell j}) dv_g \ .$$

REMARK This formula is appeared already in [6] as the first variational equation $D^* W = 0$ of the functional $\mathscr{W} : \mathscr{C}_M \longrightarrow \mathbb{R}$ (see also Lemma 1, [15]).

As a consequence of Proposition A.4

COROLLARY A.5 Let M be a K3 surface or a complex 2–torus and g be a Ricci flat (i.e., type zero) anti–self–dual metric on M. Then the second cohomology group of the complex (3.5) is $H^2_g \cong \mathbb{R}^5$. In fact $\sum a_{ij} \phi_i^+ \otimes \phi_j^+$, $a_{ij} \in \mathbb{R}$, $a_{ij} = a_{ji}$, $\sum a_{ii} = 0$, span H^2_g for self–dual harmonic (i.e., covariantly constant) 2–forms ϕ_i^+, $i = 1,2,3$.

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