Blow up formulae for Euler characteristic of moduli space of rank 3 bundles over a surface.[1]

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1. Introduction.

In this preprint we partially solve the following problem. Let X be a smooth algebraic surface over $\mathbb{C}$. We fix some ample divisor H on X and denote by $\mathfrak{M}_H(r, C, k)$ the moduli space of Mumford – Takemoto μ_H-stable rank r vector bundles on X with Chern classes C and k such that $H \cdot C$ and r are coprime. Denote by $Z_r(q)$ the following generating function:

$$Z_r(q) = \sum_{k \in \mathbb{Z}} e(\mathfrak{M}_H(r, C, k)) \cdot q^k,$$

where $e(Y)$ for a complex scheme Y is the topological Euler characteristic of Y_{red}, and $e(\mathfrak{M}_H(r, C, k)) = 0$ whenever $\mathfrak{M}_H(r, C, k)$ is empty.

Further, let $\sigma : \tilde{X} \longrightarrow X$ be the blow up of the surface X at a point x_0 and $E = \sigma^{-1}(x_0)$ be the exceptional curve. For a positive integer n by definition put $H_n = n(\sigma^* H) - E$. Denote by $\mathfrak{M}_{H_n}(r, \sigma^* C - iE, k)$ the moduli space of Mumford – Takemoto μ_{H_n}-stable rank r vector bundles V on $\tilde{X}$ with $c_1(V) = \sigma^* C - iE,\ c_2(V) = k$. It is clear that $\mathfrak{M}_{H_n}(r, \sigma^* C - iE, k) \cong \mathfrak{M}_{H_{n+1}}(r, \sigma^* C - iE, k)$ if $n \gg 0$. We denote by $\mathfrak{M}_{H_\infty}(r, \sigma^* C - iE, k)$ this moduli space. Consider the generating functions

$$Z_{r,i}(q) = \sum_{k \in \mathbb{Z}} e(\mathfrak{M}_{H_\infty}(r, \sigma^* C - iE, k)) q^k.$$

Now we can formulate the question. *How do the functions $Z_{r,i}(q)$ depend on $Z_r(q)$?*

In [7], Vafa and Witten formulated hypothetical answer for this question in the case $r = 2$ and $i = 0$ derived from the S-duality conjecture in physics. They presuppose that there exists a universal function $u(q)$ such that

$$Z_{2,0}(q) = u(q) \cdot Z_2(q).$$

Moreover, the function $u(q)$ is a modular form.

This conjecture was proved by Li and Qin in [3]. The authors obtained the following result. *For $i = 0,\ 1$ there exist universal functions*

$$u_{2,i}(q) = \sum_{n \in \mathbb{Z}} q^{n(n+i)}$$

such that $Z_{2,i}(q) = u_{2,i}(q) \cdot Z_2(q)$.

In this preprint the problem is solved in the case $r = 3$. The main result of the preprint is the following (theorem 6.1 and corollary 6.4)

For $i = 0,\ 1,\ 2$ there exist universal functions $u_{3,i}(q)$ such that $Z_{3,i}(q) = u_{3,i}(q) \cdot Z_3(q)$, and

$$u_{3,i}(q) = q^{\delta_2^i} \sum_{(x,y) \in \mathbb{Z}^2} q^{x^2 + xy + y^2 + i(x+y)},$$

[1]This work was done during the author‘s stay at the Max Plank Institute für Mathematik in 1998.

where δ_2^i *is Kronecker symbol, i.e.* $\delta_2^i = \begin{cases} 1 & \text{for} \qquad i = 2, \\ 0 & \text{otheraise} \end{cases}$.

Further we fix $r = 3$, C, and H and use the notations

$$\mathfrak{M}(k) = \mathfrak{M}_H(3, C, k), \qquad \mathfrak{M}(i, k) = \mathfrak{M}_{H_\infty}(r, \sigma^* C - iE, k)$$

The idea of proof is the same as in [3]. It is the following.

For integer $d_1 \geq -\frac{i}{3}$ and $d_1 \geq d_2 \geq -\dfrac{d_1 + i}{2}$ denote by $\mathfrak{M}^{(d_1,d_2)}(i, k)$ the subset of $\mathfrak{M}(i, k)$ consisting of all bundles V such that

$$V|_E = \mathcal{O}_E(-d_1) \oplus \mathcal{O}_E(-d_2) \oplus \mathcal{O}_E(d_1 + d_2 + i),$$

then

$$e\Big(\mathfrak{M}(i, k)\Big) = \sum_{(d_1,d_2)} e\Big(\mathfrak{M}^{(d_1,d_2)}(i, k)\Big).$$

Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{O}_E(-d_2) \oplus \mathcal{O}_E(d_1 + d_2 + i) & \longrightarrow & V|_E & \longrightarrow & \mathcal{O}_E(-d_1) & \longrightarrow & 0 \\
 & & {\scriptstyle\beta} \uparrow & & {\scriptstyle res} \uparrow & & {\scriptstyle id} \uparrow & & \\
0 & \longrightarrow & V' & \longrightarrow & V & \xrightarrow{\alpha} & \mathcal{O}_E(-d_1) & \longrightarrow & 0\ , \\
 & & \uparrow & & \uparrow & & & & \\
 & & V(-E) & = & V(-E) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $V \in \mathfrak{M}^{(d_1,d_2)}(i, k)$. Note that $V' \in \mathfrak{M}^{(m_1,m_2)}(i + 1, k - d_1 - i)$ for some pair of integers (m_1, m_2) depending on (d_1, d_2). In other words, by means of the pair (V, α) we construct the pair (V', β). Later we will show that there exists a bijective morphism from a locally trivial bundle over $\mathfrak{M}^{(d_1,d_2)}(i, k)$ with fiber $\mathbb{P}(\mathrm{Hom}\,(V, \mathcal{O}_E(-d_1)))$ to the join of locally trivial bundles over $\mathfrak{M}^{(m_1,m_2)}(i + 1, k - d_1 - i)$ with fiber

$$\{\beta\}/\mathrm{Aut}\,(\mathcal{O}_E(-d_2) \oplus \mathcal{O}_E(d_1 + d_2 + i)) = \mathrm{EP}\,{}^{-m_1,-m_2,m_1+m_2+i+1}_{-d_2,d_1+d_2+i}.$$

Thus if we know $e(\mathrm{EP}\,{}^{-m_1,-m_2,m_1+m_2+i+1}_{-d_2,d_1+d_2+i})$, we obtain recursion relations for $e\Big(\mathfrak{M}^{(d_1,d_2)}(i, k)\Big)$. Finally, the isomorphism $\mathfrak{M}(3, k) \cong \mathfrak{M}(0, k + 3)$ lets obtain the main result.

This preprint is organized as follows. In section 1 we do some calculations known to experts, which we use to prove the main result. In section 2 we define the set $\mathrm{EP}\,{}^{a,b,c}_{e,d}$ and find Euler characteristic of it. Sections 3 and 4 deal with recursion relations for $e\Big(\mathfrak{M}^{(d_1,d_2)}(i, k)\Big)$. Finally, in the last section we deduce the blowup formulae.

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2. Some calculations.

In this section we do some calculations, which we use in the text.

Let X be a smooth algebraic surface and $\sigma : \tilde{X} \longrightarrow X$ be a blowup at a point $x_0 \in X$. Denote by E the corresponding exceptional curve.

2.1. Claim. *For an exceptional curve E*

$$c_2\Big(\mathcal{O}_E(-d)\Big) = d-1.$$

Proof. Using multiplicativity of the full Chern character and the restriction exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}((d-1)E) \longrightarrow \mathcal{O}_{\tilde{X}}(dE) \longrightarrow \mathcal{O}_E(-d) \longrightarrow 0,$$

we have

$$\Big(1+(d-1)E\cdot t\Big)\cdot\Big(1+E\cdot t + c_2(\mathcal{O}_E(-d))\cdot t^2\Big) = \Big(1+dE\cdot t\Big).$$

Whence,

$$(d-1)E^2 + c_2(\mathcal{O}_E(-d)) = 0 \iff c_2(\mathcal{O}_E(-d)) = d-1.$$

2.2. Claim. *For an exceptional curve E*

$$\mathrm{Tor}_1(\mathcal{O}_E, \mathcal{O}_E(d)) = \mathcal{O}_E(d+1).$$

Proof. Tensoring the restriction sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(-(d+1)E) \longrightarrow \mathcal{O}_{\tilde{X}}(-dE) \longrightarrow \mathcal{O}_E(d) \longrightarrow 0$$

by $\mathcal{O}_E$, we obtain

$$0 \longrightarrow \mathrm{Tor}_1(\mathcal{O}_E, \mathcal{O}_E(d)) \longrightarrow \mathcal{O}_E(d+1) \longrightarrow \mathcal{O}_E(d) \longrightarrow \mathcal{O}_E(d) \longrightarrow 0.$$

Thus,

$$\mathrm{Tor}_1(\mathcal{O}_E, O_E(d)) = \mathcal{O}_E(d+1).$$

2.3. Claim. *For an exceptional curve E and a locally free sheaf V on $\tilde{X}$*

$$\mathrm{Ext}^2_{\tilde{X}}(\mathcal{O}_E(d), V) \cong \mathrm{Ext}^0_E(V|_E, \mathcal{O}_E(d-1))^*,$$
$$\mathrm{Ext}^1_{\tilde{X}}(\mathcal{O}_E(d), V) \cong \mathrm{Hom}_E(\mathcal{O}_E(d), V|_E \otimes \mathcal{O}_E(-1)).$$

Proof. By Serre's duality on the surface $\tilde{X}$, we get

$$\mathrm{Ext}^q_{\tilde{X}}(\mathcal{O}_E(d), V) \cong \mathrm{Ext}^{2-q}_{\tilde{X}}(V, \mathcal{O}_E(d)\otimes K_{\tilde{X}})^*.$$

But $K_{\tilde{X}} = \sigma^*(K_X) + E$, i.e., $E\cdot K_{\tilde{X}} = -1$. Therefore, $\mathcal{O}_E(d)\otimes K_{\tilde{X}} = \mathcal{O}_E(d-1)$.
Further, $\mathrm{Ext}^{2-q}_{\tilde{X}}(V, \mathcal{O}_E(d-1)) \cong \mathrm{Ext}^{2-q}_E(V|_E, \mathcal{O}_E(d-1))$.
In the case $q=1$ we use Serre's duality for the rational curve E:

$$\mathrm{Ext}^1_E(V|_E, \mathcal{O}_E(d-1))^* \cong \mathrm{Hom}_E(\mathcal{O}_E(d-1), V|_E \otimes \mathcal{O}_E(-2)).$$

2.4. Claim. *For an exceptional divisor E*

$$\mathcal{E}xt^1(\mathcal{O}_E(-d), \mathcal{O}_{\tilde{X}}) \cong \mathcal{O}_E(d-1).$$

Proof. Let us apply the functor $\mathcal{H}om(\,\cdot\,, \mathcal{O}_{\tilde{X}})$ to the restriction sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}((d-1)E) \longrightarrow \mathcal{O}_{\tilde{X}}(dE) \longrightarrow \mathcal{O}_E(-d) \longrightarrow 0.$$

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(-dE) \longrightarrow \mathcal{O}_{\tilde{X}}((1-d)E) \longrightarrow \mathcal{E}xt^1(\mathcal{O}_E(-d), \mathcal{O}_{\tilde{X}}) \longrightarrow 0.$$

This implies the claim.

3. The set $\mathrm{EP}\,^{a,b,c}_{e,d}$.

In this section we calculate Euler characteristic of the set $\mathrm{EP}\,^{a,b,c}_{e,d}$ of all epimorphisms

$$\beta : \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c) \longrightarrow \mathcal{O}(e) \oplus \mathcal{O}(d) \longrightarrow 0$$

up to the group $\mathrm{Aut}\,(\mathcal{O}(e) \oplus \mathcal{O}(d))$ action. Here we deal with bundles over projective line and we will denote by $\mathcal{O}$ the structure sheaf of $\mathbb{P}^1$.

For two sheaves A and B on $\mathbb{P}^1$ we denote by $U(A, B)$ the subset in $\mathrm{Hom}\,(A, B)$ consisting of all epimorphisms.

3.1. Claim. *Let a, b, c, d and e be integers such that the set*

$$U^{a,b,c}_{d,e} = U(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c), \mathcal{O}(d) \oplus \mathcal{O}(e))$$

is nonempty. Then the group $G_{d,e} = \mathrm{Aut}\,(\mathcal{O}(d) \oplus \mathcal{O}(e))$ natural action[2] *on the set $U^{a,b,c}_{d,e}$ is free, i.e., there exists a geometrical quotient*

$$\pi : U^{a,b,c}_{d,e} \longrightarrow \mathrm{EP}\,^{a,b,c}_{d,e} = U^{a,b,c}_{d,e}/G_{d,e}$$

with $\pi^{-1}([\beta]) \cong G_{d,e}$.

Moreover, there is an open (in Zariski sense) covering $\bigcup V_i = \mathrm{EP}\,^{a,b,c}_{d,e}$ such that for each V_i there exists a section $s_i : V_i \longrightarrow U^{a,b,c}_{d,e}$ with $\pi{\circ}s_i = \mathrm{id}_{V_i}$.

Proof. Each element

$$\beta \in \mathrm{Hom}\,(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c), \mathcal{O}(d) \oplus \mathcal{O}(e))$$

can be wrote as a matrix

$$B = \begin{pmatrix} p_{d-a} & p_{d-b} & p_{d-c} \\ p_{e-a} & p_{e-b} & p_{e-c} \end{pmatrix},$$

where p_k is a homogeneous polynomial of two variables $(x : y)$ and degree k. By definition put,

$$m_{12}(x:y) = \det B_{12}(x:y) = \begin{vmatrix} p_{d-a} & p_{d-b} \\ p_{e-a} & p_{e-b} \end{vmatrix}, \quad m_{13}(x:y) = \det B_{13}(x:y) = \begin{vmatrix} p_{d-a} & p_{d-c} \\ p_{e-a} & p_{e-c} \end{vmatrix},$$

$$m_{23}(x:y) = \det B_{23}(x:y) = \begin{vmatrix} p_{d-b} & p_{d-c} \\ p_{e-b} & p_{e-c} \end{vmatrix}.$$

We see, that β with the matrix B is epimorphism iff for each point $(x : y) \in \mathbb{P}^1$ the vector

$$(m_{12}(x:y), m_{13}(x:y), m_{23}(x:y))$$

is nonzero.

For $(ij) \in \{(12), (13), (23)\}$ denote by V_{ij} the open subset in $U^{a,b,c}_{d,e}$ consisting of all epimorphisms β such that m_{ij} is a nonzero polynomial.

Suppose that there is an epimorphism $\beta \in U^{a,b,c}_{d,e}$ such that $\varphi{\circ}\beta = \beta$ for all $\varphi \in H \subset G_{d,e}$, where H is some subgroup of $G_{d,e}$. Without loss of generality we can assume that $\beta \in V_{11}$, i.e., $m_{11} \not\equiv 0$. Therefore the matrix $B_{11}(x_0 : y_0)$ belongs to $\mathrm{GL}(2,\mathbb{C})$ for some point $(x_0 : y_0) \in \mathbb{P}^1$. Since $\varphi{\circ}\beta = \beta$, we see that $\Phi(x_0 : y_0) \cdot B_{11}(x_0 : y_0) = B_{11}(x_0 : y_0)$, where $\Phi(x_0 : y_0)$ is a matrix of $\varphi(x_0 : y_0)$. On the other hand, $\Phi(x_0 : y_0) \in \mathrm{GL}(2,\mathbb{C})$. Hence, $\Phi(x_0 : y_0) = E$ is identity

[2] $\varphi : \beta \mapsto \varphi{\circ}\beta$

matrix. It is sufficient to prove that $G_{d,e}$ acts freely in the case $d = e$. If $d < e$, we note that there exists an open subset $U(\beta) \subset \mathbb{P}^1$ such that $B_{11}(x:y) \in \mathrm{GL}(2,\mathbb{C})$ for each $(x:y) \in U(\beta)$.

To prove existence of local sections $s_i : V_i \longrightarrow U^{a,b,c}_{d,e}$ for open sets $V_i \subset \mathbb{EP}^{a,b,c}_{d,e}$ it is sufficient to show that there exists closed subsets $Z_i \subset \pi^{-1}(V_i) \subset U^{a,b,c}_{d,e}$ such that intersection of each $G_{d,e}$-orbit in $U^{a,b,c}_{d,e}$ with Z_i equals a point.

Note that each automorphism $\varphi(x:y) \in G_{d,e}$ is determined by its values at $e-d+1$ different points. Let d_{ij} be degree of the polynomial $m_{ij}(x:y)$ and $d'_{ij} = \max(d_{ij}+2, e-d+1)$. For $s = 1, 2, ..., d'_{ij}$ we fix different points $(x_s : y_s) \in \mathbb{P}^1$ and define an open set $W^s_{ij} \subset V_{ij}$ as the set of matrices $B \in V_{ij}$ such that $m_{ij}(x_s : y_s) \neq 0$. By definition put, $V^s_{ij} = \pi(W^s_{ij})$. We see that

$$\bigcup_{s,(ij)} V^s_{ij} = \mathbb{EP}^{a,b,c}_{d,e}.$$

The description of closed subsets $Z^s_{ij} \subset W^s_{ij}$ is easy to understanding and difficult to explanation. For example, we consider two cases: $d = e$ and $d = e-1$.

In the first case Z^s_{ij} is defined as the subset of matrices B's in W^s_{ij} such that $B_{ij}(x_s : y_s) = E$.

In the second one we introduce some more open subsets in W^s_{ij}: $W^{sk}_{ij} = W^s_{ij} \cap W^k_{ij}$ and if $B(x_s : y_s) = \begin{pmatrix} a^s_{11} & a^s_{12} \\ a^s_{21} & a^s_{22} \end{pmatrix}$, then

$$W^{sk}_{ij}(p,q) = \left\{ B \in W^{sk}_{ij} \;:\; a^s_{1p} \cdot a^k_{1q} \neq 0 \right\}.$$

Now we can define closed subsets $Z^{sk}_{ij}(p,q)$ as

$$Z^{sk}_{ij}(p,q) = \left\{ B \in W^{sk}_{ij}(p,q) : \; a^s_{2p} = 0,\; a^s_{1p} = a^s_{2(p\pm1)} = 1,\; a^k_{2q} = 0 \right\}.$$

These examples conclude the proof.

3.2. Claim. *Let X, Y be topological spaces and a group G acts on X and Y such that the action on X is free and there exists a geometrical quotient $\pi : X \longrightarrow (X/G)$ with a section $s : (X/G) \longrightarrow X$ ($\pi \circ s = id$). Assume that G acts on the direct product $X \times Y$ by the rule $g(x,y) \mapsto (gx, gy)$. Then there exists geometrical quotient $(X \times Y)/G$ and $(X \times Y)/G \cong (X/G) \times Y$.*

Proof. We prove that $(X \times Y)/G \cong (X/G) \times Y$, i.e., there are projections

$$(X/G) \xleftarrow{p_X} (X \times Y)/G \xrightarrow{p_Y} Y$$

such that for any topological space Z with maps $q_X : Z \longrightarrow X/G$, $q_Y : Z \longrightarrow Y$ there exists a map $q : Z \longrightarrow (X \times Y)/G$ with $q \circ p_X = q_X$, $q \circ p_Y = q_Y$.

The projection p_X takes each orbit $G(x,y)$ to orbit Gx.

Since the action on X is free, for each $x \in X$ we can determine an element $g_x \in G$ such that $g_x x = s(Gx)$, where Gx is the corresponding element of (X/G).

By definition put, $p_y : G(x,y) \longrightarrow g_x y$. Since $g_{g_0 x} = g_x g_0^{-1}$, we see that p_y is well defined.

Now, determining the map $q : Z \longrightarrow (X \times Y)/G$ by the rule $q(z) = G(s(q_X(z)), q_Y(z))$, we prove the claim.

To calculate Euler characteristic of the set $\mathbb{EP}^{a,b,c}_{e,d}$ we need to know Euler characteristic of the set $U(m_1, m_2, m_3)$, which is defined in the following way.

For several nonnegative integers $m_1,\ m_2,\ \ldots,\ m_s$, let $U(m_1, m_2, \ldots, m_s)$ be the subset of $\mathbb{P}\Big(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(m_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(m_s))\Big)$ parameterizing all collections $(f_1 : f_2 : \ldots : f_s)$ of coprime homogeneous polynomials with $\deg f_i = m_i,\ i = 1,\ 2,\ ...,\ s$.

It is clear, that $U(m_1, m_2, \dots, m_s)$ parameterizes all surjective maps

$$\bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^1}(-m_i) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0.$$

In the case $s = 2$ Euler characteristic was calculated by Li and Qin in [3]:

3.3. Lemma. *(Li and Qin) For nonnegative integers $m_1 \le m_2$*

$$e(U(m_1, m_2)) = \delta^0_{m_1} + \delta^0_{m_2}.$$

Here we prove the following

3.4. Lemma. *Let $m_1,\ m_2,\ m_3$ be integers such that $0 \le m_1 \le m_2 \le m_3$, then*

$$e(U(m_1, m_2, m_3)) = \delta^0_{m_1} + \delta^0_{m_2} + \delta^0_{m_3}.$$

Proof. Let us start with the case: $\underline{\ 0 = m_1 = m_2 = m_3\ }$, i.e., $U(0,0,0)$ is the set of all nonzero maps $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}$ up to a scalar multiplier. We see, that $U(0,0,0) = \mathbb{P}^2$ and the equality follows from the formulae

$$e(\mathbb{P}^n) = n + 1.$$

In the case of $\underline{\ 0 = m_1 = m_2 < m_3\ }$ we have maps

$$u : \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m_3) \longrightarrow \mathcal{O}_{\mathbb{P}^1}.$$

The morphism u is not surjective iff $u|_{\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}} = 0$. Hence,

$$U(0,0,m_3) = \mathbb{P}\Big(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m_3))\Big) \setminus \mathbb{P}\Big(\{0\} \cup \{0\} \cup \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(m_3))\Big) = \mathbb{P}^{m_3+2} \setminus \mathbb{P}^{m_3}$$

and $e(U(0,0,m_3)) = 2$ whenever $m_3 > 0$.

The set $U(0, m_2, m_3)$, when $\underline{\ 0 < m_2 \le m_3\ }$, can be produced as

$$U(0, m_2, m_3) = U_0(0, m_2, m_3) \coprod U_1(0, m_2, m_3),$$

where $U_1(0, m_2, m_3) = \{(\lambda, f_2, f_3) \text{ with } \lambda \ne 0\}$, and $U_0(0, m_2, m_3) = \{(0, f_2, f_3)\}$ with coprime $f_2,\ f_3$. It is clear, that

$$U_1(0, m_2, m_3) = \mathbb{P}\Big(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m_2) \oplus \mathcal{O}_{\mathbb{P}^1}(m_3))\Big) \setminus \mathbb{P}\Big(\{0\} \cup \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(m_2) \oplus \mathcal{O}_{\mathbb{P}^1}(m_3))\Big) =$$

$$= \mathbb{P}^{m_2+m_3+2} \setminus \mathbb{P}^{m_2+m_3+1},$$

and $U_0(0, m_2, m_3) = U(m_2, m_3)$ — the set of all coprime homogeneous polynomials of degree m_2 and m_3 respectively. By lemma 3.3, $e(U(m_2, m_3)) = 0$ whenever $0 < m_2 \le m_3$.

We have,

$$e(U(0, m_2, m_3)) = e(U_1(0, m_2, m_3)) = m_2 + m_3 + 3 - (m_2 + m_3 + 2) = 1.$$

Now, let $\underline{\ 0 < m_1 \le m_2 \le m_3\ }$. Let us represent $U(m_1, m_2, m_3)$ in the form

$$U(m_1, m_2, m_3) = U_0(0, m_2, m_3) \coprod U_1(m_1, m_2, m_3),$$

where $U_1(m_1, m_2, m_3)$ is the all triples $(f_1, f_2, f_3) \in U(m_1, m_2, m_3)$ with $f_1 \not\equiv 0$ and the set $U_0(0, m_2, m_3)$ as before.

Since $U_0(0, m_2, m_3) = U(m_2, m_3)$, we get

$$e(U(m_1, m_2, m_3)) = e(U_1(m_1, m_2, m_3)).$$

Note that

$$U_1(m_1, m_2, m_3) \subset \mathbb{P}\Big(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(m_2) \oplus \mathcal{O}_{\mathbb{P}^1}(m_3))\Big) \setminus \mathbb{P}\Big(\{0\} \cup \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(m_2) \oplus \mathcal{O}_{\mathbb{P}^1}(m_3))\Big).$$

The last set we denote by $W = \mathbb{P}^{m_1+m_2+m_3+2} \setminus \mathbb{P}^{m_2+m_3+1}$. On the other hand, any triples of polynomials (f_1, f_2, f_3) we can rewrite in the form (hg_1, hg_2, hg_3) such that $g_1,\ g_2,\ g_3$ are coprime and h is a polynomial of some degree d. Let us denote by Y_d the set of all such triples. We see that $U_1(m_1, m_2, m_3) = Y_0$.

We'll show that $e(U_1(m_1, m_2, m_3)) = 0$ for $m_1 > 0$ by induction on m_1. The set Y_d is a product $\mathbb{P}^d$ and $U_1(m_1 - d, m_2 - d, m_3 - d)$ whenever $1 \le d \le m_1$, i.e., $e(Y_d) = e(\mathbb{P}^d) \cdot e(U_1(m_1 - d, m_2 - d, m_3 - d)) = 0$ for $1 \le d < m_1$ (by inductive hypothesis).

If $d = m_1$, then

$$U_1(0, m_2 - m_1, m_3 - m_1) =$$

$$= \mathbb{P}\Big(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m_2 - m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(m_3 - m_1))\Big) \setminus \mathbb{P}\Big(\{0\} \cup \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(m_2 - m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(m_3 - m_1))\Big),$$

therefore $e(U_1(0, m_2 - m_1, m_3 - m_1)) = (m_2 - m_1 + m_3 - m_1 + 3) - (m_2 - m_1 + m_3 - m_1 + 2) = 1$, and $e(Y_{m_1}) = m_1 + 1$.

Finally, we obtain

$$e(W) = m_1 + 1 = \sum_{d=0}^{m_1} e(Y_d) = e(Y_0) + e(Y_{m_1}) = e(Y_0) + m_1 + 1.$$

Hence,

$$e(Y_0) = e(U_1(m_1, m_2, m_3)) = e(U(m_1, m_2, m_3)) = 0.$$

This complates the proof.

Since $\mathrm{EP}\,^{a,b,c}_{e,f} \cong \mathrm{EP}\,^{a-1,b-1,c-1}_{e-1,f-1}$, it can be assumed that $a = 0$ and $e = k,\ f = d - k$, i.e., we deal with $\mathrm{EP}\,^{0,b,c}_{k,d-k}$. By definition, put

$$e^{0,b,c}_{k,d-k} = e(\mathrm{EP}\,^{0,b,c}_{k,d-k}).$$

Let us determine possible values of the integers $b,\ c,\ k$ and d in the case $U^{0,b,c}_{k,d-k} \neq \emptyset$. Without loss of generality we may assume that

$$b \ge c \ge 0, \qquad d - k \ge k.$$

On the other hand, there exists an epimorphism

$$\beta : \mathcal{O} \oplus \mathcal{O}(c) \oplus \mathcal{O}(b) \longrightarrow \mathcal{O}(k) \oplus \mathcal{O}(d-k) \longrightarrow 0.$$

This yields that either $k = 0$, or $k \ge c$.

In the case $k = 0$, we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \mathcal{O} & \xrightarrow{id} & \mathcal{O} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}(c+b-d) & \xrightarrow{\alpha^*} & \mathcal{O} \oplus \mathcal{O}(c) \oplus \mathcal{O}(b) & \xrightarrow{\beta} & \mathcal{O} \oplus \mathcal{O}(d) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}(c+b-d) & \longrightarrow & \mathcal{O}(c) \oplus \mathcal{O}(b) & \xrightarrow{\beta'} & \mathcal{O}(d) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}.$$

Therefore, we obtain: either $d = c$ or $d \geq b$.

Suppose that $d - k < b$, then the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \mathcal{O} \oplus \mathcal{O}(c) & \longrightarrow & \mathcal{O}(k) \oplus \mathcal{O}(d-k) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}(c+b-d) & \longrightarrow & \mathcal{O} \oplus \mathcal{O}(c) \oplus \mathcal{O}(b) & \longrightarrow & \mathcal{O}(k) \oplus \mathcal{O}(d-k) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}(b) & \longrightarrow & \mathcal{O}(b) & \longrightarrow & 0 & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

implies, $k = 0,\ d = c$. Thus we obtain. If $b \geq c \geq 0$ and $d - k \geq k \geq 0$, then

$$U^{0,b,c}_{k,d-k} \neq \emptyset \iff \begin{cases} \text{either } d/2 \geq k \geq c,\ d \geq b + k, \\ \text{or } k = 0, \text{ and } d = c, \text{ or } d \geq b. \end{cases} \tag{3.1}$$

Suppose that $d < c + b$ (in particular, d can be equaled to c if $b > 0$), then each epimorphism $\alpha : \mathcal{O}(-b) \oplus \mathcal{O}(-c) \oplus \mathcal{O} \longrightarrow \mathcal{O}(d-b-c) \longrightarrow 0$ has the kernel $\ker \alpha = \mathcal{O} \oplus \mathcal{O}(-d)$. Therefore we have bijective correspondence between $\mathbb{EP}^{\,0,b,c}_{\,0,d}$ and the set of all epimorphisms $\alpha : \mathcal{O}(-b) \oplus \mathcal{O}(-c) \oplus \mathcal{O} \longrightarrow \mathcal{O}(d-b-c) \longrightarrow 0$ up to a scalar multiplier. The last set either is isomorphic to $U(d-b, d-c)$ whenever $d \geq b$ or consists of one point only (when $d - b) < 0$.

Thus we get the following statement

3.5. Statement. *For integers $b \geq c \geq 0$ and $0 \leq d < b + c$ the following equalities*

$$e\Big(\mathbb{EP}^{\,0,b,c}_{\,k,d-k}\Big) = e^{0,b,c}_{k,d-k} = \begin{cases} 0, & \textit{when } k > 0, \\ \delta^d_b + \delta^d_c, & \textit{when } d \geq b,\ k = 0 \\ 1, & \textit{when } b > d = c,\ k = 0. \end{cases}$$

hold.

Suppose that $d = c = b = 0$, then we see that $\mathbb{EP}^{\,0,0,0}_{\,0,0} \cong U(0,0,0)$ and

$$e^{0,0,0}_{0,0} = 3. \tag{3.2}$$

Now we assume that $d \geq b + c > c$. Hence $d \geq b + k$.

Denote by $U^k(d-b-c, d-b, d-c)$ the subset of $U(d-b-c, d-b, d-c)$, consisting of all epimorphism classes

$$\alpha : \mathcal{O}(-b) \oplus \mathcal{O}(-c) \oplus \mathcal{O} \longrightarrow \mathcal{O}(d-c-b) \longrightarrow 0$$

such that $\ker \alpha = \mathcal{O}(-k) \oplus \mathcal{O}(k-d)$.

3.6. Statement. *For integers $b,\ c,\ k$ and d, satisfying conditions (3.1) the following relations*

1. $U^k(d-b-c, d-b, d-c) \cong \mathbb{EP}^{\,0,b,c}_{\,k,d-k}$;
2. $e^{0,b,c}_{0,d} + \sum\limits_{\substack{c \leq k \leq d/2 \\ 0 < k \leq d-b}} e^{0,b,c}_{k,d-k} = e(U(d-b-c, d-b, d-c))$

hold whenever $d \geq b + c$.

Proof. Each class $[\beta] \in \mathbb{EP}^{0,b,c}_{k,d-k}$ unequally determines an epimorphism

$$\alpha : \mathcal{O}(-b) \oplus \mathcal{O}(-c) \oplus \mathcal{O} \longrightarrow \mathcal{O}(d-c-b) \longrightarrow 0$$

up to a scalar multiplier. On the other hand, $\ker \alpha$ is a rank 2 bundle of degree d, i.e., $\ker \alpha = \mathcal{O}(-k) \oplus \mathcal{O}(k-d)$ with $0 \leq k \leq d/2$. This proves the first relations.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{O}(-r) & \longrightarrow & \mathcal{O}(-b) & \longrightarrow & \mathcal{O}_\xi & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{O}(-k) \oplus \mathcal{O}(k-d) & \xrightarrow{\beta^*} & \mathcal{O}(-b) \oplus \mathcal{O}(-c) \oplus \mathcal{O} & \xrightarrow{\alpha} & \mathcal{O}(d-b-c) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{O}(r-d) & \longrightarrow & \mathcal{O}(-c) \oplus \mathcal{O} & \longrightarrow & \mathcal{O}(d-c-r) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array},$$

where $\mathcal{O}(-r)$ is the image of the map from $\mathcal{O}(-k) \oplus \mathcal{O}(k-d)$ to $\mathcal{O}(-b)$. We see that $-r \leq -b$ and (since $-k \geq k-d$) either $-k \leq -r$, or $-r = k-d$. If $-k \leq -r$, we get, that $k \geq r \geq b \geq c$, i.e., $k \geq c$. If $-r = k-d$, then the last row of the diagram has the form

$$0 \longrightarrow \mathcal{O}(-k) \longrightarrow \mathcal{O}(-c) \oplus \mathcal{O} \longrightarrow \mathcal{O}(k-c) \longrightarrow 0.$$

Therefore, either $k = 0$, or $k \geq c$. This concludes the proof.

3.7. Lemma. *Suppose that integers b, c, d and k satisfy the conditions: $d \geq b+c$, $b \geq c \geq 0$ and either $c \leq k \leq \min(d/2, d-b)$, $k < d-k$, or $k = 0$, $c > 0$. Then*

$$e^{0,b,c}_{k,d-k} = \begin{cases} \sum\limits_{\substack{m=-b \\ -c \leq m \leq (k-b-c)/2 \\ m \leq k-b-c}} e(U(d-b-c-m, d-k+m)) \cdot e^{-b,-c,0}_{m,k-b-c-m}, \text{for } k \geq c \geq 0, \\ e(U(d-c, d-b)) \cdot e^{0,b-c,b}_{0,b-c} \quad \textit{for } k = 0,\ c > 0. \end{cases}$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}(k-d) & \longrightarrow & F & \xrightarrow{\eta} & \mathcal{O}(d-b-c) & \longrightarrow & 0 \\
 & & \uparrow & & \varepsilon\uparrow & & id\uparrow & & \\
0 & \longrightarrow & \mathcal{O}(-k) \oplus \mathcal{O}(k-d) & \xrightarrow{\beta^*} & \mathcal{O}(-b) \oplus \mathcal{O}(-c) \oplus \mathcal{O} & \xrightarrow{\alpha} & \mathcal{O}(d-b-c) & \longrightarrow & 0 \\
 & & \gamma\uparrow & & \delta\uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}(-k) & \xrightarrow{id} & \mathcal{O}(-k) & \longrightarrow & 0 & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.3}$$

Recall that by statement 3.6 a class $[\beta] = \{\varphi \circ \beta\}_{\varphi \in G_k} \in \mathbb{EP}^{0,b,c}_{k,d-k}$ unequally determines a class $[\alpha] = \{\lambda \cdot \alpha\}_{\lambda \in \mathbb{C}^*} \in U^k(d-b-c, d-c, d-b)$, where $G_k = \operatorname{Aut}(\mathcal{O}(-k) \oplus \mathcal{O}(k-d))$.

If we hold fixed a nonzero map

$$\gamma \in \operatorname{Hom}(\mathcal{O}(-k), \mathcal{O}(-k) \oplus \mathcal{O}(k-d)) = \operatorname{Hom}(\mathcal{O}(-k), \mathcal{O}(-k)),$$

we have a subset

$$[\delta] = \{\beta^*\circ\varphi\circ\gamma\}_{\varphi\in G_k} \subset \mathrm{Hom}\,(\mathcal{O}(-k), \mathcal{O}(-b)\oplus\mathcal{O}(-c)\oplus\mathcal{O}).$$

Since the group G_k acts transitive on the set $\mathrm{Hom}\,(\mathcal{O}(-k), \mathcal{O}(-k)) \setminus \{0\}$, the subset $[\delta]$ does not depend on a choice of γ. Therefore we can assume that γ has the matrix $\Gamma = (1, 0)^t$. On the other hand, the map β^* can be wrote as a matrix

$$B = \begin{pmatrix} p_{k-b} & p_{d-b-k} \\ p_{k-c} & p_{d-c-k} \\ p_k & p_{d-k} \end{pmatrix},$$

where p_i is a homogeneous polynomial of two variables and degree i. An automorphism φ has a matrix

$$\Phi = \begin{pmatrix} \lambda & p_{d-2k} \\ 0 & \mu \end{pmatrix}$$

with $\lambda, \mu \in \mathbb{C}^*$. Therefore a map $\delta \in [\delta]$ has a matrix

$$D = B\cdot\Phi\cdot\Gamma = (\lambda\cdot p_{k-b}, \lambda\cdot p_{k-c}, \lambda\cdot p_k)^t,$$

i.e., each class $[\beta]$ unequally determines a map δ up to a scalar multiplier. Hence having a class $[\beta]$ via $[\delta]$ we unequally construct a bundle $F = \mathrm{coker}\,\delta$ and a class of epimorphisms $[\varepsilon]$

$$\varepsilon \in \mathrm{Hom}\,(\mathcal{O}(-b)\oplus\mathcal{O}(-c)\oplus\mathcal{O}, F)$$

up to the group $\mathrm{Aut}\,F$ action: $[\varepsilon] = \{\psi\circ\varepsilon\}_{\psi\in\mathrm{Aut}\,F}$.

Furthermore, for fixed epimorphisms α and ε there exists a unique epimorphism $\eta \in \mathrm{Hom}\,(F, \mathcal{O}(d-b-c))$ such that $\alpha = \eta\circ\varepsilon$. Moreover, if we are twisting the map ε by an automorphism $\psi \in \mathrm{Aut}\,F$, the map η is changing to $\eta\circ\psi^{-1}$. Besides, α is determined up to a scalar multiplier, consequently by means of class $[\beta] \in \mathbb{EP}^{\,0,b,c}_{k,d-k}$ we unequally construct a set

$$[\varepsilon, \eta] = \{\psi\circ\varepsilon, \lambda\cdot\eta\circ\psi^{-1}\}_{\lambda\in\mathbb{C}^*,\ \psi\in\mathrm{Aut}\,F}.$$

We see that the pair $[\varepsilon, \eta]$ is an element of quotient

$$\Big(U(\mathcal{O}(-b)\oplus\mathcal{O}(-c)\oplus\mathcal{O}, F)\times U(F, \mathcal{O}(d-b-c))\Big)/(\mathrm{Aut}\,F\times\mathbb{C}^*),$$

where the group $\mathrm{Aut}\,F\times\mathbb{C}^*$ acts by the rule

$$(\psi, \lambda) : (\varepsilon, \eta) \mapsto (\psi\circ\varepsilon, \lambda\cdot\eta\circ\psi^{-1}),$$

and $U(A, B)$ is the set of epimorphisms from A to B. Let us denote this quotient by $\mathbb{CO}(F)$. The following lemma is consequence of claims 3.1 and 3.2.

3.8. Lemma. *There exists a locally trivial bundle*

$$\mathbb{CO}(F) \longrightarrow U(\mathcal{O}(-b)\oplus\mathcal{O}(-c)\oplus\mathcal{O}, F)/\mathrm{Aut}\,F$$

with fiber $U(F, \mathcal{O}(d-b-c))/\mathbb{C}^*$.

Moreover, it is easy to see that the class $[\beta]$ is unequally reconstructed by means of a pair $[\varepsilon, \eta] \in \mathbb{CO}(F)$. Thus there exists one-to-one correspondence between $\mathbb{EP}^{\,0,b,c}_{k,d-k}$ and $\coprod_F \mathbb{CO}(F)$.

Let us determine possible F's. F is a rank 2 bundle with degree $(k-b-c)$. Therefore it has the form $F = \mathcal{O}(m)\oplus\mathcal{O}(k-b-c-m)$, where $\dfrac{k-b-c}{2} \geq m$. The bundle F is included in commutative diagram (3.3). In particular, there exists an epimorphism

$$\varepsilon : \mathcal{O}(-b)\oplus\mathcal{O}(-c)\oplus\mathcal{O} \longrightarrow \mathcal{O}(m)\oplus\mathcal{O}(k-b-c-m) \longrightarrow 0.$$

This yields that the following possibilities take place:

$$k-b-c-m \geq 0,\ m=-b,$$

$$k-b-c-m \geq 0,\ m \geq -c \geq -b,$$

$$k-b-c-m < 0 \implies k=0,\ m=-b.$$

Besides, F is included in the exact sequence

$$0 \longrightarrow \mathcal{O}(k-d) \longrightarrow \mathcal{O}(m) \oplus \mathcal{O}(k-b-c-m) \longrightarrow \mathcal{O}(d-c-b) \longrightarrow 0,$$

whence,

$$k-d \leq m \leq k-b-c-m \leq d-b-c \iff m \geq k-d,$$

or

$$m = d-b-c.$$

By our assumption, $-b \geq k-d$. Therefore, inequality $m \geq k-d$ is true.
If $k-b-c \geq m$ and $k < d-k$, the equality $m = d-b-c$ is impossible.
If $k=0$ and $c>0$, then we see that there exists only one possibilities for F
$F = \mathcal{O}(-b) \oplus \mathcal{O}(-c)$. This concludes the proof.

3.9. Corollary. *Under the conditions of lemma 3.7 we have*

$$e^{0,b,c}_{k,d-k} = \begin{cases} e^{0,b,c}_{d-b,b} = e^{0,b,b-c}_{0,d-c}, & \text{when } k=d-b,\ d<2b, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By lemma 3.7,

$$e^{0,b,c}_{k,d-k} = \sum_{\substack{m=-b \\ -c \leq m \leq (k-b-c)/2 \\ m \leq k-b-c}} e(U(d-b-c-m, d-k+m)) \cdot e^{-b,-c,0}_{m,k-b-c-m}, \quad \text{for } k \geq c \geq 0,$$

$$\text{or} \quad e(U(d-c,d-b)) \cdot e^{0,b-c,b}_{0,b-c} \ \text{ for } k=0,\ c>0.$$

In the last case $e(U(d-c,d-b))=0$, since $d \geq b+c > b \geq c$.

It follows from the assumptions that

$$2m \leq k-b-c \iff d-k+m \leq d-b-c-m.$$

Besides, $d-k+m \geq 0$ and $e(U(p,q)) = \delta^0_p + \delta^0_q$. Therefore, $e(U(d-b-c-m, d-k+m)) = 0$ iff $m = k-d$.

Suppose that $-b = m = k-d$, i.e., $k = d-b$. The inequality $k < d-k$ (by assumptions) implies that $d < 2b$. In this case $e(U(d-b-c-m, d-k+m)) = 1 + \delta^0_{d-c}$.

The equality $d=c$ contradicts the assumptions $d \geq b+c$, $b \geq c \geq 0$ of the lemma with $d<2b$. Hence $e(U(d-b-c-m, d-k+m)) = 1$.

Consider the situation $-c \leq m = k-d \leq \dfrac{k-b-c}{2}$. The left side of this inequalities with the assumption of the lemma $k \leq d-b,\ b \geq c, d \geq b+c$ give:

$$d-k=b=c, \qquad k=d-b, \qquad d \geq b+c = 2b.$$

On the other hand, by assumptions of the lemma 3.7 $k<d-k$. This with $k=d-b$ gives: $d<2b$. This contradiction proves that $e^{0,b,b}_{k,d-k}=0$ whenever $0 \leq b \leq k \leq d-b,\ k<d-k$ and $d \geq 2b$.

The above reasons yield that the sum of lemma 3.7 has one nonzero summand only: $e^{-b,-c,0}_{m,k-b-c-m}$ with $m = k - d = -b$ in the case $d < 2b$.

We see that
$$e^{-b,-c,0}_{m,k-b-c-m} = e^{-b,-c,0}_{-b,d-c-b} = e^{0,b,b-c}_{0,d-c}.$$

This completes the proof.

3.10. Proposition. *For integers $b \geq c \geq 0$, $d/2 \geq k \geq 0$ the numbers $e^{0,b,c}_{k,d-k}$ are nonzero iff either $k = 0$ or $k = c > 0$. Moreover,*
$$e^{0,b,c}_{0,d} = \begin{cases} 3, & \textit{when} \quad d = b = c = 0; \\ 2, & \textit{when} \quad d = b = c > 0, \ \textit{or } d = b > 0, \ c = 0; \\ 1, & \textit{when} \quad d = b > c > 0, \ \textit{or } b > d = c; \\ 0, & \textit{otherwise.} \end{cases}$$
$$e^{0,b,c}_{c,d-c} = \begin{cases} 1, & \textit{when} \quad d - c = b, \ c > 0; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We assume that $e^{0,b,c}_{k,d-k} = 0$ whenever $U^{0,b,c}_{k,d-k} = \emptyset$.

Suppose that $0 \leq d < b + c$. Using statement 3.5 and equality (3.2), we see that $e^{0,b,c}_{k,d-k} = 0$ if $k > 0$ and
$$e^{0,b,c}_{0,d} = \begin{cases} 3, & \text{when} \quad d = b = c = 0; \\ 2, & \text{when} \quad d = b = c > 0; \\ 1, & \text{when} \quad d = b > c > 0, \ \text{or } b > d = c; \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Suppose that $d \geq b + c$ and $k < d - k$, then by corollary 3.9, $e^{0,b,c}_{k,d-k} = 0$ unless $k = d - b$. Besides, $e^{0,b,c}_{d-b,b} = e^{0,b,b-c}_{0,d-c}$ and $d < 2b$, i.e., $d - c < b + (b - c)$. Now we use equality (3.4) and obtain that
$$e^{0,b,b-c}_{0,d-c} = \begin{cases} 3, & \text{when} \quad d - c = b = b - c = 0; \\ 2, & \text{when} \quad d - c = b = b - c > 0; \\ 1, & \text{when} \quad d - c = b > b - c > 0, \ \text{or } b > d - c = b - c; \\ 0, & \text{otherwise.} \end{cases}$$

The equalities $d - c = b = b - c = 0$ are equivalent to $d = b = c = 0$, what contradicts the inequality $d < 2b$.

The equalities $d - c = b = b - c > 0$ imply $d = b > 0, \ c = 0$.

The relations $d - c = b > b - c > 0$ yield that $d = b + c, \ c > 0$. In this situation the assumption $d < 2b$ gives $b > c$.

The relations $b > d - c = b - c$ imply $d = b, \ c > 0$, what contradicts the assumption $d \geq b + c$.

Therefore,
$$e^{0,b,c}_{d-b,d} = \begin{cases} e^{0,b,0}_{0,b} = 2, & \text{when} \quad d = b > 0, \ c = 0; \\ e^{0,b,c}_{c,b} = 1, & \text{when} \quad d = b + c, \ b > c > 0; \\ 0, & \quad\ \ \text{otherwise.} \end{cases}$$

It is remains to consider the case $k = d - k$ and $d \geq b + c$. Let us denote $d = 2n$ and calculate the numbers $e^{0,b,c}_{n,n}$.

Taking into account statement 3.6, we get
$$e^{0,b,c}_{0,2n} + \sum_{\substack{c \leq k \leq n \\ 0 < k \leq 2n-b}} e^{0,b,c}_{k,2n-k} = \delta^{2n}_{b+c} + \delta^{2n}_{b} + \delta^{2n}_{c}. \tag{3.5}$$

Since

$$e^{0,b,c}_{0,2n} = \begin{cases} 2, & \text{when} \quad b = 2n,\ c = 0; \\ 0, & \text{otherwise.} \end{cases},$$

and for integer k such that $c \le k \le 2n - b$, $0 < k < n$ the following relations hold

$$e^{0,b,c}_{k,2n-k} = \begin{cases} 1, & \text{when} \quad k = c,\ b + c = 2n,\ b > c > 0; \\ 0, & \text{otherwise.} \end{cases}$$

we see that the left side of equality (3.5) has the form

$$e^{0,b,c}_{n,n} + \begin{cases} 2, & \text{when} \quad b = 2n,\ c = 0; \\ 1, & \text{when} \quad b + c = 2n,\ b > c > 0; \\ 0, & \text{otherwise.} \end{cases}.$$

On the other hand, since $2n \ge b + c$, the right side of equality (3.5) is equal to

$$\begin{cases} 2, & \text{when} \quad b = 2n,\ c = 0; \\ 1, & \text{when} \quad b + c = 2n,\ c > 0; \\ 0, & \text{otherwise.} \end{cases}.$$

Therefore,

$$e^{0,b,c}_{n,n} = \begin{cases} 1, & \text{when} \quad b = c = n > 0; \\ 0, & \text{otherwise.} \end{cases}.$$

4. Lemma about stability.

Recall that $\sigma : \tilde{X} \longrightarrow X$ is a blow up of a smooth projective surface X at a point $x_0 \in X$ and $E = \sigma^{-1}(x_0)$ is an exceptional curve on $\tilde{X}$. We denote by $\tilde{H}$ the divisor $\sigma^*(H)$ and $H_n = n\tilde{H} - E$, where H is an ample divisor on X.

Besides, we recall the definition of stability in Mumford–Takemoto sense and some properties of stable sheaves.

A torsion free sheaf F on a surface X is called μ_H-(semi)stable w.r.t. an ample (or numerically effective) divisor H, if for any its subsheaf G with $\operatorname{rk} G < \operatorname{rk} F$ the following takes place

$$\mu_H(G) = \frac{c_1(G) \cdot H}{\operatorname{rk} G} < \frac{c_1(F) \cdot H}{\operatorname{rk} F} = \mu_H(F) \qquad (\le \text{ for semistable } F).$$

It is known (see, for example [6], [2]), that μ_H-stable sheaf F is simple, i.e., $\operatorname{End}(F) = \mathbb{C}$, and if F is locally free then the dual sheaf F^* is μ_H-stable too.

4.1. Lemma. *Assume, V is a bundle over $\tilde{X}$ such that $c_1(V) \cdot \tilde{H}$ and $\operatorname{rk}(V) = r$ are coprime. Denote by V' the kernel of an epimorphism $V \longrightarrow \mathcal{O}_E(d) \longrightarrow 0$:*

$$0 \longrightarrow V' \longrightarrow V \longrightarrow \mathcal{O}_E(d) \longrightarrow 0. \tag{4.1}$$

Then V' is a locally free sheaf and it is μ_{H_n}-stable for $n \gg 0$ iff V is μ_{H_n}-stable for $n \gg 0$ too.

Proof. At first we note that V' has no torsion, because of it is a subsheaf of torsion free sheaf V. If V' is not locally free, we have a commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & Q & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & \mathcal{O}_E(d) & \longrightarrow & 0 \\
 & & \downarrow & & || & & \downarrow & & \\
0 & \longrightarrow & (V')^{**} & \longrightarrow & (V)^{**} & \longrightarrow & Q' & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & Q & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array},$$

where $\dim \operatorname{Supp} Q = 0$. But Q is a subsheaf of a bundle over the curve, i.e., $Q = 0$.

The second statement of the lemma we prove in several steps.

STEP 1. *If a sheaf F is μ_{H_n}-stable for $n \gg 0$, then it is μ_H-semistable.* For any subsheaf G of F we have

$$\mu_{H_n}(G) < \mu_{H_n}(F) \iff \frac{nc_1(G)\cdot\tilde{H} - c_1(G)\cdot E}{r(G)} < \frac{nc_1(F)\cdot\tilde{H} - c_1(F)\cdot E}{r(F)},$$

i.e.,

$$n\left(\frac{c_1(G)\cdot\tilde{H}}{r(G)} - \frac{c_1(F)\cdot\tilde{H}}{r(F)}\right) < \frac{c_1(G)\cdot E}{r(G)} - \frac{c_1(F)\cdot E}{r(F)}$$

for $n \gg 0$. Therefore,

$$\frac{c_1(G)\cdot\tilde{H}}{r(G)} \le \frac{c_1(F)\cdot\tilde{H}}{r(F)}.$$

STEP 2. *Assume that a sheaf F is μ_{H_n}-stable for some n. Then it is μ_{H_k}-stable for all $k > n$.* The proof follows from the previous step.

STEP 3. *If V is μ_{H_n}-stable for $n \gg 0$, Then V' is μ_{H_n}-stable for $n \gg 0$.* Suppose, that V' is not μ_{H_n}-stable, then there exists a subsheaf $G \subset V'$ of rank $r(G) < r$ such that

$$\mu_{H_n}(V') \le \mu_{H_n}(G). \tag{4.2}$$

But G is a subsheaf of V too. Hence,

$$\mu_{H_n}(G) < \mu_{H_n}(V). \tag{4.3}$$

Since $c_1(V') = c_1(V) - E$ and rank of V' equals r, we have,

$$\mu_{H_n}(V') = \frac{1}{r}\Big(c_1(V')(n\tilde{H} - E)\Big) = \frac{1}{r}\Big(c_1(V)(n\tilde{H} - E) - 1\Big),$$

$$\mu_{H_n}(V) = \frac{1}{r}\Big(c_1(V)(n\tilde{H} - E)\Big), \qquad \mu_{H_n}(G) = \frac{1}{r(G)}\Big(c_1(G)(n\tilde{H} - E)\Big),$$

Combining (4.2) and (4.3), we have

$$\frac{r(G)}{r}\Big(c_1(V)(n\tilde{H} - E)\Big) - \frac{r(G)}{r} \le c_1(G)(n\tilde{H} - E) < \frac{r(G)}{r}\Big(nc_1(V)\cdot\tilde{H} - c_1(V)\cdot E\Big) \tag{4.4}$$

By assumption of the lemma $c_1(V)\cdot\tilde{H}$ and r are coprime, therefore the number

$$\frac{r(G)}{r}\Big(nc_1(V)\cdot\tilde{H} - c_1(V)\cdot E\Big)$$

is integer for $n \equiv \Big(c_1(V)\cdot E\Big)\Big(c_1(V)\cdot \tilde{H}\Big)^{-1} \pmod r$. The number $c_1(G)\Big(n\tilde{H}-E\Big)$ is integer too, and $0 < \dfrac{r(G)}{r} < 1$. Therefore, inequality (4.4) is not possible. Now the step 3 follows from the second one.

STEP 4. *If V' is μ_{H_n}-stable for $n \gg 0$, Then V is μ_{H_n}-stable for $n \gg 0$.* Let us apply the functor $\mathrm{Hom}\,(\,\cdot\,,\mathcal{O}_{\tilde{X}})$ to the sequence (4.1):

$$0 \longrightarrow V^* \longrightarrow (V')^* \longrightarrow \mathcal{E}xt^{\,1}(\mathcal{O}_E(d),\mathcal{O}_{\tilde{X}}) \longrightarrow 0.$$

It follows from claim 2.4 that $\mathcal{E}xt^{\,1}(\mathcal{O}_E(d),\mathcal{O}_{\tilde{X}}) \cong \mathcal{O}_E(1-d)$. Since the μ_{H_n}-stability of V and V^* is equivalent, our lemma follows from the step 3.

4.2. Lemma. *For a μ_{H_n}-stable bundle V on $\tilde{X}$ the space* $\mathrm{Hom}\,(V(-E),V)$ *has dimension 1.*

Proof. Suppose, that the restriction of V to E has the form $V|_E = \bigoplus\limits_{i=1}^{r}\mathcal{O}_E(d_i)$ such that $d_1 \le d_2 \le \ldots \le d_r$. We prove the lemma by induction on $\Delta(V) = d_r - d_1$.

In the case of $\Delta(V)=0$ we can assume, that $d_1 = d_2 = \ldots = d_r = 0$, since

$$\mathrm{Hom}\,(V(-E+dE),V(dE)) \cong \mathrm{Hom}\,(V(-E),V).$$

Applying the functor $\mathrm{Hom}\,(\,\cdot\,,V)$ to the restriction sequence

$$0 \longrightarrow V(-E) \longrightarrow V \longrightarrow V|_E \longrightarrow 0,$$

we obtain

$$0 \longrightarrow \mathrm{Hom}\,(V,V) \longrightarrow \mathrm{Hom}\,(V(-E),V) \longrightarrow \mathrm{Ext}^{\,1}(r\mathcal{O}_E,V),$$

where $r\mathcal{O}_E = \mathcal{O}_E\oplus\ldots\oplus\mathcal{O}_E$. Since V is μ_{H_n}-stable, it is simple, i.e., $\mathrm{Hom}\,(V,V)=\mathbb{C}$. On the other hand, by claim 2.3 it follows, that $\mathrm{Ext}^{\,1}(r\mathcal{O}_E,V) \cong \mathrm{Ext}^{\,1}_E(V|_E, r\mathcal{O}_E(-1))^* = 0$. Therefore, the base of induction is proved.

Suppose, $\Delta(V)>0$ and $d_1=\ldots=d_s<d_{s+1}$. Consider an exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow \bigoplus_{i=1}^{s}\mathcal{O}_E(d_i) \longrightarrow 0 \tag{4.5}$$

and restrict it to the curve E:

$$0 \longrightarrow \mathrm{Tor}_{\,1}(\bigoplus_{i=1}^{s}\mathcal{O}_E(d_i),\mathcal{O}_E) \longrightarrow V'|_E \longrightarrow V|_E \longrightarrow \bigoplus_{i=1}^{s}\mathcal{O}_E(d_i) \longrightarrow 0.$$

Since $\mathrm{Tor}_{\,1}(\mathcal{O}_E(d),\mathcal{O}_E) = \mathcal{O}_E(d+1)$ (see claim 2.2), we get

$$0 \longrightarrow \bigoplus_{i=1}^{s}\mathcal{O}_E(d_i+1) \longrightarrow V'|_E \longrightarrow \bigoplus_{i=s+1}^{r}\mathcal{O}_E(d_i) \longrightarrow 0.$$

Whence, $V'|_E = \bigoplus\limits_{i=1}^{r}\mathcal{O}_E(d_i')$ with $d_1 < d_1' \le d_2' \le \ldots \le d_r' \le d_r$, i.e., $\Delta(V') < \Delta(V)$. Besides, V' is μ_{H_n}-stable (see lemma 4.1). By the inductive hypothesis, $\mathrm{Hom}\,(V'(-E),V') = \mathbb{C}$.

Let us apply the functor $\mathrm{Hom}\,(V(-E),\,\cdot\,)$ to the sequence (4.5):

$$0 \longrightarrow \mathrm{Hom}\,(V(-E),V') \longrightarrow \mathrm{Hom}\,(V(-E),V) \longrightarrow \mathrm{Hom}\,(V(-E),\bigoplus_{i=1}^{s}\mathcal{O}_E(d_i)).$$

Since $\mathrm{Hom}\,(V(-E),\bigoplus\limits_{i=1}^{s}\mathcal{O}_E(d_i)) = \mathrm{Hom}_{\,E}(\bigoplus\limits_{i=1}^{r}\mathcal{O}_E(d_i+1),\bigoplus\limits_{i=1}^{s}\mathcal{O}_E(d_i)) = 0$,

$$\mathrm{Hom}\,(V(-E),V') \cong \mathrm{Hom}\,(V(-E),V).$$

Further, tensoring the sequence (4.5) by $\mathcal{O}(-E)$ and applying the functor $\mathrm{Hom}\,(\,\cdot\,, V')$ to it, we get:

$$0 \longrightarrow \mathrm{Hom}\,(V(-E), V') \longrightarrow \mathrm{Hom}\,(V'(-E), V').$$

Because of the equality $\mathrm{Hom}\,(V'(-E), V') = \mathbb{C}$, dimension of the space $\mathrm{Hom}\,(V(-E), V')$ is less or equal to 1. Finally,

$$1 \leq \dim \mathrm{Hom}\,(V(-E), V) = \dim \mathrm{Hom}\,(V(-E), V') \leq 1.$$

This concludes the proof.

5. Mutations.

Let $\mathfrak{M}(i,k)$ be a moduli space of μ_{H_n}-stable (for $n \gg 0$) rank 3 bundles V on $\tilde{X}$ with $c_1(V) = \sigma^*(C) - iE,\ c_2(V) = k$ (see Introduction). Then the restriction of a bundle $V \in \mathfrak{M}(i,k)$ to the rational curve E has the form

$$V|_E = \mathcal{O}_E(-d_1) \oplus \mathcal{O}_E(-d_2) \oplus \mathcal{O}_E(i + d_1 + d_2) \tag{5.1}$$

We may assume that $i = 0,\ 1,\ 2$ or 3 and $-\frac{i}{3} \leq d_1,\ -d_1 \leq -d_2 \leq i + d_1 + d_2$, i.e., $-\dfrac{d_1 + i}{2} \leq d_2 \leq d_1$.

Denote by $\mathfrak{M}^{(d_1,d_2)}(i,k)$ the subset of $\mathfrak{M}(i,k)$ consisting of bundles V with restriction to E in the form (5.1). Then

$$\mathfrak{M}(i,k) = \coprod_{\substack{-\frac{i}{3} \leq d_1 \\ -\frac{d_1+i}{2} \leq d_2 \leq d_1}} \mathfrak{M}^{(d_1,d_2)}(i,k).$$

5.1. Lemma. *For integers $d \geq 0$, and $i = 0,1,2$ the following formula takes place*

$$2 \cdot e\Big(\mathfrak{M}^{(d,d)}(i,k)\Big) = \sum_{-\frac{d+i+1}{2} \leq l \leq d-1} e^{0,2d+l+i+1,d-l}_{0,3d+i} \cdot e\Big(\mathfrak{M}^{(d,l)}(i+1, k-i-d)\Big)$$

unless $i = d = 0$, where $e(Y)$ is Euler characteristic of Y_{red} and

$$e^{0,2d+l+i+1,d-l}_{0,3d+i} = e(\mathbb{EP}\,^{0,2d+l+i+1,d-l}_{0,3d+i})$$

is Euler characteristic of the quotient of all epimorphism set

$$U(\mathcal{O}_E \oplus \mathcal{O}_E(2d + l + i + 1) \oplus \mathcal{O}_E(d - l), \mathcal{O}_E \oplus \mathcal{O}_E(3d + i))$$

by the group $\mathrm{Aut}\,(\mathcal{O}_E \oplus \mathcal{O}_E(3d + i))$ *action.*

Proof. Let V be a bundle from $\mathfrak{M}^{(d,d)}(i,k)$, i.e.,

$$V|_E = \mathcal{O}_E(-d) \oplus \mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d + i).$$

Consider an arbitrary epimorphism $\alpha : V \longrightarrow \mathcal{O}_E(-d)$. Note that

$$\alpha \in \mathrm{Hom}\,(V, \mathcal{O}_E(-d)) \cong \mathrm{Hom}\,_E(\mathcal{O}_E(-d) \oplus \mathcal{O}_E(-d), \mathcal{O}_E(-d)) \cong \mathbb{C}^2.$$

Therefore, α up to a scalar multiplier is element of $\mathbb{P}^1$. Denote by V'_α the kernel of α:

$$0 \longrightarrow V'_\alpha \longrightarrow V \xrightarrow{\alpha} \mathcal{O}_E(-d) \longrightarrow 0. \tag{5.2}$$

It follows from lemma 4.1, that V'_α is a μ_{H_n}-stable bundle. We see that

$$\mathrm{rk}\,(V'_\alpha) = 3, \qquad c_1(V'_\alpha) = c_1(V) - E = \sigma^*(C) - (i+1)E$$

and $c_2(V'_\alpha)$ can be calculated by the standard way, using multiplicativity of the full Chern character and exact triple (5.2):

$$\Big(1 + c_1(V'_\alpha)t + c_2(V'_\alpha)t^2\Big)\Big(1 + Et + c_2(\mathcal{O}_E(-d))t^2\Big) = 1 + c_1(V)t + k\,t^2.$$

Since $c_2(\mathcal{O}_E(-d)) = d - 1$ (see claim 2.1),

$$c_2(V'_\alpha) = k - d + 1 - c_1(V)\cdot E + E^2 = k - d - i.$$

Whence, $V'_\alpha \in \mathfrak{M}(i+1, k-d-i)$.

Exact sequence (5.2) induces the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i) & \longrightarrow & V|_E & \longrightarrow & \mathcal{O}_E(-d) & \longrightarrow & 0 \\
 & & {\scriptstyle\beta}\uparrow & & {\scriptstyle res}\uparrow & & {\scriptstyle id}\uparrow & & \\
0 & \longrightarrow & V'_\alpha & \longrightarrow & V & \xrightarrow{\alpha} & \mathcal{O}_E(-d) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & V(-E) & = & V(-E) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{5.3}$$

Thus we constructed a bundle $V'_\alpha \in \mathfrak{M}(i+1, k-d-i)$ and an epimorphism $\beta : V'_\alpha \longrightarrow \mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i) \longrightarrow 0$ by means of the bundle V and the epimorphism $\alpha : V \longrightarrow \mathcal{O}_E(-d)$. Let us show, that different α's give various bundles V'_α.

If we apply the functor $\mathrm{Hom}\,(\,\cdot\,, V)$ to the left column of the diagram, we get

$$0 \longrightarrow \mathrm{Hom}\,(V'_\alpha, V) \longrightarrow \mathrm{Hom}\,(V(-E), V).$$

Taking into account lemma 4.2, we obtain $\mathrm{Hom}\,(V'_\alpha, V) \cong \mathbb{C}$. Moreover, this equality holds for the kernel of any epimorphism $\alpha \in \mathrm{Hom}\,(V, \mathcal{O}_E(-d))$.

Assume, that $V'_{\bar\alpha}$ is the kernel of another epimorphism $\bar\alpha \in \mathrm{Hom}\,(V, \mathcal{O}_E(-d))$, and there exists an isomorphism $\varphi : V'_\alpha \longrightarrow V'_{\bar\alpha}$. Because of the bundle V is simple, the map of composition $0 \longrightarrow \mathrm{Hom}\,(V, V) \longrightarrow \mathrm{Hom}\,(V'_\alpha, V)$ is an isomorphism, and we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V'_\alpha & \longrightarrow & V & \xrightarrow{\alpha} & \mathcal{O}_E(-d) & \longrightarrow & 0 \\
 & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle\varphi_V}\downarrow & & {\scriptstyle\varphi_{\mathcal{O}}}\downarrow & & \\
0 & \longrightarrow & V'_{\bar\alpha} & \longrightarrow & V & \xrightarrow{\bar\alpha} & \mathcal{O}_E(-d) & \longrightarrow & 0
\end{array}$$

with nonzero φ_V and $\varphi_{\mathcal{O}}$. The sheaves V and $\mathcal{O}_E(-d)$ are simple, therefore, $\varphi_V = \lambda\cdot\mathrm{id}_V$ and $\varphi_{\mathcal{O}} = \mu\cdot\mathrm{id}_{\mathcal{O}_E(-d)}$. Thus bundles V'_α and $V'_{\bar\alpha}$ are isomorphic iff $\alpha = \nu\bar\alpha$ for some $\nu \in \mathbb{C}^*$.

5.2. Lemma. *Let* β *and* β' *be epimorphisms from* $\mathrm{Hom}\,(V'_\alpha, \mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i))$, *then* $\ker\beta \cong \ker\beta'$ *iff there exists an element* $\varphi \in \mathrm{Aut}(\mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i))$ *such that* $\beta' = \varphi\circ\beta$.

Proof. It is clear, that if $\beta' = \varphi\circ\beta$, then $\ker\beta \cong \ker\beta'$. Now suppose that $V(-E) = \ker\beta \cong \ker\beta' = V'(-E)$ and $\psi : V(-E) \longrightarrow V'(-E)$ is isomorphism:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V(-E) & \xrightarrow{\gamma} & V'_\alpha & \xrightarrow{\beta} & \mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i) & \longrightarrow & 0 \\
 & & {\scriptstyle\psi}\downarrow & & & & & & \\
0 & \longrightarrow & V'(-E) & \xrightarrow{\gamma'} & V'_\alpha & \xrightarrow{\beta'} & \mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i) & \longrightarrow & 0
\end{array}.$$

Therefore, there exists a map $\gamma' \circ \psi : V(-E) \longrightarrow V'_\alpha$. Let us apply the functor $\mathrm{Hom}\,(\,\cdot\,, V'_\alpha)$ to the first sequence of the diagram

$$0 \longrightarrow \mathrm{Hom}\,(V'_\alpha, V'_\alpha) \xrightarrow{p} \mathrm{Hom}\,(V(-E), V'_\alpha).$$

On the other hand, we know, that

$$0 \longrightarrow V'_\alpha \longrightarrow V \xrightarrow{\alpha} \mathcal{O}_E(-d) \longrightarrow 0,$$

and $\mathrm{Hom}\,(V(-E), V) \cong \mathbb{C}$. Hence, $\mathrm{Hom}\,(V(-E), V'_\alpha) \cong \mathbb{C}$, i.e., the map p is isomorphism. This implies that there exists the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V(-E) & \xrightarrow{\gamma} & V'_\alpha & \xrightarrow{\beta} & \mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i) & \longrightarrow & 0 \\
 & & \psi \downarrow & & \psi' \downarrow & & \varphi \downarrow & & \\
0 & \longrightarrow & V'(-E) & \xrightarrow{\gamma'} & V'_\alpha & \xrightarrow{\beta'} & \mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i) & \longrightarrow & 0
\end{array},$$

where ψ' and φ are isomorphisms. This concludes the proof.

Prove that there exists one-to-one correspondence between

$$\big\{(V, \alpha)\big\} \quad \text{and} \quad \big\{(V'_\alpha, \beta/\mathrm{mod}\,\mathrm{Aut}(\mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i))\big\}.$$

At first, the bundle V is uniquely determined by (V'_α, β) as $(\ker \beta) \otimes \mathcal{O}(E)$. Let us reconstruct commutative diagram (5.3).

Applying the functor $\mathrm{Hom}\,(\,\cdot\,, V(-E))$ to the first row of the diagram, we obtain

$$0 \longrightarrow \mathrm{Ext}^1(\mathcal{O}_E(-d), V(-E)) \xrightarrow{p} \mathrm{Ext}^1(V|_E, V(-E)) \xrightarrow{q} \mathrm{Ext}^1(\mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i), V(-E)) \tag{5.4}$$

By claim 2.3, we get

$$\mathrm{Ext}^1(\mathcal{O}_E(-d), V(-E)) \cong \mathrm{Hom}_E(\mathcal{O}_E(-d), V|_E), \qquad \mathrm{Ext}^1(V|_E, V(-E)) \cong \mathrm{End}_E(V|_E),$$

$$\mathrm{Ext}^1(\mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i), V(-E)) \cong \mathrm{Hom}_E(\mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i), V|_E).$$

Consequently triple (5.4) has the form

$$0 \longrightarrow \mathrm{Hom}_E(\mathcal{O}_E(-d), V|_E) \xrightarrow{p} \mathrm{End}_E(V|_E) \xrightarrow{q} \mathrm{Hom}_E(\mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i), V|_E).$$

By dimension consideration we get, that q is epimorphism. If the left column of the diagram is determined by an element $e \in \mathrm{Ext}^1(\mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i), V(-E))$, then taking an exact sequence corresponding to $q^{-1}(e) \in \mathrm{Ext}^1(V|_E, V(-E))$, we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i) & \longrightarrow & V|_E & \longrightarrow & \mathcal{O}_E(-d) & \longrightarrow & 0 \\
 & & \beta \uparrow & & res \uparrow & & id \uparrow & & \\
0 & \longrightarrow & V'_\alpha & \longrightarrow & \bar{V} & \xrightarrow{\bar{\alpha}} & \mathcal{O}_E(-d) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & V(-E) & = & V(-E) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Now we need to check, that $\bar{V} \cong V$ and $\bar{\alpha} = \nu\alpha$. It is sufficient to prove the first equality since the second one follows from the first equality and the fact that was proved above.

First we note that the group $\mathrm{Aut}(V|_E)$ acts on the space $\mathrm{Ext}^1(V|_E, V(-E)) \cong \mathrm{End}_E(V|_E)$ and extensions determined by elements from one orbit are isomorphic.

Let us show that elements belonging to the open orbit determines locally free extensions. Apply the functor $\mathcal{H}om\,(\,\cdot\,, \mathcal{O}_{\tilde{X}})$ to the middle column of the last diagram

$$0 \longrightarrow \bar{V}^* \longrightarrow V^*(E) \xrightarrow{f} \mathcal{E}xt^1(V|_E, \mathcal{O}_{\tilde{X}}) \longrightarrow \mathcal{E}xt^1(\bar{V}, \mathcal{O}_{\tilde{X}}).$$

The sheaf $\bar{V}$ is locally free iff $\mathcal{E}xt^1(\bar{V}, \mathcal{O}_{\tilde{X}}) = 0$, i.e., the map f is an epimorphism. Taking into account assumptions and claim 2.4, we get

$$\mathcal{E}xt^1(V|_E, \mathcal{O}_{\tilde{X}}) \cong \mathcal{O}_E(d-1) \oplus \mathcal{O}_E(d-1) \oplus \mathcal{O}_E(-2d-1).$$

On the other hand, the map f is an epimorphism iff its restriction $f|_E$ to the curve E is one. In other words, $f|_E$ is automorphism of $V^*(E)|_E$. Therefore elements of $\mathrm{Ext}^1(V|_E, V(-E)) \cong \mathrm{End}_E(V|_E)$, determining locally free extensions are automorphisms of $V|_E$.

For the same reason, the element $e \in \mathrm{Ext}^1(\mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i), V(-E))$, determining the left column of the diagram is a monomorphism $e : \mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i) \longrightarrow V|_E$, since V'_α is locally free. Hence, to reconstruct the diagram we can choose $q^{-1}(e)$ in the such way that it be an automorphism of $V|_E$. Therefore, $\bar{V}$ is locally free and is isomorphic to V.

Let us calculate the restriction of V'_α to the curve E. The left column of diagram (5.3) tensoring by $\mathcal{O}_E$ has the form:

$$0 \longrightarrow \mathrm{Tor}_1(\mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i), \mathcal{O}_E) \longrightarrow V(-E)|_E \longrightarrow V'_\alpha|_E \xrightarrow{\beta|_E} \mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i) \longrightarrow 0.$$

It follows from claim 2.2, that

$$\mathrm{Tor}_1(\mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i), \mathcal{O}_E) \cong \mathcal{O}_E(1-d) \oplus \mathcal{O}_E(2d+i+1).$$

Hence $V'_\alpha|_E$ is included in the following exact triple:

$$0 \longrightarrow \mathcal{O}_E(1-d) \xrightarrow{\gamma} V'_\alpha|_E \xrightarrow{\beta|_E} \mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i) \longrightarrow 0, \tag{5.5}$$

and

$$V'_\alpha|_E = \mathcal{O}_E(-d) \oplus \mathcal{O}_E(-l) \oplus \mathcal{O}_E(d+l+i+1),$$

with $1-d \le -l \le d+l+i+1 \le 2d+i \iff -\dfrac{d+1+i}{2} \le l \le d-1.$

5.3. Remark. *The set $\tilde{\mathfrak{M}}^{(d,d)}(i,k)$ of all pairs (V,α), where $V \in \mathfrak{M}^{(d,d)}(i,k)$ and $\alpha \in \mathbb{P}\mathrm{Hom}\,(V, \mathcal{O}_E(-d))$ is a locally trivial bundle over $\mathfrak{M}^{(d,d)}(i,k)$ with fiber $\mathbb{P}^1$, i.e.,*

$$e(\tilde{\mathfrak{M}}^{(d,d)}(i,k)) = 2e(\mathfrak{M}^{(d,d)}(i,k)).$$

Proof. Since $n \gg 0$ and $C \cdot H \not\equiv 0 \pmod 3$, there are no μ_{H_n}-semistable bundles V on $\tilde{X}$ with $c_1(V) = C - iE$, $\mathrm{rk}\,(V) = 3$ and $c_2(V) = k$. Therefore, by remark A.7 in [4] there exists a universal bundle $\mathcal{V}$ over $\tilde{X} \times \mathfrak{M}^{(d,d)}(i,k)$. Denote by p_i the projection of the direct product to the i-th factor. Consider the following sheaf on $\mathfrak{M}^{(d,d)}(i,k)$

$$\mathcal{F} = R^0_{p_{2*}}\left(\mathcal{V}^* \otimes p_1^*\mathcal{O}_E(-d)\right).$$

We see that for each bundle $V \in \mathfrak{M}^{(d,d)}(i,k)$ the fiber of $\mathcal{F}$ over V is isomorphic to

$$\mathrm{Hom}_{\tilde{X}}(V, \mathcal{O}_E(-d)) \cong \mathrm{Hom}_E(\mathcal{O}_E(-d) \oplus \mathcal{O}_E(-d) \oplus \mathcal{O}_E(2d+i), \mathcal{O}_E(-d)) \cong \mathbb{C}^2.$$

Thus, $\mathcal{F}$ is the locally free sheaf (see [5], corollary 2 on p. 50), and $\mathbb{P}(\mathcal{F})$ is the set of the pairs (V,α). This concludes the proof of the remark.

Let $W_l^{(d,d)}$ be the subset of $\tilde{\mathfrak{M}}^{(d,d)}(i,k)$ consisting of all pairs (V,α) such that the above defined V'_α belongs to $\mathfrak{M}^{d,l}(i+1,k-i-d)$. Then

$$\tilde{\mathfrak{M}}^{(d,d)}(i,k) = \coprod_{-\frac{d+i+1}{2}\le l\le d-1} W_l^{(d,d)}$$

and the following statement takes place.

5.4. Remark. *The set $W_l^{(d,d)}$ is a locally trivial bundle over $\mathfrak{M}^{(d,l)}(i+1,k-i-d)$ with the fibers isomorphic to $\mathbb{EP}^{\,-d,d+l+i+1,-l}_{\,-d,2d+i} \cong \mathbb{EP}^{\,0,2d+l+i+1,d-l}_{\,0,3d+i}$, i.e.,*

$$e(W_l^{(d,d)}) = e^{0,2d+l+i+1,d-1}_{0,3d+i}\cdot e(\mathfrak{M}^{(d,l)}(i+1,k-i-d)).$$

Proof. It follows from the previous proof that $\tilde{\mathfrak{M}}^{(d,d)}(i,k)\cong\mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is the vector bundle over $\mathfrak{M}^{(d,d)}(i,k)$ parameterizing morphisms from $V\in\mathfrak{M}^{(d,d)}(i,k)$ to $\mathcal{O}_E(-d)$. Denote by $\pi:\mathbb{P}(\mathcal{F})\longrightarrow\mathfrak{M}^{(d,d)}(i,k)$ the natural projection, by q the projection from the direct product $\tilde{X}\times\mathbb{P}(\mathcal{F})$ to the second factor, and by $\mathcal{O}_{\mathcal{F}}(-1)$ the tautological line bundle over $\mathbb{P}(\mathcal{F})$. Then on the direct product $\tilde{X}\times\mathbb{P}(\mathcal{F})$ there exists an exact sequence of sheaves ([1])

$$0\longrightarrow\mathcal{V}'_d\longrightarrow(id_{\tilde{X}}\times\pi)^*\mathcal{V}\otimes q^*(\mathcal{O}_{\mathcal{F}}(-1))\longrightarrow(id_{\tilde{X}}\times\pi)^*p_1^*\mathcal{O}_E(-d)\longrightarrow 0,$$

which restriction to $\tilde{X}\times(V,\alpha)$ has the form

$$0\longrightarrow V'_\alpha\longrightarrow V\xrightarrow{\alpha}\mathcal{O}_E(-d)\longrightarrow 0.$$

If we restrict the sheaf $\mathcal{V}'_d$ to $\tilde{X}\times W_l^{(d,d)}$, we obtain a family of μ_{H_n}-stable bundles V'_α from $\mathfrak{M}^{(d,l)}(i+1,k-i-d)$. But the moduli space $\mathfrak{M}(i+1,k-i-d)$ is fine, i.e., there exists a universal bundle $\mathcal{V}'$ over $\tilde{X}\times\mathfrak{M}(i+1,k-i-d)$ ([4]). Moreover, there exists a morphism f from the base of the family $\mathcal{V}'_d$ $\quad W_l^{(d,d)}$ to $\mathfrak{M}(i+1,k-i-d)$ such that $\mathcal{V}'_d=(id_{\tilde{X}}\times f)^*\mathcal{V}'$. It follows from the above construction that the image of f is $\mathfrak{M}^{(d,l)}(i+1,k-i-d)$ and its fiber over V'_α has the form

$$f^{-1}(V_\alpha)=U(V'_\alpha,\mathcal{O}_E(-d)\oplus\mathcal{O}_E(2d+i))/\mathrm{Aut}\,(\mathcal{O}_E(-d)\oplus\mathcal{O}_E(2d+i))=\mathbb{EP}^{\,0,2d+l+i+1,d-1}_{\,0,3d+i}.$$

This completes the proof of the remark.

Combining remarks 5.3 and 5.4, we prove lemma 5.1.

5.5. Corollary.

$$e(\mathfrak{M}^{(d,d)}(0,k))=\frac{1}{2}e(\mathfrak{M}^{(d,d-1)}(1,k-d))\text{ for }d\ge 1;$$

$$e(\mathfrak{M}^{(d,d)}(1,k))=\begin{cases} e(\mathfrak{M}^{(0,-1)}(2,k-d-1))\text{ for }d=0,\\ \frac{1}{2}e(\mathfrak{M}^{(d,d-1)}(2,k-d-1))\text{ for }d\ge 1;\end{cases}$$

$$e(\mathfrak{M}^{(d,d)}(2,k))=\frac{1}{2}e(\mathfrak{M}^{(d+1,d)}(0,k-d+1))\text{ for }d\ge 0.$$

Proof. Let us calculate nonzero values of $e^{0,2d+l+i+1,d-l}_{0,3d+i}$ for $i=0,\ 1,\ 2$, and $-\dfrac{d+i+1}{2}\le l\le d-1$.

Since $d-l \le 2d+l+i+1$, taking into account proposition 3.10 we get

$$e_{0,3d+i}^{0,2d+l+i+1,d-l} = \begin{cases} 3, & \text{if} \quad 3d+i = 2d+l+i+1 = d-l = 0; \\ 2, & \text{if} \quad 3d+i = 2d+l+i+1 = d-l > 0; \\ 2, & \text{if} \quad 3d+i = 2d+l+i+1 > 0, \ d-l = 0; \\ 1, & \text{if} \quad 3d+i = 2d+l+i+1 > d-l > 0; \\ 1, & \text{if} \quad 2d+l+i+1 > 3d+i = d-l; \\ 0, & \qquad otherwise. \end{cases}$$

It can be proved by the direct calculation that under our assumption about $l,\ d,\ i$ we have,

$$e_{0,3d+i}^{0,2d+l+i+1,d-l} = \begin{cases} 2, & \text{when} \quad d=0,\ i=1,\ l=-1; \\ 1, & \text{when} \quad d > \frac{1-i}{3},\ l = d-1; \\ 0, & \qquad otherwise. \end{cases}$$

Therefore, the sum of lemma 5.1 has only one nonzero summand. To prove the corollary it remains to note that by the next lemma there exists an isomorphism of moduli spaces

$$\mathfrak{M}^{(d_1,d_2)}(3,k) \cong \mathfrak{M}^{(d_1+1,d_2+1)}(0,k+3).$$

5.6. Lemma.
$$e(\mathfrak{M}^{(d_1,d_2)}(3,k)) = e(\mathfrak{M}^{(d_1+1,d_2+1)}(0,k+3)).$$

Proof. It is clear, that the map $\tau : \mathfrak{M}(3,k) \longrightarrow \mathfrak{M}(0,k')$, taking each bundle V to the tensor product $V(E) = V \otimes \mathcal{O}_{\tilde{X}}(E)$ is isomorphism. Recall, that $c_2(V(E)) = c_2(V) + 3E^2 + 2E\cdot c_1(V) = c_2(V)+3$, i.e. $k' = k+3$. Moreover, if $V \in \mathfrak{M}^{(d_1,d_2)}(3,k)$, then

$$V|_E = \mathcal{O}_E(-d_1) \oplus \mathcal{O}_E(-d_2) \oplus \mathcal{O}_E(3+d_1+d_2).$$

Therefore,
$$V(E)|_E = \mathcal{O}_E(-d_1-1) \oplus \mathcal{O}_E(-d_2-1) \oplus \mathcal{O}_E(2+d_1+d_2)$$

This proves that the restriction of τ to the subset $\mathfrak{M}^{(d_1,d_2)}(3,k)$ is isomorphism with the subset $\mathfrak{M}^{(d_1+1,d_2+1)}(0,k+3)$.

5.7. Lemma. *If integers $d_1,\ d_2$ satisfy the conditions: $d_1 \ge 0,\ -\frac{d_1+i}{2} \le d_2 < d_1$, with $i = 0,1,2$, then*

$$e\Big(\mathfrak{M}^{(d_1,d_2)}(i,k)\Big) = \sum_{(m_1,m_2)\in D_i(d_1,d_2)} e_{m_1-d_2,d_1+d_2+i+1+m_1}^{0,2m_1+m_2+i+1,m_1-m_2} \cdot e\Big(\mathfrak{M}^{(m_1,m_2)}(i+1,k-d_1-i)\Big),$$

where

$$D_i(d_1,d_2) = \left\{ (m_1,m_2)\in\mathbb{Z}^2 \ \middle| \ \begin{matrix} \begin{pmatrix} -\frac{i+1}{3} \le m_1 \le d_1 - 1, \\ \max(d_2, -\frac{m_1+i+1}{2}) \le m_2 \\ m_2 \le \min(m_1, d_1+d_2-m_1-1) \end{pmatrix} & \textit{when } d_2 < -\frac{i+1}{3} \\ \begin{pmatrix} d_2 < m_1 \le d_1 - 1, \\ \max(d_2, -\frac{m_1+i+1}{2}) \le m_2 \\ m_2 \le \min(m_1, d_1+d_2-m_1-1) \end{pmatrix} & \textit{when } d_2 \ge -\frac{i+1}{3} \\ \begin{pmatrix} m_1 = d_2, \\ -\frac{m_1+i+1}{2} \le m_2 \le m_1 \end{pmatrix} & \textit{when } d_2 \ge -\frac{i+1}{3} \end{matrix} \right\}$$

Proof. Let V be a bundle from $\mathfrak{M}^{(d_1,d_2)}(i,k)$ with $d_1 > d_2$, i.e.,

$$V|_E = \mathcal{O}_E(-d_1) \oplus \mathcal{O}_E(-d_2) \oplus \mathcal{O}_E(d_1+d_2+i).$$

Hence there exists a unique up to a scalar multiplier epimorphism $V \longrightarrow \mathcal{O}_E(-d_1) \longrightarrow 0$. Denoting by V' its kernel we can construct the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{O}_E(-d_2) \oplus \mathcal{O}_E(d_1+d_2+i) & \longrightarrow & V|_E & \longrightarrow & \mathcal{O}_E(-d_1) & \longrightarrow & 0 \\
 & & {\scriptstyle\beta}\uparrow & & {\scriptstyle res}\uparrow & & {\scriptstyle id}\uparrow & & \\
0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & \mathcal{O}_E(-d_1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & V(-E) & = & V(-E) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \qquad (5.6)$$

Arguing as above, we see that $V' \in \mathfrak{M}(i+1, k-i-d_1)$. Furthermore, there exists one-to-one correspondence between $\mathfrak{M}^{(d_1,d_2)}(i,k)$ and the set of pairs

$$\Big\{(V', \beta/\mathrm{mod\,Aut}(\mathcal{O}_E(-d_2) \oplus \mathcal{O}_E(d_1+d_2+i)))\Big\}.$$

Let us restrict the left column of diagram (5.6) to the curve E:

$$0 \to \mathrm{Tor}_1(\mathcal{O}_E(-d_2) \oplus \mathcal{O}_E(d_1+d_2+i), \mathcal{O}_E) \to V(-E)|_E \to$$

$$\to V'|_E \xrightarrow{\beta|_E} \mathcal{O}_E(-d_2) \oplus \mathcal{O}_E(d_1+d_2+i) \to 0.$$

Since

$$\mathrm{Tor}_1(\mathcal{O}_E(-d_2) \oplus \mathcal{O}_E(d_1+d_2+i), \mathcal{O}_E) = \mathcal{O}_E(1-d_2) \oplus \mathcal{O}_E(1+d_1+d_2+i),$$

(see claim 2.2), the restriction $V'|_E$ is included in the exact sequence

$$0 \longrightarrow \mathcal{O}_E(1-d_1) \xrightarrow{\gamma} V'|_E \xrightarrow{\beta|_E} \mathcal{O}_E(-d_2) \oplus \mathcal{O}_E(d_1+d_2+i) \longrightarrow 0. \qquad (5.7)$$

Whence,

$$V'|_E = \mathcal{O}_E(-m_1) \oplus \mathcal{O}_E(-m_2) \oplus \mathcal{O}_E(m_1+m_2+i+1),$$

that is V' belongs to $\mathfrak{M}^{(m_1,m_2)}(i+1, k-d_1-i)$. We regulate the direct summand of $V'|_E$ in the following way: $-m_1 \le -m_2 \le m_1+m_2+i+1$. This implies that

$$-\frac{i+1}{3} \le m_1; \qquad -\frac{m_1+i+1}{2} \le m_2 \le m_1.$$

Besides, it follows from exact triple (5.7), that $1-d_1 \le -m_1$, and $\beta|_E$ is an epimorphism in two cases only:

$$\text{either} \qquad -m_1 < -d_2, \qquad -m_2 \le -d_2, \qquad m_1+m_2+i+1 \le d_1+d_2+i,$$

$$\text{or} \qquad -m_1 = -d_2, \qquad -d_2 \le -m_2 \le m_1+m_2+i+1 \le d_1+d_2+i.$$

In other words, for $i = 0,\ 1,\ 2$, $d_1 \ge 0$ and $-\dfrac{d_1+i}{2} \le d_2 < d_1$ a pair (m_1, m_2) can be belonged to the set $D_i(d_1,d_2)$ from the lemma.

If we denote by $W^{(d_1,d_2)}_{(m_1,m_2)}$ the subset of $\mathfrak{M}^{(d_1,d_2)}(i,k)$ consisting of all V with

$$V'|_E = \mathcal{O}_E(-m_1)\oplus\mathcal{O}_E(-m_2)\oplus\mathcal{O}_E(m_1+m_2+i+1),$$

then

$$\mathfrak{M}^{(d_1,d_2)}(i,k) = \coprod_{(m_1,m_2)\in D_i(d_1,d_2)} W^{(d_1,d_2)}_{(m_1,m_2)}.$$

It can be proved by the same way as in remark 5.4 that $W^{(d_1,d_2)}_{(m_1,m_2)}$ is a locally trivial bundle over $\mathfrak{M}^{(m_1,m_2)}(i+1,k-d_1-i)$ with the fibers isomorphic to $\mathbb{EP}^{0,2m_1+m_2+i+1,m_1-m_2}_{m_1-d_2,d_1+d_2+i+m_1}$. This concludes the proof.

5.8. Corollary. *Under the conditions of the previous lemma*

$$e\Big(\mathfrak{M}^{(d_1,d_2)}(0,k)\Big) = \begin{cases} 2e\Big(\mathfrak{M}^{(d_1-1,d_2)}(1,k-d_1)\Big) & \text{for} \quad d_1\ge 1,\ d_2=d_1-1,\\ e\Big(\mathfrak{M}^{(d_1-1,d_2)}(1,k-d_1)\Big) & \text{for} \quad d_1\ge 2,\ d_1-1>d_2\ge -\frac{d_1}{2};\end{cases}$$

$$e\Big(\mathfrak{M}^{(d_1,d_2)}(1,k)\Big) = \begin{cases} 2e\Big(\mathfrak{M}^{(d_1-1,d_2)}(2,k-d_1-1)\Big) & \text{for} \quad d_1\ge 1,\ d_2=d_1-1,\\ e\Big(\mathfrak{M}^{(d_1-1,d_2)}(2,k-d_1-1)\Big) & \text{for} \quad d_1\ge 1,\ d_1-1>d_2\ge -\frac{d_1+1}{2};\end{cases}$$

$$e\Big(\mathfrak{M}^{(d_1,d_2)}(2,k)\Big) = \begin{cases} 3e\Big(\mathfrak{M}^{(0,0)}(0,k+1)\Big) & \text{for} \quad d_1=0,\ d_2=-1,\\ 2e\Big(\mathfrak{M}^{(d_1,d_2+1)}(0,k-d_1+1)\Big) & \text{for} \quad d_1\ge 1,\ d_2=d_1-1,\\ e\Big(\mathfrak{M}^{(d_1,d_2+1)}(0,k-d_1-1)\Big) & \text{for} \quad d_1\ge 1,\ d_1-1>d_2\ge -\frac{d_1+2}{2};\end{cases}$$

Proof. Since a pair $(m_1,m_2)\in D_i(d_1,d_2)$ satisfies the conditions $m_1-m_2\le 2m_1+m_2+i+1$ and $m_1-d_2\le d_1+d_2+i+m_1$, using proposition 3.10, we get $e^{0,2m_1+m_2+i+1,m_1-m_2}_{m_1-d_2,d_1+d_2+i+m_1}$ is nonzero iff either $m_1-d_2=0$ or $m_1-d_2=m_1-m_2>0$.

Suppose that $m_1=d_2$, then

$$e^{0,2m_1+m_2+i+1,m_1-m_2}_{m_1-d_2,d_1+d_2+i+m_1} =$$

$$= e^{0,2d_2+m_2+i+1,d_2-m_2}_{0,d_1+2d_2+i} = \begin{cases} 3, & \text{for} \quad d_1+2d_2+i=2d_2+m_2+i+1=d_2-m_2=0,\\ 2, & \text{for} \quad d_1+2d_2+i=2d_2+m_2+i+1=d_2-m_2>0,\\ 2, & \text{for} \quad d_1+2d_2+i=2d_2+m_2+i+1>0,\ d_2-m_2=0,\\ 1, & \text{for} \quad d_1+2d_2+i=2d_2+m_2+i+1>d_2-m_2>0,\\ 1, & \text{for} \quad 2d_2+m_2+i+1>d_1+2d_2+i=d_2-m_2,\\ 0, & \qquad \text{otherwise.}\end{cases}$$

In the case $m_1=d_2$ integers m_2 and d_2 satisfy the conditions: $d_1-1\ge d_2\ge -\frac{i+1}{3}$, $d_2\ge m_2\ge -\frac{d_2+i+1}{2}$. This gives,

$$e^{0,2d_2+m_2+i+1,d_2-m_2}_{0,d_1+2d_2+i} = \begin{cases} 3, & \text{when} \quad i=2,\ d_1=0,\ m_2=d_2=-1,\\ 2, & \text{when} \quad m_2=d_2=d_1-1,\ d_1>\frac{2-i}{3},\\ 0, & \qquad \text{otherwise.}\end{cases}$$

Suppose that $m_1-d_2=m_1-m_2>0$, i.e., $m_2=d_2,\ m_1>m_2$, then

$$e^{0,2m_1+m_2+i+1,m_1-m_2}_{m_1-d_2,d_1+d_2+i+m_1} =$$

$$= e^{0,2m_1+d_2+i+1,m_1-d_2}_{m_1-d_2,d_1+d_2+i+m_1} = \begin{cases} 1, & \text{for} \quad m_1=d_1-1>m_2=d_2,\\ 0, & \qquad \text{otherwise.}\end{cases}$$

Thus we see that the sum of lemma 5.7 has only one nonzero summand, that is equal to

$$m_i(d_1,d_2)e\Big(\mathfrak{M}^{(d_1-1,d_2)}(i+1,k-d_1-i)\Big),$$

where

$$m_i(d_1,d_2)=\begin{cases}3 & \text{when} \quad i=2,\ d_1=0,\ d_2=-1,\\ 2, & \text{when} \quad d_1\ge 1,\ d_2=d_1-1,\\ 1, & \qquad\qquad \text{otherwise.}\end{cases}$$

This with lemma 5.6 completes the proof.

6. Deduction of blow up formulae.

Combining corollaries 5.5 and 5.8 together we obtain.

F*or* $d_1\ge 1,\ d_1\ge d_2\ge -\dfrac{d_1}{2}$ $\quad e\Big(\mathfrak{M}^{(d_1,d_2)}(0,k)\Big)=m_0(d_1,d_2)e\Big(\mathfrak{M}^{(d_3,d_4)}(1,k-d_1)\Big),$ where

$$(d_3,d_4)=\begin{cases}(d_1-1,d_2) & \text{if} \quad d_2<d_1,\\ (d_1,d_2-1) & \text{if} \quad d_2=d_1;\end{cases}\quad m_0(d_1,d_2)=\begin{cases}2 & \text{if} \quad d_2=d_1-1,\\ 1 & \text{if} \quad d_2<d_1-1\\ \frac12 & \text{if} \quad d_2=d_1.\end{cases}$$

For $d_3\ge 0,\ d_3\ge d_4\ge -\dfrac{d_3+1}{2}$

$$e\Big(\mathfrak{M}^{(d_3,d_4)}(1,k-d_1)\Big)=m_1(d_3,d_4)e\Big(\mathfrak{M}^{(d_5,d_6)}(2,k-d_1-d_3-1)\Big),\quad \text{where}$$

$$(d_5,d_6)=\begin{cases}(d_3-1,d_4) & \text{if} \quad d_4<d_3,\\ (d_3,d_4-1) & \text{if} \quad d_4=d_3;\end{cases}\quad m_1(d_3,d_4)=\begin{cases}2 & \text{if} \quad d_4=d_3-1,\\ 1 & \text{if} \quad d_4<d_3-1,\\ 1 & \text{if} \quad d_4=d_3=0,\\ \frac12 & \text{if} \quad d_4=d_3>0.\end{cases}$$

For $d_5\ge 0,\ d_5\ge d_6\ge -\dfrac{d_5+2}{2}$

$$e\Big(\mathfrak{M}^{(d_5,d_6)}(2,k-d_1-d_3-1)\Big)=m_2(d_5,d_6)e\Big(\mathfrak{M}^{(d_7,d_8)}(0,k-d_1-d_3-d_5)\Big),\quad \text{where}$$

$$(d_7,d_8)=\begin{cases}(d_5,d_6+1) & \text{if} \quad d_6<d_5,\\ (d_5+1,d_6) & \text{if} \quad d_6=d_5;\end{cases}\quad m_2(d_5,d_6)=\begin{cases}3 & \text{if} \quad d_6=d_5-1=-1,\\ 2 & \text{if} \quad d_6=d_5-1\ge 0,\\ 1 & \text{if} \quad d_6<d_5-1,\\ \frac12 & \text{if} \quad d_6=d_5.\end{cases}$$

Therefore,

$$e(\mathfrak{M}(0,k))=\sum_{\substack{d_1\ge 0\\ d_1\ge d_2\ge -\frac{d_1}{2}}} e\Big(\mathfrak{M}^{(d_1,d_2)}(0,k)\Big)=$$

$$=\sum_{\substack{d_1\ge 0\\ d_1\ge d_2\ge -\frac{d_1}{2}}} m_0(d_1,d_2)m_1(d_3,d_4)m_2(d_5,d_6)e\Big(\mathfrak{M}^{(d_7,d_8)}(0,k-d_1-d_3-d_5)\Big)$$

Note that each pair (d_1,d_2) suah that $d_1\ge 0,\ d_1\ge d_2\ge -\frac{d_1}{2}$ determains a finite sequence of integers

$$s(d_1,d_2)=d_1,d_2,d_3,d_4,d_5,d_6,\dots,d_{6k-1},d_{6k},0,0 \tag{6.1}$$

by the following rule: if $m \equiv 0 \pmod 3$, then

$$(d_{2m+1}, d_{2m+2}) = \begin{cases} (d_{2m-1}, d_{2m}+1) & \text{when} \quad d_{2m-1} > d_{2m}, \\ (d_{2m-1}+1, d_{2m}) & \text{when} \quad d_{2m-1} = d_{2m}; \end{cases}$$

if $m \not\equiv 0 \pmod 3$, then

$$(d_{2m+1}, d_{2m+2}) = \begin{cases} (d_{2m-1}-1, d_{2m}) & \text{when} \quad d_{2m-1} > d_{2m}, \\ (d_{2m-1}, d_{2m}-1) & \text{when} \quad d_{2m-1} = d_{2m}. \end{cases}$$

Let us define some functions by means of sequence $s(d_1, d_2)$.

$S_0(d_1, d_2) = d_1 + d_3 + d_5 + \cdots + d_{6k-1}, \qquad P_0(d_1, d_2) = m_0(d_1, d_2)m_1(d_3, d_4)\cdots m_2(d_{6k-1}, d_{6k}),$

$S_1(d_3, d_4) = d_3 + d_5 + d_7 + \cdots + d_{6k-1}, \qquad P_1(d_3, d_4) = m_1(d_3, d_4)m_2(d_5, d_6)\cdots m_2(d_{6k-1}, d_{6k}),$

$S_2(d_5, d_6) = d_5 + d_7 + d_9 + \cdots + d_{6k-1}, \qquad P_2(d_5, d_6) = m_2(d_5, d_6)m_0(d_7, d_8)\cdots m_2(d_{6k-1}, d_{6k}),$

where the number k is determined by formula (6.1). In this notation we have

$$e(\mathfrak{M}(0,k)) = \sum_{\substack{d_1 \geq 0 \\ d_1 \geq d_2 \geq -\frac{d_1}{2}}} P_0(d_1, d_2) e\Big(\mathfrak{M}^{(0,0)}(0, k - S_0(d_1, d_2))\Big).$$

$$e(\mathfrak{M}(1,k)) = \sum_{\substack{d_3 \geq 0 \\ d_3 \geq d_4 \geq -\frac{d_3+1}{2}}} P_1(d_3, d_4) e\Big(\mathfrak{M}^{(0,0)}(0, k - S_1(d_3, d_4))\Big).$$

$$e(\mathfrak{M}(2,k)) = \sum_{\substack{d_5 \geq 0 \\ d_5 \geq d_6 \geq -\frac{d_5+2}{2}}} P_2(d_5, d_6) e\Big(\mathfrak{M}^{(0,0)}(0, k - S_2(d_5, d_6))\Big).$$

It is easy to see that

$$e(\mathfrak{M}(0,k)) = \sum_{n \geq 0} \left(\sum_{\substack{d_1 \geq 0 \\ d_1 \geq d_2 \geq -\frac{d_1}{2} \\ S_0(d_1,d_2)=n}} P_0(d_1, d_2) \right) e\Big(\mathfrak{M}^{(0,0)}(0, k-n)\Big).$$

$$e(\mathfrak{M}(1,k)) = \sum_{n \geq 0} \left(\sum_{\substack{d_3 \geq 0 \\ d_3 \geq d_4 \geq -\frac{d_3+1}{2} \\ S_1(d_3,d_4)=n}} P_1(d_3, d_4) \right) e\Big(\mathfrak{M}^{(0,0)}(0, k-n)\Big).$$

$$e(\mathfrak{M}(2,k)) = \sum_{n \geq 0} \left(\sum_{\substack{d_5 \geq 0 \\ d_5 \geq d_6 \geq -\frac{d_5+2}{2} \\ S_2(d_5,d_6)=n}} P_2(d_5, d_6) \right) e\Big(\mathfrak{M}^{(0,0)}(0, k-n)\Big).$$

Recall that $\mathfrak{M}^{(0,0)}(0,k) \cong \mathfrak{M}(k)$ is the moduli space of rank 3 μ_H-stable vector bundles on the surface X with the first Chern class equals $C \in \mathrm{Pic}(X)$ and the second one is k. If we denote by $Z_{3,i}(q)$ and $Z_3(q)$ the generating functions

$$Z_{3,i}(q) = \sum_{k \in \mathbb{Z}} e(\mathfrak{M}(i,k)) q^k, \qquad Z_3(q) = \sum_{k \in \mathbb{Z}} e(\mathfrak{M}(k)) q^k,$$

assuming that $e(\mathfrak{M}(i,k)) = 0$ whenever $\mathfrak{M}(i,k)$ is empty, we obtain the following theorem.

6.1. Theorem. *There exist universal functions* $u_{3,i}(q) = \sum\limits_{n\geq 0} u_{3,i,n}q^n$ *such that*

$$Z_{3,i}(q) = u_{3,i}(q)\cdot Z_3(q),$$

where $i = 0,1,2$ *and*

$$u_{3,i,n} = \sum_{\substack{d\geq 0\\ \frac{3d+i}{2}\geq k\geq 0\\ S_i(d,d-k)=n}} P_i(d,d-k).$$

Now we calculate the universal functions $u_{3,i}(q)$. At first we shall find values of the functions $P_0(d,d-k)$ and $S_0(d,d-k)$. It is clear, that $S_0(0,0) = 0$ and $P_0(0,0) = 1$.

6.2. Lemma. *For integers* $d \geq 1$, $\frac{3d}{2} \geq k \geq 0$ *the following formulae*

$$S_0(d,d-k) = 3d^2 - 3dk + k^2, \qquad P_0(d,d-k) = \begin{cases} 3, & \text{when } k = 0 \text{ or } k = \frac{3d}{2},\\ 6, & \text{otherwise} \end{cases}$$

hold.

Proof. The proof is by induction on d and k. We start with the case $k = 0$. Using formulae (6.1) and the definition of the functions P_0, S_0, $m_i(d_1,d_2)$, we have

$$s(d,d) = d,d,d,d-1,d-1,d-1,d,d-1,d-1,d-1,d-1,d-2,d-1,d-1,\dots,$$

therefore, $S_0(d,d) = 6d - 3 + S_0(d-1,d-1) = 6d - 3 + 3(d-1)^2 = 3d^2$.

$$P_0(d,d) = \left\{\begin{array}{ll} 3, & \text{if } d = 1,\\ P_0(d-1,d-1), & \text{otherwise} \end{array}\right\} = 3.$$

If $k = 1$, then

$$s(d,d-1) = d,d-1,d-1,d-1,d-1,d-2,d-1,d-1,\dots,$$

hence, $S_0(d,d-1) = 3d - 2 + S(d-1,d-1) = 3d - 2 + 3(d-1)^2 = 3d^2 - 3d + 1$.

$$P_0(d,d-1) = \left\{\begin{array}{ll} 6, & \text{if } d = 1,\\ 2P_0(d-1,d-1), & \text{otherwise} \end{array}\right\} = 6.$$

In the case $k = 2$,

$$s(d,d-2) = d,d-2,d-1,d-2,d-2,d-2,d-1,d-2,\dots,$$

whence, $S_0(d,d-2) = 3d-3+S_0(d-1,d-1-1) = 3d-3+3(d-1)^2-3(d-1)+1 = 3d^2-3\cdot 2d+2^2$. $P_0(d,d-2) = P_0(d-1,d-1-1) = 6$.

Finally suppose, $k \geq 3$, then

$$s(d,d-k) = d,d-k,d-1,d-k,d-2,d-k,d-2,d-k+1,\dots, .$$

It follows that $S_0(d,d-k) = 3d - 3 + S_0(d-2,d-2-(k-3))$. By the induction hypothesis,

$$S_0(d-2,d-2-(k-3)) = 3(d-2)^2 - 3(d-2)(k-3) + (k-3)^2,$$

i.e., $S_0(d,d-k)=3d^2-3kd+k^2$. Furthermore,

$$P_0(d,d-k)=\left\{\begin{array}{ll}3, & \text{if } d=2,\ k=3,\\ 2P_0(d-2,d-2), & \text{if } d>2,\ k=3,\\ P_0(d-2,d-2-(k-3)), & \text{otherwise}\end{array}\right\}=\left\{\begin{array}{ll}3, & \text{if } k=\frac{3d}{2},\\ 6, & \text{otherwise.}\end{array}\right.$$

This completes the proof.

By the similar calculations we get

6.3. Lemma. *For integers $d_3\geq 0,\ d_3\geq d_4\geq -\frac{d_3+1}{2}$ the following formulae take place*

$$S_1(d_3,d_4)=d_3^2+d_4^2+d_3d_4+d_3+d_4,\qquad P_1(d_3,d_4)=\left\{\begin{array}{ll}3 \ \text{when} & d_4=d_3,\ \text{or } d_4=-\frac{d_3+1}{2},\\ 6 & \text{otherwise.}\end{array}\right.$$

If integers $d_5,\ d_6$ satisfy the conditions $d_5\geq 0$, $d_5\geq d_6\geq -\frac{d_5+2}{2}$, then the functions $S_2(d_5,d_6)$ and $P_2(d_5,d_6)$ have the form

$$S_2(d_5,d_6)=d_5^2+d_6^2+d_5d_6+2(d_5+d_6)+1,\ P_2(d_5,d_6)=\left\{\begin{array}{ll}3 \ \text{when} & d_6=d_5,\ \text{or } d_6=-\frac{d_5+2}{2},\\ 6 & \text{otherwise.}\end{array}\right.$$

6.4. Corollary. *Under the assumption of theorem 6.1 we have,*

$$u_{3,0}(q)=\sum_{(x,y)\in\mathbb{Z}^2} q^{x^2+xy+y^2}.$$

$$u_{3,1}(q)=\sum_{(x,y)\in\mathbb{Z}^2} q^{x^2+xy+y^2+x+y}.$$

$$u_{3,2}(q)=\sum_{(x,y)\in\mathbb{Z}^2} q^{x^2+xy+y^2+2(x+y)+1}.$$

Proof. We prove the first equality of the corollary only, because of other are proved in the similar way. Taking into account the theorem and lemma 6.2, we obtain,

$$u_{3,0}(q)=1+\sum_{\substack{d\geq 1\\ \frac{3d}{2}\geq k\geq 0\\ S_0(d,d-k)=n}} P_0(d,d-k)=1+\sum_{d\geq 1}\left(3q^{S_0(d,d)}+3q^{S_0(d,-\frac{d}{2})}+6\sum_{\frac{3d}{2}>k>0} q^{S_0(d,d-k)}\right).$$

Note that,

$$\sum_{d\geq 1} q^{S(d,-\frac{d}{2})}=\sum_{s\geq 1} q^{S_0(2s,2s-3s)}=\sum_{s\geq 1} q^{3s^2}.$$

Therefore,

$$u_{3,0}(q)=1+6\sum_{\substack{d\geq 1\\ \frac{3d}{2}>k\geq 0}} q^{3d^2-3dk+k^2}.$$

If $c=d-k$, then

$$u_0(q)=1+6\left(\sum_{\substack{d\geq 1\\ d\geq c>-\frac{d}{2}}} q^{d^2+dc+c^2}\right).$$

The quadratic form $Q(d,c) = d^2 + dc + c^2$ has the following symmetries

$$t_0(d,c) = (d,c), \qquad t_1(d,c) = (-d,-c), \qquad t_2(d,c) = (c,d),$$

$$t_3(d,c) = (-c,-d), \qquad t_4(d,c) = (-d,d+c), \qquad t_5(d,c) = (d,-d-c).$$

In other words, we have

$$Q(d,c) = Q(c,d) = Q(-c,-d) = Q(-d,d+c) = Q(d,-d-c) = Q(-d,-c).$$

Moreover,

$$\bigcup_{i=0}^{6} t_i(A) = \mathbb{Z}^2 \setminus \mathbb{Z}\cdot(2,-1), \qquad \Big(t_2(A)\coprod t_3(A)\Big) \cap \Big(t_0(A)\coprod t_1(A)\Big) = \mathbb{Z}\cdot(1,1),$$

and other intersections are empty. Thus, for each vector $v_0 = (d,c) \in A$ with $d > c$ there exist 5 different vectors v_i such that $Q(v_0) = Q(v_i)$, $(i = 1,\ldots,6)$. In the case $v_0 = (d,c) \in A$ with $d = c$ with help of the symmetries we obtain only 3 different vectors: $v_1 = (-d,-d)$, $v_2 = (-d,2d)$, $v_3 = (d,-2d)$ such that $Q(v_0) = Q(v_i)$ $(i = 1,2,3)$. On the other hand, $Q(d,d) = Q(-2d,d) = Q(2d,-d)$ and the last two vectors belong to the line $\mathbb{Z}\cdot(2,-1)$. This proves the corollary.

It is known to experts that

$$u_{3,0}(q) = 1 + 6\sum_{n\geq 1}\left(\sum_{d|3}\left(\frac{d}{3}\right)\right)q^n$$

is Eisenstain series, where $\left(\frac{d}{3}\right)$ is Legendre symbol:

$$\left(\frac{d}{3}\right) = \begin{cases} +1 & \text{if} \quad d \equiv +1 \pmod 3 \\ -1 & \text{if} \quad d \equiv -1 \pmod 3 \\ 0 & \text{if} \quad d \equiv 0 \pmod 3 \end{cases}$$

In other words, $u_{3,0}(q)$ is a modular form of weight 1.

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