A unified approach to the four vertex theorems II

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Contents

Introduction. The present paper is a continuation of [U] by the second author. For the sake of simplicity, we will refer to that paper by I. For example, I-Theorem 2.1 means Theorem 2.1 in [U].

Our main goal is to study closed space curves. We will show that the following theorem can be used to improve some known results and reprove others. Our improvements will concern results of A. Kneser [Kn] and Segre [Se] on the osculating planes of spherical curves and Sedykh's four vertex theorem [Sd]. We will also be able to improve Ghys' theorem on extremal points of projective line diffeomorphisms and obtain some new results about them.

Theorem 0.1. *Let $\gamma : S^1 \to \mathbf{R}^3$ be a C^2-regular convex simple closed nonplanar space curve with nonvanishing curvature. Then γ has at least four sign changes of clear vertices, meaning that it has two clear maximal vertices p_1, p_3 and two clear minimal vertices p_2, p_4 such that $p_1 \succ p_2 \succ p_3 \succ p_4$ holds, where $\succ$ is the rotational order of the curve γ.*

A closed space curve is called *convex* if it lies on the boundary of its convex hull. Here a *clear maximal* (resp. *minimal*) *vertex* (cf. I-Definition 4.4) is a point which is an absolute maximum (resp. minimum) of the height function with respect to the binormal vector at that point. Moreover, if the level set of absolute maxima (resp. minima) is connected, it is called a *clean maximal* (resp. *minimal*) vertex.

If the curve γ in Theorem 0.1 satisfies one of some additional conditions, the four points can be chosen to be clean vertices so that the osculating planes of γ at these points are mutually different. (See Corollary 1.6.) To prove the theorem, we will use and further develop the method of intrinsic circle systems introduced in I-§1.

As an application of Theorem 0.1, we will improve results of A. Kneser [Kn] and Segre [Se] on spherical curves. If $\gamma : S^1 \to S^2$ is a simple closed curve, then Segre proved that if p lies in the convex hull of γ without lying on γ, then the osculating planes of at least four distinct points of γ pass through p. We will show that this is still true for any convex space curve γ with nowhere vanishing curvature and a point

p that lies in the interior of the domain containing the curve which is bounded by the osculating hyperplanes at the four clear vertices, whose existence is claimed in the theorem.

Another application is the following four vertex theorem for space curves which may not be convex: Let γ be a C^2-regular simple closed curve in $\mathbf{R}^3$ with nowhere vanishing curvature. Assume there is a point p in the interior of its convex hull such that no ray starting in p intersects γ in two or more points or is tangent to γ in some point. Then the curve γ has at least four honest vertices. Here an *honest vertex* is a point at which the curve does not cross the osculating plane. This improves Sedykh's four vertex theorem for convex space curves, see [Sd].

We will also discuss how this relates to Arnold's Tennis Ball Theorem [A1] and the theorem of Möbius [M] on inflection points of curves in the projective plane.

As a further application, we also use the abstract methods in section 1 to improve a theorem of Ghys on extremal points of projective line diffeomorphisms (see [OT] and [T]) and to arrive at new results about them.

§1. Compatible pairs of intrinsic circle systems.

We let S^1 denote the unit circle with a fixed orientation. Let $\succ$ denote the order induced by the orientation on the complement of any interval in S^1. Any two distinct points $p, q \in S^1$ divide S^1 into two closed arcs $[p,q]$ and $[q,p]$ such that on $[p,q]$ we have $q \succ p$ and on $[q,p]$ we have $p \succ q$. We let (p,q) and (q,p) denote the corresponding open arcs. We also use the notation $p \succeq q$, which means $p = q$ or $p \succ q$.

Let A be a subset of S^1 and $p \in A$. We denote by $Z_p(A)$ the connected component of A containing p. The concept of an intrinsic circle system was introduced in I-§1 as a multivalued function on S^1 satisfying certain axioms. It was used there to prove an abstract Bose type formula. Here we recall the definition.

Definition 1.1. A family of nonempty closed subsets $F := (F_p)_{p\in S^1}$ of S^1 is called an *intrinsic circle system on* S^1 if it satisfies the following three conditions for any $p \in S^1$.

(I1) If $q \in F_p$, then $F_p = F_q$.

(I2) If $q \in S^1 \setminus F_p$, then $F_q \subset Z_q(S^1 \setminus F_p)$. (Or equivalently, if $p' \in F_p$, $q' \in F_q$ and $q \succeq p' \succeq q' \succeq p (\succeq q)$, then $F_p = F_q$ holds.)

(I3) Let $(p_n)_{n\in\mathbf{N}}$ and $(q_n)_{n\in\mathbf{N}}$ be two sequences in S^1 such that $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} q_n = q$ respectively. Suppose that $q_n \in F_{p_n}$ $(n = 1, 2, 3, \ldots)$. Then $q \in F_p$ holds.

We will let $\operatorname{rank}(p)$ denote the number of connected components of F_p.

The proof of the following fundamental fact, which plays an important role in the previous paper, only uses property (I2).

Fact 1.1 (I-Lemma 1.1). *Let F be an intrinsic circle system. Let p, q be distinct points on S^1 such that $q \in F_p$. Suppose that $(p,q) \not\subset F_p$. Then there exists a point $x \in (p,q)$ such that* $\operatorname{rank}(x) = 1$.

We will now give an application of I-§1,§2 by discussing pairs of intrinsic circle systems satisfying the following compatibility condition.

Definition 1.2. A pair of intrinsic circle systems $(F^\bullet, F^\circ)$ is said to be *compatible* if it satisfies the following two conditions.

(C1) $F_p^\bullet \cap F_p^\circ = \{p\}$ for all $p \in S^1$.

(C2) Suppose that $\operatorname{rank}^\bullet(p) = 1$ (resp. $\operatorname{rank}^\circ(p) = 1$). Then there are no points of $\operatorname{rank}^\circ = 1$ (resp. $\operatorname{rank}^\bullet = 1$) in a sufficiently small neighborhood of p.

For each $p \in S^1$, we denote by $\operatorname{rank}^\bullet(p)$ (resp. $\operatorname{rank}^\circ(p)$), the number of connected components of $F_p^\bullet$ (resp. F_p°).

The following are examples of compatible pairs of intrinsic circle systems.

Example 1. Let $\gamma : S^1 \to \mathbf{R}^2$ be a C^2-regular simple closed curve which is not a circle. Let Γ be the set of all oriented circles and lines in $\mathbf{R}^2$. The curve γ separates the plane into two closed domains. We denoted by $D^\bullet(\gamma)$ the compact domain bounded by γ and by $D^\circ(\gamma)$ the noncompact one. We assume that γ is positively oriented, meaning that the compact domain $D^\bullet(\gamma)$ is on the left of γ. For each $p \in \gamma$, there is an element $C_p^\bullet \in \Gamma$ (resp. $C_p^\circ \in \Gamma$) which has the smallest (resp. largest) curvature among $C \in \Gamma$ that are tangent to γ in p and satisfy $C \subset D^\bullet(\gamma)$ (resp. $C \subset D^\circ(\gamma)$). We call $C_p^\bullet$ (resp. C_p°) the maximal (resp. minimal) circle at p. Now we set

$$F_p^\bullet := \gamma \cap C_p^\bullet, \qquad F_p^\circ := \gamma \cap C_p^\circ. \tag{1.1}$$

Then $(F^\bullet, F^\circ)$ is a pair of intrinsic circle systems. (See I-Proposition 3.1.) Condition (C1) of Definition 1.2 trivially holds. For a point p of $\operatorname{rank}^\bullet(p) = 1$ (resp. $\operatorname{rank}^\circ(p) = 1$), the osculating circle C_p at p coincides with $C_p^\bullet$ (resp. C_p°). (See I-Proposition A.5.) In particular, condition (C2) of Definition 1.2 is also satisfied. Thus $(F^\bullet, F^\circ)$ is a compatible pair. Instead of Γ, we can use a system of Minkowski circles in the plane. (See I-§3 for details.)

Example 2. An immersed closed space curve $\gamma : S^1 \to \mathbf{R}^3$ is called convex if it lies on the boundary ∂H of its convex hull H. We fix a nonplanar C^2-convex simple closed curve γ and assume that its curvature function is positive. The boundary ∂H of the convex hull is homeomorphic to a sphere and γ divides ∂H into two domains. Let $\partial H^\bullet$ (resp. ∂H°) be the left-hand (right-hand) closed domain bounded by γ. We set

$$F_p^\bullet := \{q \in \gamma\,;\, \overline{pq} \subset \partial H^\bullet\}, \quad (\text{resp. } F_p^\circ := \{q \in \gamma\,;\, \overline{pq} \subset \partial H^\circ\}). \tag{1.2}$$

By I-Theorem 4.8, $F^\bullet$ and F° are intrinsic circle systems if γ satisfies one of the following two conditions.

(1) For each point $p \in \gamma$, there exists a supporting plane U_p such that $U_p \cap \gamma = \{p\}$.

(2) γ has no planar open subarcs.

Moreover, $(F^\bullet, F^\circ)$ is a compatible pair if γ satisfies one of the following three conditions.

(a) γ satisfies (1).

(b) γ satisfies (2) and any tangent line of γ meets γ in only one point.

(c) γ is a C^3-convex space curve whose torsion function has only finitely many zeros.

In fact, (C1) of Definition 1.2 is satisfied by definition. For definitions of concepts we are now going to use, see the Introduction. When γ satisfies (a) or (b), a point p on γ is of of $\text{rank}^\bullet = 1$ (resp. $\text{rank}^\circ = 1$) if and only if it is a clean maximal (resp. minimal) vertex by I-Lemma 4.10. When γ satisfies (c), a point p of $\text{rank}^\bullet = 1$ (resp. $\text{rank}^\circ = 1$) is a clear maximal (resp. minimal) vertex by I-Proposition 4.15 and I-Proposition 4.20. A clear maximal vertex cannot be a clear minimal vertex because γ is not planar. Thus (C2) of Definition 1.2 also follows in any of these three cases.

We will give a further example in §3.

From now on, we fix a compatible pair $(F^\bullet, F^\circ)$ of intrinsic circle systems.

Definition 1.3. If $\text{rank}^\bullet(p) = 2$ (resp. $\text{rank}^\circ(p) = 2$), p is called a *•-regular (resp. ∘-regular) point.* If $\text{rank}^\bullet(p) \geq 2$ (resp. $\text{rank}^\circ(p) \geq 2$), p is called *weakly •-regular (weakly ∘-regular)*. An open arc I of S^1 is called •-regular (resp. ∘-regular) if all of its points are •-regular (resp. ∘-regular). More generally, an open arc I of S^1 is called weakly •-regular (resp. weakly ∘-regular) if all of its points are weakly •-regular (resp. weakly ∘-regular).

If (X, Γ) is a circle system as defined in I-§3, the above definitions and notations are compatible with those in I-Definition 3.6.

Definition 1.4. (1) Let I be a closed arc on S^1 and A be a subset of I. Then let $\sup_I(A)$ and $\inf_I(A)$ denote the least upper bound and the greatest lower bound of A respectively with respect to the order $\succ$ on I.

(2) Let $I = (x_1, x_2)$ be a weakly •-regular arc. For any $p \in I$, we set

$$\mu^\bullet_+(p) := \sup_{S^1 \setminus I}(Y_p), \qquad \mu^\bullet_-(p) := \inf_{S^1 \setminus I}(Y_p),$$

where $Y_p := F_p \setminus Z_p(F_p)$. (By Fact 1.1, one can easily show that $Y_p \subset S^1 \setminus I$.) Moreover, we extend the definition of $\mu^\bullet_\pm$ to the boundary of I as follows. If x_j $(j = 1, 2)$ is weakly •-regular, we set

$$\mu^\bullet_+(x_j) := \sup_{S^1 \setminus I}(Y_{x_j}), \qquad \mu^\bullet_-(x_j) := \inf_{S^1 \setminus I}(Y_{x_j}).$$

On the other hand, if x_j is not weakly regular, we set

$$\mu^\bullet_+(x_j) := \sup_{S^1 \setminus I}(F_{x_j}), \qquad \mu^\bullet_-(x_j) := \inf_{S^1 \setminus I}(F_{x_j}).$$

If I is a weakly ∘-regular arc, $\mu^\circ_\pm$ can be similarly defined on $\overline{I}$. We will refer to $\mu^\bullet_\pm$ and $\mu^\circ_\pm$ as antipodal maps.

Only parts (i), (ii) and (iii) of the following Fact 1.2 will be used in the proof of Theorem 1.4. We will not refer explicitly to the other parts and only bring them here for the sake of completeness. Notice though that they are used in the proof of Fact 1.3 below.

Fact 1.2. *Let $I = (x_1, x_2)$ be a weakly $\bullet$-regular (resp. $\circ$-regular) arc on S^1. Then the antipodal maps $\mu_\pm := \mu^\bullet_\pm$ (resp. $\mu_\pm := \mu^\circ_\pm$) satisfy the following properties.*

(i) *For $p \in I$ of rank$^\bullet(p) = 2$ (resp. rank$^\circ(p) = 2$), $[\mu_-(p), \mu_+(p)] \subset F^\bullet_p$ (resp. $[\mu_-(p), \mu_+(p)] \subset F^\circ_p$) holds (where possibly $\mu_-(p) = \mu_+(p)$).*

(ii) $\mu_\pm(\overline{I}) \subset S^1 \setminus I$.

(iii) *Let $p, q \in \overline{I}$ be such that $p \succeq q$. Then $\mu_+(q) \succeq \mu_+(p)$ and $\mu_-(q) \succeq \mu_-(p)$ with respect to the order on $S^1 \setminus I$. Moreover, if $F^\bullet_p \neq F^\bullet_q$ (resp. $F^\circ_p \neq F^\circ_q$), then $\mu_-(q) \succ \mu_+(p)$ holds on $S^1 \setminus I$.*

(iv) $\lim_{x \to p-0} \mu_+(x) = \mu_+(p)$ *holds for any $p \in (x_1, x_2]$.*

(v) $\lim_{x \to p+0} \mu_-(x) = \mu_-(p)$ *holds for any $p \in [x_1, x_2)$.*

(vi) *If I is $\bullet$-regular (resp. $\circ$-regular), then the open arc $(\mu_+(x_2), \mu_-(x_1))$ is $\bullet$-regular (resp. $\circ$-regular). Moreover, for any $q \in (\mu_-(p), \mu_+(p))$, there exists $p \in I$ such that $q \in F^\bullet_p$ (resp. $q \in F^\circ_p$).*

Property (i) follows immediately from the definition and (ii) follows directly from Fact 1.1. Properties (iii)-(vi) are proved in 1.3-1.7 in I-§1.

Each intrinsic circle system $F^\bullet$ (resp. F°) induces an equivalence relation, which in turn induces a quotient space $S^1/F^\bullet$ (resp. S^1/F°). The equivalence class containing $p \in S^1$ will be denoted by $[p]^\bullet$ (resp. $[p]^\circ$). Then rank$^\bullet([p]^\bullet) :=$ rank$^\bullet(p)$ (resp. rank$^\circ([p]^\circ) :=$ rank$^\circ(p)$) is well defined on $S^1/F^\bullet$ (resp. $S^1/F^\bullet$) by virtue of (I1). We set

$$
\begin{aligned}
S(F^\bullet) &:= \{[p]^\bullet \in S^1/F^\bullet; \operatorname{rank}^\bullet([p]^\bullet) = 1\}, \\
S(F^\circ) &:= \{[p]^\circ \in S^1/F^\circ; \operatorname{rank}^\circ([p]^\circ) = 1\}, \\
T(F^\bullet) &:= \{[p]^\bullet \in S^1/F^\bullet; \operatorname{rank}^\bullet([p]^\bullet) \geq 3\}, \\
T(F^\circ) &:= \{[p]^\circ \in S^1/F^\circ; \operatorname{rank}^\circ([p]^\circ) \geq 3\}.
\end{aligned}
$$

The set $S(F^\bullet)$ (resp. $S(F^\circ)$) is called the single tangent subset of $S^1/F^\bullet$ (resp. S^1/F°) and $T(F^\bullet)$ (resp. $T(F^\circ)$) is called the tritangent subset of $S^1/F^\bullet$ (resp. S^1/F°). Moreover, we set

$$
\begin{aligned}
s(F^\bullet) &:= \text{the cardinality of the set } S(F^\bullet), \\
s(F^\circ) &:= \text{the cardinality of the set } S(F^\circ), \\
t(F^\bullet) &:= \sum_{[p]^\bullet \in T(F^\bullet)} (\operatorname{rank}^\bullet(p) - 2), \\
t(F^\circ) &:= \sum_{[p]^\circ \in T(F^\circ)} (\operatorname{rank}^\circ(p) - 2).
\end{aligned}
$$

Definition 1.5. The single tangent set $S(F^\bullet)$ (resp. $S(F^\circ)$) is said to be *supported by a continuous function* $\tau : S^1 \to \mathbf{R}$ if for each p with rank$^\bullet(p) = 1$ (resp. rank$^\circ(p) = 1$), $F^\bullet_p$ (resp. F°_p) is a connected component of the zero set of τ.

In I-§3, the following was proved by using the properties in Fact 1.2.

Fact 1.3. *Let $(F^\bullet, F^\circ)$ be a compatible pair of intrinsic circle systems. Then the following holds.*

(i) *If $s(F^\bullet) < \infty$ (resp. $s(F^\circ) < \infty$), then $t(F^\bullet) < \infty$ (resp. $t(F^\circ) < \infty$). The converse is also true if the single tangent set $S(F^\bullet)$ (resp. $S(F^\circ)$) is supported by a continuous function $\tau : S^1 \to \mathbf{R}$.*

(ii) *Suppose that $s(F^\bullet) < \infty$ (resp. $s(F^\circ) < \infty$). Then the following Bose type formulas hold*

$$s(F^\bullet) - t(F^\bullet) = 2 \qquad (s(F^\circ) - t(F^\circ) = 2).$$

We will give an application of the formulas in (ii) in Section 3.

We now come to the main result of this section.

Theorem 1.4. *Let $(F^\bullet, F^\circ)$ be a compatible pair of intrinsic circle systems. Then there are four points $p_1, p_2, p_3, p_4 \in \gamma$ satisfying $p_1 \succ p_2 \succ p_3 \succ p_4 (\succ p_1)$ such that*

$$rank^\bullet(p_1) = rank^\circ(p_2) = rank^\bullet(p_3) = rank^\circ(p_4) = 1.$$

Proof. Suppose there are less than four sign changes of rank one points. Since the number of sign changes is even, it must be exactly two. We set

$$V^\bullet := \{x \in \gamma\,;\, \mathrm{rank}^\bullet(x) = 1\},$$
$$V^\circ := \{x \in \gamma\,;\, \mathrm{rank}^\circ(x) = 1\},$$

and denote by $\overline{V^\bullet}$ and $\overline{V^\circ}$ their closures. Let I be the connected component of $S^1 \setminus \overline{V^\circ}$ containing $\overline{V^\bullet}$. We set

$$x_1 := \sup_{\overline{I}}(\overline{V^\bullet}) \qquad x_2 := \inf_{\overline{I}}(\overline{V^\bullet}).$$

By condition (C2) of Definition 1.2, it holds that $x_1, x_2 \in I$. Then the open interval $J := (x_1, x_2)$ is a weakly $\bullet$-regular arc and so the antipodal functions $\mu^\bullet_\pm$ are defined on J. On the other hand, I is a weakly $\circ$-regular arc and so $\mu^\circ_\pm$ are defined on it. By (ii) of Fact 1.2, we have

$$\mu^\bullet_\pm(\overline{J}) \subset \overline{I}, \qquad \mu^\circ_\pm(\overline{I}) \subset \overline{J}.$$

We set

$$A = \{p \in \overline{J}\,;\, \mu^\circ_-(\mu^\bullet_-(p)) \succ p \text{ on } \overline{J}\},$$
$$B = \{p \in \overline{J}\,;\, \mu^\circ_+(\mu^\bullet_+(p)) \prec p \text{ on } \overline{J}\}.$$

We suppose that $p \in \overline{J} \setminus A$. Since $F^\bullet_{\mu^\bullet_-(p)} \cap F^\circ_{\mu^\bullet_-(p)} = \mu^\bullet_-(p)$, we have $p \neq \mu^\circ_-(\mu^\bullet_-(p))$. Then we have

$$p \succ \mu^\circ_-(\mu^\bullet_-(p)) \qquad (\text{on } \overline{J}).$$

Hence (I2) of Definition 1.1 for F° yields that

$$(p \succ)\mu_-^\circ(\mu_-^\bullet(p)) \succeq \mu_+^\circ(\mu_+^\bullet(p)) \succ \mu_+^\bullet(p) \succeq \mu_-^\bullet(p) \qquad \text{on } [\mu_-^\bullet(p), p].$$

This implies $p \in B$. Thus we have

$$\overline{J} = A \cup B. \tag{1.3}$$

We will now use Lemma 1.5 below. It says there that A is nonempty. We set

$$q := \sup_{\overline{J}}(A).$$

If $q \in A$, then Lemma 1.5 also yields that there is $y \in A$ such that $y \succ q$. Thus $q \notin A$, that is $q \in B$ by (1.3). Then by Lemma 1.5, there exists $z \in B$, $q \succ z$, such that $(z, q) \cap A = \emptyset$, contradicting that $q := \sup_{\overline{J}}(A)$. □

Lemma 1.5. *The sets A and B are nonempty subsets of the arc $\overline{J}$. Moreover, for each $x \in A$ (resp. $x \in B$), $[x, y] \subset A$ and $(x, y) \cap B = \emptyset$ (resp. $[y, x] \subset B$ and $(y, x) \cap A = \emptyset$) hold, where $y := \mu_-^\circ(\mu_-^\bullet(x))$.*

Proof. We prove the assertion for A. (The corresponding assertion for B follows if one reverses the orientation of S^1.) First we prove $x_1 \in A$. In fact, if $x_1 \notin A$, then $x_1 \in B$. Then

$$x_1 \succ \mu_+^\circ(\mu_+^\bullet(x_1)) \qquad \text{on } \overline{J}.$$

But this contradicts the fact that x_1 is the smallest point contained in $\overline{J}$. This implies $x_1 \in A$. In particular, A is not empty. Now we fix an element $x \in A$ arbitrarily. Then by definition, we have

$$y = \mu_-^\circ(\mu_-^\bullet(x)) \succ x \text{ on } \overline{J}.$$

We fix a point z on the interval (x, y) arbitrarily. Since $z \succ x$ on J, applying (iii) of Fact 1.2 we have

$$\mu_-^\bullet(x) \succeq \mu_\pm^\bullet(z) \text{ on } \overline{I}.$$

Then applying (iii) of Fact 1.2 again, we have

$$\mu_\pm^\circ(\mu_\pm^\bullet(z)) \succeq \mu_-^\circ(\mu_-^\bullet(x))(\succ z).$$

This implies $z \in A$ and $z \notin B$. □

Corollary 1.6. *Let $\gamma : S^1 \to \mathbf{R}^3$ be a C^2-regular convex simple closed nonplanar space curve with nonvanishing curvature satisfying one of the following two conditions.*

(a) *For each point $p \in \gamma$, there exists a supporting plane U_p such that $U_p \cap \gamma = \{p\}$.*
(b) *γ contains no planar open subarcs and no tangent line of γ meets γ in more than one point.*

Then γ has at least four sign changes of clean vertices. Moreover, at these four clean vertices the osculating planes of γ are mutually different.

Proof. As mentioned in Example 2, the pair of intrinsic circle systems $(F^\bullet, F^\circ)$ is compatible if γ satisfies either one of the conditions in (a) and (b). We now apply Theorem 1.4. We know that each rank one point on γ is a clean vertex by I-Lemma 4.10. From this it follows immediately that the four osculating planes are mutually distinct. □

We now can prove Theorem 0.1 in Introduction. Let $\mathfrak{C}$ be the set of nonplanar C^2-convex simple closed space curves and $\mathfrak{C}_0$ be the subset of $\mathfrak{C}$ consisting of convex space curves which satisfy condition (a) of Corollary 1.6. It is sufficient to prove the following lemma.

Lemma 1.7. *For each $\gamma \in \mathfrak{C}$, there exists a sequence $(\gamma_n)_{n\in\mathbf{N}}$ in $\mathfrak{C}_0$ converging to γ with respect to the uniform C^2-topology.*

In fact, assuming the lemma, we can prove the theorem as follows: Let γ be a C^2-regular convex nonplanar space curve with nonvanishing curvature. By Lemma, 1.7, there exists a sequence $(\gamma_n)_{n\in\mathbf{N}}$ in $\mathfrak{C}_0$ converging to γ with respect to the uniform C^2-topology. Since the curvature function of γ never vanishes, the same is true for γ_n if n is sufficiently large. By Corollary 1.6, each γ_n has two clean maximal vertices $p_1^{(n)}, p_3^{(n)}$ and two clean minimal vertices $p_2^{(n)}, p_4^{(n)}$ such that $p_1^{(n)} \succ p_2^{(n)} \succ p_3^{(n)} \succ p_4^{(n)}$ holds. If necessary, by going to a subsequence, we may assume that there exist points (p_1, p_2, p_3, p_3) such that

$$\lim_{n\to\infty}(p_1^{(n)}, p_2^{(n)}, p_3^{(n)}, p_4^{(n)}) = (p_1, p_2, p_3, p_3).$$

Since $p_1^{(n)}$ and $p_3^{(n)}$ (resp. $p_2^{(n)}$ and $p_4^{(n)}$) are clean maximal (resp. minimal) vertices, the limit points $p_1^{(n)}$ and $p_3^{(n)}$ (resp. $p_2^{(n)}$ and $p_4^{(n)}$) are clear maximal (resp. minimal) vertices. Since γ is not a plane curve, a clear maximal vertex is not a clear minimal vertex. Thus the four points are mutually distinct and satisfy $p_1 \succ p_2 \succ p_3 \succ p_4$. □

Proof of Lemma 1.7. To prove the lemma, we recall some fundamental properties of convex bodies. A convex bounded open region Ω in $\mathbf{R}^3$ is called a *convex body.* We fix an interior point o of Ω. Without loss of generality we may assume that o is the origin of $\mathbf{R}^3$. We set

$$\rho(x) := \inf\{t > 0\,;\, x \in t\Omega\} \qquad (x \in \mathbf{R}^3),$$

where $t\Omega := \{tx \,:\, x \in \Omega\}$. Notice that $\pi : \partial\Omega \to S^2$ defined by $\pi(x) := x/|x|$ is a homeomorphism. We have the expression

$$\rho(x) = |x|\rho(x/|x|) = |x|\pi^{-1}(x/|x|). \tag{1.4}$$

Thus $\rho : \mathbf{R}^3 \to \mathbf{R}$ is a continuous function. Moreover the function ρ satisfies the following properties (cf. Proposition 1.1.5 in [KR]).

(1) For each $x \in \mathbf{R}^3$, $\rho(x) \geq 0$ and $\rho(x) = 0$ if and only if $x = o$.
(2) $\rho(ax) = a\rho(x)$ for any real number $a \geq 0$.
(3) $\rho(x + y) \geq \rho(x) + \rho(y)$ for all $x, y \in \mathbf{R}^3$.

Furthermore Ω can be expressed as

$$\Omega = \{x \in \mathbf{R}^3\,;\, \rho(x) < 1\}. \tag{1.5}$$

The convex body Ω is called *strictly convex*, if for each boundary point $p \in \Omega$, there is a plane U passing through p such that $\Omega \cap U = \{p\}$. One can easily show that Ω is strictly convex if ρ satisfies the condition that

$$\rho(x+y) < \rho(x) + \rho(y) \tag{1.6}$$

for any two linearly independent $x, y \in \mathbf{R}^3$. Conversely, if there exists a continuous function $\rho : \mathbf{R}^3 \to \mathbf{R}$ satisfying the three properties (1)-(3), the open subset Ω defined in (1.5) is a convex body.

Now we fix a C^2-convex space curve $\gamma : [0,1] \to \mathbf{R}^3$. Let Ω_γ be its convex hull. Without loss of generality, we may assume that Ω_γ contains the origin o in its interior. Let ρ be the continuous function satisfying (1)-(3) associated to the convex body Ω_γ. We set

$$\tilde{\gamma} := \pi(\gamma), \tag{1.7}$$

where $\pi : \partial\Omega \to S^2$ is the projection defined above. Then $\tilde{\gamma} : [0,1] \to S^2$ is a C^2-regular embedding. By (1.4), we have $\gamma = \{\rho(\tilde{\gamma})\}\tilde{\gamma}$. We set

$$\rho_n(x) := \rho(x) + \frac{|x|}{n}.$$

Then ρ_n satisfies (1)-(3) and also (1.6). Thus the associated convex body Ω_n is strictly convex. We set

$$\gamma_n(t) = \rho_n(\tilde{\gamma}(t)) \cdot \tilde{\gamma}(t) \qquad (t \in [0,1]). \tag{1.8}$$

Notice that $\rho_n(\tilde{\gamma}(t))$ is clearly C^2 in t although ρ_n is only continuous. Thus each curve γ_n is a C^2-regular simple closed curve that lies in the boundary of Ω_n. Since Ω_n is strictly convex, γ_n satisfies condition (a) of Corollary 1.6. Moreover, it is obvious that γ_n converges to γ in the uniform C^2-topology. □

We next give an example of a plane curve γ with the following two properties:

(1) There are only four sign changes of clear vertices, although γ has more than four clean vertices.
(2) The number of clean or clear maximal vertices are not equal to the number of clean or clear minimal vertices.

In particular, this example shows that we cannot improve the number of sign changes to $2n$ in Theorem 1.4 when the curve γ meets a circle in $2n$ points such that the rotational order on γ and the circle coincide. In fact, since the number of clean vertices exceeds four, we may assume that the number of maximal clean vertices is at least three. By (ii) of Fact 1.3, we have $t(F^\bullet) \geq 1$, i.e., there is a triple tangent enclosed circle C.

Expanding C slightly by a homothety with the same center as C, we get a circle C_ε which meets γ in six points whose order on C_ε coincides with that on γ.

The curve γ is constructed as follows: Consider the ellipse

$$\gamma_1 : \quad \frac{x^2}{4} + y^2 = 1$$

that we assume to be positively oriented (i.e. with an inward pointing normal vector). We shrink it by a homothety with the dilation factor $\sqrt{1-\varepsilon}$, where $\varepsilon > 0$ is a sufficiently small number. Then we have another ellipse

$$\gamma_2 : \quad \frac{x^2}{4} + y^2 = 1 - \varepsilon$$

that we assume to be negatively oriented (i.e. with on outward pointing normal vector). We also consider the parabola

$$x = -4y^2 - 1$$

oriented in the negative direction of the y-axis.

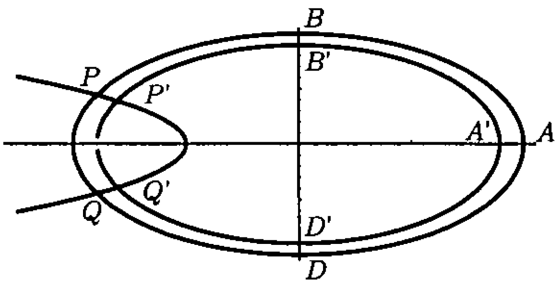

Figure 1-a.

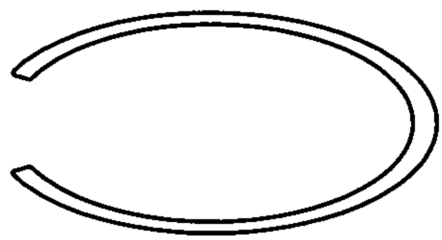

Figure 1-b.

Then γ_1 (resp. γ_2) meets the parabola in two points P, Q (resp. P', Q') as in Figure 1-a. Note that with the orientation chosen above and notation as in Figure 1-a the arcs QD, $D'Q'$ and PP' are curvature decreasing and BP, $P'B'$ and $Q'Q$ are curvature decreasing. We can round the corners at P, Q (resp. P', Q') introducing exactly one (resp. three) vertices using the method of Proposition 2.3 in [KU], and get a simple closed curve γ as in Figure 1-b with the following vertices

$$A,\ B,\ P,\ P'_{-1},\ P'_0,\ P'_1,\ B',\ A',\ D',\ Q'_{-1},\ Q'_0,\ Q'_1,\ Q,\ D.$$

Here P'_{-1}, P'_0, P'_1 (resp. Q'_{-1}, Q'_0, Q'_1) are the vertices which appear after rounding the corner at P' (resp. Q'). More precisely, P'_0, Q'_0 (resp. P'_{-1}, P'_1, Q'_{-1}, Q'_1) are local maxima (resp. minima) of the curvature function. By construction, γ has 14 vertices. The clean (clear) maximal vertices are P, P'_0, Q, Q'_0. The clean (clear) minimal vertices are B, D, A'. (In fact, if ε is sufficiently small, other vertices cannot be clear vertices.) Thus the number of clean maximal vertices is greater than the number of clean minimal vertices. Moreover, the clean vertices lie on the curve γ in the order

$$B,\ P,\ P'_0,\ A',\ Q'_0,\ Q,\ D,$$

so the number of sign changes of clean vertices is only four.

We end this section by giving a further corollary of Theorem 1.4. It is a refinement of the four vertex theorem for convex curves.

Corollary 1.8. *Let $\gamma : S^1 \to \mathbf{R}^2$ be a C^2-regular convex plane curve with length 2π, which is not a circle. Then the function $\kappa^2 - 1$ changes sign at least four times, where $\kappa : S^1 \to \mathbf{R}$ is the curvature of γ.*

Proof. We may assume that $\kappa \geq 0$. We use the notation in Example 1. Let $p_1, ..., p_4$ be points as in Theorem 1.4 and C_j $(j = 1, ..., 4)$ the osculating circle at p_j. Then as mentioned in Example 1, we have

$$C_i = C^{\bullet}_{p_i} \subset D^{\bullet}(\gamma) \qquad (i = 1, 3), \tag{1.9}$$

and

$$C_j = C^{\circ}_{p_j} \subset D^{\circ}(\gamma) \qquad (j = 2, 4). \tag{1.10}$$

Since γ is not a circle, the lengths of C_1, C_3 (resp. C_2, C_4) are less (resp. greater) than 2π by (1.9) and (1.10). In particular, $\kappa > 1$ at p_1, p_3 and $\kappa < 1$ at p_2, p_4. □

It should be remarked that there is an elementary proof of Corollary 1.8 using integration that we now want to explain.

An alternative proof of Corollary 1.8. We parametrize the curve according to arclength t $(0 \leq t \leq 2\pi)$ and write $\gamma(t) = (x(t), y(t))$. We assume that $\kappa \geq 0$. Since the length and total curvature of γ are both 2π, we have the following identity

$$\int_0^{2\pi} (\kappa(t) - 1)(ax'(t) + by'(t) + c)dt = 0, \tag{1.11}$$

where a, b, c are arbitrary real numbers. If we set $a = b = 0$ and $c = 1$, this implies that $\kappa - 1$ changes sign even number of times. Suppose that the sign changes are only two. Then we may assume that there exists $t_0 \in [0, 2\pi]$ such that $\kappa - 1 \geq 0$ on $[0, t_0]$ and $\kappa - 1 \leq 0$ on $[t_0, 2\pi]$. We can choose the numbers a, b, c so that the line $ax + by + c = 0$ passes through the two points $\gamma'(0)$ and $\gamma'(t_0)$. Then $(\kappa(t) - 1)(ax'(t) + by'(t) + c)$ is a nonpositive or nonnegative function. By (1.11), this implies κ is identically 1, which is a contradiction. □

§2. Applications to space curves.

With the methods of section 1 at hand it is remarkably easy to improve some well-known theorems and show how others follow as corollaries. In this section we will use Theorem 0.1 (which is a corollary of Theorem 1.4) to improve a result of A. Kneser [Kn] and Segre [Se] on spherical curves. We will also discuss how this improvement relates to the Tennis Ball Theorem of Arnold [A] and the Theorem of Möbius [M] that a simple closed curve in the projective plane that is not nullhomotopic has at least three inflection points. We will also prove a four vertex theorem for closed simple space curves that improves a result of Sedykh [Sd]. Instead of convexity we assume that the curve satisfies a condition of starshapedness. We do assume that the curvature does never vanish and our theorem seems to improve existing results on this class of curves.

Assume that $\gamma : S^1 \to S^2$ is simple, closed and regular. Then Segre [Se] proved the following (p. 243):

(1) If $p \in \mathbf{R}^3$ lies in the convex hull of γ without lying on γ, then the osculating planes of at least four distinct points of γ pass through p.
(2) If p lies on γ and p is not a vertex, then the osculating planes of at least three distinct points of γ, all of which differ from p, pass through p.
(3) If p is a vertex of γ, then the osculating planes of at least two distinct points of γ, both of which differ from p, pass through p.

A. Kneser proved part (2) of the above theorem as an application of the Theorem of Möbius. This was a step in his proof of the four vertex theorem for simple closed curves in the Euclidean plane. Segre does not say explicitly in the statement of his theorem that the points he finds in (2) and (3) all differ from p, but this seems to be what he proves.

Segre pointed out on p. 258 of his paper that the claim in (1) also holds for a point p in the convex hull of a space curve $\gamma : S^1 \to \mathbf{R}^3$, if $p \notin \gamma$ and no ray starting in p is tangent to γ or meets γ in two or more points. This is proved by simply projecting γ radially onto a sphere with center in p. We will also use this argument in the proof of Theorem 2.2 below.

Our improvements of the results of A. Kneser and Segre will consist in the following:

(1) We will consider more general space curves.
(2) Instead of the convex hull of the curve, we can prove (1) for points p in the interior of a certain polyhedron Δ that contains the curve and only has vertices of the curve in its boundary.
(3) We will show that the osculating planes found in the theorem of A. Kneser and Segre can under certain conditions been chosen to be at points different from the vertices of the curve. This will be needed to prove the Tennis Ball Theorem and the Möbius Theorem as corollaries. Segre also discusses such things in his proof without summarizing them in the statement of his theorem. We will give an example that shows that this is not true for all choices of p in the boundary of the convex hull of the curve. This example will also show that some of the osculating planes at the different points A. Kneser and Segre find might coincide.

We will be dealing with a C^2-regular simple closed convex nonplanar space curve $\gamma : S^1 \to \mathbf{R}^3$ with nowhere vanishing curvature. Notice that this situation is more general than the one in the theorem of A. Kneser and Segre.

We know from Theorem 0.1 that γ has at least four clear vertices $t_1 \succ t_2 \succ t_3 \succ t_4$ such that t_1 and t_3 are maxima of the height functions in the direction of the binormal vectors at the respective points, and t_2 and t_4 are minima of the corresponding height functions. Let Π_i be the osculating plane of γ at t_i. Then γ is contained in one of the closed halfspaces bounded by Π_i. We denote this halfspace by S_i. The binormal of γ at t_i points out of S_i if i is 1 or 3 and in to S_i if i is 2 or 4. We cannot exclude that $\Pi_1 = \Pi_3$ and $\Pi_2 = \Pi_4$, but it follows that $\Pi_i \neq \Pi_{i+1}$ since γ is not planar. Let Δ denote the intersection of these halfspaces or, what is the same thing, the closure of the connected component of $\mathbf{R}^3 \setminus (\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4)$ containing γ. Then Δ is a simplex if no two of the osculating planes Π_i coincide or are parallel. Otherwise Δ is a polyhedron that might be unbounded.

We denote the closure of the connected component of $\mathbf{R}^3 \setminus (\Pi_1 \cup \Pi_2)$ containing γ by S_{12}. Let Π_t denote the osculating plane of γ at t. We want to show that for every point $p \in S_{12}$ there is a $t \in [t_1, t_2]$ such that $p \in \Pi_t$. Notice that the conditions on γ as well as the conclusions we are aiming at can all be phrased in terms of projective geometry. We can therefore, if necessary, apply a projective transformation that sends the line in which Π_1 and Π_2 intersect to infinity. Hence we can assume that the planes Π_1 and Π_2 are parallel and the curve γ lies in between the planes. The set S_{12} is thus a slab. The binormal vector of γ at t_1 points out of the slab and at t_2 it points into it.

Now assume that $p \in S_{12}$ is not contained in any osculating plane Π_t for $t \in (t_1, t_2)$. Let q_t be the point on Π_t closest to p and let v_t be the unit vector at q_t pointing in the direction from p to q_t. The vector v_t is perpendicular to Π_t for all $t \in [t_1, t_2]$. Notice that v_{t_1} points out of the slab S_{12} since p lies in its interior, i.e., v_{t_1} points in the same direction as the binormal vector at $\gamma(t_1)$. By continuity, v_t points in the same direction as the binormal vector at $\gamma(t)$ for all $t \in [t_1, t_2]$. It follows that the binormal vector at $\gamma(t_2)$ points out of the slab, which is a contradiction. We have thus proved that for every point $p \in S_{12}$ there is a $t \in [t_1, t_2]$ such that $p \in \Pi_t$.

We can repeat the argument above for the pairs (t_2, t_3), (t_3, t_4) and (t_4, t_1) and prove that for every point p in the interior of $\Delta = S_{12} \cap S_{23} \cap S_{34} \cap S_{41}$, there is an $s_i \in (t_i, t_{i+1})$ such that $p \in \Pi_{s_i}$. It follows that every point in the interior of Δ lies in the osculating planes of four different points of γ.

This already improves part (1) of the theorem of Segre for the following reason: A point p in the convex hull of γ which neither lies in the interior of Δ nor on γ is in the convex hull of the vertices in one of the osculating planes Π_i. Since γ is a spherical curve in Segre's theorem we can choose the points $t_1, \ldots, t_4$ to be clean vertices. It follows that Π_i contains an open arc of vertices of γ and that Π_i is the osculating plane at each point of this arc. There are therefore infinitely many points of γ whose osculating planes contain p. Notice that we do not claim that these osculating planes are different. We will see in an example below that the number of osculating planes containing such a point p can be three. One does of course not need to assume that γ is spherical to prove the claim in part (1) of Segre's theorem. It is enough to have a condition that guarantees that the vertices $t_1, \ldots, t_4$ are clean.

Part (2) of the theorem of A. Kneser and Segre also follows since any point on γ which is not a vertex lies in the interior of Δ.

Assume that the osculating planes $\Pi_1, \dots, \Pi_4$ are all different and let p be in the interior of the face $\Delta \cap \Pi_1$. Then the above argument shows that there are $s_2 \in (t_2, t_3)$ and $s_3 \in (t_3, t_4)$ such that $p \in \Pi_{s_2}$ and $p \in \Pi_{s_3}$. Since p is also in Π_1 we see that every point in the open face in $\Delta \cap \Pi_1$ lies in the osculating planes of at least three different points of γ. This improves part (3) of the theorem of Segre since the planes $\Pi_1, \dots, \Pi_4$ are all different if γ is spherical. The argument cannot be applied to points in the open 1-simplices of Δ in $\Pi_1 \cap \Pi_3$ and $\Pi_2 \cap \Pi_4$, but they contain no points on γ.

We summarize what we have proved in the next theorem.

Theorem 2.1. *Let γ be a C^2-regular convex space curve with nowhere vanishing curvature. Let Δ be the closed polyhedron containing γ that is bounded by osculating supporting planes Π_1, Π_2, Π_3 and Π_4 at $t_1 \succ t_2 \succ t_3 \succ t_4$ respectively such that Π_1 and Π_3 are maximal and Π_2 and Π_4 are minimal. Then every point in the interior of Δ is contained in osculating planes of at least four different points of γ. If the planes Π_1, Π_2, Π_3 and Π_4 are all different, then every point in the open faces of Δ lie in the osculating planes of three different points of γ.*

As we will see below, it is quite important in applications to know that the osculating hyperplanes that we find do not belong to honest vertices. Recall that t is an *honest vertex* if the following holds: Let Π_t be the osculating plane of γ at $\gamma(t)$ and s_1 and s_2 are such that $\gamma([s_1, s_2])$ is the connected component of $\gamma \cap \Pi_t$ containing $\gamma(t)$, then there is an $\varepsilon > 0$ such that $\gamma(s_1 - \varepsilon, s_1)$ and $\gamma(s_2, s_2 + \varepsilon)$ lie on the same side of Π_t. In loose terms, t is an honest vertex if γ does not cross Π_t in t.

The question is therefore whether one can improve Theorem 2.1 and show that the osculating planes whose existence is claimed do not belong to honest vertices. We will see that the answer is yes and no. Notice that Segre discusses this question in his proof without summarizing the results in the statement of his theorem.

In part (1) of the next theorem we give an easy positive answer to this question if p is contained in the interior of the convex hull of γ. The method of proof is based on Segre's observation on p. 258 in his paper on how to use the spherical curve case to get similar informations on more general curves. In part (2) we generalize Sedykh's four vertex theorem [Sd] that says that a convex simple closed space curve with nowhere vanishing curvature has at least four honest vertices

Theorem 2.2. *Let $\gamma : S^1 \to \mathbf{R}^3$ be a C^2-regular simple closed curve in $\mathbf{R}^3$ with nowhere vanishing curvature. Assume there is a point p in the interior of the convex hull of γ such that no ray starting in p intersects γ in two or more points or is tangent to γ in some point. Then*

(1) *there are four distinct points on γ, none of which is an honest vertex, whose osculating planes contain p.*
(2) *the set of honest vertices on γ has at least four connected components.*

Proof. We project γ radially from p onto a sphere $S^2(p)$ with center in p. We denote the new curve by $\tilde{\gamma}$. It follows that p lies in the interior of the convex hull of $\tilde{\gamma}$ and that an osculating plane of $\tilde{\gamma}$ that passes through p and does or does not correspond to an honest vertex of $\tilde{\gamma}$ has the same properties with respect to γ. It is therefore enough to prove the claim in (1) for $\tilde{\gamma}$. Notice that $\tilde{\gamma}$ satisfies the conditions that we assumed in Theorem 2.1 whereas γ might not. Using the same notation as above, let t_1 and t_2 be points such that t_1 is a maximum of the height function in the direction of the binormal vector at t_1 and t_2 is a minimum of the height function in direction of the binormal vector at t_2. Let Π_i be the osculating plane of $\tilde{\gamma}$ at t_i. Then $\tilde{\gamma}$ lies between the planes Π_1 and Π_2. Denote the osculating plane at t by Π_t and let D_t^+ and D_t^- denote the disjoint open halfspaces bounded by Π_t such that the binormal of $\tilde{\gamma}$ at t points into D_t^+. We define the following sets

$$A = \{t \in [t_1, t_2];\ p \notin D_t^+\}$$

and

$$B = \{t \in [t_1, t_2];\ p \notin D_t^-\}.$$

Since p is an interior point of the convex hull, we have $t_1 \in A$ and $t_2 \in B$. Moreover, we set

$$s_1 = \sup(A), \text{ and } s_1' = \inf(B \cap [s_1, t_2]).$$

If $s_1 = s_1'$, then the osculating plane Π_{s_1} of $\tilde{\gamma}$ at s_1 passes through p and s_1 is clearly not an honest vertex since no neighborhood of s_1 maps onto one side of Π_{s_1}. If $s_1 < s_1'$, then for every $t \in [s_1, s_1']$, the osculating plane at t passes through p. Hence the osculating circles of $\tilde{\gamma}|_{[s_1, s_1']}$ on $S^2(p)$ are all great circles and therefore intersect. It follows that the torsion of $\tilde{\gamma}$ vanishes for all $t \in [s_1, s_1']$ since the osculating circles of a spherical curve segment with nonvanishing torsion do not intersect. As a consequence $\tilde{\gamma}|_{[s_1, s_1']}$ is planar. It follows that $\tilde{\gamma}|_{[s_1, s_1']}$ is a great circle arc since it is contained in the osculating plane Π_{s_1}. No neighborhood of $[s_1, s_1']$ maps onto one side of Π_{s_1}. Hence we have proved that s_1 is not an honest vertex. We repeat this argument for the segments (t_2, t_3), (t_3, t_4), (t_4, t_1) and get the four points s_1, s_2, s_3, s_4 as claimed in (1).

To prove (2) notice that the spherical curve $\tilde{\gamma}$ on $S^2(p)$ changes its convexity in s_1 and s_2. We can assume that $\tilde{\gamma}|_{(s_1, s_2)}$ is a locally convex spherical curve. Then the curve $\tilde{\gamma}$ crosses its osculating plane at s_1 in the direction of its binormal vector in s_1 and it crosses its osculating plane at s_2 in the direction opposite of the binormal vector at s_2 (or the other way around). The same holds true for the curve γ. By continuity, there is a point $r_1 \in (s_1, s_2)$ where the curve γ is locally on one side of the osculating plane. This point is an honest vertex. By repeating this argument for (s_2, s_3), (s_3, s_4) and (s_4, s_1), we find four different honest vertices, which belongs to different components of the set of honest vertices. □

Remark. Theorem 2.2 has at least two well-known results as immediate applications as we now would like to explain.

Arnold's Tennis Ball Theorem [A] says that a C^2-regular simple closed curve γ on the unit sphere S^2 that divides the area of the sphere into two equal parts has at least four inflection points. Here an inflection point is a point p on the curve with the property that locally around the connected component containing p of the intersection of γ with the tangent great circle C, the curve does not lie on one side of C. Notice that the origin must lie in the interior of the convex hull of γ since we may assume that the curve is not a great circle. In fact, one can slightly generalize the Tennis Ball Theorem by assuming that the origin lies in the interior of the convex hull of γ instead of assuming that γ divides the area of S^2 into two equal parts. There are therefore at least four different points on γ that are not honest vertices whose osculating planes pass through the origin. It follows that the osculating circles on S^2 of γ at these points are great circles. Thus they are inflection points since they are not honest vertices. This application was pointed out to us by S. Tabachnikov.

A closed curve in the projective plane which is not nullhomotopic must have at least one inflection point. Here an inflection point is defined as for spherical curves by replacing 'tangent great circles' by 'tangent projective lines'. One sees this as follows: if γ does not have an inflection point, then we have a continuous normal vector field along it pointing to the side γ is curving. This is a contradiction since a closed curve that is not nullhomotopic does not admit any such normal vector field. This argument can be used to prove that the number of inflection points of such a curve must be odd, if it is finite.

Möbius proved the following theorem in [M]: If γ is a C^2-regular simple closed curve in the projective plane, then it has at least three inflection points. This follows from Theorem 2.2 by the following argument: Let us think of the projective plane as the plane $z = -1$ in $\mathbf{R}^3$ with a line added at infinity. Let $\tilde{\gamma}$ be the curve on S^2 whose points are the intersections of the lines with S^2 that connect the origin with the points on γ. There are two points on $\tilde{\gamma}$ corresponding to a point on γ. Notice that $\tilde{\gamma}$ is connected since γ is not nullhomotopic. It is clear that the origin lies in the interior of the convex hull of $\tilde{\gamma}$, since we may assume that γ is not a line. It follows that there are at least four different points on $\tilde{\gamma}$, none of which is an honest vertex, whose osculating planes pass through the origin. These four different points correspond to at least two different points in the projective plane that are inflection points. Since γ has an odd number of inflection points, there are at least three inflection points. Of course one can also deduce the Möbius theorem from the Tennis Ball Theorem.

The next theorem is a consequence of the Theorem of Möbius. The proof is similar to an argument by A. Kneser [Kn].

Theorem 2.3. *Let γ be a C^3-regular convex simple closed curve in $\mathbf{R}^3$ with nowhere vanishing curvature and let p be a point on γ. Assume that no ray starting in p meets γ in two or more points or is tangent to γ in some point. If p is not an honest vertex of γ, then there are at least three different points on γ, all of which differ from p and none of which is an honest vertex, such that the osculating planes at these points contain p. If p is an honest vertex, there are at least two such points.*

Proof. Let P be a plane supporting γ in p and let Π be the osculating plane of γ in p. The planes P and Π can only coincide if p is an honest vertex. Let L be the tangent line of γ in p. We project γ radially from p onto a sphere $S^2(p)$ with center in p. Notice that the image curve which we denote by $\tilde{\gamma}$ lies in a hemisphere of $S^2(p)$ that is bounded by the great circle $P \cap S^2(p)$ and is not closed since p corresponds to the two antipodal points in $L \cap S^2(p)$. Let $\hat{\gamma}$ be the curve in the projective plane $\mathbf{P}^2\mathbf{R}$ that we get by composing $\tilde{\gamma}$ with $\pi : S^2(p) \to \mathbf{P}^2\mathbf{R}$ where π is the identification of antipodal points.

Without loss of generality, we may assume that the point p is the origin. Then the curve $\tilde{\gamma}$ is given by

$$\tilde{\gamma}(t) = \gamma(t)/|\gamma(t)|.$$

We assume that $\gamma(0) = p$. By the Bouquet formula, it holds that

$$\gamma(t) = \mathbf{e}t + \kappa(t)\mathbf{n}\frac{t^2}{2} + o(t^2),$$

where $\mathbf{e}$ is the unit tangent vector at p, $\mathbf{n}$ is the unit normal vector at p, $\kappa(t)$ the curvature and t is the arc length parameter. So it can be easily shown that $\hat{\gamma} : S^1 \to \mathbf{P}^2\mathbf{R}$ is an immersion (at $t = 0$), if and only if $\kappa(0) \neq 0$.

Notice that $\hat{\gamma}$ is still C^3 except in the point $\hat{p}$ corresponding to p where it might only be C^2. Let λ be the line in $\mathbf{P}^2\mathbf{R}$ that is the image of $\Pi \cap S^2(p)$. Then λ is tangent to $\hat{\gamma}$ in $\hat{p}$. If p is an honest vertex, then γ does not cross Π in p. Consequently, $\hat{\gamma}$ *crosses* λ in $\hat{p}$, and we see that $\hat{p}$ is an inflection point of $\hat{\gamma}$. Similarly we see that if p is not an honest vertex, then $\hat{p}$ is not an inflection point of $\hat{\gamma}$. Let l be the line in $\mathbf{P}^2\mathbf{R}$ that is the image of $P \cap S^2(p)$. The mod 2 intersection number between $\hat{\gamma}$ and l is equal to one, since $\hat{\gamma}$ changes sides of l locally around $\hat{p}$ and only there. It follows that $\hat{\gamma}$ is not nullhomotopic. By the Theorem of Möbius, $\hat{\gamma}$ has at least three inflection points. Notice that an inflection point of $\hat{\gamma}$ different from $\hat{p}$ corresponds to a point on $\tilde{\gamma}$ with osculating circle on $S^2(p)$ being a great circle. Hence the corresponding osculating plane of γ passes through p. It follows that if p is not an honest vertex, there are at least three points all of which are different from p whose osculating planes pass through p. It is also clear that none of these points is an honest vertex. If p is an honest vertex there are at least two such points. □

We end this section by giving an example. We will need the following lemma in which we denote the closed unit ball by B_1.

Lemma 2.4. *Let $\gamma : S^1 \to S^2$ be a smooth regular spherical curve which may have self-intersections, and let x be a point in B_1. Assume there are points $p, q \in \gamma$ such that the osculating planes at p and q pass through x. Then there is at least one honest vertex on the open arc $\gamma|_{(p,q)}$.*

Proof. We divide the proof into two parts:
(*Case 1*) If x lies on the sphere S^2, then x is an intersection point between the osculating circle C_p at p and the osculating circle C_q at q. Then the assertion follows immediately from the result of A. Kneser that the osculating circles of an arc without vertices do not intersect.
(*Case 2*) Assume that x lies in the interior of B_1. Let Π_p and Π_q be the osculating planes of the curve γ at p and q respectively. We set

$$L := \Pi_p \cap \Pi_q.$$

Since $x \in \Pi_p \cap \Pi_q$, L is not empty. Thus L is a line passing through the point x. Since x is an interior point of B_1, the line L must meet S^2 in two points. Let y be one of them. Then we have $y \in \Pi_p \cap \Pi_q$ and the assertion follows from the first case. □

Remark 1. Notice that Lemma 2.4 together with the Theorem of A. Kneser and Segre immediately implies the four vertex theorem. This is very similar to A. Kneser's original proof.

Remark 2. Using the same argument, one can easily generalize the assertion of the lemma to a simple closed C^2-curve γ on a smooth convex surface with positive Gaussian curvature and a point x in the compact region bounded by the surface. (Instead of the result of Kneser, apply I-Lemma A.9 to Example 2 in I-§2.)

We give an example of a simple closed spherical curve γ with the following property: *There is a point q in the boundary of the convex hull of γ such that there are four distinct points whose osculating planes contain q, but there are only two such points which are not honest vertices.* This example seems to contradict section 9 in Segre's paper [Se], where he claims that if p is in the convex hull of γ, but not in its image, then there must be more than two points which are not honest vertices and whose osculating planes contain p. Notice that we have proved in Theorem 2.2 that if p lies in the *interior* of the convex hull of γ, then there are at least four such points.

We first prove the following:

There is a smooth simple closed curve $\gamma : S^1 \to \mathbf{R}^2$ satisfying the following two properties:

(1) *γ has three isolated clean vertices p'_1, p'_2, p'_3 and a closed arc $I := \gamma|_{[x',y']}$ consisting of clean maximal vertices.*
(2) *The curvature function of γ never vanishes except on p'_1, p'_2, p'_3 and I.*

To see this consider the functions

$$f(x) := \begin{cases} e^{-1/(|x|-1)} & (|x| \geq 1), \\ 0 & (|x| < 1). \end{cases}$$

and

$$g(x) = -x^2 + 2.$$

The graphs of these functions meet in two points that we denote by p and q, see fig. 2.

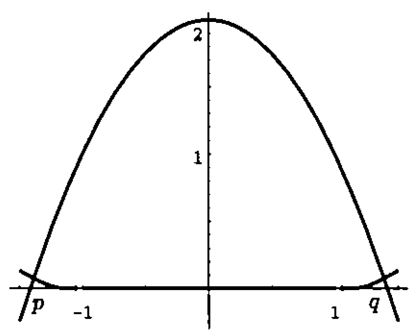

Figure 2

Denote by $\tilde{\gamma}$ the closed curve that we get by joining the graphs of f and g between p and q. Using the method of rounding corners in Proposition 2.3 in [KU], we find a curve γ' that agrees with $\tilde{\gamma}$ except close to p and q and has exactly the following vertices: one vertex close to p, another one close to q, one at the top of the graph of g and finally the whole interval $[0,1]$ on the x-axis.

Set $\gamma = \pi \circ \gamma' : S^1 \to S^2$ where $\pi : \mathbf{R}^2 \to S^2$ is the inverse of the stereographic projection and γ' is a curve with the two properties (1) and (2). Set

$$x := \pi(x'), \quad y := \pi(y'), \quad p_j := \pi(p_j') \qquad (j = 1, 2, 3).$$

Without loss of generality, we may assume that $(x \succ)p_3 \succ p_2 \succ p_1 \succ y \succ x$ holds. Then $\gamma|_{[x,y]}$ lies in a common osculating supporting plane Π_0. Let q be the midpoint of the line segment $\overline{xy}$. Then q lies in the boundary of the convex hull of γ. On $\gamma|_{(y,p_1]}$ and $\gamma|_{[p_3,x)}$, there is no point whose osculating plane passes through q. In fact, if such a point would exist, then there would be an honest vertex on $\gamma|_{(y,p_1)}$ (resp. $\gamma|_{(p_3,x)}$) by Lemma 2.4, which is a contradiction. By an intermediate argument as in the proof of Theorem 2.1, there is at least one point on $\gamma|_{(p_1,p_2)}$ (resp. $\gamma|_{(p_2,p_3)}$) whose osculating plane passes through q. We denoted it by z_1 (resp. z_2). Since $\gamma|_{(p_1,p_2)}$ and $\gamma|_{(p_2,p_3)}$ have no vertices, there is no other such point on $\gamma|_{(p_1,p_2)}$ (resp. $\gamma|_{(p_2,p_3)}$) by Lemma 2.4. Thus the points whose osculating planes pass through q are exactly z_1, z_2 and all points on $\gamma|_{[x,y]}$. Among them only z_1 and z_2 are not honest vertices.

Thus the curve γ and the point q in the boundary of its convex hull are the example we have been looking for.

§3. Extremal points of projective line diffeomorphisms.

Let $\mathbf{P}^1\mathbf{R}$ denote the real projective line. Let $f : \mathbf{P}^1\mathbf{R} \to \mathbf{P}^1\mathbf{R}$ be a diffeomorphism and $x \in \mathbf{P}^1\mathbf{R}$. Then there is a unique projective transformation $A_x : \mathbf{P}^1\mathbf{R} \to \mathbf{P}^1\mathbf{R}$ whose 2-jet at x coincides with that of f. Let us call A_x the *osculating map of f at x*. If the osculating map A_x has the same 3-jet as f in x, we call x a *projective point of f*. One owes to Ghys the beautiful theorem that any diffeomorphism $f : \mathbf{P}^1\mathbf{R} \to \mathbf{P}^1\mathbf{R}$ has at least four distinct projective points, see [OT] and [T]. In fact he proved somewhat more: f has at least four extremal points (see the definition below).

Ghys' theorem is of course reminiscent of the four vertex theorem. We will reprove it here as an application of the theory of compatible intrinsic circle systems in section 1. As a consequence we can prove a Bose type formula as well as an analogue of part (2) of the theorem of A. Kneser and Segre that was discussed in the last section.

Ghys proved his theorem by translating it into an equivalent statement about curves in the Lorentz plane. This will also be our approach. For a proof using Sturm theory see [OT] and [T]. It is interesting to notice that in one of the first papers on vertices [Kn], A. Kneser already considers curves in the Lorentz plane and arrives at a result that we will use below. More precisely, he proves that two osculating hyperbolas of a curve segment γ with no timelike tangent vectors in the Lorentz plane do not meet if the curvature of γ does not have local extrema. The corresponding statement for curves in the Euclidean plane is well-known and can be found in many elementary textbooks on differential geometry. It was also first proved by A. Kneser in the same paper.

We will always assume for simplicity that the diffeomorphism f is orientation preserving. The orientation reversing case then follows by composing f with an orientation reversing projective transformation.

As will become clear when we set up the correspondence with Lorentz geometry, the result of A. Kneser that we mentioned above implies the following fact about a diffeomorphism f: If $x \in \mathbf{P}^1\mathbf{R}$ is not a projective point of f, then the fixed point x of $f \circ A_x^{-1}$ is either an attractor (sink) or an expellor (source). Consequently, if x is neither an attractor nor an expellor of $f \circ A_x^{-1}$, then x is a projective point of f. We call a point x an *extremal point of* f, if there is a neighborhood I of the connected component F_x of the fixed point set of $f \circ A_x^{-1}$ containing x such that $f \circ A_x^{-1}$ attracts the points in I on one side of F_x towards F_x and expels those on the other side. If $f \circ A_x^{-1}$ attracts points in I on the left of F_x and expels points on the right, then we call the points in F_x *minimal points of* f, otherwise they are called *maximal points.*

Another consequence is that if x is not an extremal point, then f and A_x agree in a disconnected set. Hence a point x with the property that f and A_x only agree on an interval containing x must be an extremal point. We call such an x a *clean extremal point of* x. It is now clear what we mean by a *clean minimal* and a *clean maximal point of* f. Notice that the notions of maximal and minimal points depend on the orientation of $\mathbf{P}^1\mathbf{R}$ given by the parametrization. Changing the orientation, a minimal point becomes maximal and vice versa.

We are now in a position to state a more precise version of Ghys' theorem.

Theorem 3.1. *Let* $f : \mathbf{P}^1\mathbf{R} \to \mathbf{P}^1\mathbf{R}$ *be a diffeomorphism Then*

(1) *the number of connected components of extremal points of* f *is even if it is finite. More precisely, between any two minimal points there is a maximal point and vice versa.*
(2) f *has at least two distinct connected components of clean minimal points and at least two distinct connected components of clean maximal points.*

The proof of the theorem follows immediately when we have associated to a diffeomorphism f a pair of compatible intrinsic circle systems.

Remark. Notice that we do not claim that there are clean maximal points between two connected components of clean minimal points. This is not expected to be true, see a counterexample in the similar case of planar curves in section 1, figure 1.

Before we associate a pair of compatible intrinsic circle systems to a diffeomorphism f of $\mathbf{P}^1\mathbf{R}$, we explain how it relates to planar Lorentz geometry.

By composing f with a rotation if necessary, we can assume that f has a fixed point. We choose the fixed point as the point at infinity and restrict f to the reals, $\mathbf{R} = \mathbf{P}^1\mathbf{R} - \{\infty\}$, keeping the notation f for the restriction. We are therefore in the situation of a surjective strictly increasing function $f : \mathbf{R} \to \mathbf{R}$ with a nowhere vanishing positive derivative. The orientation preserving projective transformations of $\mathbf{P}^1\mathbf{R}$ correspond to the linear fractional transformations

$$P(x) = \frac{\alpha x + \beta}{\gamma x + \delta}$$

with $\alpha\delta - \beta\gamma = 1$. Their graphs are the nonvertical lines with positive inclination if they have ∞ for a fixed point (i.e., $\gamma = 0$), otherwise their graphs are hyperbolas of the type

$$(x-a)(y-b)=-c^2$$

with $c \neq 0$. Notice that a hyperbola satisfying the above equation is asymptotic to the perpendicular lines $x = a$ and $y = b$ and corresponds to a linear fractional transformation sending a to ∞. The branches of the hyperbola lie in the second and the fourth quadrant of $\mathbf{R}^2$ with $x = a$ and $x = b$ as axes. The osculating map of f at a point x_0 corresponds to such a line or hyperbola that has second order contact with the graph of f in (x_0, y_0) where $y_0 = f(x_0)$. To simplify the exposition we call the lines and hyperbolas of the above type *admissible hyperbolas* or only *hyperbolas.* Notice that a point x is an extremal point of f if and only if the osculating hyperbola at x is locally around the connected component containing x of the set where it coincides with the graph of f on one side of the graph of f. It is a minimal point if the osculating hyperbola is locally below the graph, otherwise it is a maximal point.

It is now easy to prove the claim in part (1) of Theorem 3.1. Let x and y be minimum points and assume that there are no extremal points between them. Then the osculating hyperbolas between x and y do not intersect by the result of A. Kneser in [Kn] that we have already mentioned. It follows that the graph of f cannot be locally above the osculating hyperbolas both in x and y. Hence there is a point z between x and y in which the graph must lie locally below the osculating hyperbola.

We now begin with the definition of the intrinsic circle systems. We choose a point (a, b) on the graph of f. The branches of the admissible hyperbolas tangent to the graph of f in (a, b) fill up the first and the third quadrant of the complement of $x = a$ and $y = b$. Furthermore, they only meet in the point (a, b). Notice that we have a one parameter family of these hyperbolas. We assume the parameter to go from minus to plus infinity. Being a diffeomorphism, f only meets the lines $x = a$ and $y = b$ in the point (a, b). We can therefore find admissible hyperbolas that are tangent to the graph of f in (a, b), lying above it and not meeting it any other point. Let $\mathcal{H}$ be the set of all such admissible hyperbolas. Then $\mathcal{H}$ either corresponds to an open or closed half line of the parameter. Denote the hyperbola that corresponds to the endpoint of $\mathcal{H}$ by H_a.

If H_a is not a line, we define the set

$$F_a^\bullet = \{x \in \mathbf{R} \mid (x, f(x)) \in H_a\}.$$

If H_a is a line, then we set

$$F_a^\bullet = \{x \in \mathbf{R} \mid (x, f(x)) \in H_a\} \cup \{\infty\}.$$

The set F_a° is defined analogously using hyperbolas that lie below the graph of f. The sets $F_a^\bullet$ and F_a° are clearly independent of the choice of the point at infinity. It follows that we can associate the sets $F_a^\bullet$ and F_a° to all points $a \in \mathbf{P}^1\mathbf{R}$, including the one that was chosen to be at infinity.

A more intrinsic way to define $F_a^\bullet$ is as follows. Let $\mathcal{P}$ denote the one-parameter family of projective transformations whose 1-jets at $a \in \mathbf{P}^1\mathbf{R}$ agree with the 1-jet of f at a. We assume that this family is parameterized by the real numbers. We consider $f \circ P_t^{-1}$ for $P_t \in \mathcal{P}$. It follows from the considerations above that there are numbers

$t_0 \leq t_1$ such that if $t \notin [t_0, t_1]$, then $f \circ P_t^{-1}$ only has a fixed point in a. We assume the parameter chosen so that $f \circ P_t^{-1}$ moves points locally on the left of a away and brings those on the right closer, if $t < t_0$. Assume that the interval $[t_0, t_1]$ is the smallest possible with the above property. Then $F_a^\bullet$ is the fixed point set of $f \circ P_{t_0}^{-1}$ and F_a° is the fixed point set of $f \circ P_{t_1}^{-1}$.

Theorem 3.2. *Let $f : \mathbf{P}^1\mathbf{R} \to \mathbf{P}^1\mathbf{R}$ be a diffeomorphism that is not a projective transformation. Then $(F_a^\bullet)_{a \in \mathbf{P}^1\mathbf{R}}$ and $(F_a^\circ)_{a \in \mathbf{P}^1\mathbf{R}}$ are compatible intrinsic circle systems.*

Proof. We first prove that $(F_a^\bullet)_{a \in \mathbf{P}^1\mathbf{R}}$ is an intrinsic circle system. Let a, b be such that $b \in F_a^\bullet$. We can assume that $a, b \in \mathbf{R}$. Then $b \in H_a$. It clearly follows that $H_b = H_a$ and hence that $F_b^\bullet = F_a^\bullet$. Hence $(F_a^\bullet)_{a \in \mathbf{P}^1\mathbf{R}}$ satisfies (I1). Property (I2) follows from the fact that two distinct admissible hyperbolas intersect at most in two points, or equivalently that two distinct projective transformations agree in at most two points. Property (I3) is clear. We have thus proved that $(F_a^\bullet)_{a \in \mathbf{P}^1\mathbf{R}}$ is an intrinsic circle system. The proof that $(F_a^\circ)_{a \in \mathbf{P}^1\mathbf{R}}$ is an intrinsic circle system is completely analogous.

We now prove that the intrinsic circle systems are compatible. Property (C1) is immediate since f is not a projective transformation. Assume that $\text{rank}^\bullet(a) = 1$ and $a \in \mathbf{R}$. Then $\text{rank}^\circ(a) > 1$ since f is not a projective transformation. Also a is isolated in $F^\circ(a)$ for the same reason. We have thus proved (C2). This completes the proof. □

In section 1 we associated two equivalence relations $\sim_1$ and $\sim_2$ to an intrinsic circle system. We denote the quotient spaces of $\mathbf{P}^1\mathbf{R}$ with respect to these relations by $\mathbf{P}^1\mathbf{R}/F^\bullet$ and $\mathbf{P}^1\mathbf{R}/F^\circ$ respectively. The sets $S(F^\bullet)$, $S(F^\circ)$, $T(F^\bullet)$ and $T(F^\circ)$ were defined in section 1 as well as the numbers $s(F^\bullet)$, $s(F^\circ)$, $t(F^\bullet)$ and $t(F^\circ)$ associated to them. Notice that $s(F^\bullet)$ (resp. $s(F^\circ)$) is the number of connected components of the set of clean maximal (resp. minimal) points of f. Hence the number $s(f)$ of connected components of clean extremal points of f equals $s(F^\bullet) + s(F^\circ)$.

We would like to interpret the meaning of the sets $T(F^\bullet)$ and $T(F^\circ)$ in terms of properties of the diffeomorphism f. The equivalence class of a is in $T(F^\bullet)$ if and only if there is a projective transformation P such that the fixed point set of the mapping $f \circ P^{-1}$ has at least three connected components, one of them containing a, and $f \circ P^{-1}$ moves all points in the complement of the fixed point set of $f \circ P^{-1}$ against the orientation of $\mathbf{P}^1\mathbf{R}$. Similarly, the equivalence class of a lies in $T(F^\circ)$ if and only if there is a projective transformation P such that the fixed point set of the mapping $f \circ P^{-1}$ has at least three connected components, one of them containing a, and $f \circ P^{-1}$ moves all points in the complement of the fixed point set of $f \circ P^{-1}$ with the orientation of $\mathbf{P}^1\mathbf{R}$. We set $t(f) = t(F^\bullet) + t(F^\circ)$. It follows from section 1 that $t(f)$ is finite if $s(f)$ is finite.

The following Bose type formula follows from section 1. Notice that it has part (2) of Theorem 3.1 as in immediate corollary.

Theorem 3.3. *Let $f : \mathbf{P}^1\mathbf{R} \to \mathbf{P}^1\mathbf{R}$ be a diffeomorphism that is not a projective transformation and assume that $s(f)$ is finite. Then $s(f) - t(f) = 4$.*

Remark. It follows from Lemma 1.2 in [OT] that the sets $S(F^{\bullet})$ and $S(F^{\circ})$ are supported by a certain continuous function. So by (i) of Fact 1.3, $s(f) < \infty$ if and only if $t(f) < \infty$.

A further consequence of section 1 is the following theorem.

Theorem 3.4. *Let $f : \mathbf{P}^1\mathbf{R} \to \mathbf{P}^1\mathbf{R}$ be a diffeomorphism that is not a projective transformation. Then there are four points $a_1 \succ a_2 \succ a_3 \succ a_4$ on $\mathbf{P}^1\mathbf{R}$ such that a_1 and a_3 are clean maximal points of f and a_2 and a_4 are clean minimal points of f.*

We would now like to prove as a corollary of Theorem 3.4 a result that is similar to part (2) of the Theorem of A. Kneser and Segre that we quoted in section 2: If p is a point on a simple closed curve γ on the unit sphere S^2 that is not a vertex, there are at least three distinct points p_1, p_2, p_3 on γ, all of which are different from p, with the property that the osculating planes at p_1, p_2, and p_3 pass through p. If p is a vertex, one can find two such points.

Theorem 3.5. *Let $f : \mathbf{P}^1\mathbf{R} \to \mathbf{P}^1\mathbf{R}$ be a diffeomorphism and $a \in \mathbf{P}^1\mathbf{R}$. If a is not a clean extremal point of f, then there are three distinct points b_1, b_2 and b_3, all different from a and non of which is an extremal point, such that the osculating maps of f at b_1, b_2 and b_3 all agree with f in a. If a is a clean extremal point, then we can find at least two such points b_1 and b_2.*

Proof. The idea of the proof is exactly the same as the one we used to prove Theorem 2.1. Assume that a is not a clean extremal point. By Theorem 3.4 there are four points $a_1 \succ a_2 \succ a_3 \succ a_4$ on $\mathbf{P}^1\mathbf{R}$ such that a_1 and a_3 are clean maximal points of f and a_2 and a_4 are clean minimal points of f. We choose the point at infinity between a_4 and a_1 and assume that it is different from a. We also assume that a lies between a_4 and a_1. The the graph of f lies between its osculating hyperbola at a_1 and its osculating hyperbola at a_2. As we move from a_1 to a_2 we must by continuity go through a point b_1 such that the osculating hyperbola of the graph of f at b_1 passes through the point $(a, f(a))$. We can use exactly the same argument as in the proof of Theorem 2.2 to show that b_1 can be chosen such that it is not an extremal point of f. We apply exactly the same argument to the intervals (a_2, a_3) and (a_3, a_4) to find the points b_2 and b_3. The case that a is a clean maximal point is similar. This proves the theorem. □

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