Clifford analysis for Dirac operators on manifolds with boundary

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Abstract

Dirac operators are well known to provide an elegant generalisation of complex analysis both to domains in higher dimensional Euclidean space (Clifford analysis) and to closed manifolds (spin geometry). This paper is concerned with the meeting point of these areas: Dirac operators on manifolds with boundary. The aim is to demonstrate that many of the ideas from function theory in the plane have natural analogues on Riemannian (or conformal spin) manifolds by providing, as far as possible, elementary proofs of the main analytical results about the boundary behaviour of Dirac operators. Emphasised throughout are the conformally invariant aspects of the theory, and also the usefulness of the Clifford algebra formalism. A number of classical results from complex analysis, and their counterparts in Clifford analysis, are extended to Dirac operators on manifolds, including the Cauchy integral formula, the Plemelj formula, the Kerzman-Stein formula, and the L^2-boundedness of the Cauchy and Hilbert transforms.

Finally, the null space of the Dirac operator on a conformal spin manifold is shown to define a conformally invariant Hilbert space of boundary values, such that the norm of the pointwise evaluation of solutions on the interior gives rise to a conformally invariant metric which is complete and has negative scalar curvature.

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INTRODUCTION

The idea of finding first order systems of equations which factor a second order linear differential operator is certainly a venerable one, but over the past thirty years or so, a number of different threads have been pulling together around what is now widely known as a Dirac operator. Although it is possible to define Dirac operators quite directly, a deeper understanding is acquired when these interrelated threads are brought into play. Relevant ideas include: Clifford algebras, which provide the right setting and language for Dirac operators just as the complex numbers do for the Cauchy-Riemann operator; harmonic analysis, which expresses the relationship, with its analytical ramifications, of Dirac operators to the representation theory of the spin group; and conformal differential geometry, which enters the picture as soon as one observes that the Dirac operator is conformally invariant, and provides a context for studying Dirac operators in a manifestly invariant way.

The Clifford algebraic, function theoretic, and harmonic analytical aspects of Dirac operators have been brought together very fruitfully through the work of many people within the rapidly developing field of Clifford analysis [12, 19, 28, 37, 46]. Similarly, the use of Dirac operators in differential geometry is widespread in areas as diverse as Riemannian geometry, index theory, noncommutative geometry, general relativity and elliptic cohomology [1, 5, 7, 11, 35, 43, 50]. At present, however, the Clifford analysis and differential geometry of Dirac operators are developed largely along separate lines. Yet these lines run very close at times: in Clifford analysis, certain Dirac operators on submanifolds of $\mathbb{R}^n$ are being studied [37, 47, 48, 51, 52] and the geometry of Dirac operators is playing an increasingly important role [19, 44, 47], while in differential geometry, knowledge of hard analytical properties of Dirac operators can be invaluable [7, 26, 43]. Hence it seems worthwhile to build more bridges between these areas—this paper is intended as a step towards that end. In [24], Gilbert and Murray described the analysis of Dirac operators both on domains in $\mathbb{R}^n$, as in Clifford analysis, and also on compact boundaryless manifolds, the usual setting in differential geometry. The focus here will be on the most obvious meeting point: compact manifolds with boundary. The aim is to present a thorough treatment of the analysis of Dirac operators on such manifolds, with particular reference to conformal invariance and also to the potency of Clifford algebra as a language in which to express the results.

Much of the analysis studied here is already known in the context of elliptic pseudodifferential operators—for example, some of the results are special cases of those of Seeley [49]. However, the theory of pseudodifferential operators on manifolds with boundary is more complicated than on closed manifolds, and the material is often too technical for a wide audience. In [11], Booß and Wojciechowski observe that many simplifications can be made when one restricts attention to Dirac operators. Nonetheless, for some crucial steps in their approach, they follow the technical computations of Seeley.

Here I wish to show that a function theoretic point of view provides an alternative, more elementary approach to the analysis of Dirac operators on manifolds with boundary. Indeed, using only integral Sobolev spaces, and no pseudodifferential operators or Fourier analysis, proofs of the main analytical results are given. As in Clifford analysis, one pleasant aspect of these proofs, is that the arguments are recognisable *even in some of the details* as generalisations of complex analytical methods. Consequently, many of the results obtained are direct analogues of classical theorems in complex analysis, and so I shall refer to them by their classical names. The results established, for arbitrary Dirac operators on arbitrary (Riemannian or conformal spin) manifolds with boundary, include Cauchy's theorem, the Cauchy integral formula, the Pompeiu representation formula, the Plemelj formula and the L^2-boundedness of the Cauchy and Hilbert transforms. To obtain such generalisations, complex and Clifford analytical techniques must be supplemented by potent tools, such as the Bochner-Weitzenböck formula, and formulated in a geometric context. This context, I believe, sheds light even upon the two dimensional results. Furthermore, I claim that these methods are not only illuminating, but also useful, in that they provide tools which are easy to apply. To illustrate this, some function theoretic aspects of boundary problems for Dirac operators are developed, and an application in conformal geometry is presented.

Since this paper is aimed at several audiences, I have tried to keep it reasonably self-contained, which partly accounts for its length. There is consequently a certain amount of well-established material. To Clifford analysts, I am labouring a familiar point when I emphasise that Dirac operators are a generalisation of complex analysis, while to other analysts, the avoidance of powerful pseudodifferential operator methods may seem perverse. Also, the differential geometer will find herein yet another summary of the elliptic theory of Dirac operators. I crave the indulgence of all these readers. I should also remark that it has been necessary at times to choose between conflicting notation and terminology; I have tended to use geometrically invariant notation, but occasionally adopt analytical language and conventions.

In the first two sections I briefly review the algebraic material used throughout, before presenting, in sections 3–6, the analytical tools and elliptic theory of Dirac operators on closed manifolds. Here I follow [7, 24, 35, 45], although Bochner-Weitzenböck integral formulae are established for a wider class of Dirac operators than is usual. In section 4, I also recall (essentially from [32]) the important fact that the Dirac operator associated to a spin structure is conformally invariant, in the sense that it is defined intrinsically on any conformal spin manifold. Most of the formulae obtained in later sections are explicitly conformally invariant in this case.

In section 7, I present the generalisation of the Cauchy integral formula to manifolds with boundary, the highlight being the analogue of the Pompeiu representation formula. Applications of this Cauchy integral to mean value inequalities, removable singularities and residues are then discussed in section 8.

The heart of the paper lies in section 9, where the Hardy space H of L^2 boundary data (the boundary values of sections in the null space of the Dirac operator) is introduced. As observed in [11], much of the analysis of Dirac operators follows from a twisted orthogonality property of this boundary data, and the aim here is to prove this property. This is done by using the Cauchy transform and a generalisation of the Kerzman-Stein formula [34] to establish L^2-boundedness results directly, thus bypassing a lot of the technical analytical theory of elliptic boundary problems. In fact, the line of proof in this section is based quite closely upon Bell's monograph [6] on the Cauchy transform in the plane. In section 10, further analytical aspects of Dirac operators on manifolds with boundary are discussed, such as the Bergman kernel and the Dirichlet problem for the square of the Dirac operator.

The final section is devoted to an application of these tools in conformal geometry, which was in fact the original motivation for much of the work presented here. The key point is that, for the conformally invariant Dirac operator, the L^2-norm on the Hardy space H depends only on the conformal structure, and so a canonical norm is obtained 'for free' on a space of spinors with well defined interior values. This norm then trivialises the density bundle. In other words, given a conformal structure on a compact spin manifold with boundary, the Cauchy integral defines a conformally invariant metric on the interior. The metric is complete with negative scalar curvature, and generalises the Poincaré metric on the unit ball. Such a result was first obtained, in the Euclidean case, by Hitchin [33], who observed that the Cauchy integral (as found in Gay and Littlewood [23]) is conformally invariant and bounded. His results on the completeness and scalar curvature of this metric generalise readily to conformal spin manifolds, except that the negativity (rather than just nonpositivity) of the scalar curvature relies on an integrability result for the Dirac equation whose proof uses the full machinery developed here.

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I. ALGEBRAIC PRELIMINARIES

1 Clifford algebras

I work throughout with the algebras introduced by W. K. Clifford [17, 18] and H. Grassmann [27]. There are many ways to define these Clifford algebras; I will use the following:

1.1 Definition. Let V be a linear space. Then a *Clifford algebra* for V is an extension of $\mathbb{R} \oplus V$ to an associative algebra A with identity $1 \in \mathbb{R}$ such that

(i) A is generated (as a ring) by $\mathbb{R} \oplus V$

(ii) $v^2 \in \mathbb{R}$ for all $v \in V$.

It follows that $v \mapsto v^2$ defines a quadratic form on V. I will say that the Clifford algebra is *nondegenerate* or *positive/negative definite* iff this quadratic form is. If q is any quadratic form on V, then A will be called a Clifford algebra *for* (V, q) if $v^2 = q(v)$.

REMARK. Clifford algebras are also sometimes defined by the relation $v^2 = -q(v)$. This may seem a trivial difference; however in Euclidean and Riemannian geometry it is usual to work with positive definite quadratic forms, and so this extra minus sign leads to a negative definite Clifford algebra. The focus herein will be on the positive definite case.

It is not immediately clear from the above definition that a Clifford algebra for (V, q) always exists, so I will briefly recall a couple of constructions. A *Clifford map* on (V, q) is defined to be a linear map ι from V to an associative algebra A such that $\iota(v)^2 = q(v)1$. (If ι is injective then the subalgebra of A generated by the image is a Clifford algebra.) There is a standard algebraic construction of a universal Clifford map $\pi\colon V \to Cl(V, q)$.

1.2 Definition. Define $Cl(V,q) := \bigotimes V / \langle v \otimes v - q(v)1 \rangle$, the quotient of the tensor algebra $\bigotimes V$ by the relation $v \otimes v = q(v)1$, and let π be induced by the inclusion $V \hookrightarrow \bigotimes V$.

This is clearly a Clifford map; the universal property is a consequence of the following:

1.3 Proposition. *Let (V, q) and (W, r) be quadratic spaces, $\iota\colon W \to A$ a Clifford map, and $T\colon V \to W$ an isometry (that is, $r(Tv) = q(v)$ for all $v \in V$). Then there is a unique algebra homomorphism $T_*\colon Cl(V, q) \to A$ extending T, in the sense that $T_* \circ \pi = \iota \circ T$. Note also that if A is a Clifford algebra and T is surjective, then so is T_*.*

Proof: $\iota \circ T\colon V \to A$ is a linear map from V into an associative algebra, and so by the universal property of the tensor algebra, there is a unique extension of $\iota \circ T$ to an algebra homomorphism $T_*\colon \bigotimes V \to A$. Since T is an isometry and ι a Clifford map, $T_*(v \otimes v) = \iota(T(v))^2 = r(Tv) = q(v) = T_*(q(v)1)$, so T_* descends to the quotient $Cl(V, q)$. □

Similarly there is a unique algebra *anti*homomorphism from $Cl(V, q)$ to A extending T.

If there exists a Clifford algebra A for (V, q), then, taking T to be the identity map in the above proposition, it immediately follows that $Cl(V, q)$ is also a Clifford algebra. One way to obtain existence is as follows (see for example [7, 24]). Let $\Lambda(V)$ be exterior algebra of V and let $A(V, q)$ be the subalgebra of $\operatorname{End}\Lambda(V)$ generated by $\{c(v) = \varepsilon_v + \iota_v : v \in V\}$, where $\varepsilon_v(x) = v \wedge x$ and ι_v is contraction by v (with respect to q).

1.4 Proposition. *$A(V, q)$ is a Clifford algebra for (V, q).*

Proof: First note that $c(v)(1) = v$, so $\mathbb{R} \oplus V$ embeds into and generates $A(V, q)$. Now $\varepsilon_v^2 = 0$, $\iota_v^2 = 0$ and $\iota_v\varepsilon_v = -\varepsilon_v\iota_v + q(v)$, by an easy computation, and so $c(v)^2 = q(v)$. □

1.5 Proposition. *For finite dimensional V, the evaluation map ev_1 (at $1 \in \mathbb{R}$) is a linear isomorphism from $A(V,q)$ to $\Lambda(V)$. There is also a natural algebra isomorphism between $A(V,q)$ and $Cl(V,q)$, and a basis is given by $\mathcal{S} = \{e_1^{m_1} \dots e_n^{m_n} : m_j = 0, 1\}$, where $e_1, \dots e_n$ is any orthogonal basis of V.*

Proof: A simple inductive argument shows that ev_1 is surjective, since $\operatorname{im} ev_1$ contains $\mathbb{R} = \Lambda^0(V)$, and the highest degree part of $ev_1(c(v_1)\dots c(v_k))$ is $v_1 \wedge \dots \wedge v_k$. Also, by 1.3, there is a surjective algebra homomorphism from $Cl(V,q)$ to $A(V,q)$. Since $e_j e_k = -e_k e_j$ for all $j \neq k$, $\mathcal{S}$ is a spanning set for any Clifford algebra, and so $2^n \geqslant \dim Cl(V,q) \geqslant \dim A(V,q) \geqslant \dim \Lambda(V) = 2^n$. Hence equality holds all the way through and the surjective linear maps are all bijective. □

Henceforth, $A(V,q)$ will be identified with $Cl(V,q)$ and called *the* Clifford algebra of (V,q). Its elements are sometimes called *Clifford numbers* or *multivectors.* $Cl(V,q)$ is a graded algebra: it may be written as a direct sum $Cl(V,q) = Cl(V,q)^{ev} \oplus Cl(V,q)^{od}$, with $Cl(V,q)^{ev}$ a subalgebra, the *even subalgebra.* Frequent use will be made of the decomposition $vw = \langle v\,, w\rangle + v \wedge w$ of a product of vectors into its symmetric and skew parts, where $\langle .\,,.\rangle$ denotes the induced inner product on V.

The Clifford algebra has several involutions, the most important being the *chirality, grading, twisting* or *principal* automorphism $x \mapsto x^*$ induced by the isometry $v \mapsto -v$; its fixed point set is the even subalgebra. The antiautomorphism $x \mapsto \tilde{x}$ induced by the identity on V maps $v_1 \dots v_k$ to $v_k \dots v_1$ and so is called *reversion.*

From now on, only the nondegenerate Clifford algebras $Cl_{p,m}$ or Cl_n will be considered; here (p,m) is the signature of the inner product on V, and $Cl_n = Cl_{n,0}$.

2 Spin groups and Clifford modules

2.1 Definitions. Let $Cl^*_{p,m}$ be the Lie group of invertible elements of $Cl_{p,m}$ and let $\mathfrak{cl}^*_{p,m}$ be its Lie algebra (which is $Cl_{p,m}$ with bracket $[x,y] = xy - yx$). The *adjoint action* $\operatorname{Ad}\colon Cl^*_{p,m} \to \operatorname{Aut}(Cl_{p,m})$ is given by $\operatorname{Ad}_x(y) = xyx^{-1}$, but if x is odd, it is often useful to incorporate the grading of $Cl_{p,m}$ and define the *twisted adjoint action* $\operatorname{Ad}^*\colon Cl^{ev*}_{p,m} \cup Cl^{od*}_{p,m} \to \operatorname{Aut}(Cl_{p,m})$ by $\operatorname{Ad}^*_x = \operatorname{Ad}_x$ for x even, but $\operatorname{Ad}^*_x(y) = xy^*x^{-1}$ for x odd.

2.2 Proposition. *For $x \in Cl_{p,m}$ the following hold:*

(i) if $xv = vx^*$ $\forall v \in V$ then $x \in \mathbb{R}$.

(ii) if x is invertible and $\forall v \in V$ $xv(x^{-1})^* \in V$, then $v \mapsto xv(x^{-1})^*$ is an isometry of V.

(iii) if x is a non-null vector then $v \mapsto -xvx^{-1}$ is a reflection in the hyperplane $x^\perp \leqslant V$.

This is straightforward, as is the next proposition, which is a consequence of the definitions below and the fact that any isometry can be written as a composite of reflections.

2.3 Definition. The *Clifford semigroup* $\Lambda_{p,m}$ consists of those elements of $Cl_{p,m}$ which can be written as a product of vectors, the *Clifford group* $\Gamma_{p,m}$ being the invertible elements (with $\tilde{x}x \neq 0$). Define the *Pin, Spin* and *Spin$_+$ groups* by:

$$\begin{aligned} \operatorname{Pin}(p,m) &= \{x \in \Lambda_{p,m} : \tilde{x}x = \pm 1\} \\ \operatorname{Spin}(p,m) &= \{x \in \Lambda^{ev}_{p,m} : \tilde{x}x = \pm 1\} \\ \operatorname{Spin}_+(p,m) &= \{x \in \Lambda^{ev}_{p,m} : \tilde{x}x = 1\}. \end{aligned}$$

2.4 Proposition. Ad^* *defines an action of* $\Gamma_{p,m}$ *on* V *by isometries and on* $Cl_{p,m}$ *by automorphisms. The homomorphism from* $\Gamma_{p,m}$ *to the group of isometries of* V *is surjective with kernel* $\mathbb{R}^*$ *and restricts to a two-fold cover of* $\mathrm{O}(p,m)$ *by* $\mathrm{Pin}(p,m)$, $\mathrm{SO}(p,m)$ *by* $\mathrm{Spin}(p,m)$ *and* $\mathrm{SO}_+(p,m)$ *by* $\mathrm{Spin}_+(p,m)$.

The Lie algebra of these double covering groups will be denoted $\mathfrak{spin}(p,m)$. It consists of the bivectors in $Cl_{p,m} = \mathfrak{cl}^*_{p,m}$, and the Lie algebra map ad: $\mathfrak{spin}(p,m) \to \mathfrak{so}(p,m)$ is an isomorphism. More precisely:

2.5 Proposition. *Given* $x, y \in V$, *define a skew endomorphism* $x\Delta y$ *of* V *by* $x\Delta y(w) = \langle x\,,w\rangle y - \langle y\,,w\rangle x$. *Then* $\mathrm{ad}^{-1}(x\Delta y) = -\frac{1}{2}x\wedge y = -\frac{1}{4}(xy - yx)$.

Proof: The action of $a \in \mathrm{Spin}(p,m)$ on V is given by $v \mapsto ava^{-1}$, and so the action of $\xi \in \mathfrak{spin}(p,m)$, obtained by differentiating, is $v \mapsto \xi v - v\xi$. After substituting $\xi = xy - yx$, a simple computation using the Clifford relation establishes the result. □

The convention $v^2 = -\langle v\,, v\rangle$ would give the opposite sign in this formula.

2.6 Definition. A *Clifford module* for $Cl_{p,m}$ is a vector space $\mathbb{E}$ on which $Cl_{p,m}$ acts as an algebra; that is, an algebra homomorphism $Cl_{p,m} \to \mathrm{End}(\mathbb{E})$ is given. Elements of a Clifford module are often called *spinors.* A Clifford module is said to be *graded* if it has a direct sum decomposition $\mathbb{E} = \mathbb{E}^- \oplus \mathbb{E}^+$ preserved by $Cl^{ev}_{p,m}$, and such that $Cl^{od}_{p,m}$ exchanges the summands. By restriction, any Clifford module (and also either component of a graded Clifford module) is a representation of $\mathrm{Spin}(p,m)$. Such a representation will be called a *spin representation.*

Often $\mathbb{E}$ is equipped with an inner product such that Clifford multiplication by vectors is either symmetric or skew. In the symmetric case, the inner product can only be definite if the Clifford algebra is positive definite, whereas in the skew case, the Clifford algebra must be negative definite. This is the main difference between the positive and negative definite Clifford algebras, and has the consequence that Dirac operators are skew-adjoint in the positive definite case, and self-adjoint in the negative definite case.

The most natural example of a Clifford module is $Cl_{p,m}$ acting on itself by left multiplication. Since $Cl_{p,m}$ is $\Lambda(V)$ as a vector space, this may also be viewed as the natural action of $Cl_{p,m}$ on $\Lambda(V)$. An inner product on $Cl_{p,m}$ is given by $\langle x\,,y\rangle = \langle \tilde{x}y\rangle$, where $\langle . \rangle$ denotes the scalar part, although it is also of interest to work with the inner product $\tilde{x}y$ taking values in $Cl_{p,m}$ (see [12]). For $v \in V$, $\langle vx\,,y\rangle = \langle \tilde{x}vy\rangle = \langle x\,,vy\rangle$, so vectors are symmetric, and for the positive definite algebra Cl_n, this inner product is positive definite.

The case of irreducible (graded) Clifford modules is also of some importance. The corresponding irreducible spin representations are often simply called *the* spin representations. In fact there are, up to isomorphism, only two such representations in even dimensions, and one in odd dimensions. Other Clifford modules must decompose into a direct sum of these, although the decomposition is not canonically defined. For further details see [3, 16, 30, 35].

3 Dirac operators and Bochner-Weitzenböck formulae

Let M be a (semi)Riemannian manifold. Then the Clifford algebra bundle $Cl(M)$ is the vector bundle whose fibre at $x \in M$ is the Clifford algebra $Cl(T_xM)$. Using the metric this is isomorphic to $Cl(T_x^*M)$ and hence, as a vector space, it is isomorphic to ΛT_x^*M.

Now suppose E is a Clifford module bundle on M, with covariant derivative D^E. Then for each $x \in M$ there is a Clifford action $c\colon T_x^*M \otimes E_x \to E_x$, written $c(\alpha \otimes s) = c(\alpha)s$.

3.1 Definition. The (generalised) *Dirac operator* associated to (E, D^E) is the differential operator $\nabla\!\!\!\!/ = c \circ D^E\colon C^\infty(M,E) \to C^\infty(M,E)$. A section in the kernel of a Dirac operator will be called *monogenic*. (Other terms in common use are *Clifford analytic functions* and *harmonic spinors*—however, for general Dirac operators, monogenic sections may not be analytic, and on nonclosed manifolds, the kernels of $\nabla\!\!\!\!/$ and $\nabla\!\!\!\!/^2$ no longer agree.)

3.2 Historical remarks. This definition has a long and complicated history. After the Cauchy-Riemann operator, the first Dirac operator to be introduced was the quaternionic ∇ operator of Hamilton and Tait (a Dirac operator in 3 dimensions). In a remarkable paper [21], Dixon studied "Hamiltonian functions" and gave an analogue of Cauchy's integral formula for Hamilton's operator. The Dirac operator in $(3,1)$-dimensional space-time was introduced by Dirac [20], and here the spinor transformation law was also identified. The elliptic analogue in higher dimensions was described in Moisil [41], while Brauer and Weyl [13] gave the general setting for the Dirac construction. Quaternionic function theory was explored by Fueter and his school in the thirties, and later they extended their methods to higher dimensions (see [29]). In the sixties Dirac operators were studied more intensively, when they were rediscovered by Delanghe, Gay and Littlewood, Hestenes, Iftimie, and Stein and Weiss—see [12] or [46] for a thorough bibliography. Around the same time, the Dirac operator began to play an important role in differential geometry through the work of Atiyah and Singer [5], and Lichnerowicz [38].

3.3 Examples. A basic way to obtain examples of Dirac operators is from the representation theory of the spin group. More precisely, let $\mathbb{E}$ be simultaneously a Clifford module for $Cl_{p,m}$ and a representation of $\mathrm{Spin}(p,m)$, such that the actions c and . are compatible, in the sense that $c(axa^{-1})a.\psi = a.c(x)\psi$ for all $a \in \mathrm{Spin}(p,m)$, $x \in Cl_{p,m}$ and $\psi \in \mathbb{E}$. Let M be a Riemannian manifold equipped, if necessary, with a spin structure (see section 4—this is only needed if the representation of $\mathrm{Spin}(p,m)$ on $\mathbb{E}$ does not descend to $\mathrm{SO}(p,m)$). Then $\mathbb{E}$ gives rise, via the associated bundle construction, to a Clifford module bundle E with a covariant derivative induced by the Levi-Civita connection. The most important cases of this are as follows.

(i) If $\mathrm{Spin}(p,m)$ acts on $Cl_{p,m}$ by conjugation, then E is the bundle $\Lambda T^*M \cong Cl(M)$, and the Clifford module structure is given by the action of the Clifford algebra bundle on itself by left multiplication. The induced Dirac operator on E is then the Hodge-Kähler $d + \delta$ operator, where d is the exterior derivative, and $\delta = -d^*$ is the exterior divergence.

(ii) If $\mathbb{E}$ is any Clifford module for $Cl_{p,m}$, then by restriction it is a spin representation of $\mathrm{Spin}(p,m)$—a simple example is the action of $\mathrm{Spin}(p,m)$ on $Cl_{p,m}$ by left multiplication. Now on any Riemannian spin manifold, $\mathbb{E}$ induces a *spinor bundle* E, and a special Dirac operator, the Atiyah-Singer or spinor Dirac operator, on E. This operator will simply be referred to as *the* Dirac operator on M (associated to $\mathbb{E}$). Its significance, over more general Dirac operators, is that it is conformally invariant in a very interesting way—see section 4.

In practice Dirac operators are often "chiral" in the sense that $E = E^- \oplus E^+$ and the Dirac operator is given by $\not{\nabla}^{\pm}\colon C^\infty(M, E^\mp) \to C^\infty(M, E^\pm)$; in other words, the Clifford module bundle is graded and the Dirac operator is odd. This is clearly the case in the first example above, and in any such example if $\mathbb{E}$ is a graded Clifford module. Even if E is ungraded, the notation $E^\pm = E$ provides a useful way of differentiating between the domain and codomain of a Dirac operator. The Dirac operators between $E^\pm$ are then "nonchiral" if there is a distinguished equivariant isomorphism $E^+ \cong E^-$ identifying them.

A central property of Dirac operators is that their square is a *Laplacian* on E; in other words, its symbol is scalar, and is given by a nondegenerate bilinear form on M (the metric). This follows immediately from the Clifford relation $v^2 = \langle v\,, v\rangle$, and is the key to the analysis of Dirac operators. It also leads to the alternative definition [7]:

3.4 Definition. Let M be a manifold and E a graded vector bundle. Then a *Dirac operator* on E is an odd first order linear differential operator $\not{\nabla}$ on E such that $\not{\nabla}^2$ is a Laplacian.

From this definition it is immediate that the symbol of $\not{\nabla}^2$ defines a (semi)Riemannian metric on M, and that the symbol of $\not{\nabla}$ defines a graded Clifford module structure on E (see [7] for details). It is also easy to see that any Dirac operator is transitive in the sense that there exists a covariant derivative D^E on E such that $\not{\nabla} = c \circ D^E$, as in 3.1. Note, however, that D^E is not uniquely determined by $\not{\nabla}$.

3.5 Examples. Some examples illustrating the scope of this definition are as follows.

(i) Suppose that Δ is a second order linear differential operator with nondegenerate scalar principal symbol, acting on a bundle E^- and equipped with a given factorisation $\Delta = \mathcal{D}_1 \circ \mathcal{D}_2$ into first order linear operators between E^- and another bundle E^+. Now if $\mathcal{D}_2 \circ \mathcal{D}_1$ has the same scalar principal symbol on E^+, then $\mathcal{D}_1$ and $\mathcal{D}_2$ define a Dirac operator on $E^- \oplus E^+$. The Cauchy-Riemann equations and the Dirac equations fit this pattern.

(ii) Let M be a submanifold of a (semi)Riemannian manifold X such that the pullback metric is nondegenerate. On M the tangent bundle of X splits into a direct sum $T \oplus N$, where T is the tangent bundle of M. The Levi-Civita derivative of X pulls back to a covariant derivative on $T \oplus N$, given by the Levi-Civita derivative D^T on T, a metric compatible derivative D^N on N and the second fundamental form II acting between T and N. If E is a Clifford module bundle associated to $T \oplus N$ (as in 3.3) then there are two induced covariant derivatives on E, one coming from $D^{T\oplus N} = D^T \oplus D^N$, the other from $D^{T\oplus N} + \mathrm{II}$. The second of these gives rise to an interesting *submanifold Dirac operator*, which for a spacelike hypersurface in a Lorentzian spin manifold, is the hypersurface Dirac operator used in Witten's proof of the positive energy theorem [43]. This is also the type of Dirac operator which is sometimes studied on submanifolds of $\mathbb{R}^n$ or $\mathbb{C}^n$ in Clifford analysis [12, 47, 51], and used to analyse the Cauchy transform on a Lipschitz surface [37, 40, 42].

It is of crucial importance to have an explicit "Weitzenböck" formula for the Laplacian $\not{\nabla}^2$. The basic way of obtaining such a formula is to compare $\not{\nabla}^2$ to the Bochner Laplacian $\Delta^E = \operatorname{tr} D^{T^*\otimes E} \circ D^E$ of some covariant derivative D^E on E. Any covariant derivative can be used, but simpler formulae are obtained if the derivative is related to the Dirac operator.

3.6 Definition. Let $\not{\nabla}$ be a Dirac operator on E and D^E a covariant derivative. Then D^E is called a *Clifford derivative* iff $D^E c = 0$, in the sense that

$$D^E_X(c(\alpha \otimes \phi)) = c(D_X\alpha \otimes \phi) + c(\alpha \otimes D^E_X\phi),$$

where D is the Levi-Civita derivative. I will also say that D^E is a *Dirac compatible derivative* iff $\not{\nabla} = c \circ D^E$. Finally, $\not{\nabla}$ will be called a *Clifford* Dirac operator iff there is a compatible Clifford derivative D^E on E; that is, $\not{\nabla} = c \circ D^E$ and $D^E c = 0$.

The following theorem is now very well known [24, 35, 45], although the short global approach to the proof given below seems little used in the literature. The product rule reduces this result to its essence: a decomposition into skew and symmetric parts. However, despite its apparent simplicity, it proves to be extremely powerful tool.

3.7 Theorem (Bochner-Weitzenböck). *Let $\not{\nabla}$ be a Clifford Dirac operator. Then*

$$\not{\nabla}^2\phi = \Delta^E\phi + c^{(2)}R^E\phi,$$

where Δ^E is the Bochner Laplacian of the compatible Clifford derivative D^E, $R^E\phi$ is the curvature $\mathrm{Alt}(D^{T^*\otimes E} \circ D^E\phi)$ and $c^{(2)}$ is the Clifford action of Λ^2T^*M on E.

Proof: Since $D^E c = 0$, $D^E \circ c = (id \otimes c) \circ D^{T^*\otimes E}$ and so $\not{\nabla}^2 = c \circ (id \otimes c) \circ D^{T^*\otimes E} \circ D^E$. Now split this into skew and symmetric parts. On the one hand, $c \circ (id \otimes c)(\frac{1}{2}\,\mathrm{Alt}(D^{T^*\otimes E} \circ D^E\phi)) = c^{(2)}(R^E\phi)$, while on the other hand, because $c \circ (id \otimes c)(\frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha) \otimes \phi) = \langle\alpha\,,\beta\rangle\phi$, it follows that $c \circ (id \otimes c)(\frac{1}{2}\,\mathrm{Sym}(D^{T^*\otimes E} \circ D^E\phi)) = \mathrm{tr}\, D^{T^*\otimes E} \circ D^E\phi$. □

If D^E were not Clifford, then there would be a first order term in the above formula, given by $(c \circ D^E c) \circ D^E$. More generally, for any covariant derivative D^E on E, one can write $\not{\nabla} = c \circ D^E + A^E$, with A^E an endomorphism of E, and so obtain a Bochner-Weitzenböck formula with first order term $(c \circ D^E c + A^E \circ c + c \circ A^E) \circ D^E$.

It is well known [7, 25] that for *any* Laplacian, there is a unique covariant derivative whose Bochner Laplacian differs from the given Laplacian by a zero order term. In the case of $\not{\nabla}^2$ there is the following interesting description of this derivative.

3.8 Proposition. *Let $\not{\nabla}$ be any Dirac operator and D^E any covariant derivative on E with $\not{\nabla} = c \circ D^E + A^E$. Then*

$$\not{\nabla}\big(c(X)\phi\big) + c(X)\not{\nabla}\phi - c(\nabla X)\phi = 2D^E_X\phi + \big(c \circ D^E c(X) + A^E \circ c(X) + c(X) \circ A^E\big)\phi,$$

where ∇ denotes the $d + \delta$ operator applied to vector fields (curl plus divergence).

This expression, when divided by 2, is therefore a covariant derivative $\hat{D}^E$ on E (right hand side) defined purely in terms of $\not{\nabla}$ (left hand side). $\hat{D}^E$ defines an endomorphism A of E by $\not{\nabla} = c \circ \hat{D}^E + A$, and is characterised by the formula $c \circ \hat{D}^E c + c \circ A + A \circ c = 0$.

Proof: By the product rule $D^E\big(c(X)\phi\big) = (D^E c)(X)\phi + c(DX)\phi + c(X)D^E\phi$. The result follows by applying Clifford multiplication to this, using the Clifford relation to compute $c \circ c(X)D^E\phi = 2D^E_X\phi - c(X)(c \circ D^E)\phi$, and adding the A^E terms. □

$\hat{D}^E$ will be called the *associated derivative* of $\not{\nabla}$. Its characterisation immediately gives the following Bochner-Weitzenböck formula:

3.9 Theorem. *Let $\hat{D}^E$ be the associated derivative of a Dirac operator $\not{\nabla} = c \circ \hat{D}^E + A$. Then $\not{\nabla}^2 = \Delta^{\hat{D}^E} - K$, where K is the zero order operator $-\big(c^{(2)}\hat{R}^E + c \circ \hat{D}^E A + A^2\big)$.*

A similar result appears in the paper [1] of Ackermann and Tolksdorf, although they write the curvature terms differently by introducing additional covariant derivatives. The above presentation arose from joint work with Tammo Diemer aimed at understanding [1] in an explicitly invariant way. Various parts of 3.8 have appeared in other places. For example, in [7], Berline, Getzler and Vergne give a formula for the derivative associated to $\not{D}^2$ corresponding to the left hand side, which shows that $\hat{D}^E$ is the supercommutator of $\not{D}$ with Clifford multiplication by vector fields. Different formulations of the right hand side of 3.8 can be found in [1, 26]. In practice it is most useful to compute $\hat{D}^E$ (and then K) in terms of a Clifford derivative. It follows in particular that if a Dirac operator is Clifford, then the associated derivative is the only compatible Clifford derivative on E. The Dirac operators in 3.3 are all Clifford, but submanifold Dirac operators in general are not.

The following property of $\hat{D}^E$ is easily deduced from the Clifford relation $c(X)^2 = g(X,X)$, either by using the characterisation or by introducing a Clifford derivative.

3.10 Proposition. *Let $\hat{D}^E$ be the associated derivative of a Dirac operator. Then $\hat{D}^E c$ is skew, in the sense that $(\hat{D}^E_X c)(Y) = -(\hat{D}^E_Y c)(X)$ for all vector fields X, Y.*

In specific examples, it is interesting to calculate the Bochner-Weitzenböck curvature term K more explicitly. In particular, for the Dirac operator on the spinor bundle E (associated to $\mathbb{E}$ using a spin structure), there is the following result of Lichnerowicz [38]:

3.11 Theorem. *The square of the Dirac operator on a spin manifold is given by the formula $\not{D}^2 = \Delta^E - \frac{1}{4}\kappa$, where κ is the scalar curvature of the metric. In other words $K = \frac{1}{4}\kappa$.*

Since $K = -c^{(2)}R^E$, where R^E is the action on E of the curvature of the Levi-Civita derivative, this computation is a simple consequence of the Bianchi symmetry (see [35]).

4 Conformal invariance

Conformal geometry is central to section 11, and so I will review the basic notions and give a proof of conformal invariance for the Dirac operator on a spin manifold.

4.1 Definition. Two inner products g_1, g_2 on a vector space V are said to be *conformally equivalent* iff there is a nonzero real number λ such that for all vectors v, w, $g_1(v,w) = \lambda^2 g_2(v,w)$. A *conformal inner product* on V is an equivalence class of inner products. Given an inner product g on V, a *conformal linear map* with scale factor $\lambda \in \mathbb{R}^+$ is an invertible linear map T such that $g(Tv, Tw) = \lambda^2 g(v,w)$. A *conformal frame* is a basis of orthogonal vectors of V which all have the same length with respect to the inner product.

Clearly the notions of conformal linear map and conformal frame depend only on the conformal equivalence class of the inner product, and the conformal linear maps act freely and transitively on the conformal frames. Any vector v in an conformal inner product space is an element of a conformal frame and this defines an element $CV(v)$ of $\Lambda^n(V)$ which depends (up to a sign) only on v and the conformal inner product.

4.2 Definition. A *density* ρ on an n-dimensional vector space V is a map from $\Lambda^n(V)$ to $\mathbb{R}$ such that $\rho(\lambda\omega) = |\lambda|\rho(\omega)$ for all $\lambda \in \mathbb{R}$ and $\omega \in \Lambda^n(V)$. The densities on V form a one dimensional linear space denoted $|\Lambda^n V^*|$, and $\rho(v_1, \ldots v_n)$ is written for $\rho(v_1 \wedge \cdots \wedge v_n)$. An inner product on V induces a nonzero density on V, the *volume element*. Finally, define $L = L(V)$ to be the space of maps ρ from $\Lambda^n(V)$ to $\mathbb{R}$ such that $\rho(\lambda\omega) = |\lambda|^{1/n}\rho(\omega)$.

Note that $L^n = |\Lambda^n V^*|$, so the density bundle of $L \otimes V$ is canonically trivial. An inner product on $L \otimes V$ will be called *normalised* iff its volume element is the canonical one.

4.3 Proposition. *There is a one to one correspondence between conformal inner products on V and normalised inner products on $L \otimes V$.*

Proof: Given any inner product $\langle .\,,.\rangle$ on $L\otimes V$, define the conformal class of inner products on V to consist of those g for which there is an element l of L such that $g(v,w) = \langle l\otimes v\,, l\otimes w\rangle$ for all $v, w \in V$. In the converse direction it suffices to define $\langle l \otimes v\,, l \otimes v\rangle$. To do this form $CV(v) \in \Lambda^n V$ and $l^n \in L^n$. These are not uniquely defined, but it is easy to see that the real number obtained by evaluating l^n on $CV(v)$ and squaring is well defined, and that taking the positive nth root gives a normalised quadratic form. □

4.4 Definition. Let M be a smooth manifold. Then the *weightless tangent bundle* is defined to be the bundle $L \otimes TM$ where L is the trivialisable line bundle whose fibre at $x \in M$ is $L_x = L(T_xM)$. A *conformal structure* on M is a normalised metric on the weightless tangent bundle. This defines a conformal class of inner products on each tangent space. M is then said to be a *conformal manifold.* Such a structure is equivalently given by the principal $\mathrm{CO}(p,m)$ bundle of conformal frames. Note that a trivialisation of L defines a Riemannian metric on M and that L^n is the density bundle of M.

A conformal manifold does not have a canonical Levi-Civita derivative. Instead there is a distinguished family of torsion free covariant derivatives on the tangent bundle called *Weyl derivatives*, which are those derivatives compatible with the metric on the weightless tangent bundle. For example, the Levi-Civita derivative of any metric in the conformal class of inner products is a Weyl derivative. The difference between any two Weyl derivatives $\tilde{D}$ and D, as an endomorphism valued 1-form, must be a section of $T^*M\otimes \mathfrak{co}(TM) \cap S^2T^*M \otimes TM$. This bundle is isomorphic to T^*M—indeed there is a (scalar valued) 1-form γ with $\tilde{D}_X - D_X = X\triangle\gamma - \gamma(X)id$, where $X\triangle\gamma\colon M \to \mathfrak{so}(TM)$ is the skew endomorphism given by X and γ using the conformal structure.

The theory of general Dirac operators in section 3 may equally be developed on a conformal manifold equipped with a Weyl derivative. A Dirac operator on E is then a first order odd operator $E \to L \otimes E$ whose symbol is a weightless Clifford action $c\colon T^*M \otimes E \to L \otimes E$. The associated derivative may then be defined using the chosen Weyl derivative instead of the Levi-Civita derivative of a (semi)Riemannian metric.

The rest of this section is devoted to the special case of the (Atiyah-Singer) Dirac operator on a spin manifold.

4.5 Definition. A *spin structure* on an oriented manifold M is a principal $\widetilde{\mathrm{GL}}^+(n)$-bundle, together with a 2-fold cover of the bundle of oriented frames compatible with the (nontrivial) 2-fold cover $\widetilde{\mathrm{GL}}^+(n) \to \mathrm{GL}^+(n)$. A manifold with a spin structure is called a *spin manifold.*

Not every manifold admits a spin structure, although it is possible to relax the orientability requirement by considering "pin structures". This will not be done here, nor will topological obstructions be discussed (see [35] for a full treatment), but instead it will now be assumed that M is conformal spin manifold. Therefore, the principal $\mathrm{CO}_+(p,m)$ bundle of oriented conformal frames has a double cover, a principal $\mathrm{Spin}_+(p,m)\times\mathbb{R}^+$ bundle $\Gamma(M)$. The aim is to show that this structure is sufficient to define the Dirac operator.

4.6 Definitions. Let M be a conformal spin manifold, and $\mathbb{E}$ be a Clifford module. Then associated to $\Gamma(M)$ are the following vector bundles:

(i) the *tangent bundle*, $TM \cong \Gamma(M) \times_{\rho_1} \mathbb{R}^n$,
where ρ_1 is the standard representation of $\mathrm{Spin}(p,m)\times\mathbb{R}^+$ on $\mathbb{R}^n$ (i.e., $(a,\lambda)\colon x \mapsto \lambda axa^{-1}$)

(ii) the *weightless tangent bundle*, $L\otimes TM \cong \Gamma(M) \times_{\rho_2} \mathbb{R}^n$,
where ρ_2 is the standard representation with $\mathbb{R}^+$ acting trivially (i.e., $(a,\lambda)\colon x \mapsto axa^{-1}$)

(iii) the *Clifford algebra bundle*, $Cl(M) \cong \Gamma(M) \times_{\rho_3} Cl_{p,m}$,
where ρ_3 is the extension of ρ_2 to $Cl_{p,m}$ (i.e., the adjoint action of $\mathrm{Spin}(p,m)$ on $Cl_{p,m}$)

(iv) the *density bundle* with weight w, $L^w \cong \Gamma(M) \times_{\mu_w} \mathbb{R}$,
where μ_w is the action $(a,\lambda)\colon \alpha \mapsto \lambda^{-w}\alpha$

(v) the *spinor bundles* with weight w, $E^\pm_w \cong \Gamma(M) \times_{\sigma_w} \mathbb{E}^\pm$,
where σ_w is the weight w spin representation $(a,\lambda)\colon \psi \mapsto \lambda^{-w}a\psi$.

Note that $Cl(M)$ is the conformal version of the Clifford algebra bundle defined earlier; its fibre $Cl(M)_x$ is the Clifford algebra of $L_x\otimes T_xM$ with its normalised inner product. The Clifford action on $\mathbb{E}$ is spin invariant, and so, for each w, E_w is a bundle of modules for $Cl(M)$, and therefore there is a Clifford action $c_w\colon T^*M\otimes E^\mp_w \to E^\pm_{w+1}$.

Given a Weyl derivative D, a Dirac operator may be defined for each weight w as the operator $c_w \circ D^E$ from E_w to E_{w+1}, where D^E is the induced covariant derivative on E_w and the symbol c_w is independent of the Weyl derivative. The important fact is that, provided the weight w is chosen correctly, the Dirac operator itself is independent of the Weyl derivative, and is therefore canonically associated to the conformal spin manifold.

4.7 Theorem. *The Dirac operator $c_w \circ D^E$ does not depend upon the choice of the Weyl derivative D iff $w = \frac{n-1}{2}$.*

Proof: (c.f. Hitchin [32].) It must be shown that $c_w \circ (\tilde{D}^E - D^E) = 0$ (for all possible choices) iff $w = \frac{n-1}{2}$. The difference between any two Weyl derivatives on the tangent bundle is given by $\tilde{D}_X - D_X = X\Delta\gamma - \gamma(X)id$ for some 1-form γ. It then follows from Proposition 2.5 that the corresponding section of $\mathfrak{spin}(M)\oplus\mathbb{R}$ is $\frac{1}{4}(\gamma X - X\gamma) - \gamma(X)$, where γX and $X\gamma$ denote weightless Clifford multiplication and contraction of the weights. Now the action of $(\xi,\mu) \in \mathfrak{spin}(p,m)\oplus\mathbb{R}$ on $\mathbb{E}_w$ (from the weight w representation of $\mathrm{Spin}(p,m)\times\mathbb{R}^+$) is $\psi \mapsto \xi\psi - w\mu\psi$. Therefore

$$(\tilde{D}^E_X - D^E_X)\phi = \frac{1}{4}c(\gamma X - X\gamma)\phi + w\gamma(X)\phi = \frac{1}{2}c\big(\gamma(X) - X\gamma\big)\phi + w\gamma(X)\phi,$$

and so (contracting the X variable with c_w)

$$c_w \circ (\tilde{D}^E - D^E)\phi = \frac{1}{2}c_w(\gamma - n\gamma)\phi + wc_w(\gamma)\phi = \left(w - \frac{n-1}{2}\right)c_w(\gamma)\phi.$$

This is zero for all γ iff $w = \frac{n-1}{2}$. □

5 Inner products and the Green formula

Henceforth, I restrict attention to definite Dirac operators, i.e., the Clifford algebra bundle on M will be definite. The two cases (positive or negative definite) are very similar, and so only the positive definite case will be treated, partly in order to demonstrate that the theory is just as pleasant as the negative definite case (which is more widely considered), and partly in order to emphasise the link between the Green formula and the product rule. The negative definite version is easily obtained by judicious insertion of minus signs.

The reason for the restriction to definite Dirac operators is that they are elliptic, and so have a simpler and better-developed analytical theory. More naïvely, the analysis of definite Dirac operators is easier because the Clifford module bundle may be given an invariant definite inner product by averaging over the (compact) Spin group. To be precise, it will now be assumed that the bundles $E^{\pm}$ are equipped with inner products whose real parts are positive definite, and such that the Clifford action of any (co)tangent vector is symmetric with respect to the induced inner product on E. It will also be assumed that the Dirac operator is *uncharged* with respect to the inner product, in the sense that the associated derivative $\hat{D}^E$ is compatible with the inner product (i.e., the inner product is covariant constant) and $A = \not{\nabla} - c \circ \hat{D}^E$ is a skew endomorphism—it follows that K is a symmetric endomorphism. More general Dirac operators can be decomposed into an uncharged part and a contracted potential, but this will be discussed elsewhere.

In the case of the conformally invariant Dirac operator associated to a Clifford module $\mathbb{E}$, such an inner product bundle is easily obtained by equipping $\mathbb{E}$ with a definite inner product such that vectors are symmetric (by averaging over the Pin group). The equivariance of this inner product under $\mathrm{Spin}(n) \times \mathbb{R}^+$ ensures that it induces inner products on the spinor bundles, such that the inner product of a section of E_{w_1} with a section of E_{w_2} is a section of $L^{w_1+w_2}$. Any Weyl derivative D induces a covariant derivative D^E on E with respect to which this inner product is automatically parallel. Note also that D^E is then the associated derivative of the Dirac operator (computed using the given Weyl derivative D). From time to time it will be necessary, during computation, to make such a choice of Weyl derivative, for example by choosing a metric in the conformal class.

The aim now is to develop the analytical properties of general Dirac operators, in such a way that in the case of the Dirac operator on a conformal spin manifold, the formulae obtained are manifestly conformally invariant. To this end some notation will be useful.

5.1 Notation. In the conformally invariant case, $E^{\pm}$ will be used for the weight $\frac{n-1}{2}$, and $\hat{E}^{\pm}$ for the weight $\frac{n+1}{2}$ (so the Dirac operator acts from E^- to $\hat{E}^+$). The L^{n-1} valued inner product on E will be denoted $(.,.)$. For more general Dirac operators, $\hat{E}^{\pm} = L \otimes E^{\pm}$, and the chosen inner product $(.,.)$ on E will be assumed to take values in L^{n-1} as in the conformal invariant case. On a Riemannian manifold the line bundles L^w are each trivialised by a natural section, and so may be ignored when conformal aspects are not being considered.

As in [7], integration will be defined in terms of densities, rather than n-forms, although they are equivalent in the orientable case. Integration over M is then a linear functional $\int_M : C_c^\infty(M, L^n) \to \mathbb{R}$, where $C_c^\infty(M, L^n)$ the space of smooth compactly supported sections ρ of L^n. Given such a ρ and a vector field X, define $\mathrm{div}(X \otimes \rho)$ to be the Lie derivative $\mathcal{L}_X \rho$. It is easy to see that this is well defined (indeed it is the trace of $D(X \otimes \rho)$ for any torsion-free derivative D) and that the **Divergence formula**

$$\int_M \mathrm{div}(X \otimes \rho) = \int_{\partial M} \langle X, \rho \rangle$$

holds. Here the boundary integrand is the *contraction of X with ρ along ∂M*, a section of L^{n-1} over ∂M which may be defined as follows. Let v be any outward pointing vector field along ∂M, and α the section of T^*M along ∂M such that $\alpha(v) = 1$ and $\ker\alpha = T(\partial M)$. Then $\langle X, \rho\rangle = \alpha(X)\rho(v, -)|_{\partial M}$. If M is a Riemannian (or conformal) manifold then this equals $\langle X, \nu\rangle\rho(\nu, -)$, where ν is the (weightless) outward unit normal.

5.2 Theorem. *Let $\not{\nabla}$ be any (uncharged) Dirac operator on E, and let ϕ, ψ be sections of E^- and E^+. Then:*

$$\operatorname{div}(c(.)\phi, \psi) = (\not{\nabla}^+\phi, \psi) + (\phi, \not{\nabla}^-\psi),$$

where $(c(.)\phi, \psi)$ is a vector field density; that is, an L^n valued linear map on T^*M.

Proof: The divergence will be calculated directly—in the conformal case a Weyl derivative needs to be chosen for this computation. By assumption, $\hat{D}^E$ is compatible with $(.,.)$ and so the product rule gives the following formula for the Levi-Civita or Weyl derivative D applied to $(c(.)\phi, \psi)$:

$$D(c(.)\phi, \psi) = ((\hat{D}^E c)(.)\phi, \psi) + (c(.)\hat{D}^E\phi, \psi) + (\phi, c(.)\hat{D}^E\psi).$$

The divergence is obtained by taking the trace of this equation. To do this, observe that $\operatorname{tr} c(.)\hat{D}^E\phi = c(\hat{D}^E\phi)$ and $\operatorname{tr}\hat{D}^E c = 0$ (by 3.10). Since $\not{\nabla} = c \circ \hat{D}^E + A$ with A skew, the right hand side of the stated formula is obtained. □

5.3 Corollary (Green formula). *If ϕ and ψ are compactly supported sections, then there is the following integration by parts formula for $\not{\nabla}$:*

$$\int_{\partial M} (c(\nu)\phi, \psi) = \int_M (\not{\nabla}^+\phi, \psi) + \int_M (\phi, \not{\nabla}^-\psi).$$

(The integrals are well defined in the conformal case, since $(\not{\nabla}^+\phi, \psi)$ and $(\phi, \not{\nabla}^-\psi)$ are sections of L^n, and $(c(\nu)\phi, \psi)$ is a section of L^{n-1}, as ν is weightless.)

5.4 Corollary (Cauchy's theorem). *If $\not{\nabla}\phi = \not{\nabla}\psi = 0$ on M, then $\int_{\partial M}(c(\nu)\phi, \psi) = 0$.*

REMARK. The proof of theorem 5.2 makes essential use of the compatibility of $\hat{D}^E$ with the inner product. In fact it is easy to see that if the divergence formula in 5.2 holds then $\hat{D}^E$ is compatible with the inner product and A is skew.

The Green formula is a formal skew-adjointness result. One way to interpret this is by means of distributions. Let $C_0^\infty(M, V)$ denote the space of smooth compactly supported sections of V vanishing (to infinite order) on the boundary of M.

5.5 Definition. The space of *distributional* sections of a bundle V, denoted $\mathcal{D}(M, V)$ is defined to be the continuous dual of $C_0^\infty(M, V^* \otimes L^n)$ with respect to the C^∞-*topology* of uniform smooth convergence on compact subsets. Note that any $s \in C^\infty(M, V)$ determines the functional $\int_{y\in M}\langle s(y), .\rangle$ on $C_0^\infty(M, V^*\otimes L^n)$, where $\langle.,.\rangle$ denotes the L^n valued contraction of $V^*\otimes L^n$ with V. For each $y \in \operatorname{int} M$ and $\theta_y \in V_y^*$, the functional $\theta_y \circ \delta_y : f \mapsto \theta_y(f(y))$ is continuous and so is a distribution. Thus the *delta function* δ_y is in $\mathcal{D}(M, V^* \otimes L^n) \otimes V_y$, where $\mathcal{D}(M, V^* \otimes L^n)$ is the continuous dual of $C_0^\infty(M, V)$.

5.6 Proposition. *In the case of a spinor bundle, $(E^-)^* \otimes L^n \cong \hat{E}^-$, and so the dual of $C_0^\infty(M, E^-)$ is $\mathcal{D}(M, \hat{E}^-)$, with $C^\infty(M, \hat{E}^-)$ embedded into $\mathcal{D}(M, \hat{E}^-)$ as the linear functionals $\int_M(\phi, .)$, and similarly for the positive spinors.*

Hence the Dirac operators $\not{\nabla}^\pm\colon C_0^\infty(M, E^\mp) \to C_0^\infty(M, \hat{E}^\pm)$ are formally skew-adjoint, in the sense that the adjoint (transpose) of $\not{\nabla}^+$, when restricted to the smooth spinor fields (of weight $\frac{n-1}{2}$) vanishing on the boundary, is $-\not{\nabla}^-$.

This also shows that $\bar{\partial}^+ : C^\infty(M, E^-) \to C^\infty(M, \hat{E}^+)$ extends to a continuous linear operator $\mathcal{D}(M, E^-) \to \mathcal{D}(M, \hat{E}^+)$, namely $(-\bar{\partial}^-)^*$, the transpose of the formal adjoint.

I now turn to the Bochner-Weitzenböck formula of section 3—since this involves $\bar{\partial}^2$, a metric on M (or at least a Weyl derivative) is required. The Bochner-Weitzenböck formula is a powerful tool for establishing important properties of $\bar{\partial}$, particularly when it is reexpressed using the inner product on E. Such a formulation is easily obtained using the Green formula for the Dirac operator and a well known Green formula for a covariant derivative on a vector bundle with compatible inner product. For the associated derivative on E, the latter Green formula arises from the following:

$$\operatorname{div}(\langle \phi , \psi \rangle \sharp \alpha) = \langle \operatorname{tr} \hat{D}^E(\alpha \otimes \phi) , \psi \rangle + \langle \alpha \otimes \phi , \hat{D}^E \psi \rangle.$$

The operator $\operatorname{tr} \hat{D}^E$ is often called the covariant divergence. Since it is the operator appearing in the Bochner Laplacian, it is now a straightforward matter to prove:

5.7 Theorem (Inner product form of Bochner-Weitzenböck). *For sections ϕ, ψ of E, there are the following equalities between pointwise inner products:*

$$\langle \bar{\partial}\phi , \bar{\partial}\psi \rangle - \operatorname{div}\langle c(.)\bar{\partial}\phi , \psi \rangle \;=\; \langle -\bar{\partial}^2\phi , \psi \rangle \;=\; \langle \hat{D}^E\phi , \hat{D}^E\psi \rangle + \langle K\phi , \psi \rangle - \operatorname{div}\langle \hat{D}^E\phi , \psi \rangle.$$

5.8 Corollary (Integral form of Bochner-Weitzenböck). *For $\phi, \psi \in C^\infty_c(M, E)$,*

$$\int_M \langle \hat{D}^E\phi , \hat{D}^E\psi \rangle + \int_M \langle K\phi , \psi \rangle = \int_M \langle \bar{\partial}\phi , \bar{\partial}\psi \rangle - \int_{\partial M} \langle (c(\nu)\bar{\partial} - \hat{D}^E_\nu)\phi , \psi \rangle.$$

The operator in the boundary integrand is $c(\nu)\bar{\partial}^T$ where $\bar{\partial}^T$ is a Dirac operator on ∂M (the tangential part of $\bar{\partial}$). Since K is a symmetric endomorphism, the above formula implies $c(\nu)\bar{\partial}^T$ is a formally self-adjoint differential operator on ∂M (in fact it is an example of a Dirac operator with a negative definite Clifford algebra bundle).

There are two immediate and important consequences of the Bochner-Weitzenböck integral formula: an L^2 estimate and a vanishing theorem. Both require M to be compact.

5.9 Gårding's inequality. *For M compact and $\phi \in C^\infty(M, E)$,*

$$\|\hat{D}^E\phi\|^2_{L^2} \leqslant \|\bar{\partial}\phi\|^2_{L^2} + (\sup |K|)\|\phi\|^2_{L^2} - \int_{\partial M} \langle c(\nu)\bar{\partial}^T\phi , \phi \rangle,$$

where $|K|$ denotes the pointwise operator norm.

On a closed manifold, or more generally if $\phi|_{\partial M}$ is in the span of the positive spectrum of $c(\nu)\bar{\partial}^T$, the boundary term disappears.

5.10 Theorem. *Suppose that M is compact, and the symmetric endomorphism K is nonnegative. Then every monogenic spinor (satisfying the above spectral boundary condition if M is not closed) is $\hat{D}^E$-parallel, and identically zero if K is somewhere (strictly) positive.*

Proof: By 5.8, $\int_M \langle \hat{D}^E\phi , \hat{D}^E\phi \rangle + \int_M \langle K\phi , \phi \rangle$ is nonpositive, since $\bar{\partial}\phi = 0$ and the boundary integrand is nonpositive. But K is nonnegative, so $\hat{D}^E\phi = 0$ and $\langle K\phi , \phi \rangle = 0$. □

This type of vanishing result (on closed manifolds) goes back to Bochner [8]. In the case of the Dirac operator on a conformal spin manifold, the Lichnerowicz formula 3.11, with respect to any chosen metric in the conformal class, gives $K = \frac{1}{4}\kappa$, and so the above theorem reduces to the Lichnerowicz vanishing theorem [38]. The extension of Lichnerowicz's theorem to spectral boundary conditions was given in [10], where it was used to study the moduli space of metrics of positive scalar curvature.

6 Elliptic theory on closed manifolds

This section summarises the well known elliptic theory of Dirac operators on closed manifolds. This will be done within the framework of (integer) L^2 Sobolev spaces of sections of a vector bundle V, denoted $L^2_j(M,V)$. Roughly speaking, a section s is in $L^2_j(M,V)$ iff its derivatives up to order j are all in $L^2(M,V)$. The derivatives can be defined with respect to a covariant derivative D^V on V. L^2_j is a Hilbert space with norm

$$\|s\|_{L^2_j} = \left(\sum_{j=0}^{k} \|(D^V)^j s\|^2_{L^2}\right)^{1/2}.$$

Any differential operator of order k is continuous from $L^2_j(M,V)$ to $L^2_{j-k}(M,V)$ for $j \geqslant k$.

Most of the properties of elliptic operators can be deduced from elliptic estimates for the Sobolev norms. A notable feature of Dirac operators is that these estimates are easy to establish, requiring no local computations with pseudodifferential operators or Fourier analysis, and no parametrix machinery. The proof below is based on Roe [45].

6.1 Proposition. *Let M be a closed manifold. Then for each $j \in \mathbb{N}$ there is a constant C_j such that for any $\phi \in C^\infty(M,E)$, the inequality $\|\phi\|_{L^2_{j+1}} \leqslant C_j(\|\phi\|_{L^2_j} + \|\not{D}\phi\|_{L^2_j})$ holds. Thus if $\phi, \not{D}\phi \in L^2_j$ then in fact $\phi \in L^2_{j+1}$.*

Proof: For $j = 0$, this is immediate from 5.9. Now use induction on j. To estimate the L^2_{j+1}-norm of ϕ, it suffices to estimate the L^2_j-norm of $\hat{D}^E_X\phi$ for any vector field X, which, by induction, is bounded by $C_{j-1}(\|\hat{D}^E_X\phi\|_{L^2_{j-1}} + \|\not{D}\hat{D}^E_X\phi\|_{L^2_{j-1}})$. Since both $\hat{D}^E_X$ and $[\not{D}, \hat{D}^E_X] = \not{D}\hat{D}^E_X - \hat{D}^E_X\not{D}$ are first order operators, the L^2_{j-1}-norms of $\hat{D}^E_X\phi$, $\hat{D}^E_X\not{D}\phi$ and $[\not{D}, \hat{D}^E_X]\phi$ are bounded by L^2_j-norms of ϕ and $\not{D}\phi$, and the required estimate follows. □

From these estimates and some elementary functional analysis (the Sobolev embedding theorem, the Rellich compactness theorem, and an abstract closed range theorem) the following properties of Dirac operators on closed manifolds are easily deduced:

6.2 Local elliptic regularity. *Let U be an open subset of M, and suppose that $\phi \in L^2(M,E)$ with $\not{D}\phi$ (represented by) a smooth function on U. Then ϕ is smooth on U.*

Secondly, suppose $\not{D}\phi_j = 0$ on U and $\phi_j \to \phi$ in $L^2(W,E)$ for all compact subsets W of U. Then $\phi_j \to \phi$ locally uniformly in all derivatives on U, and hence $\not{D}\phi = 0$ on U.

6.3 Theorem. *On a closed manifold, $\not{D}^\pm\colon C^\infty(M,E^\mp) \to C^\infty(M,\hat{E}^\pm)$ has a finite dimensional kernel and a closed range, and the orthogonal complement of $\ker\not{D}^-$ in $C^\infty(M,E^+)$ is $\operatorname{im}\not{D}^+$, and similarly for the negative spinors. (More precisely $\ker\not{D}^-$ and $\operatorname{im}\not{D}^+$ are mutual annihilators with respect to the pairing of $C^\infty(M,E^+)$ and $C^\infty(M,\hat{E}^+)$.)*

There is a similar result for the Dirac operator acting between Sobolev spaces. Note also, that in the case of the $d+\delta$ operator, this gives a straightforward proof of the Hodge decomposition: $C^\infty(M,\Lambda T^*M) = \operatorname{im} d \oplus (\ker d \cap \ker\delta) \oplus \operatorname{im}\delta$.

In order to study Dirac operators on manifolds with boundary, it is convenient to use an extension of the given Dirac operator to an invertible operator on a closed manifold containing the given manifold with boundary. In fact it is always possible to construct such an extension, thanks to the following unique continuation property.

6.4 Theorem. *Let Ω be a connected open set and $\phi\colon \Omega \to E$ a section with $\not{D}\phi = 0$. Suppose ϕ vanishes on an open subset of Ω. Then ϕ vanishes on Ω.*

One way to prove this is to apply a result of Aronszajn [2] to the square of the Dirac operator—for another proof, see [11]. The theorem below only uses the weaker continuation property that on a connected manifold with nonempty boundary, a monogenic function vanishing on the boundary vanishes identically.

6.5 Theorem. *Let M be a compact connected Riemannian manifold with nonempty boundary and a Dirac operator $\not\nabla^+ \colon \mathrm{C}^\infty(M, E^-) \to \mathrm{C}^\infty(M, E^+)$. Then there is a closed manifold $\tilde{M}$ containing M as a submanifold of the same dimension, and an extension of $\not\nabla^+$ to a Dirac operator on $\tilde{M}$ which is invertible.*

The proof of this theorem (see [11] or [14] for the details) involves doubling the manifold M and gluing E^+ to E^- across the boundary using Clifford multiplication by the unit normal. This twist gives the Clifford module bundle on $\tilde{M}$ a special global structure, and it is this which accounts for the invertibility of the Dirac operator.

III. Dirac Operators on Manifolds with Boundary

7 The Cauchy integral formula

I will now turn to the analysis of $\not\nabla$ on a manifold M with boundary, using an invertible extension of $\not\nabla$ to a closed manifold $\tilde{M}$, and the restriction map $r \colon \mathrm{C}^\infty(\tilde{M}, E^-) \to \mathrm{C}^\infty(\partial M, E^-)$. The main object of study is the following (see Seeley [49]):

7.1 Definition. The *Cauchy integral* is the operator

$$C^+ = \left(c(\nu) \circ r \circ (\not\nabla^-)^{-1}\right)^* = (\not\nabla^-)^{*-1} \circ r^* \circ c(\nu) \colon \mathrm{C}^\infty(\partial M, E^-) \to \mathcal{D}(\tilde{M}, E^-),$$

where (i) $c(\nu) \colon \mathrm{C}^\infty(\partial M, E^-) \to \mathrm{C}^\infty(\partial M, E^+)$ is the action of the (weightless) unit normal,
(ii) $r^* \colon \mathrm{C}^\infty(\partial M, E^+) \to \mathcal{D}(\tilde{M}, \hat{E}^+)$ is given by $r^*\phi[\psi] = \int_{\partial M} (\phi, r\psi)$, and
(iii) $(\not\nabla^-)^{*-1} \colon \mathcal{D}(\tilde{M}, \hat{E}^+) \to \mathcal{D}(\tilde{M}, E^-)$ is the inverse of the transpose $(\not\nabla^-)^* = -\not\nabla^+$.

Similarly, there is a Cauchy integral C^- for E^+. Since $(\not\nabla^-)^{-1}$ is bounded from $\mathrm{L}^2(\tilde{M}, E^-)$ to $\mathrm{L}^2_1(\tilde{M}, E^+)$ and r is bounded from $\mathrm{L}^2_1(\tilde{M}, E^+)$ to $\mathrm{L}^2(\partial M, E^+)$, it follows that:

7.2 Proposition. *The Cauchy integral is a bounded linear map $\mathrm{L}^2(\partial M, E^-) \to \mathrm{L}^2(M, E^-)$.*

This simple result will not be used until much later. Instead some more informative expressions for the Cauchy integral on smooth functions will be developed, starting with:

7.3 Proposition. *The Cauchy integral is given by the formulae*

$$C^+\phi[\psi] = \int_{\partial M} (c(\nu)\phi, (\not\nabla^-)^{-1}\psi) = \int_M (\phi, \psi) + \int_M (\not\nabla^+\phi, (\not\nabla^-)^{-1}\psi),$$

where in the last expression ϕ has been extended to M. Hence if ϕ is monogenic on M then $C^+(r\phi) = \phi$ as distributions on $\operatorname{int} M$. Also note that $\not\nabla^+(C^+\phi) = 0$ on $\tilde{M} \setminus \partial M$.

Proof: The first expression is a matter of unravelling the definition:

$$C^+\phi[\psi] = \left((\not\nabla^-)^{*-1} \circ r^* \circ c(\nu)\, \phi\right)[\psi] = \left(r^*(c(\nu)\phi)\right)\left[(\not\nabla^-)^{-1}\psi\right] = \int_{\partial M} \left(c(\nu)\phi, (\not\nabla^-)^{-1}\psi\right).$$

The second expression then follows from the Green formula, and hence if $\not\nabla^+\phi = 0$, $C^+(r\phi)$ and ϕ agree on test functions ψ. For the last part it must be shown that $(\not\nabla^+ \circ C^+\phi)[\psi] = 0$

for any test function $\psi \in C^\infty(\tilde{M}, \hat{E}^+)$ supported in int M. But $\not{\nabla}^+$ on distributions is given by $-(\not{\nabla}^-)^*$, so $(\not{\nabla}^+ \circ C^+\phi)[\psi] = -r^*(c(\nu)\phi)[\psi] = -\int_{\partial M} (c(\nu)\phi, r\psi) = 0$ since $r\psi = 0$. □

This proposition is already a distributional version of the Cauchy integral formula. It gives direct expression to the fact that a monogenic function is determined by its boundary values. However, it is of little use unless the Cauchy integral is described more explicitly. In particular, since $C\phi$ smooth away from ∂M (by elliptic regularity), it should be possible to give an expression for its point values. This can be done using the fundamental solution.

7.4 Definition. Recall that for each $x \in \tilde{M}$ there is a delta function $\delta_x \in \mathcal{D}(\tilde{M}, \hat{E}^-) \otimes E_x^-$. Define the distribution G_x^+ by $G_x^+ = (\not{\nabla}^+)^{*-1}\delta_x \in \mathcal{D}(\tilde{M}, E^+) \otimes E_x^-$, so that $G_x^+[\psi] = ((\not{\nabla}^+)^{*-1}\delta_x)[\psi] = \delta_x[(\not{\nabla}^+)^{-1}\psi] = ((\not{\nabla}^+)^{-1}\psi)(x)$. Now $(\not{\nabla}^+)^*$ is the action of $-\not{\nabla}^-$ on distributions, and so $\not{\nabla}^- G_x^+ = 0$ outside $\{x\}$. Hence over $\{(x, y) \in \tilde{M} \times \tilde{M} : x \neq y\}$, one can define the *fundamental solution* of $\not{\nabla}^+$ to be the point values $G^+(x, y) \in E_y^+ \otimes E_x^-$ of G_x^+.

Likewise $\not{\nabla}^-$ has a fundamental solution $G^-(x, y)$.

An important fact to be established is that the distributions $G_x^\pm$ are actually represented by the fundamental solutions $y \mapsto G^\pm(x, y)$, since a priori $G_x^\pm$ may harbour a genuine distribution on the diagonal. However, it is at least clear that if $\psi = 0$ near x, then

$$((\not{\nabla}^\pm)^{-1}\psi)(x) = \int_{y \in \tilde{M}} (G^\pm(x, y), \psi(y)).$$

7.5 Proposition. *For $x \neq y$, $G^-(y, x)^\tau = -G^+(x, y)$, where $^\tau$ denotes transposition of tensors: $E_x^- \otimes E_y^+ \cong E_y^+ \otimes E_x^-$. Hence the fundamental solutions are smooth in both variables.*

Proof: It follows from the Green formula that $\int_{\tilde{M}} (\phi, (\not{\nabla}^-)^{-1}\psi) + \int_{\tilde{M}} ((\not{\nabla}^+)^{-1}\phi, \psi) = 0$. For ϕ, ψ with disjoint support, this implies:

$$\int_{y \in \tilde{M}} \left(\phi(y), \int_{x \in \tilde{M}} (G^-(y, x), \psi(x))\right) + \int_{x \in \tilde{M}} \left(\int_{y \in \tilde{M}} (G^+(x, y), \phi(y)), \psi(x)\right) = 0.$$

Since this holds for all such ϕ, ψ, the equality follows. □

It is now possible to describe the Cauchy integral operator more explicitly.

7.6 Theorem (Point values of the Cauchy integral). *Away from ∂M, $C^+\phi$ is given by the smooth function*

$$C^+\phi(x) = \int_{\partial M} (-G_x^+, c(\nu)\phi).$$

Proof: For ψ supported away from ∂M,

$$C^+\phi[\psi] = \int_{y \in \partial M} \left(c(\nu)\phi(y), \int_{x \in \tilde{M}} (G^-(y, x), \psi(x))\right)$$
$$= \int_{x \in \tilde{M}} \int_{y \in \partial M} \Big((-G^+(x, y), c(\nu)\phi(y)), \psi(x)\Big),$$

using 7.5 and the continuity of the integrand. This gives the stated formula. □

7.7 Corollary (Boundedness of the Cauchy integral). *In the oriented line L_x^{n-1},*

$$(C^+\phi(x), C^+\phi(x)) \leqslant \left(\int_{\partial M} (\phi, \phi)\right) \left(\int_{\partial M} (G_x^+, G_x^+)\right)_x,$$

where the last integral is contracted to lie in L_x^{n-1}.

Therefore the Cauchy integral extends to a continuous linear map from the (conformally invariant) Hilbert space $L^2(\partial M, E)$ *to* $C^\infty(\operatorname{int} M, E)$.

Proof: This is just the Cauchy-Schwarz inequality for the E_x^- valued pairing of G_x^+ and ϕ, dressed up in conformally invariant language. It immediately follows that the pointwise Cauchy integral is continuous, but also since $\int_{\partial M} (G_x^+, G_x^+)$ is smooth for $x \in \operatorname{int} M$, it is in L^2 on compact subsets. Now the Cauchy integral is monogenic, and so the continuity (on the dense subspace of smooth boundary functions) follows from 6.2. □

7.8 Corollary (Cauchy integral formula). *If* ϕ *is smooth on* M *and* $\not{\nabla}^+\phi = 0$ *on* $\operatorname{int} M$ *then*

$$\phi(x) = \int_{\partial M} \left(-G_x^+, c(\nu)\phi\right)$$

for $x \in \operatorname{int} M$. *Hence the Cauchy integral on boundary values of smooth monogenic functions is an evaluation map.*

This generalises the standard formula in Clifford analysis [12, 24] (and thence the classical formula), as can be seen by computing the fundamental solution on $\mathbb{R}^n$. Since such computation is also essential in order to understand the behaviour of the fundamental solution on a general manifold more concretely, I will recall it here.

7.9 Proposition. *The inverse of the Dirac operator on* S^n *is represented at* $x \in \mathbb{R}^n$ *by the fundamental solution* $G(x,y) = \frac{1}{\omega_n}\frac{x-y}{|x-y|^n}$, *where* ω_n *is the area of* S^{n-1} *and* $x-y$ *acts from* $\hat{E}^+$ *to* E^-, *or from* $\hat{E}^-$ *to* E^+.

Proof: It must be verified that $\not{\nabla}^* G(x,.) = \delta_x$, with $G(x,y)$ as stated. In other words that $G(x,.)[\not{\nabla}\phi] = \phi(x)$ for test functions ϕ. The left hand side may be written as

$$\lim_{r\to 0}\int_{y\in \tilde{M}\setminus B_r(x)} (G(x,y), \not{\nabla}\phi(y)) = \lim_{r\to 0}\int_{y\in\partial B_r(x)} (c(-\nu)G(x,y), \phi(y)),$$

since $G(x,y) = \frac{1}{\omega_n}\frac{x-y}{|x-y|^n}$ is monogenic in y for $y \neq x$ (a straightforward verification—also see below). Here $\nu = \frac{y-x}{|y-x|}$ is the outward normal on $\partial B_r(x)$, and so $-c(\nu)G(x,y) = \frac{1}{\omega_n |x-y|^{n-1}}$. Therefore the integral is the average of ϕ over a small sphere centred at x, which tends to $\phi(x)$ as $r \to 0$, since ϕ is continuous. □

REMARK. The conformal invariance of the Dirac operator on S^n suggests that the fundamental solution should be viewed as being "constant" away from the singularity. More precisely, by the vanishing theorem 5.10, constant spinors on $\mathbb{R}^n$ do not extend to monogenic spinors over S^n. Thus, if a constant spinor ψ on $\mathbb{R}^n$ is transformed by the conformal map $x \mapsto x/|x|^2$ of S^n, the result is a monogenic function on $S^n \setminus \{0\}$. This is easily computed to be $\frac{x}{|x|^n}\psi$, which is a component of the fundamental solution at 0. This gives an easy way of seeing that the fundamental solution is monogenic away from the singularity.

An important aspect of the above proposition is that the inverse of the Dirac operator on the sphere is represented by its fundamental solution. One way to establish the same result more generally is to show that the inverse of a Dirac operator on a general manifold can be approximated by the fundamental solution on the sphere. To see this, it is necessary to calculate.

7.10 Lemma. *Let ϕ be a smooth spinor valued function on $\mathbb{R}^n \setminus \{0\}$ of bounded support, such that $|\not\nabla\phi(x)| \leqslant C/r^k$ with $k < n$ (here $r = |x|$). Then $|\phi(x)| \leqslant \tilde{C}/r^{k-1}$ plus a logarithmic term if $k = 1$.*

Proof: Writing $\psi = \not\nabla\phi$ and $\Omega = \operatorname{supp}\psi$ and applying the fundamental solution, it suffices to establish that

$$\int_{y\in\Omega} \frac{1}{\omega_n|x-y|^{n-1}}|\psi(y)|$$

has the stated growth at the origin. In order to estimate the integral when $|x| = r$, split it into integrals over $|y| < r/2$, $r/2 < |y| < 3r/2$ and $|y| > 3r/2$, and integrate in polar coordinates (around the origin). The integral over $|y| < r/2$ is bounded by

$$\int_{|y|=0}^{r/2} \omega_n|y|^{n-1}\frac{2^{n-1}}{\omega_n r^{n-1}}\frac{C}{|y|^k} \leqslant \frac{\tilde{C}_1}{r^{k-1}},$$

since $k < n$. Similarly for $|y| > 3r/2$, one can estimate

$$\int_{|y|>3r/2} \omega_n|y|^{n-1}\frac{2^{n-1}}{\omega_n(|y|-r)^{n-1}}\frac{C}{|y|^k} \leqslant \frac{\tilde{C}_2}{r^{k-1}} + \log \text{ term if } k = 1.$$

For the integral over $r/2 < |y| < 3r/2$, it is necessary to separate out the pole of the fundamental solution in $|y - x| < r/2$. For the integral without the pole, there is the bound,

$$\int_{y=r/2}^{3r/2} \omega_n|y|^{n-1}\frac{2^{n-1}}{\omega_n r^{n-1}}\frac{C}{|y|^k} \leqslant \frac{\tilde{C}_3}{r^{k-1}},$$

since $|y|$ is approximately r. It remains to estimate

$$\int_{|y-x|<r/2} \frac{1}{\omega_n|x-y|^{n-1}}|\psi(y)|.$$

Now this can be integrated in polar coordinates around x, which removes the singularity of the fundamental solution, leaving $|\psi(y)|r/2 \leqslant \tilde{C}_4/r^{k-1}$, since again $|y|$ is approximately r. Putting these estimates together completes the proof. □

This shows that if $\not\nabla\phi = O(r^{-k})$ then $\phi = O(r^{-(k-1)})$ (plus a possible log term), where $O(r^{-k})$ is the usual notation for a function which, when multiplied by r^k, is bounded near $r = 0$. A similar argument shows that if $\not\nabla\phi$ is logarithmic, then ϕ is bounded.

These results will be used to construct a correction term for the following parametrix. At each $x \in \tilde{M}$ introduce Riemannian normal coordinates, and trivialise E locally by radial parallel transport. Using a bump function on $\mathbb{R}^n$ which is identically 1 near the origin, it is then straightforward to lift the Euclidean fundamental solution to produce a function $\tilde{G}(x, y)$ which is smooth off the diagonal, equal to the Euclidean fundamental solution in normal coordinates (at x) when y is sufficiently close to x, and zero when y is far from x.

7.11 Proposition. *The inverse of the Dirac operator on $\tilde{M}$ is represented by a fundamental solution $G(x,y) = \tilde{G}(x,y) + O\big(dist(x,y)^{-(n-3)}\big)$. Since $\tilde{G}(x,y) = O\big(dist(x,y)^{-(n-1)}\big)$, it follows that $G(x,y)dist(x,y)^{n-1}$ is bounded. From the construction of $\tilde{G}$ it also follows that in a normal coordinate chart*

$$G(x,y) = \frac{1}{\omega_n}\frac{x-y}{|x-y|^n} + O\left(\frac{1}{|x-y|^{n-3}}\right).$$

Proof: The function $O\big(dist(x,y)^{-(n-3)}\big)$ must be constructed so that $\not{\nabla}^* G(x,.) = \delta_x$. For y close to x introduce normal coordinates centred at x and trivialise the spinor bundles using radial parallel transport. Then $g_{ij}(y) = \delta_{ij} + O(r^2)$, where $r = |y|$. Also, the symbol of the Dirac operator on $\tilde{M}$ differs from the Euclidean Clifford multiplication by a term of order r, and the connection on E differs from the flat connection by a 1-form of order r. It follows that the Dirac operator differs from the Euclidean Dirac operator by a zero order operator Θ of order r. In these coordinates, $\tilde{G}(x,y)$ is the Euclidean fundamental solution near the origin and so if $\not{\nabla}^* = -\not{\nabla}$ is applied, the result is $\delta_x + \Theta\tilde{G}(x,y)$ up to a bounded term, and so the delta distribution is obtained with error $e_1 = O\big(r^{-(n-2)}\big)$. Applying the (truncated) Euclidean fundamental solution gives a first correction term c_1, which is $O\big(r^{-(n-3)}\big)$ by the lemma. Applying the Dirac operator to c_1 corrects the error e_1, but there remains an error $e_2 = \Theta c_1$ (up to a bounded term), which is $O\big(r^{-(n-4)}\big)$. Repeating this process gives further corrections $c_k = O\big(r^{-(n-2-k)}\big)$ to the $O\big(r^{-(n-3)}\big)$ function and the error is reduced to a bounded term. Extending this $O\big(r^{-(n-3)}\big)$ function to $\tilde{M}$, adding it to $\tilde{G}(x,.)$ and applying the Dirac operator gives the delta distribution with a bounded error. The final correction is obtained by applying $\not{\nabla}^{-1}$ to this. □

7.12 Corollary. *If $\psi \in C^\infty(\tilde{M}, \hat{E}^\pm)$ then*

$$((\not{\nabla}^\pm)^{-1}\psi)(x) = \lim_{r\to 0} \int_{y\in\tilde{M}\setminus B_r(x)} \big(G^\pm(x,y)\,,\psi(y)\big)\,,$$

where $B_r(x)$ denotes the ball of radius r (using a metric near x).

In fact it is not necessary to write the integral as a limit, since the integrand is integrable over the n-manifold $\tilde{M}$.

7.13 Theorem. *The Cauchy integral of the restriction of $\phi \in C^\infty(\tilde{M}, E^-)$ to ∂M is given, for $x \in \operatorname{int} M$, by the following formula:*

$$C^+\phi(x) = \phi(x) - \int_M \big(G_x^+\,,\not{\nabla}^+\phi\big)\,.$$

Proof: By 7.3, the Cauchy integral paired with a test function ψ supported in $\operatorname{int} M$, is given by

$$C^+\phi[\psi] = \int_M (\phi\,,\psi) + \int_M \big(\not{\nabla}^+\phi\,,(\not{\nabla}^-)^{-1}\psi\big)\,.$$

Substituting the formula from 7.12 for the inverse of $\not{\nabla}^-$, and changing the order of integration gives

$$C^+\phi[\psi] = \int_{x\in M} (\phi(x)\,,\psi(x)) - \int_{x\in M}\left(\int_M \big(G_x^+\,,\not{\nabla}^+\phi\big)\,,\psi(x)\right),$$

which establishes the result. □

Combining this with the Cauchy integral theorem 7.6 gives:

7.14 The Pompeiu representation formula. *Any smooth spinor field ϕ on M is given at x by the formula*

$$\phi(x) = \int_{\partial M} \big(-G_x^+\,,c(\nu)\phi\big) + \int_M \big(G_x^+\,,\not{\nabla}^+\phi\big)$$

on $\operatorname{int} M$.

This result was obtained in the Euclidean case by Moisil [41], but see also [12, 31].

8 Applications of the Cauchy integral formula

8.1 Proposition (Mean value inequalities). *If ϕ is monogenic near x, then in a normal coordinate chart at x, and for all r sufficiently small,*

$$|\phi(x)| \leqslant \frac{C}{r^{n-1}} \int_{\partial B(x,r)} |\phi|.$$

Integrating $r^{n-1}|\phi(x)|$ from 0 to r gives

$$|\phi(x)| \leqslant \frac{nC}{r^n} \int_{B(x,r)} |\phi|.$$

Proof: Apply the boundedness of $|G(x,y)|\,|x-y|^{n-1}$ to the Cauchy integral formula. □

The next result concerns the extension of monogenic functions to submanifolds. Such removable singularity results are known to exist for arbitrary differential operators (see for example Bochner [9]), but the proof below is interesting, because it is a simple application of the Cauchy integral formula, exactly as in complex analysis.

8.2 Proposition (Removable singularities). *Let S be a compact submanifold of M of codimension $k \geqslant 2$ and suppose ϕ is a smooth function on $M \setminus S$ which is monogenic on* $\operatorname{int} M \setminus S$. *If $\phi(x) dist(x, S)^{k-1} \to 0$ as $x \to S$ then ϕ extends smoothly to S and is monogenic on* $\operatorname{int} M$.

Proof: The idea is to extend ϕ to S using the Cauchy integral formula. To do this the boundary of M must be nonempty, but this is not really a restriction, since there is certainly a manifold with boundary containing S. Let S_ε be a ε-tubular neighbourhood of S in M so that the area of ∂S_ε is bounded by a constant times ε^{k-1} (S has finite volume). Then for $x \in \operatorname{int} M \setminus S$ choose $\delta \leqslant 1$ such that $x \in \operatorname{int}(M \setminus S_\delta)$. Now for any $\varepsilon < \delta$,

$$\phi(x) = \int_{\partial M \cup \partial S_\varepsilon} (-G_x\,, c(\nu)\phi)$$

by the Cauchy integral formula on $M \setminus S_\varepsilon$. But, for fixed x, G_x is bounded on ∂S_ε independently of ε, and so the integrand is of order $o(\varepsilon^{-(k-1)})$. But the area of ∂S_ε is $O(\varepsilon^{k-1})$ and so the integral over ∂S_ε can be made arbitrarily small for small ε. Now the rest of the expression is independent of ε, and so

$$\phi(x) = \int_{\partial M} (-G_x\,, c(\nu)\phi)\,.$$

But this formula defines a monogenic extension of ϕ to S. □

This suggests that a monogenic spinor which does not extend to a surface of codimension $\geqslant 2$ has some sort of "pole" there. Indeed, in the Euclidean case, a residue theory has been developed by Delanghe, Sommen and Souček [19], using the Leray-Norguet residue. There is a simple and direct generalisation to arbitrary Dirac operators (see also [52]).

8.3 Definition. Let S be a closed submanifold of M. Then the space $H(S)$ of monogenic functions on S is defined to be the direct limit of the spaces of smooth monogenic functions on neighbourhoods of S in M; that is, the space of germs of monogenic functions near S.

The idea is to define the residue on S of function ϕ monogenic on $M \setminus S$ as a linear functional on $H(S)$ in such a way that for any ψ monogenic on M,

$$\int_{\partial M} (c(\nu)\phi, \psi) = (\operatorname{Res}_S \phi)\, [\underline{\psi}] \,,$$

where $\underline{\psi}$ is the germ of ψ along S. One can almost take this as the definition of the residue, the main point being to show that the left hand side depends only on the germ of ψ. More precisely let $U \subset\subset V$ be any open neighbourhoods of S in M with smooth boundaries. Now if ϕ is monogenic on $\overline{V} \setminus S$ and ψ is monogenic on $\overline{V}$, then both are monogenic on $\overline{V} \setminus U$ and so by Cauchy's theorem

$$\int_{\partial U} (c(\nu)\phi, \psi) = \int_{\partial V} (c(\nu)\phi, \psi) \,,$$

where ν denotes the outward normal to V and U. It follows that

$$(\operatorname{Res}_S \phi)\, [\underline{\psi}] = \int_{\partial U} (c(\nu)\phi, \psi)$$

is well defined, independent of the choice of a (sufficiently small) neighbourhood U of S and the extension of the germ $\underline{\psi}$ to a monogenic function on U. The notion of residue simply formalises the idea that the bad behaviour of ϕ is local to S, and the following theorem is immediate:

8.4 Residue theorem. *Let S_1, S_2 be disjoint closed submanifolds of* $\operatorname{int} M$ *and suppose that ϕ is monogenic on $M \setminus S_1$ and ψ is monogenic on $M \setminus S_2$. Then*

$$\int_{\partial M} (c(\nu)\phi, \psi) = (\operatorname{Res}_{S_1} \phi)\, [\underline{\psi}] + (\operatorname{Res}_{S_2} \psi)\, [\underline{\phi}] \,.$$

As an example, observe that the residue of G_x on the submanifold $\{x\}$ is just the delta function δ_x. This is just a reformulation of the Cauchy integral formula.

9 Hardy space theory for Dirac operators

Let $\mathcal{M}^\infty(M, E^\mp)$ be the space of $\phi \in C^\infty(M, E^\mp)$ with $\not{\nabla}^\pm \phi = 0$ on $\operatorname{int} M$. Such sections ϕ have boundary values in $C^\infty(\partial M, E^\mp)$. This section is devoted to the following:

9.1 Definition. The *Hardy space* $H^\pm$ is defined to be the closure of the space of boundary values of elements of $\mathcal{M}^\infty(M, E^\mp)$ in the boundary L^2-norm. The orthogonal projection from $L^2(\partial M, E^\mp)$ to $H^\pm$ will be denoted $P^\pm$.

Note that these definitions are intrinsic to M, and that in the conformally invariant case, the boundary L^2-norm (and hence $H^\pm$) are defined without choosing a particular metric.

Cauchy's theorem 5.4 states that H^+ and $c(\nu)H^-$ are orthogonal in $L^2(\partial M, E^-)$. The main goal of this section is to prove that they are orthogonal complements (9.19). This is done by studying the boundary values of the Cauchy integral of $\phi \in C^\infty(\partial M, E^-)$. If $\tilde{\phi}$ is any extension of ϕ to M and $x \in \operatorname{int} M$ then by 7.13,

$$C^+\phi(x) = \tilde{\phi}(x) - \int_M \left(G_x^+, \not{\nabla}^+ \tilde{\phi} \right).$$

However, it is not a priori clear that the integral has a limit as $x \to \partial M$.

9.2 Proposition. *Let $\psi \in \mathrm{L}^2(M, E^+)$, extended by zero to $\tilde{M}$. Then $(\not{\nabla}^+)^{-1}\psi$ is in $\mathrm{L}^2_1(\tilde{M}, E^-)$, and for ψ smooth on M, $(\not{\nabla}^+)^{-1}\psi$ is smooth on $\operatorname{int} M$, where it is given by*

$$(\not{\nabla}^+)^{-1}\psi(x) = \int_{y \in M} (G^+(x,y), \psi(y)) .$$

This is immediate from local elliptic regularity and the representation of $\not{\nabla}^{-1}$ by the fundamental solution. Taking $\psi = \not{\nabla}^+\tilde{\phi}$ on M (and extending by zero) shows that the Cauchy integral of a smooth ϕ is in L^2_1, which at least gives an L^2 trace on the boundary.

It is possible to do much better than this by exploiting the freedom in the choice of the extension $\tilde{\phi}$. In order to find a good extension some technical tools are needed, but these tools are entirely elementary. In fact this approach follows the book of Bell [6] on the Cauchy integral in two dimensions.

First of all a defining function ρ for ∂M needs to be chosen. This is a function on $\tilde{M}$ such that $\rho \neq 0$ on $\operatorname{int} M$ and $\rho = 0, d\rho(\nu) > 0$ on ∂M. Such a function is easily constructed using a partition of unity. The following lemma is the main technical computation (see [6]).

9.3 Lemma. *Let $\psi \in \mathrm{C}^\infty(\tilde{M}, E)$. Then for each $k \geqslant 0$ there is a smooth section ϕ_k which vanishes on ∂M, but such that $\psi - \not{\nabla}\phi_k$ vanishes to order k on ∂M in the sense that $\psi - \not{\nabla}\phi_k = \rho^{k+1}\theta_k$ for some smooth θ_k.*

Proof: Let η be a smooth function which is identically 1 on a neighbourhood of ∂M but vanishes on a neighbourhood of the critical points of ρ and define $c(d\rho)^{-1}$ to be zero on the critical points. Write $\phi_0 = \rho\chi_0$ so that $\not{\nabla}\phi_0 = \rho\not{\nabla}\chi_0 + c(d\rho)\chi_0$. Hence if $\chi_0 = \eta c(d\rho)^{-1}\psi$ then $\psi - \not{\nabla}\phi_0 = \rho\theta_0$ with $\theta_0 = (1-\eta)\psi - \not{\nabla}\chi_0$. Now, continuing by induction on k, write $\phi_k = \phi_{k-1} + \rho^{k+1}\chi_k$. Then $\not{\nabla}\phi_k = \not{\nabla}\phi_{k-1} + \rho^{k+1}\not{\nabla}\chi_k + (k+1)\rho^k c(d\rho)\chi_k = \psi + \rho^k\theta_{k-1} + \rho^{k+1}\not{\nabla}\chi_k + (k+1)\rho^k c(d\rho)\chi_k$. Defining $\chi_k = \frac{1}{k+1}\eta c(d\rho)^{-1}\theta_{k-1}$ gives $\psi - \not{\nabla}\phi_k = (1-\eta)\rho^k\theta_{k-1} - \rho^{k+1}\not{\nabla}\chi_k = \rho^{k+1}\theta_k$ for some θ_k, which proves the lemma. □

9.4 Proposition. *If $\psi \in \mathrm{C}^\infty(M, E^+)$ and $\phi \in \mathrm{C}^\infty(\partial M, E^-)$, then $(\not{\nabla}^+)^{-1}\psi$ and $C^+\phi$ are smooth on M. More precisely, for each $k \geqslant 0$ there are smooth sections ϕ_k vanishing on ∂M and smooth extensions $\tilde{\phi}_k$ of ϕ, such that $\psi - \not{\nabla}^+\phi_k$ and $\not{\nabla}^+\tilde{\phi}_k$ vanish to order k on ∂M. The formulae*

$$\phi_k(x) + \int_{y \in M} (G^+(x,y), (\psi - \not{\nabla}^+\phi_k)(y))$$

and

$$\tilde{\phi}_k(x) - \int_{y \in M} \left(G^+(x,y), \not{\nabla}^+\tilde{\phi}_k(y)\right)$$

then define L^2_{k+1} extensions of $(\not{\nabla}^+)^{-1}\psi$ and $C^+\phi$ from $\operatorname{int} M$ to $\tilde{M}$. (Of course the extensions are arbitrary on $\tilde{M} \setminus M$.)

Proof: The existence of ϕ_k was given in the lemma, and the first formula follows by extending $\not{\nabla}\phi_k - \psi$ by zero to $\tilde{M}$, giving a C^k integrand on $\tilde{M}$. It is therefore in L^2_k and so applying $\not{\nabla}^{-1}$ gives a function is L^2_{k+1}. Next, if $\tilde{\phi}$ is any extension of ϕ then taking $\psi = \not{\nabla}\tilde{\phi}$ it follows that $\tilde{\phi}_k = \tilde{\phi} - \phi_k$ are also extensions of ϕ and the rest easily follows. □

9.5 Definition. The *Cauchy transform* on ∂M is the linear map

$$C^\pm : \mathrm{C}^\infty(\partial M, E^\mp) \to \mathrm{C}^\infty(\partial M, E^\mp)$$

given by restricting the Cauchy integral to the boundary.

Since $C^\infty(\partial M, E^-)$ has a canonical inner product, it is natural to ask whether the Cauchy transform has a formal adjoint.

9.6 Proposition. *Define* $(\mathcal{C}^+)^*\psi = \psi - c(\nu)\mathcal{C}^-\big(c(\nu)\psi\big)$. *Then* $(\mathcal{C}^+)^*$ *is formally adjoint to* $\mathcal{C}^+$. *The analogous result holds for* $\mathcal{C}^-$.

Proof: Let $\tilde{\phi}$ be a (sufficiently good) extension of ϕ to M. Then, omitting the tilde:

$$\begin{aligned}
\int_{\partial M} (\mathcal{C}^+\phi\,,\psi) &= \int_{x\in\partial M}\Big(\phi(x) - \int_{y\in M}\big(G^+(x,y)\,,\not\nabla^+\phi(y)\big)\,,\psi(x)\Big)\\
&= \int_{\partial M}(\phi\,,\psi) - \int_{x\in\partial M}\int_{y\in M}\Big(\big(G^+(x,y)\,,\not\nabla^+\phi(y)\big)\,,\psi(x)\Big)\\
&\overset{(*)}{=} \int_{\partial M}(\phi\,,\psi) - \int_{y\in M}\int_{x\in\partial M}\Big(\not\nabla^+\phi(y)\,,\big(-G^-(y,x)\,,\psi(x)\big)\Big)\\
&= \int_{\partial M}(\phi\,,\psi) - \int_{y\in M}\Big(\not\nabla^+\phi(y)\,,\mathcal{C}^-\big(c(\nu)\psi\big)(y)\Big)\\
&= \int_{\partial M}(\phi\,,\psi) - \int_{y\in\partial M}\Big(c(\nu)\phi(y)\,,\mathcal{C}^-\big(c(\nu)\psi\big)(y)\Big)
\end{aligned}$$

which establishes the proposition, provided that the change of order of integration at (*) is justified. To see this, choose the extension $\tilde{\phi}$ such that $\not\nabla\tilde{\phi}$ vanishes on ∂M. This ensures that the singularity of the fundamental solution does not cause problems as $y \to \partial M$. $\square$

At present $\mathcal{C}^\pm\phi$ has only been defined for smooth ϕ, but for such ϕ the following formula is now straightforward. It will be seen shortly that it holds for all ϕ in L^2.

9.7 Theorem (Kerzman-Stein formula). *For* $\phi \in C^\infty(\partial M, E)$, $\mathcal{C}\phi = P(\phi + (\mathcal{C} - \mathcal{C}^*)\phi)$.

Proof: Simply check $\phi + \mathcal{C}\phi - \mathcal{C}^*\phi = \mathcal{C}\phi + c(\nu)\mathcal{C}\big(c(\nu)\phi\big)$. Now $\mathcal{C}\phi$ is in H and $c(\nu)\mathcal{C}\big(c(\nu)\phi\big)$ is in $H^\perp$ by Cauchy's theorem, and so the theorem is proven. $\square$

The beauty of the Kerzman-Stein formula is that $\mathcal{C} - \mathcal{C}^*$ is a much better behaved operator than $\mathcal{C}$. This will turn out to be a consequence of the following piece of abstract functional analysis (see for example Folland [22]).

9.8 Proposition. *Let* V, W *be vector bundles on a closed* m *dimensional manifold* X, *and* $K(x,y) \in L(V_y, W_x)$ *be a function continuous off the diagonal, such that for some* $\alpha < m$, $K(x,y)dist(x,y)^\alpha$ *is bounded. Such a* K *defines in an obvious way an operator* T_K *from* V *to* W *by integration of the* y *variable against a section of* V. *Then* T_K *is a compact operator from* L^2 *to* L^2, *and* K *is called an integral kernel of order* α.

To apply this result, an analogue of the classical Plemelj formula will be used to give a (singular) integral kernel for $\mathcal{C}$, and then the integral kernel of $\mathcal{C} - \mathcal{C}^*$ will be computed.

The fundamental solution is an integral kernel of order $n-1$ and defines the compact operator $\not\nabla^{-1}$ on $\tilde{M}$. On ∂M, the closely related Cauchy kernel can only define a singular integral operator.

9.9 Definition. The *Hilbert transform* $\mathcal{H}^+$ on $C^\infty(\partial M, E^-)$ is given by the singular integral

$$\mathcal{H}^+\phi(x) = 2\lim_{r\to 0}\int_{\partial M\setminus B_r(x)}\big(-G_x^+\,,c(\nu)\phi\big)\,,$$

where $x \in \partial M$. Similarly one can define $\mathcal{H}^-$.

Of course it is not immediate that $\mathcal{H}^+\phi$ exists as a smooth function on M. The integral kernel used here is twice the Cauchy kernel (used in the Cauchy integral), so naïvely one might expect to obtain twice the Cauchy transform. This is not the case, as the following result on the boundary behaviour of the Cauchy kernel shows (compare Folland [22]).

9.10 Proposition. *If $\phi \in \mathcal{M}^\infty(M, E^-)$ then*

$$\int_{\partial M} (-G_x^+, c(\nu)\phi) = \begin{cases} 0 & \text{for } x \in \tilde{M} \setminus M \\ \phi(x) & \text{for } x \in \operatorname{int} M \end{cases}$$

and for $x \in \partial M$,

$$\lim_{r\to 0} \int_{\partial M \setminus B_r(x)} (-G_x^+, c(\nu)\phi) = \frac{1}{2}\phi(x).$$

Proof: The integral is zero outside M by Cauchy's theorem, and the integral for $x \in \operatorname{int} M$ is $\phi(x)$ by the Cauchy integral formula, so it remains to calculate the singular integral.

Choose a metric near x (if necessary). Since the boundary is differentiable at x, for any $\varepsilon > 0$ there is a $\delta > 0$ such that the image Y of $T_x\partial M$ under the exponential map is close to ∂M in the sense that for all $y \in Y \cap B_\delta(x)$, $dist(y, \partial M) < \varepsilon r$, where $r = dist(x, y)$. Hence $M \cap B_r(x) = \frac{1}{2}B_r(x)$, with an error of order ε for $r < \delta$. Now the integral over $\partial M \setminus B_r(x)$ can be replaced by the integral over $\partial(M \setminus B_r(x))$ provided the integral over $M \cap \partial B_r(x)$ is subtracted. The integral over $\partial(M \setminus B_r(x))$ vanishes by Cauchy's theorem, because G_x^+ and ϕ are both monogenic on $M \setminus B_r(x)$. (The lack of smoothness of the boundary does not cause any problems.) It remains to compute $\lim_{r\to 0} \int_{M\cap\partial B_r(x)} (-G_x^+, c(\nu)\phi)$, where ν is the inward normal to $B_r(x)$. By estimating the integral in normal coordinates using the Euclidean fundamental solution, the limit is easily seen to be $\frac{1}{2}\phi(x)$. □

The last part of this proposition is used to prove the following important result.

9.11 Theorem (Plemelj formula). *For $\phi \in C^\infty(\partial M, E^-)$, $\mathcal{C}^+\phi = \frac{1}{2}(\phi + \mathcal{H}^+\phi)$.*

Proof: This formula can be verified at a point $x \in \partial M$, by finding a monogenic function ϕ_0 with $\phi_0(x) = \phi(x)$. To do this, observe that for $\tilde{x} \in \tilde{M} \setminus M$ close to x, the fundamental solution $G^-(\tilde{x}, x)$ is nondegenerate and so, by contracting with a spinor in $E_{\tilde{x}}^-$, ϕ_0 can be found such that $\not{\nabla}^+\phi_0 = 0$ on M and $\phi_0(x) = \phi(x)$. Consequently $|\phi(y) - \phi_0(y)| \leqslant \text{const.}|y - x|$ for y near x in local coordinates on ∂M. Therefore $\mathcal{C}^+(\phi - \phi_0)(x) = \int_{\partial M} (-G_x^+, c(\nu)(\phi - \phi_0))$ because the integrand is locally integrable, and so

$$\mathcal{C}^+\phi(x) = \mathcal{C}^+\phi_0(x) + \lim_{r\to 0}\Big(\int_{\partial M\setminus B_r(x)} (-G_x^+, c(\nu)\phi) - \int_{\partial M\setminus B_r(x)} (-G_x^+, c(\nu)\phi_0)\Big).$$

Now $\mathcal{C}^+\phi_0(x) = \phi_0(x) = \phi(x)$ and by the lemma, the second integral converges to $\frac{1}{2}\phi_0(x) = \frac{1}{2}\phi(x)$. Hence the first integral converges and the result follows. □

It follows from the Plemelj formula that $\mathcal{C}\phi - \mathcal{C}^*\phi = \frac{1}{2}\big(\mathcal{H}\phi + c(\nu)\mathcal{H}(c(\nu)\phi)\big)$, which is an (a priori singular) integral operator with kernel $A(x, y) = \big(c(\nu_y)G(x, y) + G(x, y)c(\nu_x)\big)$.

9.12 Proposition. *$A(x, y)dist(x, y)^n$ is twice differentiable as a function of y at $y = x$. It vanishes, together with its first derivative at $y = x$, and the second is given by*

$$D_{u,u}^2 A = \frac{1}{\omega_n}\big((Su)u - u(Su)\big),$$

where $Su = D_u\nu$ is the Weingarten map applied to u.

Proof: To compute the limiting behaviour of $\big(c(\nu_y)G(x,y)+G(x,y)c(\nu_x)\big)\mathit{dist}(x,y)^n$ as y approaches x, introduce normal coordinates for M at x, and note that it suffices to work with the Euclidean fundamental solution and the Euclidean distance function, since the error terms are of higher order. Thus the function to be computed as $y \to x$ is $f_x(y) = \frac{1}{\omega_n}\big(\nu_y(x-y)+(x-y)\nu_x\big)$. Now to second order, a point y on a geodesic (in ∂M) starting at x in direction $u \in T_x\partial M$ is given by $y = x+\varepsilon u - \frac{1}{2}\varepsilon^2\langle u\,, D_u\nu\rangle\nu_x + o(\varepsilon^2)$, where ν_x is the normal at x. Also $\nu_y = \nu_x + \varepsilon D_u\nu + o(\varepsilon)$. Therefore:

$$\begin{aligned}
f_x(y) &= \frac{1}{\omega_n}\Big((\nu_x+\varepsilon D_u\nu)(-\varepsilon u + \tfrac{1}{2}\varepsilon^2\langle u\,, D_u\nu\rangle\nu_x) + \big(-\varepsilon u + \tfrac{1}{2}\varepsilon^2\langle u\,, D_u\nu\rangle\nu_x\big)\nu_x\Big) + o(\varepsilon^2)\\
&= \frac{\varepsilon}{\omega_n}\Big(-(\nu_x u + u\nu_x) + \varepsilon\big(\langle u\,, D_u\nu\rangle - (D_u\nu)u\big)\Big) + o(\varepsilon^2)\\
&= \frac{\varepsilon^2}{2\omega_n}\big(u(D_u\nu)-(D_u\nu)u\big) + o(\varepsilon^2),
\end{aligned}$$

since $\nu_x u + u\nu_x = 2\langle\nu_x\,, u\rangle = 0$. This shows that f_x and its first derivative vanish at x, with the second derivative as stated. □

From this the analogue of the theorem of Kerzman and Stein [34] is immediate.

9.13 Theorem. *$\mathcal{C}-\mathcal{C}^*$ is a compact operator on the inner product space $C^\infty(\partial M, E)$.*

Proof: It suffices to prove that $\mathcal{C}-\mathcal{C}^*$ is given by an integral kernel of order $n-2$. As observed above,

$$(\mathcal{C}\phi - \mathcal{C}^*\phi)(x) = -\lim_{r\to 0}\int_{y\in\partial M\setminus B_r(x)} (A(x\,,y),\phi(y))\,.$$

Hence it must be shown that $A(x,y)\mathit{dist}(x,y)^{n-2}$ is bounded, which is only in doubt for y close to x. But the boundedness as $y \to x$ follows from the above proposition, so the integral is not singular, and $\mathcal{C}-\mathcal{C}^*$ is a compact operator. □

REMARK. Booß and Wojciechowski [11] base their analysis of Dirac operators on a very similar result, namely that that $\mathcal{C}-P$ is a compact operator. This is essentially equivalent to the above, since by the Kerzman-Stein formula, $\mathcal{C}-P = P(\mathcal{C}-\mathcal{C}^*)$. However, their proof of this fact involves some delicate estimates following closely the paper of Seeley [49].

9.14 Theorem. *The Cauchy transform $\mathcal{C}^+$ extends to a bounded operator on $L^2(\partial M, E^-)$, with image H^+, and L^2-adjoint $(\mathcal{C}^+)^*$. Hence the Kerzman-Stein formula is valid for any L^2 section, and by the Kerzman-Stein theorem, the Cauchy transform is essentially self-adjoint.*

Proof: Since $\mathcal{C}-\mathcal{C}^*$ is compact, it extends to a bounded operator on $L^2(\partial M, E^-)$. Therefore $P(id+(\mathcal{C}-\mathcal{C}^*))$ is also a bounded operator. By the Kerzman-Stein formula, this defines an extension of the Cauchy transform, and the image is H by definition. It is now immediate that the adjoint of $\mathcal{C}$ is $\mathcal{C}^*$ since they are formally adjoint on the dense subspace of smooth spinor fields. □

9.15 Corollary. *The Hilbert transform is a bounded operator on $L^2(\partial M, E^-)$ and so the Plemelj formula is valid for arbitrary L^2 sections over the boundary.*

Proof: By the Plemelj formula, $\mathcal{H}^+\phi = 2\mathcal{C}^+\phi - \phi$, which is a bounded operator. □

9.16 Proposition. $\operatorname{im} P|_{\ker\mathcal{C}} \cong (\ker\mathcal{C})/(H^\perp\cap\ker\mathcal{C})$ *is finite dimensional.*

Proof: $P - C$ is a compact operator on L^2 and so $P|_{\ker C}$ is also compact. Let $K_C = H^\perp \cap \ker C$ be its kernel. Then $\operatorname{im}(P|_{\ker C}) = \operatorname{im}(P|_{\ker C \cap K_C^\perp})$. Now P is a projection injective on this closed subspace of L^2, so it is bounded below. Therefore $\operatorname{im}(P|_{\ker C})$ is a closed subspace of L^2. Since P is a compact operator this must be finite dimensional. □

These results, while interesting, are not intrinsic to M in that the Cauchy and Hilbert transforms involve the fundamental solution of $\not{\nabla}$ on the closed manifold $\tilde{M}$. The intrinsic analysis of M is captured by the Hardy spaces $H^\pm$ with their associated projections $P^\pm$. The above work establishes three important properties of H and P.

Firstly, functions in H have well defined interior values, given by the Cauchy integral. It was shown in 7.2 that the Cauchy integral is bounded from $L^2(\partial M, E)$ to $L^2(M, E)$. There is also the following result, a simple case of a nice argument in Seeley [49].

9.17 Proposition. *The Cauchy integral is bounded from $H \cap L^2_1(\partial M, E)$ to $L^2_1(M, E)$.*

Proof: It suffices to establish the result for $\psi \in \mathcal{M}^\infty(M, E)$, so that $\psi|_{\partial M}$ is smooth. (Note that $C(\psi|_{\partial M}) = \psi$ on M.) To do this choose a smooth extension $\tilde{\psi}$ of $\psi|_{\partial M}$ to $\tilde{M}$. It is possible to do this so that the L^2_1-norm of $\tilde{\psi}$ is controlled by the L^2_1-norm of $\psi|_{\partial M}$. Now let ϕ equal ψ on M and $\tilde{\psi}$ on $\tilde{M} \setminus M$. This will not be smooth, but in fact lies in $L^2_1(\tilde{M}, E)$ with $\not{\nabla}\phi = 0$ on $\operatorname{int} M$ and equal to $\not{\nabla}\tilde{\psi}$ on $\tilde{M} \setminus \operatorname{int} M$. One way to see this is to observe that ϕ is certainly in L^2 and compute its weak derivative, using small neighbourhoods U_δ of ∂M in $\tilde{M}$, and the Green formula on $\tilde{M} \setminus U_\delta$. In any case, it follows that the L^2-norm of $\not{\nabla}\phi$ on $\tilde{M}$ is bounded by the L^2_1-norm of $\tilde{\psi}$, and hence the L^2_1-norm of ψ is controlled by the L^2_1-norm of $\psi|_{\partial M}$. □

The second result is a regularity result for P:

9.18 Theorem. *If ϕ is smooth on ∂M, then so is $P\phi$.*

Proof: By the Kerzman-Stein formula $PC = C$ and $P(id - C^*) = 0$. Therefore $C, id - C^*$ have orthogonal images and so $\|(id + C - C^*)\phi\|^2 = \|C\phi\|^2 + \|(id - C^*)\phi\|^2$. This is zero iff $C\phi = 0$ and $C^*\phi = \phi$, which only holds if $\langle \phi, \phi \rangle = \langle \phi, C^*\phi \rangle = \langle C\phi, \phi \rangle = 0$, and so $\mathcal{F} = id + C - C^*$ is injective. But $C - C^*$ is compact, and so $\mathcal{F}$ is Fredholm of index zero on L^2, and hence is invertible. Now $\mathcal{F}$ and $\mathcal{F}^*$ both map smooth functions to smooth functions, and hence $\mathcal{F}$ is an invertible map on smooth functions. The result now follows because $P = C\mathcal{F}^{-1}$. □

Finally, there is the theorem whose proof was the main goal of this section:

9.19 Theorem. *The spaces H^+ and $c(\nu)H^-$ are orthogonal complements in $L^2(\partial M, E^-)$.*

Proof: By Cauchy's theorem, these spaces are orthogonal, so suppose $\phi \in H^\perp$. Then $0 = \langle C\psi, \phi \rangle = \langle \psi, C^*\phi \rangle$ for all ψ, and so $\phi = c(\nu)C\big(c(\nu)\phi\big)$ by definition of C^*. Thus $\phi \in c(\nu)H$. □

Booß and Wojciechowski [11] refer to this as the "twisted orthogonality of the boundary data" and use it to present an extensive survey of global elliptic boundary value problems for Dirac operators. The prototype is the following:

9.20 Proposition. *For $\psi \in L^2(M, \hat{E}^+)$, the equation $\not{\nabla}^+\phi = \psi$ and $P^+(\phi|_{\partial M}) = 0$ has a unique solution $\phi \in L^2_1(M, E^-)$, and there is a bound $\|\phi\|_{L^2_1} \leqslant \text{const.}\|\psi\|_{L^2}$. Also, if $\psi \in C^\infty(M, E)$ then so is ϕ.*

Proof: For ψ smooth, let $\tilde{\psi}$ be a smooth extension to $\tilde{M}$, otherwise extend ψ by zero. Let $\phi_0 = \not{\nabla}^{-1}\tilde{\psi}$. The solution ϕ, which is clearly unique, is obtained by subtracting from ϕ_0 the monogenic extension of $P(\phi_0|_{\partial M})$. If ψ is smooth, so is $P(\phi_0|_{\partial M})$ and hence the monogenic extension is in $\mathcal{M}^\infty(M,E)$. Therefore $\phi \in \mathrm{C}^\infty(M,E)$. In general, $\phi_0 = \not{\nabla}^{-1}\tilde{\psi} \in \mathrm{L}^2_1(\tilde{M}, E^-)$, where $\tilde{\psi}$ is now the extension by zero. It follows that $C(\phi_0|_{\partial M}) = 0$ (using the formula 7.13) and so $\phi_0|_{\partial M}$ lies in $\ker C$. But P is a smoothing operator on this space (by 9.16 and 9.18), and so it is bounded from L^2 to L^2_1. The Cauchy integral is bounded on L^2_1 by 9.17, and so $\phi \in \mathrm{L}^2_1$ with the bound as stated. □

Although the proof of this result uses $\not{\nabla}^{-1}$ on $\tilde{M}$, it is clearly intrinsic to M. In fact, it provides sufficient analytical information to remove $\tilde{M}$ from the picture. In this spirit, the Cauchy kernel and the fundamental solution on $\tilde{M}$ will be replaced by two integral kernels canonically associated to M, the Szegő kernel and the Green kernel.

For $x \in \operatorname{int} M$, the Cauchy kernel $-c(\nu)G^+_x$ represents the Cauchy integral $\phi \mapsto C^+\phi(x)$ of $\phi \in \mathrm{L}^2(\partial M, E^-)$. For $\phi \in H^+$, this reproduces the value at x of the monogenic extension of ϕ, which is intrinsic to M. Define the *Szegő kernel* by $\mathcal{S}^+_x = P^+\big(-c(\nu)G^+_x\big)$, Then:

9.21 Proposition. *The Szegő kernel represents the functional $\mathcal{S}^+_x[\phi] = C^+(P^+\phi)(x)$. It is smooth on ∂M and lies in H^+, so it has interior values, $\mathcal{S}^+_x \in \mathcal{M}^\infty(M, E^-)$ given by $\mathcal{S}^+_x(y) = \int_{\partial M} \big(\mathcal{S}^+_y\,, \mathcal{S}^+_x\big)$. Thus $\mathcal{S}^+(x,y) = \mathcal{S}^+_x(y)$ is monogenic in $x, y \in \operatorname{int} M$, and $\mathcal{S}^+(y,x)^\tau = \mathcal{S}^+(x,y)$. (Note, though, that $\mathcal{S}^+(x,x)$ becomes singular on the boundary.)*

From the definition, $\mathcal{S}^+_x = -c(\nu)G^+_x - c(\nu)\Phi^+_x$ (on ∂M) for some smooth $\Phi^+_x \in H^-$, which therefore extends to a monogenic function on M. Define the *Green kernel* on M by $\mathcal{G}^+_x = G^+_x + \Phi^+_x$, so that $-\not{\nabla}^-\mathcal{G}^+_x = \delta_x$ on $\operatorname{int} M$ and on the boundary $\mathcal{G}^+_x = c(\nu)\mathcal{S}^+_x \in c(\nu)H^+$.

9.22 Proposition. *For $x \neq y$ in $\operatorname{int} M$, $\mathcal{G}^-(y,x)^\tau = -\mathcal{G}^+(x,y)$.*

Proof: Observe that $\int_{\partial M} \big(\mathcal{G}^+_x\,, c(\nu)\mathcal{G}^-_y\big) = 0$ and apply the residue theorem 8.4: the residue of $\mathcal{G}^+_x$ at x applied to $\mathcal{G}^-_y$ gives $\mathcal{G}^-_y(x)$ and similarly the residue at y gives $\mathcal{G}^+_x(y)$. □

9.23 Corollary. *For each fixed $y \in \operatorname{int} M$, $\mathcal{G}^+_x(y)$, as a function of $x \in \partial M$, lies in $(H^+)^\perp$. Therefore if $\phi(x) = \mathcal{G}^+_x[\psi]$ then $\not{\nabla}^+\phi = \psi$ and $P^+(\phi|_{\partial M}) = 0$. In other words $\mathcal{G}^+[\phi]$ solves the boundary problem 9.20.*

I will finish this section by giving a more concrete description of the Hardy space H, showing how it generalises the two dimensional theory. So far H has only been described as an L^2-closure, whereas one would like to see that it is a space of boundary values of suitably well behaved monogenic functions on $\operatorname{int} M$, and give some sort of characterisation. To do this, using a metric near ∂M, introduce the normal geodesic flow from the boundary (a local 1-parameter family of diffeomorphisms), which identifies $\partial M \times [0,\delta]$ with a neighbourhood of ∂M in M, for some small δ. Trivialise E in the normal direction by using parallel transport along normal geodesics. Let $M_\varepsilon = M \setminus (\partial M \times [0,\varepsilon])$ and let r_ε denote the restriction map from functions on M to functions on ∂M given by restricting to ∂M_ε and identifying with ∂M. The main result to be established is the following.

9.24 Theorem. *For $\phi \in \mathcal{M}^\infty(M,E)$*

$$\int_{\partial M_\varepsilon} (\phi\,,\phi) \leqslant \text{const.} \int_{\partial M} (\phi\,,\phi)\,,$$

for some constant independent of ϕ and ε.

Proof: The integral of (ϕ, ϕ) over ∂M_ε, denoted $I(\varepsilon)$, is smooth with respect to ε, for $\varepsilon \in [0, \delta]$. It will be shown that $I'(\varepsilon) \leqslant \lambda I(\varepsilon)$, for a constant λ independent of ϕ and ε. Integrating this inequality from 0 to ε gives $I(\varepsilon) \leqslant e^{\lambda\varepsilon} I(0) \leqslant e^{\lambda\delta} I(0)$. To estimate $I'(\varepsilon)$, identify ∂M_ε with ∂M and let vol_ε be the volume form on ∂M_ε pulled pack to ∂M. Note that the outward normal to ∂M_ε is identified with the outward normal at ∂M. Therefore

$$\begin{aligned}\frac{d}{d\varepsilon} \int_{\partial M_\varepsilon} (\phi, \phi) &= \frac{d}{d\varepsilon} \int_{\partial M} (r_\varepsilon \phi, r_\varepsilon \phi)\, vol_\varepsilon \\ &= \int_{\partial M_\varepsilon} -2(\hat{D}^E_\nu \phi, \phi) + \int_{\partial M_\varepsilon} (\phi, \phi) \frac{vol'_\varepsilon}{vol_\varepsilon} \\ &= \int_{M_\varepsilon} -2(\hat{D}^E \phi, \hat{D}^E \phi) + \int_{M_\varepsilon} -2\,(K\phi, \phi) + \int_{\partial M_\varepsilon} (\phi, \phi) \frac{vol'_\varepsilon}{vol_\varepsilon},\end{aligned}$$

by the Bochner-Weitzenböck integral formula. The first integral is negative, and the second is bounded in terms of $\int_{M_\varepsilon} (\phi, \phi)$. But ϕ on M_ε is given by its Cauchy integral, which is L^2-bounded by 7.2. Therefore the second and third integrals are bounded by $I(\varepsilon)$. □

9.25 Corollary. *The Cauchy integral of a function ϕ in H (which exists as an L^2 monogenic function on* $\operatorname{int} M$ *by 7.2) is a smooth function ψ on* $\operatorname{int} M$ *with $r_\varepsilon \psi$ bounded in $L^2(\partial M, E)$ independent of ε. Furthermore $r_\varepsilon \psi \to \phi$ in L^2 as $\varepsilon \to 0$.*

Proof: Approximate ϕ by boundary values of $\phi_k \in \mathcal{M}^\infty(M, E)$. It is immediate then that the L^2 estimate applies to ϕ. Therefore it also applies to $\phi - r_0\phi_k$ and so in the estimate

$$\|r_\varepsilon C\phi - \phi\| \leqslant \|r_\varepsilon(C\phi - \phi_k)\| + \|r_\varepsilon \phi_k - r_0\phi_k\| + \|r_0\phi_k - \phi\|,$$

the first term is bounded by a constant multiple of $\|\phi - r_0\phi_k\|$. Hence, like the last term, it can be made arbitrarily small for large k. Now $\|r_\varepsilon\phi_k - r_0\phi_k\|$ approaches zero with ε since it is a continuous function of $\varepsilon \geqslant 0$ (ϕ_k being continuous on M). □

Conversely there is the following result.

9.26 Proposition. *Suppose that ψ is monogenic on* $\operatorname{int} M$ *with $\int_{\partial M} (r_\varepsilon \psi, r_\varepsilon \psi)$ is bounded independent of ε. Then ψ is a Cauchy integral of a function ϕ on the boundary with $\phi \in H$, and so $r_\varepsilon \psi \to \phi$ in norm.*

Proof: Since every bounded sequence in L^2 has a weakly convergent subsequence (Banach-Alaoglu), there is a sequence of values of ε with $r_\varepsilon \psi$ converging weakly to a function ϕ in $L^2(\partial M, E)$. Now ψ is monogenic on $\operatorname{int} M$ and so

$$\begin{aligned}C\phi - \psi &= C\phi - C_\varepsilon(\psi|_{\partial M_\varepsilon}) \\ &= C(\phi - r_\varepsilon\psi) + C(r_\varepsilon \psi) - C_\varepsilon(\psi|_{\partial M_\varepsilon})\end{aligned}$$

The first term can be made arbitrarily small by weak convergence, while the remaining terms are small for fixed x in $\operatorname{int} M$ because G_x on ∂M_ε converges uniformly to G_x on ∂M. To see that $\phi \in H$ it suffices to show that $\int_{\partial M} (\phi, c(\nu)\theta) = 0$ for all $\theta \in \mathcal{M}^\infty(M, E)$. But this follows from $\int_{\partial M_\varepsilon} (\psi, c(\nu)\theta) = 0$, by taking a weakly convergent subsequence, and using the uniform convergence of G_x again. Although only weak convergence of a subsequence has been used so far, the strong convergence now follows from 9.25. □

To summarise, H is a space of L^2 boundary values of monogenic functions on the interior, and the Cauchy integral is an isomorphism between H and the space of monogenic functions on $\operatorname{int} M$ with uniformly bounded L^2-norm on hypersurfaces near ∂M.

10 The Green kernel and boundary value problems

The relationship between a spinor field and its boundary values is central in this section.

10.1 Theorem. *Suppose* $\phi \in \mathrm{L}^2_1(M, E^-)$. *Then for any* θ *in* $\mathcal{M}^\infty(M, E^+)$,

$$\int_{\partial M} (c(\nu)\phi\,,\theta) = \int_M (\not{\nabla}^+\phi\,,\theta)\,.$$

Hence for $\psi \in \mathrm{L}^2(M, E^+)$,

$$\int_{\partial M} (c(\nu)\mathcal{G}^+[\psi]\,,\theta) = \int_M (\psi\,,\theta)\,.$$

Proof: θ is monogenic, so this is just the Green formula. □

10.2 Corollary. *For* $\psi \in \mathrm{L}^2(M, \hat{E}^+)$ *and* $\chi \in \mathrm{L}^2(\partial M, E^-)$ *the equation* $\not{\nabla}^+\phi = \psi$ *on* M, $\phi = \chi$ *on* ∂M, *has a solution iff* $\int_{\partial M} (c(\nu)\chi\,,\theta) = \int_M (\psi\,,\theta)$ *for all* θ *in* $\mathcal{M}^\infty(M, E^+)$. *Note that in general, the boundary values are attained in an* L^2 *sense.*

Proof: By the first part of the theorem, the compatibility condition on (ψ, χ) is necessary. Conversely take $\phi_0 = \mathcal{G}[\psi]$. Then it suffices to show that $\phi_0 - \chi$ is a boundary value of a monogenic function, which by 9.19 follows if $\phi_0 - \chi$ is orthogonal to $c(\nu)H$ on the boundary. But by the second part of the theorem, this is precisely the compatibility condition. □

10.3 Definition. The *Bergman space* $H^2(M, E^\pm)$ is defined as the closure of $\mathcal{M}^\infty(M, E^\pm)$ in $\mathrm{L}^2(M, E^\pm)$. By 6.2, its elements lie in the kernel of $\not{\nabla}^\mp$. The orthogonal projection $\mathcal{B}^\pm$ onto the Bergman space is called the *Bergman projection*.

10.4 Corollary (to 10.1). *If* $\phi|_{\partial M} \in H^+$ *then* $\not{\nabla}^+\phi \perp H^2(M, E^+)$, *and if* $\psi \perp H^2(M, E^+)$ *then* $\mathcal{G}^+[\psi]|_{\partial M} = 0$. *Therefore the orthogonal complement to the Bergman space is the image of* $\not{\nabla}^+$ *on the space of* L^2_1 *functions vanishing on* ∂M. *Consequently, smooth sections are dense in* $H^2(M, E^+)^\perp$ *and so any* L^2 *solution of* $\not{\nabla}^-\phi = 0$ *is in* $H^2(M, E^+)$.

10.5 Proposition. *The image of the Szegő integral* $\mathcal{S}^+ \colon \phi \mapsto C^+(P^+\phi)$ *lies in the Bergman space, and its adjoint is* $\psi \mapsto c(\nu)(\mathcal{G}^+[\psi])|_{\partial M}$, *with image in the Hardy space.*

The Bergman projection is related to boundary value problems for the Laplacian $\not{\nabla}^2$.

10.6 Proposition. *The solution (in* L^2_1*) to the problem* $\not{\nabla}^2\phi = \psi$, $\phi|_{\partial M} = 0$ *(where* $\psi \in \mathrm{L}^2$*) is given by* $\phi = \mathcal{G}\big[(id - \mathcal{B})\mathcal{G}[\psi]\big]$.

Proof: Clearly $\not{\nabla}\phi = (id - \mathcal{B})\mathcal{G}[\psi]$, and applying $\not{\nabla}$ again kills $\mathcal{B}\mathcal{G}[\psi]$ leaving ψ. The boundary condition $\phi|_{\partial M} = 0$ holds because $(I - \mathcal{B})\mathcal{G}[\psi]$ is orthogonal to $H^2(M, E)$. □

Strictly speaking, 10.6 only gives a distributional solution, but by local elliptic regularity ϕ is in L^2_2 on compact subsets of $\operatorname{int} M$. One would like to see directly that $\phi \in \mathrm{L}^2_2(M, E)$, but it has not been shown that $\mathcal{B}$ is bounded on L^2_1. A priori then, it is possible that $\not{\nabla}\phi$ behaves badly near ∂M. However, by the Bochner-Weitzenböck integral formula, ϕ satisfies $\int_M (\hat{D}^E\phi\,, \hat{D}^E\theta) + \int_M (K\phi + \psi\,, \theta) = 0$ for all $\theta \in \mathrm{C}^\infty_0(M, E)$, and so standard arguments (such as difference quotients) give a global L^2_2 bound for ϕ. In other words:

10.7 Proposition. *For any* $\psi \in \mathrm{L}^2$, *let* ϕ *denote a solution to* $\not{\nabla}^2\phi = \psi$ *and* $\phi|_{\partial M} = 0$. *Then* $\|\phi\|_{\mathrm{L}^2_2} \leqslant \mathrm{const.}\|\psi\|_{\mathrm{L}^2}$, *the constant being independent of* ψ.

It is now clear that there is a well defined solution operator $\mathcal{Q}\colon \mathrm{L}^2(M,E) \to \mathrm{L}^2_2(M,E)$, and $id - \mathcal{B} = \not\nabla \circ \mathcal{Q} \circ \not\nabla$, either by 10.6, or by noting that $\mathcal{Q}\not\nabla\phi|_{\partial M} = 0$ and using 10.4. Regularity results for $\mathcal{B}$ can now be deduced from regularity results for $\mathcal{Q}$. It can be shown, in fact, that if $\psi \in \mathrm{L}^2_j(M,E)$ then $\mathcal{Q}\psi \in \mathrm{L}^2_{j+2}(M,E)$, and hence if $\psi \in \mathrm{C}^\infty(M,E)$ then $\mathcal{Q}\psi$ is also smooth up to the boundary, not just on int M. Consequently:

10.8 Proposition. *$\mathcal{B}$ is bounded on L^2_j and maps $\mathrm{C}^\infty(M,E)$ into itself.*

Next suppose that $\chi \in \mathrm{L}^2(\partial M, E)$ has a L^2_2 extension. Then, by the regularity of $\mathcal{Q}$, it has unique Poisson extension $\mathcal{P}\chi$ satisfying $\not\nabla^2\mathcal{P}\chi = 0$. In fact a Poisson extension exists for more general χ. For example, if χ is in H then the Cauchy (or Szegő) integral gives the required extension. Now suppose that ϕ is in $\mathrm{L}^2_1(M,E)$. If $\phi|_{\partial M} \in H$ then $\mathcal{B}\not\nabla\phi = 0$ by 10.4. More generally, $\mathcal{G}[\mathcal{B}\not\nabla\phi]|_{\partial M} = \mathcal{G}[\not\nabla\phi]|_{\partial M}$, which gives the orthogonal projection of $\phi|_{\partial M}$ onto $H^\perp$. Consequently if $\phi|_{\partial M} \in H^\perp$ then $\mathcal{G}[\mathcal{B}\not\nabla\phi]$ gives a Poisson extension independent of the chosen L^2_1 extension ϕ.

10.9 Theorem. *$\mathcal{Q} = \mathcal{G} \circ (id - \mathcal{B}) \circ \mathcal{G}$ and for $\phi \in \mathrm{L}^2_1(M,E)$:*

$$\begin{aligned}
\mathcal{P}(\phi|_{\partial M}) &= \mathcal{S}[\phi|_{\partial M}] + \mathcal{G}[\mathcal{B}\not\nabla\phi] \\
\mathcal{B}\phi &= \not\nabla\mathcal{P}(\mathcal{G}[\phi]|_{\partial M}) = \phi - \not\nabla\mathcal{Q}(\not\nabla\phi) \\
\not\nabla\mathcal{P}(\phi|_{\partial M}) &= \mathcal{B}\not\nabla\phi \\
\mathcal{G}[\mathcal{B}\phi] &= \mathcal{P}(\mathcal{G}[\phi]|_{\partial M}) = \mathcal{G}\phi - \mathcal{Q}(\not\nabla\phi).
\end{aligned}$$

The integral kernels of these operators will now be briefly studied. Firstly, it is clear from the mean value inequalities that $\phi \mapsto \mathcal{B}\phi(x)$ is continuous for each $x \in \operatorname{int} M$, and so:

10.10 Proposition. *$\phi \mapsto \mathcal{B}\phi(x)$ is represented by an integral kernel $\mathcal{B}_x \in \mathrm{C}^\infty(M,E)$.*

This is the *Bergman kernel* and is a reproducing kernel on $H^2(M,E)$. Similarly $\mathcal{Q}$ is represented by $\mathcal{Q}_x = \mathcal{Q}\delta_x$, the Green function for $\not\nabla^2$ on M, and the Poisson extension is represented by the Poisson kernel $\mathcal{P}_x$. From the Green formula

$$\int_M \left((\not\nabla^2\phi,\psi) - (\phi,\not\nabla^2\psi)\right) = \int_{\partial M} \left((c(\nu)\not\nabla\phi,\psi) - (\phi, c(\nu)\not\nabla\psi)\right)$$

and the fact that $\mathcal{Q}_x|_{\partial M} = 0$, it easily follows that $\mathcal{Q}(x,y)$ is symmetric, and that $\mathcal{P}_x = c(\nu)(\not\nabla\mathcal{Q}_x|_{\partial M})$, which is a normal derivative of $\mathcal{Q}_x$. Using the formulas in 10.9, one can establish other identities between the kernels. For example:

10.11 Theorem. *The Bergman kernel is given by $\mathcal{B}_x = \not\nabla^{(x)}(\mathcal{G}_x + \not\nabla\mathcal{Q}_x) = \not\nabla(\mathcal{S}[c(\nu)\mathcal{P}_x])$.*

In this way, many results from potential theory in the plane can be seen to have direct generalisations to Dirac operators on manifolds with boundary.

Rather than develop these ideas further, an approximation result for $\not\nabla$ will be deduced from 9.20 together with unique continuation, following Lax [36].

10.12 Theorem. *Let Ω be an open subset of M. Then any monogenic function on Ω may be approximated (locally uniformly in all derivatives) by monogenic functions on M.*

Proof: By 6.2 it suffices to prove approximation in L^2 for any Ω_0 compactly contained in Ω. Let V_1 be the space of monogenic functions on Ω and V_2 the space of restrictions to Ω of monogenic functions on M. The idea is to suppose $\psi \perp V_2$ in $\mathrm{L}^2(\Omega_0, E^-)$, and show that

$\psi \perp V_1$. To do this, extend ψ by zero to M and solve the (adjoint) equation $\not{\nabla}^- \phi = \psi$ on M with $\phi|_{\partial M} = 0$. This is possible by 9.20 and 10.1, and by the unique continuation property, $\phi = 0$ on $M \setminus \Omega_0$. Now apply the Green formula on any smoothly bounded domain Ω_1 sandwiched between Ω and Ω_0, and using any monogenic θ on Ω:

$$0 = \int_{\partial\Omega_1} (c(\nu)\phi, \theta) = \int_{\Omega_1} (\not{\nabla}^- \phi, \theta) + \int_{\Omega_1} (\phi, \not{\nabla}^+ \theta) = \int_{\Omega_0} (\psi, \theta)$$

and so ψ is orthogonal to V_2 as required. □

10.13 Theorem (Integrability of the Dirac equation). *Suppose $\not{\nabla} = c \circ D^E$. Then for any $x \in \operatorname{int} M$, $\xi \in E_x$ and $\alpha \in \ker c \leqslant T_x^* \otimes E_x$ there is a monogenic ϕ on M with $\phi(x) = \xi$ and $D^E \phi(x) = \alpha$.*

Proof: Let ψ be any spinor field with $\psi(x) = \xi$ and $D^E \psi(x) = \alpha$, so $\not{\nabla}\psi(x) = 0$. Firstly it will be shown that ψ can be approximated by monogenic functions. To do this, first work on a small ball $B_r(x)$ around x, and for $y \in B_r(x)$, construct the Cauchy integral

$$\phi(y) = \int_{\partial B_r(x)} (-G_y, c(\nu)\psi) = \psi(y) - \int_{z \in B_r(x)} (G_y(z), \not{\nabla}\psi(z)).$$

Since $\not{\nabla}\psi(z)$ vanishes at x it may be written locally as $(z - x)\chi(z)$ for some bounded χ, and so the integrand is approximately $\frac{\chi(z)}{|z-x|^{n-2}}$. The integral over $B_r(x)$ is therefore order r^2, with derivative of order r. Hence both ϕ and its covariant derivative are close to those of ψ at x and the approximation is arbitrarily close on a small enough neighbourhood. But for each such neighbourhood, ϕ may be approximated arbitrarily closely by a monogenic function on M, by 10.12, hence so can ψ.

Now apply this approximation result to a basis for $E_x \oplus \ker c$. For a sufficiently good approximation, the corresponding monogenic functions will also form a basis, and the result now follows from the linearity of the Dirac operator. □

This is in marked contrast to closed manifolds, on which $\ker \not{\nabla}$ is only finite dimensional.

11 The Szegő kernel and conformal geometry

The analytical results will now be applied to the particular case of the conformally invariant Dirac operator on a manifold with boundary. The aim is to show that the Cauchy integral formula defines a conformally invariant metric on the interior of M, which is complete and has negative scalar curvature. This was established by Hitchin [33] in the Euclidean case, using arguments which easily generalise to arbitrary spin manifolds with boundary. However, the proof that this metric has negative (rather than nonpositive) scalar curvature uses the integrability result 10.13, and this relies upon the full analytical theory developed above. Also the analysis adds some flesh to the constructions below, by identifying the conformally invariant Hilbert space H as a Hardy space, rather than an abstract L^2-closure, and providing useful information about the intimately related Szegő kernel.

The conformally invariant metric arises as follows. By 7.7 the evaluation map $ev_x \colon H \to E_x$ at each $x \in \operatorname{int} M$ (given on smooth functions by the Cauchy integral) is bounded. Now because E_x has an L_x^{n-1} valued inner product, the norm squared of ev_x is an element of L_x^{n-1}, rather than $\mathbb{R}$. If it can be shown that this defines a (smooth) trivialisation of L^{n-1}

then this trivialisation equips $\operatorname{int} M$ with a (smooth) metric defined canonically in terms of the conformal structure.

A simple way to obtain a smoothly varying norm is to observe that the evaluation map ev_x on H is represented by the Szegő kernel $S_x \in H \otimes E_x$. The L^2-norm of this, contracted to lie in L_x^{n-1}, is clearly smooth in x, since by the reproducing property of the Szegő kernel, it is given by $\langle S(x,x) \rangle$, where the angle brackets denote the the contraction in E_x.

11.1 Proposition. *The norm $\langle S(x,x)\rangle$ of ev_x is nonzero for all $x \in \operatorname{int} M$.*

Proof: It suffices to show that for each $x \in \operatorname{int} M$ there is a monogenic spinor on M which is nonvanishing at x. This follows from 10.13, but there is a more explicit approach given by reintroducing the closed manifold $\tilde{M}$. The claim is that for each $x \in \operatorname{int} M$ there is a $y \in \tilde{M} \setminus M$ such that $G_y(x) \neq 0$. Now by 7.5 it suffices to show that there is such a y with $G_x(y) \neq 0$. But if G_x is zero on an open subset of $\tilde{M} \setminus M$, it must be zero on $M \setminus \{x\}$ by unique continuation. This contradicts the fact that it is the fundamental solution at x. □

Thus $\langle S(x,x)\rangle$ is a smooth trivialisation of L^{n-1} over $\operatorname{int} M$ and so defines a conformally invariant metric there. Following Hitchin [33], a slightly different point of view will be adopted in order to establish that this metric has negative scalar curvature.

11.2 Proposition. *The L^2-norm $\langle S(x,x)\rangle$ gives the Hilbert-Schmidt norm of ev_x, defined by $\|ev_x\|_{HS}^2 = \operatorname{tr}(ev_x \circ ev_x^*) = \operatorname{tr}(ev_x^* \circ ev_x)$, where $ev_x^* \colon E_x^* \to H$ is the transpose of ev_x. Hence if $\phi_k \in \mathcal{M}^\infty(M,E)$ form an orthonormal basis for H, then $\|ev_x\|_{HS}^2 = \sum (\phi_k(x), \phi_k(x))$.*

Proof: The equality of the two traces is elementary since E_x is finite dimensional. Now for $\psi_x \in E_x$, the transpose $ev_x^*(\psi_x)$ is given by $\sum \langle \psi_x, ev_x \phi_k \rangle \phi_k$, and so $\operatorname{tr}(ev_x \circ ev_x^*)$, is given by $\sum \langle \psi_j, ev_x \phi_k \rangle^2$, where ψ_j form an orthonormal basis for E_x. This is clearly the L^2-norm of S. The final expression is the other trace, namely $\operatorname{tr}(ev_x^* \circ ev_x)$. □

11.3 Corollary. *If $(.\,,.)$ denotes the conformally invariant metric, $\sum (\phi_k\,, \phi_k) = 1$.*

Proof: The metric was defined by identifying the norm in L^{n-1} with $1 \in \mathbb{R}$. □

11.4 Proposition. *$\sum (\phi_k\,, \phi_k)$ converges in $C^\infty(\operatorname{int} M, L^{n-1})$; that is, all derivatives converge uniformly on compact subsets.*

Proof: The sum converges pointwise to a continuous limit, so by Dini's theorem it converges locally uniformly, and hence locally in L^2. Convergence in all derivatives can be established by the same technique as is often used to prove local elliptic regularity. Namely, it suffices to show that $\sum \rho\,(\phi_k\,, \phi_k)$ converges in L^2_j for all j and all bump functions ρ. This follows by induction on j, using the elliptic estimate for each ϕ_k. □

The analytical tools are now in place to prove the following theorem, due to Hitchin in the Euclidean case—the proof on general spin manifolds is not materially different.

11.5 Theorem. *The conformally invariant metric has negative scalar curvature.*

Proof: Let the smooth sections ϕ_k form an orthonormal basis of H. Then by the Lichnerowicz formula (see 3.11 and 5.7) the following holds for each k and at each $x \in \operatorname{int} M$:

$$(D^E\phi_k(x)\,, D^E\phi_k(x)) + \tfrac{1}{4}\kappa(x)(\phi_k(x)\,, \phi_k(x)) = \operatorname{div}(D^E\phi_k\,, \phi_k)(x) = \operatorname{div}(\phi_k\,, D^E\phi_k)(x).$$

But $(D^E\phi_k\,, \phi_k) + (\phi_k\,, D^E\phi_k) = d(\phi_k\,, \phi_k)$, so

$$(D^E\phi_k(x)\,, D^E\phi_k(x)) + \tfrac{1}{4}\kappa(x)(\phi_k(x)\,, \phi_k(x)) = \tfrac{1}{2}\Delta(\phi_k\,, \phi_k)(x).$$

Now sum this formula over k. Since $\sum\langle\phi_k(x)\,,\phi_k(x)\rangle = 1$ (locally uniformly in all derivatives), the second term is summable, and the third term sums to $\Delta 1 = 0$. Hence:

$$\frac{1}{4}\kappa(x) = -\sum\langle D^E\phi_k(x)\,, D^E\phi_k(x)\rangle \leqslant 0.$$

Therefore the scalar curvature is negative at a point x iff there is a monogenic spinor on M with nonvanishing covariant derivative at x. In the Euclidean case, the monogenic affine spinors will do. More generally, the integrability result 10.13 ensures such spinors exist. □

It now remains to discuss the completeness of this metric. Since M is compact, it suffices to show that the conformally invariant metric blows up sufficiently fast close to the boundary with respect to any metric (on all of M) in the conformal class. Fixing such a metric, it must be shown that the norm of the evaluation map (with respect to this metric) blows up close to the boundary. Certainly $\|ev_x\|^2$ is less than $|\int_{\partial M}\langle G_x\,,G_x\rangle|$, but here a lower bound is needed. Let y be a point on ∂M and $\varepsilon > 0$ be so small that y is the closest point to $x, z = y \pm \varepsilon\nu(y)$ (so $x \in M$ and $z \in \tilde{M} \setminus M$). Now G_z is monogenic on M and so $G_z(x) = ev_x(G_z) = \int_{\partial M}\langle c(\nu)G_x\,, G_z\rangle$. Thus $\|ev_x\|^2 \geqslant |G_z(x)|^2/|\int_{\partial M}\langle G_z\,,G_z\rangle|$. The denominator is can be seen to have order $1/\varepsilon^{n-1}$, by the asymptotic behaviour of G_z, while the numerator is clearly of order $1/\varepsilon^{2n-2}$. Thus $\|ev_x\|^2 \geqslant \text{const.}/\varepsilon^{n-1}$, and so the corresponding section of L^2 is grows as fast as $1/\varepsilon^2$, which is sufficient to ensure completeness by standard arguments.

To summarise, the following theorem has been established:

11.6 Theorem. *Let M be a spin manifold with nonempty boundary and a conformal structure $[g]$. Let S be the Szegő kernel of the Dirac operator on M. Then $[g]\langle S(x,x)\rangle^{2/(n-1)}$ is a conformally invariant metric on* $\operatorname{int} M$ *which is complete and has negative scalar curvature.*

An example of this metric is the following:

11.7 Proposition. *On the unit ball in S^n with the standard conformal structure, the metric defined by the evaluation map (expressed in terms of the flat metric δ_{ij}) is given by:*

$$g_{ij}(z) = \frac{1}{\omega_n^{2/(n-1)}(1-|z|^2)^2}\delta_{ij}.$$

This is the Poincaré metric, and is complete with constant negative scalar curvature.

Since conformal transformations act transitively on the unit ball, the above metric is characterised, up to a constant, by its conformal invariance. This gives a way of computing it, and indeed also the Szegő kernel. Alternatively, the Szegő kernel can be obtained directly, by observing that for $|x| < 1, |y| = 1$ the Cauchy kernel is

$$-c(\nu_y)G_x(y) = \frac{y(y-x)}{\omega_n|y-x|^n} = \frac{1-yx}{\omega_n|1-yx|^n},$$

which extends to a monogenic function of y for $|y| < 1$. Therefore the Cauchy kernel is a boundary value of a monogenic spinor, and so the final expression actually *is* the Szegő kernel for $|x| \leqslant 1, |y| \leqslant 1$ (with a singularity on the boundary diagonal). The formula $S(x,x)^{2/(n-1)}$ then gives the Poincaré metric.

It is interesting to see the form of this kernel in the conformal chart on S^n which maps the unit disc (the lower hemisphere in S^n) to a half plane $\langle x\,,e_n\rangle > 0$. Using either the transformation law for spinors, or direct inspection of the Cauchy kernel on the half space, the following formula for the Szegő kernel is obtained:

$$\mathcal{S}(x,y) = \frac{e_n x + y e_n}{\omega_n |e_n x + y e_n|^n}.$$

This immediately gives the half space model for the hyperbolic metric.

Another interpretation of the Szegő kernel on the disc or half plane is given by the method of images. From this point of view, the identity

$$\frac{x/r^2 - y}{\omega_n |x/r^2 - y|^n} \frac{x}{r^n} = \frac{1 - yx}{\omega_n |1 - yx|^n}$$

(where $r = |x|$) means that the Szegő kernel at a point x in the disc is given by the Green kernel at the image point x/r^2, transformed appropriately. One advantage of this viewpoint is that, with a little thought, it leads to a power series for the Szegő kernel on an annulus.

11.8 Proposition. *Let M be an annulus in $\mathbb{R}^n$ between spheres of radii 1 and $\lambda < 1$, centred at the origin. Then the Szegő kernel is given by*

$$\mathcal{S}(x,y) = \frac{1}{\omega_n} \sum_{k \in \mathbf{Z}} (-1)^k \frac{\lambda^{-k} - \lambda^k y x}{|\lambda^{-k} - \lambda^k y x|^n},$$

which is a sum of image charges.

Proof: The series is monogenic in both x and y, and is equal, for $y \in \partial M$, to $-c(\nu_y)\mathcal{G}(x,y)$ where $c(\nu_y)$ is the outward normal to the annulus, and

$$\mathcal{G}(x,y) = \frac{1}{\omega_n} \sum_{k \in \mathbf{Z}} (-1)^k \frac{\lambda^k x - \lambda^{-k} y}{|\lambda^k x - \lambda^{-k} y|^n}.$$

This is the fundamental solution at x plus a function monogenic on the annulus. □

By conformal invariance this gives a power series for the Szegő kernel on a finite cylinder.

Conclusion

A number of the ideas outlined in this paper would benefit from further exploration, and so a couple of final remarks are in order.

Firstly, a problem in analysis: to establish the boundedness of the Cauchy transform under minimal smoothness assumptions. The results in this paper have all been stated for smooth manifolds with smooth boundaries, but it is quite straightforward to make weaker assumptions, provided that the conclusions are appropriately weakened. However, there are limits as to how far some of the methods in this paper can be pushed. They should apply, for instance, if the boundary is C^1 with a Lipschitz normal vector field, but if the boundary itself is merely Lipschitz, then the Kerzman-Stein theorem fails and the methods of section 9 collapse. In the Euclidean case, Murray and McIntosh have used different Clifford analysis methods to establish boundedness results for the Cauchy transform on Lipschitz domains [37, 40, 42]. It remains to be seen whether these methods can be applied to arbitrary Dirac operators on manifolds.

Secondly, more detailed properties and examples of the conformally invariant metric of section 11 are needed. One immediate question is: what differential equation does it satisfy? In [39], Loewner and Nirenberg used the conformal Laplacian to construct a conformally invariant metric on the interior of a conformal manifold with boundary. It is then immediate from their construction that their metric satisfies a differential equation: it has constant scalar curvature. However, for the metric described here, a computation of the asymptotics near the boundary shows that its scalar curvature is not constant in general. The unit disc, because it admits a transitive group of conformal transformations, is exceptional in this regard. Clearly explicit descriptions of further examples would be helpful. In particular, computations on a non-conformally flat manifold have yet to be carried out: perhaps the easiest case to try is $S^2 \times S^2$.

A third area which needs further exploration in this framework, is the topic of global elliptic boundary problems. Section 9 essentially established the well known fact that the Caldéron projection onto the Cauchy data gives a well posed global elliptic boundary problem [11], while section 10 discussed some simple local boundary value problems. However, it is geometrically more interesting to study spectral boundary conditions [4, 10, 26] and the associated η invariant of the boundary.

This third topic is not at all independent from the others. As was briefly mentioned at the end of section 5, spectral boundary conditions enter in a very natural way via the Bochner-Weitzenböck formula, the boundary operator then being a submanifold Dirac operator to which the methods of this paper apply. The functional calculus of such an operator on a Lipschitz surface also lies at the heart of the results of [40, 42]—see also [37]. Finally the asymptotics of the conformally invariant metric produce invariants of the embedded boundary, which should be closely related to the η-invariant.

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