Newforms, geodesic periods and modular forms of half-integral weight

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Introduction

In [3] an identity is given which relates the integral of a newform f of even integral weight and odd squarefree level N along a geodesic period on the modular curve $X_0(N)$ to the Fourier coefficients of a modular form g of half-integral weight and level 4N associated with f under the Shimura correspondence. This formula contains as a special case a refinement of a result of Waldspurger [8] about the special values of L-series attached to f at the central point.

The proof strongly depends on a "strong multiplicity 1" theorem for a certain subspace of forms of half-integral weight, which so far is known only when N is odd and squarefree, and therefore our identity could be stated only in this case.

The main purpose of this note is to show that the restriction to N squarefree can be lifted and that our identity in the more general case is essentially a consequence of results of Waldspurger [6,7,9].

Certainly our formula should be valid also for N even and probably could be proved in a similar way as here.

1.Notations

We let $\Gamma(1)=SL_2(\mathbf{Z})$ operate on integral binary quadratic forms $[a,b,c](x,y)=ax^2+bxy+cy^2$ by

$$[a,b,c]\circ\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}(x,y) = [a,b,c](\alpha x+\beta y, \gamma x+\delta y).$$

The symbol $\mathfrak{H}$ denotes the upper half-plane. For $z\in\mathfrak{H}$ we write $q=e^{2\pi i z}$.

The letters k and N denote positive integers, N is always assumed to be odd.

We write $N'\|N$ if $N'|N$ and $(N',\frac{N}{N'})=1$.

We let $M_{2k}(N)(S_{2k}(N))$ be the space of modular forms (cusp forms) of weight 2k on the group $\Gamma_0(N)=\{(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix})\in\Gamma(1)\mid N|\gamma\}$ and $S_{2k}^{new}(N)\subset S_{2k}(N)$ be the subspace of cuspidal newforms.

For a prime p we denote by $T_{2k}(p)$ the Hecke operator acting on $S_{2k}(N)$ by

$$T_{2k}(p)\sum_{n\geq 1} a(n)q^n = \sum_{n\geq 1}\left(a(pn)+(\frac{N^2}{p})p^{2k-1}a(\frac{n}{p})\right)q^n$$

(with the convention $a(\frac{n}{p})=0$ if $p\nmid n$). The Hecke operators leave $S_{2k}^{new}(N)$ stable.

For $f,f'\in S_{2k}(N)$ we write

$$\langle f,f'\rangle = \frac{1}{[\Gamma(1):\Gamma_0(N)]}\int_{\Gamma_0(N)\backslash\mathfrak{H}} f(z)\overline{f'(z)}y^{2k-2}dxdy \qquad (z=x+iy)$$

for the Petersson product of f and f'.

We let $\mathfrak{S}_{k+1/2}(4N)$ be the space of cusp forms of weight k+1/2 on $\Gamma_0(4N)$ ([4]) and $S_{k+1/2}(N)$ be the subspace of forms whose n^{th} Fourier coefficients at infinity vanish for $(-1)^k n\equiv 2,3(4)$ ([2]). For a prime p we write $\mathcal{T}_{k+1/2}(p^2)$ and $T_{k+1/2}(p)$, respectively, for the Hecke operators acting on $\mathfrak{S}_{k+1/2}(4N)$ and $S_{k+1/2}(N)$ by

$$\mathcal{T}_{k+1/2}(p^2)\sum_{n\geq 1} c(n)q^n = \sum_{n\geq 1}\Big(c(p^2n)+(\frac{4N^2}{p})(\frac{(-1)^k n}{p})p^{k-1}c(n) + (\frac{4N^2}{p})p^{2k-1}c(n/p^2)\Big)q^n$$

and

$$T_{k+1/2}(p)\sum_{n\geq 1,\,(-1)^k n\equiv 0,1(4)} c(n)q^n = \sum_{n\geq 1,\,(-1)^k n\equiv 0,1(4)}\Big(c(p^2n)+(\frac{N^2}{p})(\frac{(-1)^k n}{p})p^{k-1}c(n)+(\frac{N^2}{p})p^{2k-1}c(n/p^2)\Big)q^n,$$

respectively (cf. [4], §1. and [2], §3., Propos. and Remark, p.46).

For $g,g'\in\mathfrak{S}_{k+1/2}(4N)$ we denote by

$$\langle g,g'\rangle = \frac{1}{[\Gamma(1):\Gamma_0(4N)]}\int_{\Gamma_0(4N)\backslash\mathfrak{H}} g(z)\overline{g'(z)}y^{k-3/2}dxdy \qquad (z=x+iy)$$

the Petersson product of g and g'.

2. Statement of results

In [3] for every fundamental discriminant D with $(-1)^k D>0$ we defined a Shimura lifting $\mathcal{S}_D$ mapping $S_{k+1/2}(N)$ to $M_{2k}(N)$ (to $S_{2k}(N)$ if $k\geq 2$ or if N is cubefree) and a Shintani lifting $\mathcal{S}_D^*$ mapping $S_{2k}(N)$ to $S_{k+1/2}(N)$, and $\mathcal{S}_D$ and $\mathcal{S}_D^*$ were shown to be adjoint maps with respect to the Petersson products. Explicitely, for $g = \sum_{n\geq 1,\, (-1)^k n\equiv 0,1(4)} c(n)q^n \in S_{k+1/2}(N)$ one has

$$\mathcal{S}_D g = \sum_{n\geq 1} \Big(\sum_{d|n,\,(d,N)=1} \left(\tfrac{D}{d}\right) d^{k-1} c(|D|n^2/d^2) \Big) q^n,$$

and for $f\in S_{2k}^{new}(N)$ a newform (the case we will be interested in) one has

$$\mathcal{S}_D^* f = (-1)^{[k/2]} 2^k \sum_{m\geq 1,\, (-1)^k m\equiv 0,1(4)} r_{k,N,D}(f;|D|m) q^m.$$

Here for any positive integer Δ satisfying $\Delta\equiv 0,1(4)$ and $D|\Delta$ we have put

(1) $$r_{k,N,D}(f;\Delta) = \sum_{Q\in \mathcal{Q}_{N,\Delta}/\Gamma_0(N)} \omega_D(Q) \int_{C_Q} f(z)Q(z,1)^{k-1} dz,$$

where $\mathcal{Q}_{N,\Delta}/\Gamma_0(N)$ is the set of $\Gamma_0(N)$-classes of integral binary quadratic forms $Q=[a,b,c]$ with $b^2-4ac=\Delta$ and $N|a$, and where C_Q is the image in $X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash \mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q})$ of the semicircle $a|z|^2+b\mathrm{Re}z+c=0$ oriented from $\frac{-b-\sqrt{\Delta}}{2a}$ to $\frac{-b+\sqrt{\Delta}}{2a}$, if $a\neq 0$ or of the vertical line $b\mathrm{Re}z+c=0$, oriented from $-\frac{c}{b}$ to $i\infty$ if $b>0$ and from $i\infty$ to $-\frac{c}{b}$ if $b<0$, if $a=0$. Furthermore, ω_D is the genus character given by

$$\omega_D(Q) = \begin{cases} 0 & \text{if } (a,b,c,D)\neq 1 \\ \left(\frac{D}{n}\right) & \text{if } (a,b,c,D)=1, \text{ where } Q \text{ represents } n,\ (n,D)=1. \end{cases}$$

<u>Remark</u>. In [3], p.240 the factor $(-1)^{[k/2]}2^k$ is missing in the definition of $\mathcal{S}_D^*$, and the orientation of C_Q should be the opposite one for $a=0$, $b<0$.

Recall that for any positive number N' with $N'\|N$ we have an Atkin-Lehner involution $W_{N'}$ on $S_{2k}(N)$ leaving $S_{2k}^{new}(N)$ stable and defined by

$$W_{N'}f = N'^{k}(Nz+N'\delta)^{-2k}f\left(\frac{N'z+\beta}{Nz+N'\delta}\right) \qquad (\beta,\delta\in\mathbb{Z},\ N'^2\delta - N\beta = N').$$

Suppose $(D,N)=1$. Then using the fact that $Q(x,y)\mapsto Q\circ W_{N'}(x,y):=\frac{1}{N'}Q(N'x+\beta y, Nx+N'\delta y)$ induces a bijection of $\mathcal{Q}_{N,\Delta}/\Gamma_0(N)$ and that $\omega_D(Q\circ W_{N'})=(\frac{D}{N'})\omega_D(Q)$ it is easy to see that $\mathcal{S}_D^{*}f=0$ for a normalized Hecke eigenform f in $S_{2k}^{new}(N)$ unless we have $W_{N'}f=(\frac{D}{N'})f$ for all N' and hence in particular $W_{N'}f=f$ whenever N' is a square.

We define

(2) $$S_{2k}^{new}(N)^{+} = \left\{f\in S_{2k}^{new}(N) \,\middle|\, W_{N'}f=f \text{ whenever } N' \text{ is a square}\right\}.$$

The main result of this paper then is:

Theorem. Let $k\geq 1$ and N odd, and let f be a normalized Hecke eigenform in the space $S_{2k}^{new}(N)^{+}$ defined by (2). Then:

i) The subspace $S_{k+1/2}(N)_f\subset S_{k+1/2}(N)$ generated by the functions $\mathcal{S}_D^{*}f$, where D runs through all fundamental discriminants with $(-1)^kD>0$ and $(D,N)=1$, is of dimension 1.

ii) Let $g = \sum_{m\geq 1,\,(-1)^k m\equiv 0,1(4)} c(m)q^m$ be a generator of $S_{k+1/2}(N)_f$. Then for all positive integers m and n with $(-1)^k m\equiv 0,1(4)$, $(-1)^k n\equiv 0,1(4)$ and $(-1)^k n$ a fundamental discriminant we have

(3) $$\frac{c(m)\overline{c(n)}}{\langle g,g\rangle} = \frac{(-1)^{[k/2]}2^k}{\langle f,f\rangle}\, r_{k,N,(-1)^k n}(f;mn),$$

where $r_{k,N,(-1)^k n}(f;mn)$ is the cycle integral defined by (1).

As already mentioned in the Introduction, the proof which will be given in the next section, strongly depends on results of Waldspurger: from the "weak multiplicity 1" theorem for $\bar{\mathfrak{S}}_{k+1/2}(4N)$ ([6,7]) it follows that $S_{k+1/2}(N)_f$ is of dimension ≤ 1, and from the non-vanishing results for L-series at the central point ([5,9]) that, in fact, it is of dimension 1. Assertion ii) then can be deduced as in [3], using the fact the Shimura liftings and the Shintani liftings are adjoint maps with respect to the Petersson products.

For a fundamental discriminant D with (D,N)=1 let

(4) $$L(f,D,s) = \sum_{n\geq 1} \left(\frac{D}{n}\right) a(n) n^{-s} \qquad (\mathrm{Re}\ s \gg 0)$$

be the L-series of $f(z)=\sum_{n\geq 1} a(n)q^n$ twisted with the quadratic character $\left(\frac{D}{\cdot}\right)$. Recall that L(f,D,s) has a holomorphic continuation to $\mathbb{C}$ and that

$$L^*(f,D,s) = (2\pi)^{-s}(ND^2)^{s/2}\Gamma(s)L(f,D,s)$$

satisfies the functional equation

$$L^*(f,D,s) = (-1)^k\left(\frac{D}{-N}\right)L^*(W_N f,D,2k-s).$$

As in [3], setting m=n in (3) we can deduce a refined version of a result of Waldspurger ([8]):

Corollary. Let f and g be as in the Theorem, and let D be a fundamental discriminant with $(-1)^k D>0$ and (D,N)=1. Suppose that for all positive integers N′ with N′|| N we have $W_{N'}f = \left(\frac{D}{N'}\right)f$. Then

$$\frac{|c(|D|)|^2}{\langle g,g\rangle} = 2^{\nu(N)}\,\frac{(k-1)!}{\pi^k}\,|D|^{k-1/2}\,\frac{L(f,D,k)}{\langle f,f\rangle},$$

where $\nu(N)$ denotes the number of different prime divisors of N.

Of course, Corollaries 2-6 in [3] also have natural generalizations to the more general situation here. However, we leave their explicit formulation to the reader.

3. Proofs

Proposition. For all primes p and all fundamental discriminants D with $(-1)^k D>0$, (D,N)=1 one has

(5) $$\mathscr{S}_D^* T_{2k}(p) f = T_{k+1/2}(p)\mathscr{S}_D^* f \qquad (\forall f\in S_{2k}^{new}(N)).$$

Remark. Formula (5) is also true if (D,N)>1, the proof is somewhat more tedious.

Proof of Proposition. We may assume that f is a normalized Hecke eigenform.

If $p\nmid N$, then $T_{2k}(p)$ and $T_{k+1/2}(p)$ are hermitean, and since $\mathcal{S}_D$ and $\mathcal{S}_D^*$ are adjoint maps and $\mathcal{S}_D$ commutes with the action of Hecke operators (immediate verification), identity (5) is obvious in this case.

Next assume $p|N$. Then by definition of $\mathcal{S}_D^*$ and $T_{k+1/2}(p)$ we must show that

$$r_{k,N,D}(f;|D|mp^2) = r_{k,N,D}(T_{2k}(p)f;|D|m) \quad (\forall\, m\geq 1 \text{ with } (-1)^k m\equiv 0,1(4)).$$

According to Propos.7 in [3]

$$r_{k,N,D}(f;|D|m) = \alpha_k(|D|m)^{k-1/2}\langle f,f_{k,N,D,(-1)^k m}\rangle ,$$

where α_k is a constant depending only on N and k and $f_{k,N,D,(-1)^k m}(z)$ $(m\geq 1,(-1)^k m\equiv 0,1(4);z\in\mathfrak{H})$ is the modular form (cusp form if N is cubefree or if $k\geq 2$) in $M_{2k}(N)$ defined by

$$f_{k,N,D,(-1)^k m}(z) = \begin{cases} \sum_{Q\in\mathcal{Q}_{N,|D|m}} \omega_D(Q)Q(z,1)^{-k} & \text{if } k>1 \\ \lim_{s\searrow 0} \sum_{Q\in\mathcal{Q}_{N,|D|m}} \omega_D(Q)Q(z,1)^{-1}|Q(z,1)|^{-s} & \text{if } k=1 \end{cases}$$

(cf.[3],§1.). In the following we will assume $k\geq 2$ (the case k=1 is entirely similar). We will distinguish two cases.

<u>Case i)</u>: $p^2|N$. Then $T_{2k}(p)f=0$ by [1], hence we must show

(6) $\quad \langle f,f_{k,N,D,(-1)^k mp^2}\rangle = 0.$

Since $p^2|N$, the conditions $b^2-4ac=|D|mp^2$, $N|a$ imply $p^2|a$, $p|b$, hence

$$f_{k,N,D,(-1)^k mp^2}(z) = \sum_{[a,b,c]\in\mathcal{Q}_{N',|D|m}} \omega_D(p^2a,pb,c)(p^2az^2+pbz+c)^{-k}$$

where $N'=N/p^2$.

Since $p\nmid D$, we have $\omega_D(p^2a,pb,c)=\omega_D(a,b,c)$; in fact, $(a,b,c,D)=1$ is equivalent to $(p^2a,pb,c,D)=1$, and if $[a,b,c]$ represents n, then $[p^2a,pb,c]$ represents p^2n.

Hence

$$f_{k,N,D,(-1)^k mp^2}(z) = f_{k,N',D,(-1)^k m}(pz),$$

and since f is in $S_{2k}^{new}(N)$, (6) follows.

<u>Case ii)</u>: $p\|N$. By [1] we have $T_{2k}(p)f=-p^{k-1}W_pf$, hence we have to show that

$$(7)\qquad \langle f, f_{k,N,D,(-1)^k mp^2}\rangle = -p^{-k}\langle W_p f, f_{k,N,D,(-1)^k m}\rangle .$$

If $a,b,c\in\mathbb{Z}$ with $b^2-4ac=|D|mp^2$ and $N|a$, then it follows that $p|b$ and either $p^2|a$ or $p\|a$ and $p|c$. Hence setting $N'=N/p$ we have

$$f_{k,N,D,(-1)^k mp^2}(z) = \sum_{[a,b,c]\in\mathcal{Q}_{N',|D|m}} \omega_D(p^2a,pb,c)(p^2az^2+pbz+c)^{-k}$$

$$+p^{-k}\sum_{\substack{a,b,c\in\mathbb{Z},N'|a,p\nmid a\\ b^2-4ac=|D|m}} \omega_D(pa,pb,pc)(az^2+bz+c)^{-k}.$$

As in case i) the first sum equals $f_{k,N,D,(-1)^k m}(pz)$. Since $\omega_D(pa,pb,pc) = (\frac{D}{p})\omega_D(a,b,c)$ the second sum equals

$$p^{-k}(\tfrac{D}{p})(\ f_{k,N',D,(-1)^k m}(z)-f_{k,N,D,(-1)^k m}(z)\ ),$$

hence since $f\in S_{2k}^{new}(N)$ it follows that

$$\langle f, f_{k,N,D,(-1)^k mp^2}\rangle = -p^{-k}(\tfrac{D}{p})\langle f, f_{k,N,D,(-1)^k m}\rangle.$$

By definition

$$W_p f_{k,N,D,(-1)^k m}(z) = \sum_{Q\in Q_{N,|D|m}} \omega_D(Q\circ W_p)(Q\circ W_p)(z,1)^{-k}.$$

Since acting by W_p is a permutation of $\mathcal{Q}_{N,|D|m}$ and $\omega_D(Q\circ W_p) = (\frac{D}{p})\omega_D(Q)$ for $p\nmid D$ we conclude that

$$W_p f_{k,N,D,(-1)^k m} = (\tfrac{D}{p}) f_{k,N,D,(-1)^k m},$$

and (7) follows, since W_p is hermitean. This concludes the proof of the Proposition.

We shall now prove assertion i) of the Theorem. First let us show that $\dim_{\mathbb{C}} S_{k+1/2}(N)_f \leq 1$. Put $g_D=\mathcal{S}_D^* f$. Assume that $T_{2k}(p)f=\lambda_p f$ for all primes p, and let α_2 be a solution of the equation $X^2-\lambda_2 X+2^{2k-1}=0$. Put

$$G_D = (-\alpha_2+\mathcal{T}_{k+1/2}(4))g_D.$$

Then from the Proposition we see that $\mathcal{T}_{k+1/2}(p^2)G_D=\lambda_p G_D$ for all primes $p\neq 2$. Furthermore $\mathcal{T}_{k+1/2}(4)G_D = \frac{2^{2k-1}}{\alpha_2} G_D$ (observe that $\alpha_2\neq 0$): in fact, it is easy to verify that the latter equation is equivalent to the equation $\lambda_2\mathcal{T}_{k+1/2}(4)f=\mathcal{T}_{k+1/2}(4)^2 f+2^{2k-1}f$, which in turn follows from $T_{k+1/2}(2)g_D=\lambda_2 g_D$ by applying the operator $\mathcal{T}_{k+1/2}(4)$ on both sides.

From the "weak multiplicity 1" theorem for the space $\mathfrak{S}_{k+1/2}(4N)$ ([6,7]) it now follows that the $\mathbb{C}$-linear span of all the functions G_D is of dimension ≤ 1.

Let pr be the orthogonal projection from $\mathfrak{S}_{k+1/2}(4N)$ onto $S_{k+1/2}(N)$ ([2], §§2.,3.). Then

$$\begin{aligned} \mathrm{pr}\ G_D &= -\alpha_2 g_D + \mathrm{pr}\,\mathcal{T}_{k+1/2}(4)g_D \\ &= -\alpha_2 g_D + \tfrac{2}{3}T_{k+1/2}(2)g_D \qquad ([2],\S 3.,p.42) \\ &= (\tfrac{2}{3}\lambda_2-\alpha_2)g_D. \end{aligned}$$

If $\frac{2}{3}\lambda_2-\alpha_2$ were equal to zero, then $\lambda_2=\pm 3\cdot 2^{k-1}$, which contradicts Deligne's theorem, previously the Ramanujan-Petersson conjecture. Hence pr G_D is a non-zero multiple of g_D and we see that $\dim_{\mathbb{C}} S_{k+1/2}(N)_f \leq 1$, too.

To show that actually $\dim_{\mathbb{C}} S_{k+1/2}(N)_f=1$, we compute the D^{th}-Fourier coefficient of $\mathcal{S}_D^{*}f$. By definition it equals

$$r_{k,N,D}(f;D^2) = \sum_{Q\in\mathcal{Q}_{N,D^2}/\Gamma_0(N)} \omega_D(Q) \int_{C_Q} f(z)Q(z,1)^{k-1}dz.$$

If we suppose that $W_{N'}f=(\frac{D}{N'})f$ for all N' with $N' \| N$, then the same computations as in [3],p.243 show that

(8) $$r_{k,N,D}(f;D^2) = 2^{\nu(N)}(-1)^{[k/2]}(2\pi)^{-k}\Gamma(k)|D|^{k-1/2}L(f,D,k),$$

where $L(f,D,s)$ is the L-function defined for Re $s \gg 0$ by (4) and $\nu(N)$ is the number of different prime factors of N.

But according to [9], Thm.4 and [5], Thm.2.3. there is a fundamental discriminant D_0 with $(-1)^kD_0>0$, $W_{N'}f=(\frac{D_0}{N'})f$ for all N' and $L(f,D_0,k)\neq 0$. Hence $S_{D_0}^{*}f$ is a non-zero function in $S_{k+1/2}(N)_f$.

Let us now prove ii). Since f and $\mathcal{S}_{(-1)^kn}g$ have the same eigenvalues under the Hecke operators, $\mathcal{S}_{(-1)^kn}g$ is a cusp form and is a multiple of f by the "multiplicity 1" theorem for $S_{2k}^{new}(N)$. Comparing the Fourier coefficients at q we find that

(9) $$\mathcal{S}_{(-1)^kn}g=c(n)g.$$

From i) we have

$$\mathcal{S}^*_{(-1)^k n} f = \beta_n g$$

for some $\beta_n \in \mathbb{C}$.

Now

$$\beta_n c(m) = \text{coefficient of } q^m \text{ in } \mathcal{S}^*_{(-1)^k n} f$$
$$= (-1)^{[k/2]} 2^k r_{k,N,(-1)^k n}(f; mn).$$

On the other hand

$$\beta_n c(m) \langle g,g \rangle = c(m) \langle \mathcal{S}^*_{(-1)^k n} f, g \rangle$$
$$= c(m) \langle f, \mathcal{S}_{(-1)^k n} g \rangle$$
$$= c(m) \overline{c(n)} \langle f,f \rangle ,$$

where in the last line we have used (9). Comparing these two formulas we obtain (3).

The Corollary to the Theorem, of course, follows from (8).

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