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Abstract. We find an explicit expression for the zeta-regularized determinant of (the Friedrichs extensions) of the Laplacians on a compact Riemann surface of genus one with conformal metric of curvature 1 having a single conical singularity of angle 4π.

We discuss a generalization of this result to the case of metrics of curvature 1 with conical singularities on hyperelliptic curves of genus $g \geq 2$.

1 Introduction

Let X be a compact Riemann surface of genus one and let $P \in X$. According to [1], Cor. 3.5.1, there exists *at most* one conformal metric on X of constant curvature 1 with a (single) conical point of angle 4π at P. The following simple construction shows that such a metric, $m(X, P)$, in fact always exists (and due to [1] is unique).

Consider the spherical triangle $T = \{(x_1, x_2, x_3) \in S^2 \subset \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$ with all three angles equal to $\pi/2$. Gluing two copies of T along their boundaries, we get the Riemann sphere $\mathbb{P}$ with metric m of curvature 1 and three conical points P_1, P_2, P_3 of conical angle π. Consider the two-fold covering

$$\mu : X(Q) \to \mathbb{P} \tag{1.1}$$

ramified over P_1, P_2, P_3 and some point $Q \in \mathbb{P} \setminus \{P_1, P_2, P_3\}$. Lifting the metric m from $\mathbb{P}$ to the compact Riemann surface $X(Q)$ of genus one via μ, one gets the metric $\mu^* m$ on $X(Q)$ which has curvature 1 and the unique conical point of angle 4π at the preimage $\mu^{-1}(Q)$ of Q. Clearly, any compact surface of genus one is (biholomorphically equivalent to) $X(Q)$ for some $Q \in \mathbb{P} \setminus \{P_1, P_2, P_3\}$. Now let X be an arbitrary compact Riemann surface of genus one and let P be any point of X. Take $Q \in \mathbb{P}$ such that $X = X(Q)$ and consider the automorphism $\alpha : X \to X$ (the translation) of X sending P to $\mu^{-1}(Q)$. Then

$$m(X,P) = \alpha^*(\mu^*(m)) = (\mu \circ \alpha)^*(m)\,.$$

Introduce the scalar (Friedrichs) self-adjoint Laplacian $\Delta(X,P) := \Delta^{m(X,P)}$ on X corresponding to the metric $m(X,P)$. For any P and Q from X the operators $\Delta(X,P)$ and $\Delta(X,Q)$ are isospectral and, therefore, the ζ-regularized (modified, i. e. with zero modes excluded) determinant $\det\Delta(X,P)$ is independent of $P \in X$ and, therefore, is a function on moduli space $\mathcal{M}_1$ of Riemann surfaces of genus one. The main result of the present work is the following explicit formula for this function:

$$\det\Delta(X,P) = C_1\,|\Im\sigma||\eta(\sigma)|^4 F(t) = C_2 \det\Delta^{(0)}(X)F(t), \tag{1.2}$$

where σ is the b-period of the Riemann surface X, C_1 and C_2 are absolute constants, η is the Dedekind eta-function, $\Delta^{(0)}$ is the Lapalacian on X corresponding to the flat conformal metric of unit volume, the surface X is represented as the two-fold covering of the Riemann sphere $\mathbb{C}P^1$ ramified over the poits $0, 1, \infty$ and $t \in \mathbb{C} \setminus \{0,1\}$, and

$$F(t) = \frac{|t|^{\frac{1}{24}}|t-1|^{\frac{1}{24}}}{(|\sqrt{t}-1| + |\sqrt{t}+1|)^{\frac{1}{4}}}\,. \tag{1.3}$$

As it is well known, the moduli space $\mathcal{M}_1$ coincides with the quotient space

$$\left(\mathbb{C} \setminus \{0,1\}\right)/G\,,$$

where G is a finite group of order 6, generated by transformations $t \to \frac{1}{t}$ and $t \to 1-t$. A direct check shows that $F(t) = F(\frac{1}{t})$ and $F(t) = F(1-t)$ and, therefore, the right hand side of (1.2) is in fact a function on $\mathcal{M}_1$.

Remark 1. *Using the classical relation (see, e. g. [2] (3.35))*

$$t = -\left(\frac{\Theta[^1_0](0\,|\,\sigma)}{\Theta[^0_1](0\,|\,\sigma)}\right)^4,$$

one can rewrite the right hand side as a function of σ only.

The well known (see [9]) relation $\det\Delta^{(0)} = C\,|\Im\sigma||\eta(\sigma)|^4$ used in (1.2), implies that (1.2) can be considered as a version of Polyakov's formula (relating determinants of the Laplacians corresponding to two *smooth* metrics in the same conformal class) for the case of two conformally equivalent metrics on a torus: one of them is smooth and flat, another is of curvature one and has one (very special) singular point.

The above construction can be generalized to hyperelliptic surfaces of genus $g \geq 2$. Namely, choose $2g-1 = (2g+2)-3$ distinct points $Q_1, Q_2, \dots, Q_{2g-1}$ in $\mathbb{P}\setminus\{P_1,P_2,P_3\}$ and consider the two-fold covering

$$\mu_g : X(Q_1, Q_2, \dots, Q_{2g-1}) \to \mathbb{P}$$

ramified over $Q_1, \dots, Q_{2g-1}, P_1, P_2, P_3$. Lifting the metric m from $\mathbb{P}$ to the hyperelliptic curve $X(Q_1, Q_2, \dots, Q_{2g-1})$ of genus g one gets a metric μ_g^*m of constant curvature 1 with conical points of angle 4π at $2g-1$ Weierstrass points of the curve $X(Q_1, Q_2, \dots, Q_{2g-1})$ (the hyperelliptic curve has $2g+2$ Weierstrass points, three remaining Weierstrass points are nonsingular points of the metric). Using the same methods as in the genus 1 case, one can derive an explicit expression for the determinant of the Laplacian in the metric μ_g^*m as a function on moduli space of hyperelliptic curves of genus g. In the last section we write down such an expression for genus two case.

2 Metrics on the base and on the covering

Here we find an explicit expression for the metric m on the Riemann sphere $\mathbb{P} = \mathbb{C}P^1$ of curvature 1 and with three conical singularities at $P_1 = 0$, $P_2 = 1$ and $P_3 = \infty$.

The stereographic projection (from the south pole) maps the spherical triangle T onto quarter of the unit disk $\{z \in \mathbb{C}; |z| \leq 1,\ 0 \leq \operatorname{Arg} z \leq \pi/2\}$. The conformal map

$$z \mapsto w = \left(\frac{1+z^2}{1-z^2}\right)^2 \tag{2.1}$$

sends this quarter of the disk to the upper half-plane H; the corner points $i, 0, 1$ go to the points $0, 1$ and ∞ on the real line. The push forward of the standard round metric

$$\frac{4|dz|^2}{(1+|z|^2)^2}$$

on the sphere by this map gives rise to the metric

$$m = \frac{|dw|^2}{|w||w-1|(|\sqrt{w}+1| + |\sqrt{w}-1|)^2} \tag{2.2}$$

on H; clearly, the latter metric can be extended (via the same formula) to $\mathbb{C}P^1$. The resulting curvature one metric on $\mathbb{C}P^1$ (also denoted by m) has three conical singularities of angle π: at $w = 0$, $w = 1$ and $w = \infty$.

Consider a two-fold covering of the Riemann sphere by a compact Riemann surface $X(t)$ of genus 1

$$\mu : X(t) \to \mathbb{C}P^1 \tag{2.3}$$

ramified over four points: $0, 1, \infty$ and $t \in \mathbb{C} \setminus \{0, 1\}$. Clearly, the pull back metric $\mu^* m$ on $X(t)$ is a curvature one metric with exactly one conical singularity. The singularity is a conical point of angle 4π located at the point $\mu^{-1}(t)$.

3 Determinant of Laplacian as function of critical value t

The analysis from [3] in particular implies that one can introduce the standard Ray-Singer ζ-regularized determinant of the (Friedrichs) self-adjoint Laplacian $\Delta^{\mu^* m}$ in $L_2(X(t), \mu^* m)$

$$\det \Delta^{\mu^* m} := \exp\{-\zeta'_{\Delta^{\mu^* m}}(0)\}\,,$$

where $\zeta'_{\Delta^{\mu^* m}}$ is the operator zeta-function. In this section we establish a formula for the variation of $\zeta'_{\Delta^{\mu^* m}}(0)$ with respect to the parameter t (the fourth ramification point of the covering (2.3). The derivation of this formula coincides almost verbatim with the proof of [3, Proposition 6.1], therefore, we will give only few details.

For the sake of brevity we identify the point t of the base $\mathbb{C}P^1$ with its (unique) preimage $\mu^{-1}(t)$ on $X(t)$.

Let $Y(\lambda; \cdot)$ be the (unique) special solution of the Helmholz equation (here λ is the spectral parameter) $(\Delta^m - \lambda)Y = 0$ on $X \setminus \{t\}$ with asymptotics $Y(\lambda)(x) = \frac{1}{x} + O(x)$ as $x \to 0$, where $x(P) = \sqrt{\mu(P) - t}$ is *the distinguished holomorphic local parameter* in a vicinity of the ramifiction point $t \in X(t)$ of the covering (2.3). Introduce the complex-valued function $\lambda \mapsto b(\lambda)$ as the coefficient near x in the asymptotic expansion

$$Y(x, \bar{x}; \lambda) = \frac{1}{x} + c(\lambda) + a(\lambda)\bar{x} + b(\lambda)x + O(|x|^{2-\epsilon}) \text{ as } x \to 0.$$

The following variational formula is proved in [3, Propositon 6.1]:

$$\partial_t(-\zeta'_{\Delta^{\mu^*m}}(0)) = \frac{1}{2}\left(b(0) - b(-\infty)\right). \tag{3.1}$$

The value $b(0)$ is found in [3, Lemma 4.2]: one has the relation

$$b(0) = -\frac{1}{6} S_{Sch}(x)\Big|_{x=0}, \tag{3.2}$$

where S_{Sch} is the Schiffer projective connection on the Riemann surface $X(t)$.

Since $\lambda = -\infty$ is a local regime, in order to find $b(-\infty)$ the solution Y can be replaced by a local solution with the same asymptotic as $x \to 0$. A local solution $\widehat{Y}$ with asymptotic

$$\widehat{Y}(u, \bar{u}; \lambda) = \frac{1}{u} + \hat{c}(\lambda) + \hat{a}(\lambda)\bar{u} + \hat{b}(\lambda)u + O(|u|^{2-\epsilon}) \text{ as } u \to 0$$

in the local parameter $u^2 = z - s$ was constructed in [3, Lemma 4.1] by separation of variables; here z and $w = \mu(P)$ (resp. s and t) are related by (2.1) (resp. by (2.1) with $z = s$ and $w = t$) and $\hat{b}(-\infty) = \frac{1}{2}\frac{\bar{s}}{1+|s|^2}$. One can easily find the coefficients $A(t)$ and $B(t)$ of the Taylor series $u = A(t)x + B(t)x^3 + O(x^5)$. As a local solution replacing Y we can take $A(t)\widehat{Y}$. This immediately implies $b(-\infty) = A^2(t)\hat{b}(-\infty) - B(t)/A(t)$. A straightforward calculation verifies that

$$b(-\infty) = \partial_t \log\left(|t||t-1|(|\sqrt{t}+1| + |\sqrt{t}-1|)^2\right)^{1/4}. \tag{3.3}$$

Observe that the right hand side in (3.3) is actually the value of $\partial_w \log \rho(w, \bar{w})^{-1/4}$ at $w = t$, where $\rho(w, \bar{w})$ is the conformal factor of the metric (2.2); this is also a direct consequence of [8, Lemma 4].

Using (3.1) together with (3.2) and (3.3), we are now able to derive an explicit formula for $\det\Delta^{\mu^*m}$.

4 Explicit formula for the determinant

Equations (3.1), (3.2) and (3.3) imply that the determinant of the Laplacian $\det \Delta^{\mu^*m} = \exp\{-\zeta'_{\Delta^{\mu^*m}}(0)\}$ can be represented as a product

$$\det \Delta^{\mu^*m} = C\,|\Im\sigma||\tau(t)|^2 \left|\frac{1}{|t||t-1|(|\sqrt{t}+1| + |\sqrt{t}-1|)^2}\right|^{1/8} \tag{4.1}$$

where $\tau(t)$ is the value of the Bergman tau-function (see [4], [5], [6]) on the Hurwitz space $H_{1,2}(2)$ of two-fold genus one coverings of the Riemann sphere, having ∞ as a ramification point at the covering, ramified over $1, 0, \infty$ and t. More specifically, τ is a solution of the equation

$$\frac{\partial \log \tau}{\partial t} = -\frac{1}{12} S_B(x)|_{x=0}$$

where S_B is the Bergman projective connection on the covering Riemann surface $X(t)$ of genus one and x is the distinguished holomorphic parameter in a vicinity of the ramification point t of $X(t)$. We remind the reader that the Bergman and the Schiffer projective connections are related via the equation

$$S_{Sch}(x) = S_B(x) - 6\pi(\Im\sigma)^{-1}v^2(x)$$

where v is the normalized holomorphic differential on $X(t)$ and that the Rauch variational formula (see, e. g., [4]) implies the relation

$$\frac{\partial \log \Im\sigma}{\partial t} = \frac{\pi}{2}(\Im\sigma)^{-1}v^2(x)|_{x=0}\,.$$

The needed explicit expression for τ can be found e. g. in [6, f-la (18)] (it is a very special case of the explicit formula for the Bergman tau-function on general coverings of arbitrary genus and degree found in [5] as well as of a much earlier formula of Kitaev and Korotkin for hyperelliptic coverings [7]). Namely, [6, f-la (18)] implies that

$$\tau = \eta^2(\sigma)\left[\frac{v(\infty)^3}{v(P_1)v(P_2)v(Q)}\right]^{\frac{1}{12}}, \tag{4.2}$$

where P_1 and P_2 are the points of the $X(t)$ lying over 0 and 1, Q is a point of $X(t)$ lying over t and ∞ denotes the point of the covering curve $X(t)$ lying over the point at infinity of the base $\mathbb{C}P^1$; v is an arbitrary nonzero holomorphic differential on $X(t)$; and, say, $v(P_1)$ is the value of this differential in the distinguished holomorphic parameter at P_1. (One has to take into account that $\tau = \tau_I^{-2}$, where τ_I is from [6].) Taking

$$v = \frac{dw}{\sqrt{(w(w-1)(w-t)}}\,,$$

and using the following expressions for the distinguished local parameters at P_1, P_2, Q and ∞

$$x = \sqrt{w};\quad x = \sqrt{w-1};\quad x = \sqrt{w-t};\quad x = \frac{1}{\sqrt{w}}$$

one arrives at the relations (where $\sim$ means $=$ up to insignificant constants like ± 2, etc.)

$$v(P_1) \sim \frac{1}{\sqrt{t}};\quad v(P_2) \sim \frac{1}{\sqrt{t-1}};\quad v(Q) \sim \frac{1}{\sqrt{t(t-1)}};\quad v(\infty) \sim 1.$$

These relations together with (4.2) and (4.1) imply (1.2).

5 Genus two case

Let

$$\mu_2 : X(t_1, t_2, t_3) \to \mathbb{P} = \mathbb{C}P^1$$

be the two-fold covering ramified over (distinct) points t_1, t_2, t_3, 0, 1 and ∞ of $\mathbb{C}P^1$. The same arguments as above lead to the following expression for the determinant of Laplacian in the conical metric $\mu_2^* m$ on the genus 2 curve $X(t_1, t_2, t_3)$:

$$\det\Delta^{\mu_2^* m} = C \det\Im\mathbb{B}|\tau|^2 \prod_{k=1}^{3} \{\rho(t_k, \bar{t}_k)\}^{1/8}$$

where $\mathbb{B}$ is the matrix of the b-periods of the curve $X(t_1, t_2, t_3)$ and τ is the Bergman tau-function on the Hurwitz space $H_{2,2}(2)$ of meromorphic functions on Riemann surfaces of genus 2 of degree *two* and having one double pole. According to [4] (see formulas (2.40), (2.36) and (2.37)), one has

$$\tau = \left\{ \prod_{\beta} \Theta[\beta]((0|\mathbb{B}) \right\}^{\frac{1}{5}} \prod_{m<n} (\lambda_m - \lambda_n)^{\frac{1}{20}},$$

where β runs over the set *of* 10 *even characteristics* and $\lambda_1 = t_1$, $\lambda_2 = t_2$, $\lambda_3 = t_3$, $\lambda_4 = 0$, λ_5=1. Thus,

$$\det\Delta^{\mu_2^* m} = C \det \Im\mathbb{B} \left\{ \prod_{\beta} \left| \Theta[\beta](0|\mathbb{B}) \right| \right\}^{\frac{2}{5}} \times$$

$$\prod_{m<n} |\lambda_m - \lambda_n|^{\frac{1}{10}} \prod_{k=1}^{3} \frac{1}{|t_k|^{1/8}|t_k - 1|^{1/8}(|\sqrt{t_k} - 1| + |\sqrt{t_k} + 1|)^{1/4}}.$$

This implies the final expression for the determinant

$$\det\Delta^{\mu_2^* m} = C\, \mathcal{F}^{2/5} \Phi(t_1, t_2, t_3), \tag{5.1}$$

where

$$\mathcal{F} = (\det \Im\mathbb{B})^{5/2} \prod_{\beta} |\Theta[\beta](0|\mathbb{B})|$$

is the Petersson norm $||\Delta_2||$ of the Siegel cusp form $\Delta_2 = \prod_\beta \Theta[\beta](0|\mathbb{B})$ and

$$\Phi(t_1, t_2, t_3) = \frac{|t_1 t_2 t_3 (t_1 - 1)(t_2 - 1)(t_3 - 1)|^{-\frac{1}{40}} |t_1 - t_2|^{\frac{1}{10}} |t_1 - t_3|^{\frac{1}{10}} |t_2 - t_3|^{\frac{1}{10}}}{\prod_{k=1}^{3} (|\sqrt{t_k} + 1| + |\sqrt{t_k} - 1|)^{\frac{1}{4}}}$$

It is straightforward to check that the right hand side of (5.1) is a function on the moduli space $\mathcal{M}_2$ of compact Riemann surfaces of genus 2 (i. e. that the relations $\Phi(t_1, t_2, t_3) = \Phi(\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}) = \Phi(1 - t_1, 1 - t_2, 1 - t_3)$ hold true).

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