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DERIVATIVES OF CIRCLE ROTATIONS AND SIMILARITY OF ORBITS

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ABSTRACT. It is proved that a derivative of a circle rotation on any interval is either a circle rotation or a non-cyclic 3-interval exchange transformation. In the first case we calculate all possible values of new rotation angle. Also we prove that the restriction of an orbit of the circle rotation on proper intervals is similar to the orbit of other circle rotation.

1. INTRODUCTION

In the work we study arithmetics and geometry of an irrational rotation $R_\alpha = x \to x - \alpha \pmod 1$. A derivative of the transformation on a set Y is the first returning map, defined by the equality

$$d_Y R_\alpha = R_\alpha^{n_Y(x)}(x),$$

where

$$n_Y(x) = min\{n \in \mathbb{N} : R_\alpha^N(x) \in Y\}.$$

The sets Y are intervals of the unit circle. The basic result of this article is the following theorem.

Theorem. *Let $R_\alpha : x \to x - \alpha \pmod 1$ be a rotation on the unit circle. If $I = I_m$ is a proper differentiation interval (see section 2), then the derivative $d_I R_\alpha$ is a circle rotation again. If $I \neq I_m$ is an improper interval, then $d_I R_\alpha$ is a non-cyclic 3-interval exchange transformation.*

Geometrical corollary of the above theorem is the similarity of orbits of circle rotations with different α. Denote by $Orb(1, \alpha, I_0)$ the sequence which consists of the elements $-n\alpha \pmod 1, n = 0, 1, 2, \ldots$. A derivative $d^m Orb(1, \alpha, I_0)$ is the sequence with elements from $Orb(1, \alpha, I_0) \cap I_m$ numbered in the order of their hitting in the proper interval I_m. Then the Theorem implies the existence of the homothety h_m with center at the point 1 such that

$$d^m Orb(1, \alpha, I_0) = h_m Orb(1, d^m\alpha, I_0)$$

for some $d^m\alpha$. Let $\alpha = [(q_1, \cdots, q_r)]$ be a quadratic irrationality with purely periodic continued fraction expansion, and

$$T = \begin{cases} q_1 + \ldots + q_r, & \text{if } r \text{ is even,} \\ 2(q_1 + \ldots + q_r), & \text{if } r \text{ is odd.} \end{cases}$$

Then

$$d^T\alpha = \alpha$$

and

$$d^T Orb(1, \alpha, I_0) = h_T Orb(1, \alpha, I_0).$$

So, in this case we have the self-similarity of the orbits.

[1]Date: 25 June, 2004.

Differentiation of orbits induces a map

$$\alpha \to d^1\alpha$$

from the unit circle onto itself. This map has some interesting properties admitting natural interpretation in terms of symbolic dynamics. In section 5 we correspond to any $\alpha \in [0;1)$ its code, i.e. the infinite sequence of zeroes and units. The action of the map d^1 on this coding is equivalent to the action of the Bernoulli shift (see Theorem 5.2). It is unexpected that the constructed coding is the well-known decomposition of α into the Stern-Brocot sequence.

Historical remarks. The concept of differenciation map arose in the theory of dynamic systems as a discrete analogue of the Poincare sections. Rauzy [7] and Ferenczi [1],[2] began to study of circle rotation derivatives in connection with the bounded remainder sets problem. In this article we use the approach of Zhuravlev [9] based on the Fibonacci tilings. The article [9] contains the general theory of the Fibonacci tilings for $\tau = \frac{1+\sqrt{5}}{2}$. Manuylov [6] transferred these results on the case $\tau_2 = 1+\sqrt{2}$.

Acknowledgment. The author is grateful to the Max Planck Institute of Mathematics (Bonn) for hospitality during the preparation of this paper.

The research was partly supported by Russian Fund of Fundamental Researches (Grant 03-01-00368).

2. The generalized Fibonacci tilings

We define here generalized Fibonacci tilings and state some their properties. Proofs can be found in [9],[5],[8].

Let $\mathbb{T}$ be a set of all tilings of the unit circle $\mathbb{S}^1 = [0;1)$ into a finite number of intervals. Let us define an operator

$$B : \mathbb{T} \to \mathbb{T}.$$

Its action is defined as follows. Let $Til \in \mathbb{T}$ be a tiling of $\mathbb{S}^1$ which consists of a finite number of intervals. Let $|X|$ be the least length of the tiling's intervals. Then in the tiling $B(Til)$ intervals from Til with the lengths $|X|$ are kept. The intervals with the lengths $|Y| > |X|$ splits into two intervals with the lengths $|X|$ and $|Y| - |X|$ respectively. In other words, the action of the B-operator means a putting off intervals of the least length on the left-hand side of all intervals in the tiling Til.

Suppose a tiling $Til_0(\alpha)$ consists of two intervals

$$Til_0(\alpha) = [0;\alpha) \oplus [\alpha;1),$$

where we denote by $\oplus$ a noncommutative operation of applying of the intervals $[0;\alpha)$ and $[\alpha;1)$.

The generalized Fibonacci tiling of the order $m = 0,1,2,\dots$ is defined by the relation

$$Til_m(\alpha) = B^m(Til_0(\alpha)).$$

It is obvious $Til_m(\alpha) \neq Til_n(\alpha)$ for an irrational α , if $m \neq n$. It is easy to show that the tiling $Til_m(\alpha)$ consists of intervals which have two different lengths. It also cannot ends by two intervals of the same length. Consider the map $Col_{m,\alpha}$, corresponding to each interval of $Til_m(\alpha)$ color labels E and G under the following conditions.

1) Intervals $Til_m(\alpha)$ with the identical lengths have the same labels. Intervals of

different lengths have different labels.
2) The extreme right interval in $Til_m(\alpha)$ has the label E.
As a result, we get the color Fibonacci tilings

$$CTil_m(\alpha) = Col_{m,\alpha}(Til_m(\alpha))$$

for $m = 0, 1, 2, \ldots$
We will use the following notations:
L^m, S^m for long and short intervals in the tiling $Til_m(\alpha)$;
$l_m(\alpha)$, $s_m(\alpha)$ for lengths of intervals L^m, S^m;
$L_m(\alpha)$, $S_m(\alpha)$ are cardinalities of the sets of intervals L^m, S^m in $Til_m(\alpha)$;
E^m, G^m for E- and G-intervals in the tiling $CTil_m(\alpha)$;
$e_m(\alpha)$, $g_m(\alpha)$ for for lengths of intervals E^m, G^m ;
$E_m(\alpha)$, $G_m(\alpha)$ are cardinalities of the sets of intervals E^m, G^m in $Til_m(\alpha)$.
Everywhere, where it will not call ambiguities, the index α will be omitted.

To each interval in the tiling $CTil_m(\alpha)$ we correspond its number by the following rule. The tiling $CTil_0(\alpha)$ consist of two intervals

$$CTil_0(\alpha) = G_0^0 \oplus E_0^0 = [0; \alpha) \oplus [\alpha; 1)$$

which have number 0. Further interval numbers are defined by the induction on m. If $s_m(\alpha) = e_m(\alpha)$, then

$$\begin{aligned} E_i^m &\to E_i^{m+1}, \\ G_i^m &\to E_{i+E_m}^{m+1} \oplus G_i^{m+1}. \end{aligned}$$

If $s_m(\alpha) = g_m(\alpha)$, then

$$\begin{aligned} E_i^m &\to G_i^{m+1} \oplus E_i^{m+1}, \\ G_i^m &\to G_{i+E_m}^{m+1}. \end{aligned}$$

We use the following facts about $CTil_m(\alpha)$.

Proposition 2.1. *There is the equality*

$$CTil_m(1-\alpha) = \overline{CTil_m(\alpha)}^{(+)}.$$

Here we denote by $*^{(+)}$ *an operation of exchanging of two extreme right intervals in a tiling,* $\overline{CTil_m(\alpha)}$ *is a tiling obtaining from* $CTil_m(\alpha)$ *by replacement all* E *and* G *labels on* G *and* E *the labels, respectively.*

Proposition 2.2.

$$\begin{aligned} \langle E_m\alpha \rangle &= g_m, \\ \langle G_m\alpha \rangle &= 1 - e_m. \end{aligned} \tag{2.1}$$

Proposition 2.3.

$$\begin{aligned} E_i^m - \alpha &\equiv E_{i+1}^m \pmod 1, \\ G_i^m - \alpha &\equiv G_{i+1}^m \pmod 1. \end{aligned} \tag{2.2}$$

Proposition 2.4.

$$CTil_m(\alpha) - \alpha \equiv CTil_m(\alpha)^{(+)} \pmod 1. \tag{2.3}$$

3. DERIVATIVES OF THE CIRCLE ROTATIONS

Let X be a set and $Y \subset X$ its subset. Let $A : X \to X$ be a map of the set X to itself. We define a derivative (or the first returning map)

$$d_Y A : Y \to Y$$

by the relation

$$d_Y A = A^{n_Y(x)}(x), \tag{3.1}$$

where

$$n_Y(X) = \min\{n \in \mathbb{N} : A^n(x) \in Y\}.$$

Let's denote by R_α the rotation

$$R_\alpha : x \to x - \alpha \pmod 1 \tag{3.2}$$

of the unit cicle $\mathbb{S}^1$. We are interested in the case when $X = \mathbb{S}^1$ is the unit circle, $A = R_\alpha$, and $Y = I$ is an interval in $\mathbb{S}^1$. Without loss of generality, we can suppose that on the right-hand site of the interval I is a point 1.

We put $I_m = G_0^m \oplus E_0^m$, $i_m = |I_m|$ and call the intervals I_m as the proper differentiation intervals. First, we calculate the derivatives $d_I R_\alpha$ on the proper intervals I_m.

For this purpose we we consider the tiling $CTil_m(\alpha)$. From (2.2) it follows that under the action of the rotation R_α intervals in $CTil_m(\alpha)$ exchange by the following rules

$$\begin{array}{c} G_0^m \to G_1^m \to \ldots \to G_{G_m}^m, \\ E_0^m \to E_1^m \to \ldots \to E_{E_m}^m. \end{array} \tag{3.3}$$

From Proposition 2.4 it follows that

$$I_m = G_0^m \oplus E_0^m = E_{E_m}^m \oplus G_{G_m}^m.$$

Thus, the action of $d_{I_m} R_\alpha$ is equivalent to exchanging of the intervals G_0^m and E_0^m. If we glue the right- and left-hand sides of the interval I_m, we obtain that $d_{I_m} R_\alpha$ is the map

$$I_m \to I_m : x \to x - g_m \pmod{I_m}. \tag{3.4}$$

So, any derivative or restriction of a circle rotation onto the proper differenciation interval is a rotation of the circle again.

Now we consider an improper interval I. Assume that

$$I_{m+1} \subset I \subset I_m. \tag{3.5}$$

From (2.1) it follows that $E_0^m \subset I$. Let's split the interval G_0^m into two intervals

$$G_0^m = G_{0,1}^m \oplus G_{0,2}^m$$

such that $I = G_{0,2}^m \oplus E_0^m$. From (3.5) it follows that $|G_{0,1}^m| < |E_0^m|$. We represent the interval E_0^m as the sum

$$E_0^m = E_{0,1}^m \oplus E_{0,2}^m,$$

where $|E_{0,1}^m| = |G_{0,1}^m|$. The map $d_{I_m} R_\alpha$ exchanges intervals by the following rules

$$\begin{array}{c} G_{0,1}^m \to G_{G_m,1}^m, \\ G_{0,2}^m \to G_{G_m,2}^m, \\ E_{0,1}^m \to E_{E_m,1}^m = G_{0,1}^m \to G_{G_m,1}^m, \\ E_{0,2}^m \to E_{E_m,2}^m. \end{array}$$

Thus, we have

$$E^m_{E_m,2} \oplus G^m_{G_m,1} \oplus G^m_{G_m,2} = E^m_{E_m,2} \oplus G^m_{G_m} = I,$$

as

$$|I| = e_m + g_m - |G^m_{0,1}|$$

and

$$|E^m_{E_m,1}| + |G^m_{G_m}| = e_m + g_m - |G^m_{0,1}|.$$

We see that the derivative $d_I R_\alpha$ is the non-cyclic 3-intervals exchange transformation

$$G^m_{0,2} \oplus E^m_{0,1} \oplus E^m_{0,2} \to E^m_{E_m,2} \oplus G^m_{G_m,1} \oplus G^m_{G_m,2}$$

or, otherwise,

$$G^m_{0,2} \oplus E^m_{0,1} \oplus E^m_{0,2} \to d_I R_\alpha(E^m_{0,2}) \oplus d_I R_\alpha(E^m_{0,1}) \oplus d_I R_\alpha(G^m_{0,2}).$$

So, we obtain the following theorem.

Theorem 3.1. *Let*

$$R_\alpha : x \to x - \alpha \pmod 1$$

be a rotation of the unit circle $\mathbb{S}^1 = [0; 1)$. If $I = I_m$ is a proper differentiation interval, then the derivative $d_I R_\alpha$ is a circle rotation again. If $I \neq I_m$ is an improper interval, then $d_I R_\alpha$ is a non-cyclic 3-interval exchange transformation.

The first part of this theorem can be reformulated using the orbit language. Let $Orb(a, \alpha, I)$ be an orbit of the point a under the map $x \to x - \alpha \pmod I$. By definition, we have the following sequence of intervals

$$I_0 \supset I_1 \supset \ldots \supset I_m \supset \ldots$$

Let's put $S_l = (S, I_l)$, where $S = (a_i)_{i=0}^{\infty}$ is a sequence in the interval I_l,

$$d^k S_l = (S', I_{l+k}),$$

where $S' = S \cap I_{l+k}$ and elements of the sequence $S' = (a_{i_j})_{j=0}^{\infty}$ are ordered by the induction:

$$i_0 = \min\{i : a_i \in I_{l+k}\},$$

$$i_{n+1} = \min\{i > i_n : a_i \in I_{l+k}\}.$$

Theorem 3.2. *Let*

$$\begin{aligned} O^-_m &= Orb(1, -g_m(\alpha), I_m), \\ O^+_m &= Orb(1, g_m(\alpha), I_m), \\ \widetilde{O}^-_m &= Orb(1, -e_m(\alpha), I_m), \\ \widetilde{O}^+_m &= Orb(1, e_m(\alpha), I_m). \end{aligned}$$

Then

$$\begin{aligned} d^k O^-_m &= O^-_{m+k}, \\ d^k O^+_m &= O^-_{m+k}, \\ d^k \widetilde{O}^-_m &= \widetilde{O}^-_{m+k}, \\ d^k \widetilde{O}^+_m &= \widetilde{O}^+_{m+k}. \end{aligned}$$

Proof. The first relation is actually proved in Theorem 3.1. The third relation follows from the first relation for $1-\alpha$ with the help of the equalities $g_m(\alpha) = e_m(1-\alpha)$, $e_m(\alpha) = g_m(1-\alpha)$. Other two relations follow from the proved relations with the help of the equality $e_m(\alpha)+g_m(\alpha) = |I_m|$.

Let h_m be a homothety with the center at the point 1, mapping the interval I_m to $I_0 = [0;1)$. Consider the orbits $O_m = h_m(O^-_m)$. Then

$$O_0 = Orb(1, -\alpha, \mathbb{S}^1),$$
$$O_m = Orb(1, -d^m\alpha, \mathbb{S}^1)$$

for the some $d^m\alpha$. The map d^m has the property

$$d^{m_1}(d^{m_2}\alpha) = d^{m_1+m_2}\alpha.$$

From Theorem 3.2 it follows that

$$d^m\alpha = \frac{g_m(\alpha)}{g_m(\alpha)+e_m(\alpha)} = \frac{g_m(\alpha)}{i_m(\alpha)}. \tag{3.6}$$

Using (2.1), we may rewrite formula (3.6) as

$$d^m\alpha = \frac{\langle E_m\alpha\rangle}{\langle E_m\alpha\rangle - \langle G_m\alpha\rangle + 1}.$$

To receive explicit formulaes for $d^m\alpha$ is our further purpose.

4. Formulaes for $d^m\alpha$

We obtain a formula for $d^1\alpha$ by direct calculation.

Proposition 4.1.

$$d^1\alpha = \begin{cases} \frac{\alpha}{1-\alpha}, & \text{if } \alpha < \frac{1}{2}, \\ \frac{2\alpha-1}{\alpha}, & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

Proposition 4.2. *Suppose that* $\alpha = [0; q_1, q_2, q_3, \ldots]$, *where there is a continued fraction expansion of* α *in the square brackets. Then*

$$d^1\alpha = \begin{cases} [0; q_1-1, q_2, q_3, \ldots], & \text{if } q_1 > 1, \\ [0; 1, q_2-1, q_3, \ldots], & \text{if } q_1 = 1, q_2 > 1, \\ [0; q_3+1, q_4, q_5, \ldots], & \text{if } q_1 = q_2 = 1. \end{cases}$$

To find $d^m\alpha$ we need to calculate $e_m(\alpha)$ and $g_m(\alpha)$. First we obtain formulaes for $s_m(\alpha)$ and $l_m(\alpha)$.

We note that

$$d^m\alpha + d^m(1-\alpha) = 1.$$

Therefore we may consider only the case $\alpha \in (0; \frac{1}{2})$.

Number i is called α-regular, if $l_{m-1}(\alpha) > l_m(\alpha)$. Let $\{m_i\}$ be a sequence of α-regular numbers,

$$n_i = m_i - m_{i-1},$$
$$\overline{s}_i = s_{m_i}(\alpha),$$
$$\overline{l}_i = l_{m_i}(\alpha).$$

The transition

$$Til_{m-1}(\alpha) \to Til_m(\alpha)$$

transforms long and short intervals by the following rules:

$$S^m \to S^{m+1},$$
$$L^m \to S^{m+1} \oplus L^{m+1},$$

if m is not α-regular; and

$$\begin{aligned} S^m &\to L^{m+1}, \\ L^m &\to L^{m+1} \oplus S^{m+1}, \end{aligned}$$

if m is α-regular. The transition

$$Til_{m_{i-1}}(\alpha) \to Til_{m_i}(\alpha)$$

is defined by the substituons

$$\begin{aligned} S^{m_{i-1}} &\to L^{m_i}, \\ L^{m_{i-1}} &\to n_i L^{m_i} \oplus S^{m_i}. \end{aligned} \tag{4.1}$$

From (4.1) it follows that

$$\overline{s}_i = \overline{l}_{i+1}, \tag{4.2}$$

$$\overline{l}_{i-1} = n_i \overline{l}_i - \overline{l}_{i+1}. \tag{4.3}$$

Proposition 4.3. *Let's put $G(\alpha) = \langle \frac{1}{\alpha} \rangle$. Then*

$$n_1 = [\frac{1}{\alpha}] - 1,$$

$$\overline{l}_0 = 1 - \alpha,$$

$$\overline{s}_0 = \alpha.$$

For $k \geq 1$ we have

$$n_k = \left[\frac{1}{G^{k-1}(\alpha)} \right], \tag{4.4}$$

$$\overline{l}_k(\alpha) = \prod_{i=0}^{k-1} G^i(\alpha). \tag{4.5}$$

Proof. One can proof the Proposition by the induction on k using relations (4.2), (4.3), and the equalities

$$n_k = [\frac{\overline{l}_k}{\overline{s}_k}] = [\frac{\overline{l}_{k-1}}{\overline{l}_k}].$$

The map $G(\alpha) = \langle \frac{1}{\alpha} \rangle$ is known as the Gauss-Kuzmin transformation [4]. The following properties of this map are well-known. Suppose that the continued fraction expansion of α is equal to $\alpha = [0; q_1, q_2, \ldots, q_n, r_n(\alpha)]$, where $r_n(\alpha)$ is a remainder. Then

$$q_i = \left[\frac{1}{G^{i-1}(\alpha)} \right], \tag{4.6}$$

$$r_n(\alpha) = G^n(\alpha). \tag{4.7}$$

Here is supposed that $G^0(\alpha) = \alpha$.

Comparing (4.4) with (4.6) and (4.5) with (4.7), we see that

$$n_k = q_k \text{ for } k > 1 \text{ and } n_1 = q_1 - 1, \tag{4.8}$$

$$\overline{l}_k = \prod_{i=0}^{k-2} r_i(\alpha). \tag{4.9}$$

We consider the problem of evaluation $l_i(\alpha)$ and $s_i(\alpha)$ in the case when m is not α-regular. Assume that $m_{i-1} < m < m_i$. We represent m as $m = m_{i-1} + t$, where $0 < t < n_i$. A transition

$$Til_{m_{i-1}}(\alpha) \to Til_m(\alpha)$$

is defined by the substitutions

$$\begin{array}{l} S^{m_{i-1}} \to S^m, \\ L^{m_{i-1}} \to tS^m \oplus L^m. \end{array} \tag{4.10}$$

Hence,

$$s_m(\alpha) = \overline{s}_i, \tag{4.11}$$

$$l_m(\alpha) = \overline{l}_i - t\overline{s}_i. \tag{4.12}$$

Combining (4.2), (4.9) - (4.12), we get

Proposition 4.4. *Suppose that* $m = m_i + t$, $0 \le t < n_{i+1}$. *Then*

$$l_m(\alpha) = \prod_{j=0}^{i} r_j(\alpha),$$

$$s_m(\alpha) = \prod_{j=0}^{i-1} r_j(\alpha)(1 - tr_i(\alpha)).$$

Secondly, we calculate $e_m(\alpha)$ and $g_m(\alpha)$. Now we need a formula for $Col_{m,\alpha}$. As $\alpha \in (0; \frac{1}{2})$, we have $Til_0(\alpha) = S^0 \oplus L^0$. Hence,

$$\begin{array}{l} Col_{0,\alpha}(L) = E, \\ Col_{0,\alpha}(S) = G. \end{array} \tag{4.13}$$

Formula (4.10) means that the transition

$$Til_{m_{i-1}}(\alpha) \to Til_{m_{i-1}+t}(\alpha)$$

does not vary the last interval in the tiling. So, we get

$$Col_{m_{i-1}+t,\alpha} \equiv Col_{m_{i-1},\alpha}. \tag{4.14}$$

On the other hand, substitutions (4.1) imply

$$Col_{m_i,\alpha} \equiv \overline{Col_{m_{i-1},\alpha}}. \tag{4.15}$$

Combining (4.13) - (4.15), we obtain

Proposition 4.5. *Suppose that* $m = m_i + t$, $0 \le t < n_{i+1}$. *Then*

$$Col_{m,\alpha}(L) = \begin{cases} E, & \text{if } i \text{ is even,} \\ G, & \text{if } i \text{ is odd,} \end{cases}$$

$$Col_{m,\alpha}(S) = \begin{cases} G, & \text{if } i \text{ is even,} \\ E, & \text{if } i \text{ is odd.} \end{cases}$$

The inverse map is equal to

$$\begin{array}{l} Col^{-1}_{m,\alpha}(E) = \begin{cases} L, & \text{if } i \text{ is even,} \\ S, & \text{if } i \text{ is odd,} \end{cases} \\ Col^{-1}_{m,\alpha}(G) = \begin{cases} S, & \text{if } i \text{ is even,} \\ L, & \text{if } i \text{ is odd.} \end{cases} \end{array} \tag{4.16}$$

Combining Propositions 4.4, 4.5 with formula (3.6), we obtain a formula for $d^m\alpha$ in terms of the continued fraction expansion of α.

Proposition 4.6. *Suppose that* $\alpha \in (0; \frac{1}{2})$, $m = m_i + t$, $0 \le t < n_{i+1}$. *Then*

$$d^m\alpha = \begin{cases} \frac{r_i(\alpha)}{1+(1-t)r_i(\alpha)}, & \text{if } i \text{ is even,} \\ \frac{1-tr_i(\alpha)}{1+(1-t)r_i(\alpha)}, & \text{if } i \text{ is odd.} \end{cases}$$

5. Derivatives and Bernoulli shifts

We correspond to any number α the sequence

$$w(\alpha) = (w_0, w_1, w_2, \ldots),$$

where

$$w_m = \begin{cases} 0, & \text{if } g_m(\alpha) < e_m(\alpha), \\ 1, & \text{if } g_m(\alpha) > e_m(\alpha). \end{cases}$$

The sequence $w(\alpha)$ is finite only in the case $g_m(\alpha) = e_m(\alpha)$. It is possible only for rational number α. From (4.16) it follows

$$w_m = \begin{cases} 0, & \text{if } i \text{ is even}, \\ 1, & \text{if } i \text{ is odd}. \end{cases}$$

for $m = m_i + t$, $0 \le t < n_{i+1}$

We denote by

$$x^n = (x, \ldots, x)$$

the sequence of n elements x. We see that a number α with the expansion $\alpha = [0; q_1, q_2, q_3, \ldots]$ has the sequence

$$w(\alpha) = 0^{q_1-1}1^{q_2}0^{q_3}1^{q_4}\ldots \tag{5.1}$$

Therefore, the correspondence

$$\alpha \mapsto w(\alpha)$$

between $\alpha \in [0; 1]$ and $w(\alpha)$ is one-to-one.

Proposition 5.1. *$\{0, w(\alpha)\}$ is an expansion α in the Stern-Brocot sequence.*

Proof. Let's recall the definition of the Stern-Brocot sequence. For this purpose we consider two fractions $\frac{0}{1}$ and $\frac{1}{0}$, and repeat the following operation:

insert the fraction $\frac{m+m'}{n+n'}$ between two neigbour fractions $\frac{m}{n}$ and $\frac{m'}{n'}$.

As a result, we get a tree of fractions which is called a Stern - Brocot tree. Binary searching of α in this tree gives an expansion of α in the Stern - Brocot sequence. Label 0 corresponds to left branches in the tree, but label 1 corresponds to right branches [3]. Comparing (5.1) with the expansion of α in the Stern - Brocot sequence, reduced in [3], we complete the proof of the Proposition.

Now we find the action of the map d^1 on the sequence $w(\alpha)$.

Theorem 5.2. *Suppose that $w(\alpha) = (w_0, w_1, w_2, \ldots)$. Then*

$$w(d^1\alpha) = (w_1, w_2, w_3, \ldots).$$

Thus, the action of the map d^1 is equivalent to the Bernoulli shift on the set of 0/1-sequences.

Proof. By definition, we have

$$g_0(d^1\alpha) = \frac{g_1(\alpha)}{i_1(\alpha)},$$

$$e_0(d^1\alpha) = \frac{e_1(\alpha)}{i_1(\alpha)}.$$

Using the induction on m, it is easy to obtain that

$$g_m(d^1\alpha) = \frac{g_{m+1}(\alpha)}{i_1(\alpha)},$$

$$e_m(d^1\alpha) = \frac{e_{m+1}(\alpha)}{i_1(\alpha)}.$$

Hence, $g_m(d^1\alpha) > e_m(d^1\alpha)$ if and only if $g_{m+1}(\alpha) > e_{m+1}(\alpha)$. Similarly, $g_m(d^1\alpha) < e_m(d^1\alpha)$ if and only if $g_{m+1}(\alpha) < e_{m+1}(\alpha)$.

Corollary 5.3. *For any $\alpha \in [0;1)$ the equation*

$$d^m\beta = \alpha$$

has 2^m solutions.

Proof. For the proof it is enough to notice that $w(\beta) = (w_0, \dots, w_{m-1}, w(\alpha))$, where $(w_0, \dots, w_{m-1})$ is one of 2^m possible 0/1 sequences.

Corollary 5.4. *Suppose that α is irrational. Then the relation*

$$d^{m+T}\alpha = d^m\alpha \tag{5.2}$$

holds for the some m and α if and only if α is a quadratic irrationality.

This corollary is equivalent to the fact that the Stern - Brocot sequence of α is periodic if and only if α is a quadratic irrationality. This fact is equivalent to the Lagrange theorem about continued fraction expansions of quadratic irrationalities.

The geometrical sence of relation (5.2) means a similarity of orbits and their derivatives. In other words, there exists a homothety $h_{m,T}$ such that

$$h_{m,T}O_m^- = d^T O_m^-. \tag{5.3}$$

Using Theorem 3.2, we may rewrite relation (5.3) as the following relation

$$h_{m,T}O_m^- = O_{m+T}^-.$$

It means the similarity of the orbits.

If relation (5.2) is satisfied for some $m = m_0$, then it is also satisfied for all $m > m_0$. In this case the sequence of derivatives is periodic. Let $T_0(\alpha)$ be the least period.

Corollary 5.5. *Let α be a quadratic irrationality. Suppose that the continued fraction exprasion of α is equal to*

$$\alpha = [0; q_1, \dots, q_k, (q_{k+1}, \dots, q_{k+r})].$$

Here in the interior brackets the period of the expansion is written. Then

$$T_0(\alpha) = \begin{cases} q_{k+1} + \dots + q_{k+r}, & \text{if } r \text{ is even,} \\ 2(q_{k+1} + \dots + q_{k+r}), & \text{if } r \text{ is odd.} \end{cases}$$

Proof. The proof is obvious.

Corollary 5.6. *The number of all solutions of the equation $T_0(\alpha) = n$ is equal to*

$$\sum_{d|n} \mu(\frac{n}{d})(2^d - 1).$$

Here $\mu(x)$ is the Möbius function.

Proof. The number of solutions of the equation $d^n\alpha = \alpha$ is equal to $2^n - 1$. It follows from the fact that the number of 0/1-sequences of the length n is equal to 2^n, and the sequence $(1, \dots, 1)$ corresponds to $\alpha = 1$. Further, it is enough to use the standard formula of transition from periodic points to primitive periodic points.

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