Max-Planck-Institut für Mathematik
Bonn

On topological type of periodic self-homeomorphisms of closed non-orientable surfaces

by

Grzegorz Gromadzki
Błażej Szepietowski

Max-Planck-Institut für Mathematik
Preprint Series 2014 (25)

On topological type of periodic self-homeomorphisms of closed non-orientable surfaces

Grzegorz Gromadzki
Błażej Szepietowski

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Institute of Mathematics
Gdańsk University
Wita Stwosza 57
80-952 Gdańsk
Poland

MPIM 14-25

ON TOPOLOGICAL TYPE OF PERIODIC SELF-HOMEOMORPHISMS OF CLOSED NON-ORIENTABLE SURFACES

GRZEGORZ GROMADZKI, BŁAŻEJ SZEPIETOWSKI

Abstract. Let S_g denote a closed non-orientable surface of genus $g \geq 3$. At the beginning of 1980s E. Bujalance showed that the maximum order of a periodic self-homeomorphism of S_g is equal to $2g$ or $2(g-1)$ for g odd or even respectively, and this upper bound is attained for all $g \geq 3$. In this paper we investigate rigidity of topological type of cyclic group actions on S_g of order $N > g-2$, with prescribed ramification data. As an application, we compute, for N between $\max\{g, 3(g-2)/2\}$ and $2g$, the numbers of different topological types of actions of $\mathbb{Z}_N$ on S_g. This is an analogue, for non-orientable surfaces, of a result of S. Hirose which was the original motivation for this paper, along with a connection with topological properties of moduli spaces of purely imaginary real algebraic curves.

1. Introduction

By an effective action of a finite group G on a closed surface S we understand an embedding of G into the group Homeo(S) of homeomorphisms of S. Two such actions are *topologically equivalent* if the images of G are conjugate in Homeo(S). The topological classification of finite group actions on closed surfaces is a classical problem going back to Nielsen [14].

Let $\mathcal{M}_g$ denote the moduli space of complex algebraic curves of genus $g \geq 2$ and consider its subset $\mathcal{M}_g(G)$ consisting of points representing curves with a finite group G of birational automorphisms. It is intuitively plausible, and Teichmüller-Royden theory provides a more precise justification, that $\mathcal{M}_g(G)$ is smaller for bigger G. In other words, a curve is better described by its group of automorphims when this group is bigger. By the famous Hurwitz bound its order does not exceed $84(g-1)$.

Particularly interesting are the cases when a curve X is determined, up to birational equivalence, by the topological type of the action of G, or only by its ramification data, by which we understand the genus of the orbit space X/G and the branching indices of the projection $X \to X/G$, or even only by the order of G. More specifically, when $|G| > 12(g-1)$, then by the Hurwitz-Riemann formula and elementary Teichmüller theory, $\mathcal{M}_g(G)$ is finite and topological and birational types of the action coincide, see [8, 11, 16, 15]. For example, the main discovery of the paper [12] by Hirose, translated to the language of complex algebraic curves, asserts that with a few exceptions, a complex curve of genus $g \geq 2$ having an automorphism of order $N \geq 3g$ is determined (up to birational equivalence) by N. The condition $N \geq 3g$ turns out to be quite restrictive, as it forces N to be one of $4g+2$, $4g$,

Both authors supported by NCN 2012/05/B/ST1/02171. G. Gromadzki supported also by Max-Planck Mathematical Institute in Bonn.

$3g+3$ or $3g$ for $g > 12$. On the other hand, there are infinitely many rational numbers q and r, such that for infinitely many $g \geq 2$ there is a homeomorphism of a closed orientable surface of genus g having order $N = qg + r$ satisfying $2(g-1) < N < 3g$. In [1] a more general situation is studied, when the order of a cyclic group of automorphisms of a compact Riemann surface of genus $g \geq 2$, or the ramification data of the action, determine its topological type. Importance of such results follows from their connection with topology of the singular locus of the moduli space of complex algebraic curves, see [10].

Motivated by [12] and [1], in this paper we consider analogous problems for purely imaginary real curves, which can also be seen as compact, unbordered, non-orientable surfaces with dianalitic structure (see [5] for a definition). The study of dianalitic automorphisms of such surfaces is equivalent to the study of their periodic self-homeomorphisms, because every periodic homeomorphism of a surface S_g of topological genus $g \geq 3$ is a dianalitic automorphism with respect to some dianalitic structure on S_g. Bujalance showed in [2] (see also later paper of S. Wang [18]) that the maximal order of such automorphism of a non-orientable surface of genus $g \geq 3$ is equal to $2g$ or $2(g-1)$ for g odd or even respectively, and this upper bound is attained for all $g \geq 3$. The case $g = 3$ is well understood, as the mapping class group of S_3 is isomorphic to $\mathrm{GL}_2(\mathbb{Z})$ (see [7]), and the classification of conjugacy classes of torsion elements in the latter group is known. Another interesting problem concerning cyclic periodic actions on non-orientable surfaces was considered in recent paper [3], where the authors investigated such actions which can not be extended to any bigger group.

Throughout the whole paper we denote by S or S_g a closed non-orientable surface of genus $g \geq 3$. In this paper we study the extent to which the order N or the ramification data of a cyclic group G acting on S_g determine the topological type of the action, which is important in virtue of the connection with topological properties of moduli spaces of purely imaginary real algebraic curves, similar as in the case of orientable surfaces. More specifically, in Section 3 we investigate rigidity of topological type of cyclic group actions of order $N > g-2$ with prescribed ramification data. We consider a quite large family of actions, where the order N has the form $N = qg + r$, for infinitely many rational q and r. Furthermore, for each such pair q, r, there is an action of $\mathbb{Z}_N$ on S_g for infinitely many genera g. As an application, in Section 4 we calculate, for N between $\max\{g, \frac{3}{2}(g-2)\}$ and $2g$, the numbers of topological types of cyclic actions of order N on S_g. This should be seen as an analogue, for a non-orientable surface, of the main result of [12].

2. Preliminaries

2.1. **Principal definitions.** Our approach is based on algebraic properties of the discrete subgroups of isometries of the hyperbolic plane $\mathcal{H}$, called NEC-groups. We refer the reader to the monograph [5] for an extensive exposition of the theory.

Suppose that a finite group G acts effectively by homeomorphisms on a closed non-orientable surface S_g of genus $g \geq 3$. Fix a dinanalitic structure on S, with respect to which

G acts by dianalitic automorphisms. Then S is conformally isomorphic to the orbit space $\mathcal{H}/\Gamma$ for a torsion-free NEC group Γ isomorphic to $\pi_1(S)$. Such Γ is called *non-orientable surface group*. Furthermore, G is isomorphic to the quotient Λ/Γ, for some other NEC-group Λ, a subgroup of the normalizer of Γ in the group of all isometries of $\mathcal{H}$ (Λ is equal to that normaliser if and only if G is the full group of dianalitic automorphisms of S). Equivalently, there is an epimorphism $\theta\colon \Lambda \to G$ with kernel Γ, usually called *smooth epimorphism* to underline the fact that its kernel is torsion-free. This motivates the following definition.

Definition 2.1. Suppose that Λ is a NEC group, G is a finite group, and $\theta\colon \Lambda \to G$ is an epimorphism. We say that θ is a *NSK-map* (non-orientable-surface-kernel-map) if and only if $\ker\theta$ is a non-orientable surface group.

Two effective actions of G on S_g are topologically conjugate (by a homeomorphism of S_g) if and only if the associated NSK-maps are equivalent in the sense of the next definition (see [4, Proposition 2.2] and its proof; the same argument applies to closed surfaces).

Definition 2.2. We say that two NSK-maps $\theta_i\colon \Lambda_i \to G$, $i = 1, 2$, are *equivalent* if and only if there exist isomorphisms $\phi\colon \Lambda_1 \to \Lambda_2$ and $\alpha\colon G \to G$ such that the following diagram is commutative.

$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{\ \phi\ } & \Lambda_2 \\ {\scriptstyle\theta_1}\downarrow & & \downarrow{\scriptstyle\theta_2} \\ G & \xrightarrow{\ \alpha\ } & G \end{array} \tag{2.1}$$

The ramification data of G is encoded in the signature $\sigma(\Lambda)$ of Λ, which in our case has the form

$$(h; \pm; [m_1, \ldots, m_r]; \{(-) \overset{k}{\ldots}, (-)\}), \tag{2.2}$$

where $k > 0$ if the sign is "+" (see [2]). The orbit space $S/G = \mathcal{H}/\Lambda$ has genus h and k boundary components, and it is orientable if and only if the sign is "+". From the signature one can also read a presentation of Λ in terms of canonical generators and defining relations as follows. The generators are:

x_i	for $1 \le i \le r$
c_j, e_j	for $1 \le j \le k$
a_l, b_l	for $1 \le l \le h$ if the sign is "+"
d_l	for $1 \le l \le h$ if the sign is "-"

The defining relations are:

$x_i^{m} = 1$	for $1 \le i \le r$
$c_j^2 = 1, \quad [e_j, c_j] = 1$	for $1 \le j \le k$
$x_1 \cdots x_r e_1 \cdots e_k [a_1, b_1] \cdots [a_h, b_h] = 1$	if the sign is "+"
$x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_h^2 = 1$	if the sign is "-"

Note that Λ is a proper NEC group, i.e. it contains orientation-reversing isometries. Among the canonical generators, c_j and d_l are orientation-reversing, the remaining ones are orientation-preserving. We denote by Λ^+ the canonical Fuchsian subgroup of Λ, consisting of all orientation-preserving elements of Λ. Finally, we have the Hurwitz-Riemann ramification formula

$$g - 2 = |G|\mu(\sigma),$$

where

$$\mu(\sigma) = \alpha h - 2 + k + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)$$

is the normalized hyperbolic area of a (arbitrary) fundamental region for Λ. Here $\alpha = 2$ if the sign of the signature is "+" and $\alpha = 1$ otherwise.

The following lemma, which is a special case of [2, Proposition 3.2], provides an effective criterion for an NSK-map.

Lemma 2.3. *Suppose that Λ is a NEC group with signature* (2.2). *A group homomorphism $\theta\colon \Lambda \to G$ is an NSK-map if an only if*

(1) *$\theta(x_i)$ has order m_i for $1 \le i \le r$,*
(2) *$\theta(c_j)$ has order 2 for $1 \le i \le k$,*
(3) *$\theta(\Lambda^+) = G$.*

In this paper we are interested in the case where G is a cyclic group $\mathbb{Z}_N$.

Definition 2.4. Suppose that σ is an NEC signature (2.2) and N is a positive integer. We say that the pair (σ, N) is *admissible* if there exists a NSK-map $\theta\colon \Lambda \to \mathbb{Z}_N$ with $\sigma(\Lambda) = \sigma$. If, furthermore, such θ is unique up to equivalence, then we say that (σ, N) is *rigid*.

For two integers a, b we denote by (a, b) their greatest common divisor and we use the additive notation for cyclic groups throughout the whole paper.

2.2. **Automorphisms of NEC-groups vs mapping class groups.** In this subsection we recall the relationship between the outer automorphism group of an NEC-group Λ and the mapping class group of the orbit space $\mathcal{H}/\Lambda$. For simplicity we assume that the signature of Λ has the form

$$(h; -; [m_1, \dots, m_r]; \{-\}),$$

where the periods m_i are all different. For a discussion of the general case see [4, Section 4].

Set $S = \mathcal{H}/\Lambda$ and note that S is a non-orientable surface of genus h with r distinguished points, over which the projection $p\colon \mathcal{H} \to S$ is ramified. Let $\mathcal{P}$ denote the set of distinguished points, $\mathcal{U} = \mathcal{H}\backslash p^{-1}(\mathcal{P})$ and $S_0 = S\backslash\mathcal{P}$. Then $p\colon \mathcal{U} \to S_0$ is a regular covering and Λ is its deck group isomorphic to $\pi_1(S_0)/p_*(\pi_1(\mathcal{U}))$. The canonical generators $x_1, \dots, x_r$ and $d_1, \dots, d_h$ of Λ correspond to standard generators of $\pi_1(S_0)$ and $p_*(\pi_1(\mathcal{U}))$ is normally generated by $x_i^{m_i}$ for $i = 1, \dots, r$.

We denote by $\mathrm{Mod}(S,\mathcal{P})$ the mapping class group of S relative to $\mathcal{P}$, defined as the group of isotopy classes of homomorphism of S preserving $\mathcal{P}$. The pure mapping class group is the subgroup $\mathrm{PMod}(S,\mathcal{P})$ of $\mathrm{Mod}(S,\mathcal{P})$ consisting of the isotopy classes of homomorphism fixing each element of $\mathcal{P}$. The groups $\mathrm{PMod}(S,\mathcal{P})$ and $\mathrm{Out}(\Lambda)$ are isomorphic by a generalisation, for non-orientable S, of [11, Theorem 1] (see [6, Section 3]). Given an element of $\mathrm{PMod}(S,\mathcal{P})$ one can find its image in $\mathrm{Out}(\Lambda)$ as follows. Represent this element by a homeomorphism $f\colon S_0 \to S_0$ fixing some base point. Then $f_*\colon \pi_1(S_0) \to \pi_1(S_0)$ preserves $p_*(\pi_1(\mathcal{U}))$, hence it induces an automorphism of the quotient $\pi_1(S_0)/p_*(\pi_1(\mathcal{U})) \cong \Lambda$.

3. Rigid pairs

In this section we determine some rigid pairs (σ, N) and compute the numbers of equivalence classes of NSK-maps for a family of signatures σ and $N > g-2$. The results presented here are of two types. The first type concerns necessary and sufficient conditions for (σ, N) to be admissible - most of them follows from a more general result in [9], where such conditions are given for arbitrary NEC signature σ. The other type of results concerns automorphisms of NEC-groups and related mappings class groups of certain surfaces of low genus and small numbers of boundary components and punctures. Some of these results are borrowed from [4] and some of them are new. They will play a key role in Section 4 and we believe that they are of independent interest.

3.1. **Signature $(\mathbf{2}, -, [\mathbf{m}], \{-\})$.** In this subsection we fix Λ with such signature. We also fix canonical generators d_1, d_2 of Λ, satisfying single defining relation $(d_1^2 d_2^2)^m = 1$.

Lemma 3.1. *Let $y = d_1$, $z = d_1 d_2$. $\mathrm{Out}(\Lambda)$ is generated by classes of automorphisms α, β, γ defined by*

$$\alpha\colon \begin{cases} y \mapsto yz \\ z \mapsto z \end{cases} \qquad \beta\colon \begin{cases} y \mapsto y^{-1} \\ z \mapsto z \end{cases} \qquad \gamma\colon \begin{cases} y \mapsto y \\ z \mapsto z^{-1} \end{cases}$$

Proof. Set $S = \mathcal{H}/\Lambda$ and note that S is a Klein bottle with one distinguished point $x \in S$. Recall from Subsection 2.2 that $\mathrm{Out}(\Lambda)$ is isomorphic to the mapping class group $\mathrm{Mod}(S, \{x\})$. By [13, Theorem 4.9] the last group is generated by 4 elements: Dehn twist, crosscap slide and 2 boundary slides. However, only one boundary slide is needed, as the second one can be expressed in terms of the remaining 3 generators (see [17, Theorem A.5], where it is proved that the mapping class group of once-punctured Klein bottle is isomorphic to $(\mathbb{Z} \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$). By computing the automorphisms of $\pi_1(S \setminus \{x\})$ induced by the generators of $\mathrm{Mod}(S, \{x\})$ we deduce our lemma. □

The following lemma is a particular case of Lemma 5.9 in [9] for $r = 1$.

Lemma 3.2. *(σ, N) is admissible if and only if either*

- *$m|N$ and N is odd, or*
- *$2m|N$ and $\frac{N}{2m}$ is odd.*

Proposition 3.3. *Suppose that N is odd and (σ, N) is admissible. Then (σ, N) is rigid if and only if $(m, \frac{N}{m}) \leq 3$.*

Proof. Let y and z be the generators of Λ from Lemma 3.1. By Lemma 2.3, a homomorphism $\theta\colon \Lambda \to \mathbb{Z}_N$ is an NSK-map if and only if $\theta(z)$ has order m and if n is the order of $\theta(y)$, then $\mathrm{lcm}(m, n) = N$.

Set $d = (m, \frac{N}{m})$ and suppose $d \in \{1, 3\}$. Let $\theta_2\colon \Lambda \to \mathbb{Z}_N$ be any NSK-map. We are going to show that θ_2 is equivalent to θ_1 defined by $\theta_1(y) = 1$, $\theta_1(z) = \frac{N}{m}$. By post-composing θ_2 with an automorphism of $\mathbb{Z}_N$, we may assume $\theta_2(z) = \frac{N}{m}$. Set $a = \theta_2(y)$. Note that $\frac{N}{m}$ and a are coprime, in particular $d \nmid a$. We have $a = kd \pm 1$ and $d = lm + k'\frac{N}{m}$ for some integers l, k and k'. Let α be the automorphism of Λ from Lemma 3.1. We have

$$\theta_2(\alpha^{-kk'}(y)) = \theta_2(y) - kk'\theta_2(z) = a - kk'\frac{N}{m} = klm \pm 1$$

Hence, by pre-composing θ_2 with $\alpha^{-kk'}$ we may assume $a = \pm 1 \pmod{m}$. Since a is coprime to m and to $\frac{N}{m}$, thus $a \in \mathbb{Z}_N^*$. We have $\theta_2(y) = a$ and $\theta_2(z) = \pm a\frac{N}{m}$, hence either $\theta_2 = a\theta_1$ or $\theta_2\gamma = a\theta_1$. Thus θ_2 and θ_1 are equivalent.

Now suppose that $d = (m, \frac{N}{m}) > 3$ and consider two NSK-maps θ_i, $i = 1, 2$, defined by $\theta_i(z) = \frac{N}{m}$ and $\theta_i(y) = i$. We claim that θ_1 and θ_2 are not equivalent. For suppose there exist $\phi \in \mathrm{Aut}(\Lambda)$ and $a \in \mathbb{Z}_N^*$ such that $\theta_2\phi = a\theta_1$. By Lemma 3.1 we have $\theta_2\phi(y) = \pm 2 + k\frac{N}{m}$ for some k and $\theta_2\phi(z) = \pm\frac{N}{m}$. From the last equality we have $a = \pm 1 \pmod{m}$. It follows that $1 = \pm 2 \pmod{d}$, a contradiction. ☐

Proposition 3.4. *Suppose that N is even and (σ, N) is admissible. Then (σ, N) is rigid if and only if $(m, \frac{N}{2m}) \leq 3$.*

Proof. By Lemma 2.3, a homomorphism $\theta\colon \Lambda \to \mathbb{Z}_N$ is an NSK-map if and only if $\theta(z)$ has order $2m$ and if n is the order of $\theta(y)$, then $\mathrm{lcm}(2m, n) = N$.

Set $d = (m, \frac{N}{2m})$ and suppose $d \in \{1, 3\}$. Since $\frac{N}{2m}$ is odd, we have $d = (2m, \frac{N}{2m})$. If $\theta_2\colon \Lambda \to \mathbb{Z}_N$ is any NSK-map, then by the same argument as in the proof of Proposition 3.3, we may assume $\theta_2(z) = \frac{N}{2m}$ and $\theta_2(y) = \pm 1 \pmod{2m}$. It follows that θ_2 is equivalent to θ_1 defined by $\theta_1(y) = 1$, $\theta_1(z) = \frac{N}{2m}$.

Conversely, if $(m, \frac{N}{m}) > 3$, then by the same argument as in the proof of Proposition 3.3, it can be shown that θ_i defined for $i = 1, 2$ by $\theta_i(z) = \frac{N}{2m}$ and $\theta_i(y) = i$ are not equivalent. ☐

3.2. **Signatures $(\mathbf{0}, +, [\mathbf{m}], \{(-), (-)\})$ and $(\mathbf{1}, -, [\mathbf{m}], \{(-)\})$.** Here we fix NEC groups Λ_1 and Λ_2 with such signatures and fix canonical generators x, e, c_1, c_2 of Λ_1, satisfying the following defining relations:

$$x^m = c_1^2 = c_2^2 = 1, \quad ec_1 = c_1e, \quad xec_2 = c_2xe,$$

and canonical generators x, d, c, of Λ_2, satisfying the following defining relations:

$$x^m = c^2 = 1, \quad d^2xc = cd^2x.$$

The following two lemmas are proved in [4, Lemma 4.6, Proposition 4.10 and Proposition 4.12].

Lemma 3.5. $\mathrm{Out}(\Lambda_1)$ *is isomorphic to the Klein four-group and is generated by classes of automorphisms α, β defined by*

$$\alpha\colon \begin{cases} x \mapsto e^{-1}x^{-1}e \\ e \mapsto e^{-1} \\ c_1 \mapsto c_1 \\ c_2 \mapsto c_2 \end{cases} \qquad \beta\colon \begin{cases} x \mapsto e^{-1}xe \\ e \mapsto (xe)^{-1} \\ c_1 \mapsto c_2 \\ c_2 \mapsto c_1 \end{cases}$$

Lemma 3.6. $\mathrm{Out}(\Lambda_2)$ *is isomorphic to the Klein four-group and is generated by classes of automorphisms γ, δ defined by*

$$\gamma\colon \begin{cases} x \mapsto x^{-1} \\ d \mapsto x^{-1}d^{-1}x \\ c \mapsto c \end{cases} \qquad \delta\colon \begin{cases} x \mapsto x \\ d \mapsto (dx)^{-1} \\ c \mapsto (dx)^{-1}c(dx) \end{cases}$$

Lemma 3.7. *For $i = 1, 2$, (σ_i, N) is admissible if and only if N is even and m divides N .*

Proof. The "only if" part follows immediately for Lemma 2.3. For the "if" part, assume $2 \mid N$, $m \mid N$ and define $\theta_i\colon \Lambda_i \to \mathbb{Z}_N$ for $i = 1, 2$ by

$$\theta_1(x) = \frac{N}{m}, \quad \theta_1(e) = 1, \quad \theta(c_j) = \frac{N}{2} \text{ for } j = 1, 2$$
$$\theta_2(x) = \frac{N}{m}, \quad \theta_2(d) = 1 + \frac{N}{2}, \quad \theta(c) = \frac{N}{2}$$

Note that Λ_i^+ is generated by conjugates of x, e and c_1c_2 if $i = 1$, and by conjugates of x and cd if $i = 2$. It follows from Lemma 2.3 that θ_i are NSK-maps. □

Remark 3.8. Similarly as a few other signatures consider in this section, the above signature σ_2 is a special case of the one from Lemma 5.14 in [9] for $r = 1$. Unfortunately however, there is an error in the statement of that lemma, and we take the opportunity to correct it here: namely, the condition *"and some of $N/2$, $m_1, \ldots, m_r$ is even"* must be deleted. In the proof, the authors failed to observe that $c_0d \in \Lambda^+$ at the very end of page 182. Consequently, assertion (iv) of Theorem 6.4 in [9] also has to be modified. Its final part should read *"where $\alpha = 0$ if $\mathrm{lcm}(N/N_1, \ldots, N/N_r) = N$, and $\alpha = 1$ otherwise."*

Proposition 3.9. *Suppose that (σ_i, N) is admissible for $i = 1, 2$. Then it is rigid if and only if*

(1) $m \in \{2, 3, 4, 6\}$ *and*
(2) *if $m > 2$ then $m^2 | N$.*

Proof. **Case $i = 1$.** Observe that for each $a \in \mathbb{Z}_N^*$, the assignment $\theta_a(x) = \frac{N}{m}$, $\theta_a(c_j) = \frac{N}{2}$ for $j = 1, 2$, and $\theta_a(e) = a$ defines NSK-map $\theta_a\colon \Lambda_1 \to \mathbb{Z}_N$.

Suppose that (σ, N) is rigid and fix $a \in \mathbb{Z}_N^*$. Since θ_a is equivalent to θ_1, it is of the form $\theta_a = b\theta_1\phi$, where $b \in \mathbb{Z}_N^*$ and $\phi \in \mathrm{Aut}(\Lambda_1)$. By Lemma 3.5 we may assume $\phi \in \{1, \alpha, \beta, \alpha\beta\}$.

It follows that $\frac{N}{m} = b'\frac{N}{m}$, where $b' = \pm b$, and $a = b'$ or $a = b'(-1 - \frac{N}{m})$. We have $b' = 1 \pmod m$ and $a = 1 \pmod m$ or $a = -1 - \frac{N}{m} \pmod m$. Therefore, if $|\mathbb{Z}_m^*| > 2$ then we can find $a \in \mathbb{Z}_N^*$ such that θ_a is not equivalent to θ_1. The order of $\mathbb{Z}_m^*$ is given by Euler's totient function $\varphi(m)$. Since (σ, N) is rigid, $\varphi(m) \leq 2$, hence condition (1). Furthermore, if $m > 2$ then for because θ_{-1} and θ_1 are equivalent, we must have $-1 - \frac{N}{m} = -1 \pmod m$, hence condition (2).

Suppose that (1) and (2) are satisfied and $\theta\colon \Lambda_1 \to \mathbb{Z}_N$ is any NSK-map. We have $\theta(c_i) = \frac{N}{2}$ for $i = 1, 2$, and by post-composing θ with an automorphism of $\mathbb{Z}_N$ we may assume $\theta(x) = \frac{N}{m}$. Set $a = \theta(e)$. Observe that $\theta(\Lambda_1^+)$ is generated by N/m and a. Since $\theta(\Lambda_1^+) = \mathbb{Z}_N$ by Lemma 2.3, a is coprime to $\frac{N}{m}$. Suppose $m > 2$. Then (2) implies $(a, m) = 1$ and by (1) we have $a = \pm 1 \pmod m$. In particular $a \in \mathbb{Z}_N^*$. By pre-composing θ with β if necessary, we may assume $a = 1 \pmod m$. But then $\theta = a\theta_1$, hence θ is equivalent to θ_1. Suppose $m = 2$. If a is odd, then $a \in \mathbb{Z}_N^*$ and $\theta = a\theta_1$. If a is even, then $\frac{N}{2}$ must be odd, and $\theta\beta = a'\theta_1$, where $a' = -(a + \frac{N}{2}) \in \mathbb{Z}_N^*$.

Case $i = 2$**.** Suppose that $2|N$ and $m|N$. Set $y = cd$. Every NSK-map $\theta\colon \Lambda_2 \to \mathbb{Z}_N$ is equivalent (by multiplication by an element of $\mathbb{Z}_N^*$) to some θ_a defined by $\theta_a(x) = \frac{N}{m}$, $\theta_a(c) = \frac{N}{2}$ and $\theta_a(y) = a$, where $(a, \frac{N}{m}) = 1$ (note that $\theta_a(\Lambda_2^+)$ is generated by $\frac{N}{m}$ and a, hence the last equality is equivalent to $\theta_a(\Lambda_2^+) = \mathbb{Z}_N$). Suppose that (σ, N) is rigid and fix $a \in \mathbb{Z}_N^*$. Since θ_a is equivalent to θ_1, it is of the form $\theta_a = b\theta_1\phi$, where $b \in \mathbb{Z}_N^*$ and $\phi \in \mathrm{Aut}(\Lambda_2)$. By Lemma 3.6 we may assume $\phi \in \{1, \gamma, \delta, \gamma\delta\}$. It follows that $\frac{N}{m} = b'\frac{N}{m}$, where $b' = \pm b$, and $a = b'$ or $a = b'(-1 - \frac{N}{m})$. The rest of the proof follows as in the case $i = 1$. □

3.3. **Signature** $(\mathbf{0}, +, [\mathbf{m_1}, \mathbf{m_2}], \{(-)\})$**.** We fix an NEC group Λ with such signature and generators x_1, x_2, c, satisfying the following defining relations:

$$x_1^{m_1} = x_2^{m_2} = c^2 = 1, \quad x_1x_2c = cx_1x_2.$$

The following lemma is proved in [4, Proposition 4.10]

Lemma 3.10. *If $m_1 \neq m_2$ then $\mathrm{Out}(\Lambda)$ has order 2 and is generated by the class of automorphism α, defined by*

$$\alpha\colon \begin{cases} x_1 \mapsto x_1^{-1} \\ x_2 \mapsto x_1x_2^{-1}x_1^{-1} \\ c \mapsto c \end{cases}$$

The next one is proved in [2, Theorem 3.5 and Corollary 3.3] and it is also particular case of Lemma 5.16 in [9] for $r = 2$.

Lemma 3.11. *(σ, N) is admissible if and only if $N = \mathrm{lcm}(m_1, m_2)$ and N is even.*

Proposition 3.12. *Suppose that (σ, N) is admissible, $m_1 \neq m_2$ and $k = (m_1, m_2)$. There are exactly $\varphi(k)$ equivalence classes of NSK-maps $\Lambda \to \mathbb{Z}_N$. In particular, (σ, N) is rigid if and only if $k \leq 2$.*

Proof. Every NSK-map $\theta\colon \Lambda \to \mathbb{Z}_N$ is equivalent (by multiplication by an element of $\mathbb{Z}_N^*$) to θ_a defined by $\theta_a(c) = \frac{N}{2}$, $\theta_a(x_1) = \frac{N}{m_1}$ and $\theta_a(x_2) = a\frac{N}{m_2}$ for some $a \in \mathbb{Z}_{m_2}^*$. We are going to show that θ_a is equivalent to $\theta_{a'}$ if and only if $a = a' \pmod{k}$.

Suppose that θ_a is equivalent to $\theta_{a'}$. Then $\theta_a = b\theta_{a'}\phi$ for some $b \in \mathbb{Z}_N^*$ and $\phi \in \mathrm{Aut}(\Lambda)$. By Lemma 3.10, for every $\phi \in \mathrm{Aut}(\Lambda)$ either $\phi(x_i)$ is conjugate to x_i for $i = 1, 2$, or $\phi(x_i)$ is conjugate to x_i^{-1} for $i = 1, 2$. It follows that $\theta_a(x_i) = b'\theta_{a'}(x_i)$ for $i = 1, 2$, where $b' = b$ or $b' = -b$. We have $b' = 1 \pmod{m_1}$ and $b'a' = a \pmod{m_2}$, hence $a' = a \pmod{k}$.

Conversely, suppose that $a' = a \pmod{k}$. By Chinese Reminder Theorem, there exists unique $b \in \mathbb{Z}_N^*$ such that $b = 1 \pmod{m_1}$ and $b = (a')^{-1}a \pmod{m_2}$, where $(a')^{-1}$ is the inverse of a' in $\mathbb{Z}_{m_2}^*$. We have $\theta_a = b\theta_{a'}$.

To finish the proof it suffices to note that for each $d \in \mathbb{Z}_k^*$ there exists $a \in \mathbb{Z}_{m_2}^*$ such that $d = a \pmod{k}$. Hence, equivalence classes of NSK-maps $\Lambda \to \mathbb{Z}_N$ are in one to one correspondence with elements of $\mathbb{Z}_k^*$. □

Remark 3.13. If $m_1 = m_2 = N$ then there exists $\phi \in \mathrm{Aut}(\Lambda)$ which swaps the conjugacy classes of x_1 and x_2. Consequently, θ_a and $\theta_{a'}$ are equivalent if and only if either $a = a'$ or $aa' = 1$. Hence, the number of equivalence classes of NSK-maps $\Lambda \to \mathbb{Z}_N$ is $\frac{\varphi(N)}{2} + z$, where z is the number of elements of order 2 in $\mathbb{Z}_N^*$. Note that z depends on the number of different prime divisors of N.

3.4. **Signature $(\mathbf{1}, -, [\mathbf{m_1}, \mathbf{m_2}], \{-\})$.** In this subsection we fix Λ with such signature and canonical generators x_1, x_2, d of Λ, satisfying the following defining relations:

$$x_1^{m_1} = x_2^{m_2} = 1, \quad x_1x_2d^2 = 1.$$

Lemma 3.14. *If $m_1 \neq m_2$, then $\mathrm{Out}(\Lambda)$ is isomorphic to the Klein four-group and is generated by classes of automorphisms α, β defined by*

$$\alpha\colon \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto (x_2d)x_2^{-1}(x_2d)^{-1} \\ d \mapsto x_2d \end{cases} \qquad \beta\colon \begin{cases} x_1 \mapsto (x_2d)^{-1}x_1^{-1}(x_2d) \\ x_2 \mapsto x_2 \\ d \mapsto (dx_2)^{-1} \end{cases}$$

Proof. Set $S = \mathcal{H}/\Lambda$ and note that S is a projective plane with 2 distinguished points $x_1, x_2 \in S$. Recall from Subsection 2.2 that $\mathrm{Out}(\Lambda)$ is isomorphic to the pure mapping class group $\mathrm{PMod}(S, \{x_1, x_2\})$. By [13, Corollary 4.6] the last group is generated by 2 boundary slides and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. By computing the automorphisms of $\pi_1(S \backslash \{x\})$ induced by the generators of $\mathrm{PMod}(S, \{x_1, x_2\})$ we deduce our lemma. □

The next lemma is a special case of [9, Lemma 5.8] for $r = 2$.

Lemma 3.15. *(σ, N) is admissible if and only if $N = \mathrm{lcm}(m_1, m_2)$ and $\frac{N}{m_i}$ are odd for $i = 1, 2$.*

Proposition 3.16. *Suppose that (σ, N) is admissible, N is odd, $m_1 \neq m_2$ and $k = (m_1, m_2)$. There are exactly $\left\lceil \frac{\varphi(k)}{2} \right\rceil$ equivalence classes of NSK-maps $\Lambda \to \mathbb{Z}_N$. In particular, (σ, N) is rigid if and only if $k \in \{1, 3\}$.*

Proof. Every NSK-map $\theta\colon \Lambda \to \mathbb{Z}_N$ is equivalent (by multiplication by an element of $\mathbb{Z}_N^*$) to θ_a such that $\theta_a(x_1) = \frac{N}{m_1}$ and $\theta_a(x_2) = a\frac{N}{m_2}$ for some $a \in \mathbb{Z}_{m_2}^*$. Since N is odd, $\theta_a(d)$ is determined by the relation $2\theta_a(d) = -(\theta_a(x_1) + \theta_a(x_2))$. Similarly as in the proof of Proposition 3.12 it can be shown that θ_a is equivalent to $\theta_{a'}$ if and only if $a = \pm a' \pmod k$ (the only difference is that now Λ admits an automorphism, e.g. α from Lemma 3.14, such that $\alpha(x_1)$ and $\alpha(x_2)$ are conjugate respectively to x_1 and x_2^{-1}). If $k > 1$, then it is impossible that $a = -a \pmod k$ for $a \in \mathbb{Z}_{m_2}^*$, because k is odd. Hence, there are $\frac{\varphi(k)}{2}$ equivalence classes of NSK-maps if $k > 1$, and one class if $k = 1$. □

Proposition 3.17. *Suppose that (σ, N) is admissible, N is even, $m_1 \neq m_2$ and $k = (m_1, m_2)$. There are exactly $\varphi(k)$ equivalence classes of NSK-maps $\Lambda \to \mathbb{Z}_N$.*

Proof. Let $\theta\colon \Lambda \to \mathbb{Z}_N$ be a NSK-map. After multiplication by an element of $\mathbb{Z}_N^*$ we may assume that $\theta(x_1) = \frac{N}{m_1}$ and $\theta(x_2) = a\frac{N}{m_2}$ for some $a \in \mathbb{Z}_{m_2}^*$. We have $2\theta(d)+\theta(x_1)+\theta(x_2) = 0$, and since N is even, $\theta(d)$ is determined by $\theta(x_1)$ and $\theta(x_2)$ only modulo $\frac{N}{2}$. Suppose that $\theta'\colon \Lambda \to \mathbb{Z}_N$ is another NSK-map, such that $\theta'(x_1) = \frac{N}{m_1}$ and $\theta'(x_2) = a'\frac{N}{m_2}$ for some $a' \in \mathbb{Z}_{m_2}^*$. We claim that θ and θ' are equivalent if and only if either

(1) $a = a' \pmod k$ and $\theta'(d) = b\theta(d)$, where b is the unique element of $\mathbb{Z}_N^*$ satisfying $b = 1 \pmod{m_1}$ and $ba = a' \pmod{m_2}$, or
(2) $a = -a' \pmod k$ and $\theta'(d) = b(\theta(d) + \theta(x_1))$, where b is the unique element of $\mathbb{Z}_N^*$ satisfying $b = -1 \pmod{m_1}$ and $ba = a' \pmod{m_2}$.

To prove the claim we note θ and θ' are equivalent if and only if $\theta' = b\theta\phi$ for some $b \in \mathbb{Z}_N^*$ and $\phi \in \mathrm{Aut}(\Lambda)$. By Lemma 3.14 we may suppose that $\phi \in \{1, \alpha, \beta, \alpha\beta\}$. If $\phi = 1$ or $\phi = \alpha\beta$, then after replacing b by $-b$ in the latter case, we have $\theta'(x_i) = b\theta(x_i)$ for $i = 1, 2$ and $\theta'(d) = b\theta(d)$. Thus b satisfies $b = 1 \pmod{m_1}$ and $ba = a' \pmod{m_2}$. By Chinese Remainder Theorem, such (unique) b exists if and only if $a = a' \pmod k$. Similarly, if $\phi = \beta$ or $\phi = \alpha$, then after replacing b by $-b$ in the latter case, we have $\theta'(x_1) = -b\theta(x_1)$, $\theta'(x_2) = b\theta(x_2)$ and $\theta'(d) = b(\theta(d) + \theta(x_1))$. Such (unique) b exists if and only if $a = -a' \pmod k$. This completes the proof of the claim.

Suppose $k > 2$. It follows from the previous paragraph that there is a surjection ρ from the set of equivalence classes of NSK-maps onto $\mathbb{Z}_k^*/\{-1, 1\}$, defined by $\rho[\theta] = [a \bmod k]$, where θ is as above, $[\theta]$ is its equivalence class, and $[a \bmod k]$ is the element of $\mathbb{Z}_k^*/\{-1, 1\}$ represented by $(a \bmod k)$. We note that ρ is two-to-one. Indeed, take θ and θ' as above and suppose that they are not equivalent, but $\rho[\theta] = \rho[\theta']$. Then either aaa

(1′) $a = a' \pmod k$ and $\theta'(d) = b\theta(d) + \frac{N}{2}$, where b is as in (1) above, or
(2′) $a = -a' \pmod k$ and $\theta'(d) = b(\theta(d) + \theta(x_1)) + \frac{N}{2}$, where b is as in (2) above.

In case (1') θ' is equivalent (by multiplication by b^{-1}) to θ_1 defined by $\theta_1(x_1) = \frac{N}{m_1}$, $\theta_1(x_2) = a\frac{N}{m_2}$, $\theta_1(d) = \theta(d) + \frac{N}{2}$. In case (2') θ' is equivalent (by multiplication by $-b^{-1}$) to θ_2 defined by $\theta_2(x_1) = \frac{N}{m_1}$, $\theta_2(x_2) = -a\frac{N}{m_2}$, $\theta_2(d) = -\theta(d) - \frac{N}{m_1} + \frac{N}{2}$. Observe that θ_1 and θ_2 are equivalent (by (2) with $b = -1$), hence they represent the unique class $[\theta_1]$, such that $[\theta_1] \neq [\theta]$ and $\rho[\theta_1] = \rho[\theta]$. It follows that the number of equivalence classes of NSK-maps $\Lambda \to \mathbb{Z}_N$ is $2\,|\mathbb{Z}_k^*/\{-1,1\}| = \varphi(k)$.

Suppose $k = 2$. By (1) every NSK map is equivalent to $\theta\colon \Lambda \to \mathbb{Z}_N$ such that $\theta(x_1) = \frac{N}{m_1}$ and $\theta(x_2) = \frac{N}{m_2}$. Fix such θ and define θ' by $\theta'(x_i) = \theta(x_i)$ for $i = 1, 2$ and $\theta'(d) = \theta(d) + \frac{N}{2}$. We have to show that θ and θ' are equivalent. Let b be the unique element of $\mathbb{Z}_N^*$ such that $b = -1 \pmod{m_1}$ and $b = 1 \pmod{m_2}$. By (2) it suffices to show that $\theta'(d) = b(\theta(d) + \theta(x_1))$. We have

$$2b\theta(d) = -b(\theta(x_1) + \theta(x_2)) = \theta(x_1) - \theta(x_2) = 2(\theta(x_1) + \theta(d))$$

Either $b\theta(d) = \theta(d) + \theta(x_1)$ or $b\theta(d) = \theta(d) + \theta(x_1) + \frac{N}{2}$. The former equality is not possible, because $\theta(x_1) = \frac{N}{m_1}$ is odd (by Lemma 3.15) and $\theta(d)(b-1)$ is even. Hence

$$b(\theta(d) + \theta(x_1)) = b\theta(d) - \theta(x_1) = \theta(d) + \frac{N}{2} = \theta'(d)$$

It follows that all NSK-maps $\Lambda \to \mathbb{Z}_N$ are equivalent. $\square$

4. Automorphisms with large periods

Let $C(g, N)$ denote the number of topological types of actions of $\mathbb{Z}_N$ on a closed non-orientable surface of genus $g \geq 3$. It is proved in [2] that $C(g, N) = 0$ for $N > 2g$, and if g is even then $C(g, N) = 0$ for $N > 2(g-1)$. In this section we compute $C(g, N)$, for all $g \geq 3$ and $N > \max\{g, \frac{3}{2}(g-2)\}$.

For an admissible pair (σ, N), let $c(\sigma, N)$ denote the number of equivalence classes of NSK-maps $\theta\colon \Lambda \to \mathbb{Z}_N$, where Λ is an NEC-group with $\sigma(\Lambda) = \sigma$ (see Section 2 for definitions). Then $C(g, N)$ is the sum of all $c(\sigma, N)$ such that (σ, N) is an admissible pair satisfying $N\mu(\sigma) = g - 2$.

We begin by determining the possible signatures σ of the form (2.2) satisfying $0 < \mu(\sigma) < \frac{2}{3}$. By a straightforward calculation we obtain the following list.

$\sigma_0 = (0; +; [2, 2, 2], \{()\})$
$\sigma_1 = (1; -; [2, 2, 2], \{-\})$
$\sigma_2 = (0; +; [2, m], \{()\})$ for $m > 2$
$\sigma_3 = (1; -; [2, m], \{-\})$ for $m > 2$
$\sigma_4 = (0; +; [3, m], \{()\})$ for $m > 2$
$\sigma_5 = (1; -; [3, m], \{-\})$ for $m > 2$
$\sigma_6 = (0; +; [4, m], \{()\})$ for $11 \geq m \geq 4$
$\sigma_7 = (1; -; [4, m], \{-\})$ for $11 \geq m \geq 4$
$\sigma_8 = (0; +; [5, m], \{()\})$ for $7 \geq m \geq 5$
$\sigma_9 = (1; -; [5, m], \{-\})$ for $7 \geq m \geq 5$

Signature σ	Parameter m	Order N	Genus g
σ_2	g	$2g$	odd
σ_2	$2(g-1)$	$2(g-1)$	arbitrary
σ_3	$2(g-1)$	$2(g-1)$	even
σ_4	$3(g-1)/2$	$3(g-1)/2$	$g \equiv 1 \pmod 4$
σ_4	$(g+1)/2$	$3(g+1)/2$	$g \equiv 3$ or $g \equiv 7 \pmod{12}$
σ_5	$3(g-1)/2$	$3(g-1)/2$	$g \equiv 3 \pmod 4$
σ_5	$(g+1)/2$	$3(g+1)/2$	$g \equiv 1$ or $g \equiv 9 \pmod{12}$
σ_6	6	12	9
σ_6	7	28	19
σ_6	8	8	7
σ_6	9	36	25
σ_6	10	20	15
σ_6	11	44	31
σ_8	6	30	21
σ_9	7	35	25
σ_{10}		$2(g-2)$	$g > 4$
σ_{11}		$2(g-2)$	$g > 4$
σ_{12}		$2(g-2)$	$g > 4$ and $4 \mid g$

TABLE 1. All admissible pairs (σ, N) such that $N > \max\{g, 3(g-2)/2\}$

$$\sigma_{10} = (0; +; [2], \{(), ()\})$$
$$\sigma_{11} = (1; -; [2], \{()\})$$
$$\sigma_{12} = (2; -; [2], \{-\})$$

In the cases where a signature σ_i depends on the parameter m, we will also denote it as $\sigma_i(m)$. Now, for each of the above signatures we determine all admissible pairs (σ_i, N), such that $N > g$, where $g = N\mu(\sigma_i) + 2$.

Theorem 4.1. *Let $g \geq 4$ and suppose that (σ, N) is an admissible pair, such that $N > \max\{g, \frac{3}{2}(g-2)\}$, where $g = N\mu(\sigma_i) + 2$. Then σ, N and g are as in Table 1. Furthermore, (σ, N) is rigid except for the following two cases*

$$c\big(\sigma_4(3(g-1)/2), 3(g-1)/2\big) = 2 \quad \textit{for } g \equiv 1 \pmod 4,$$
$$c(\sigma_6(8), 8) = 2 \quad for\ g = 7.$$

Proof. For $i \in \{0, 1\}$ the only admissible pair is $(\sigma_i, 2)$ by [2, Theorem 3.5 and Theorem 3.6], which does not satisfy $N > g$ as $N = 2$ and $g = 3$.

By Lemma 3.11 a pair (σ_2, N) is admissible if and only if $N = \mathrm{lcm}(2, m)$. Such pair is rigid for every m by Proposition 3.12. We have $\mu(\sigma_2) = \frac{1}{2} - \frac{1}{m}$. If m is odd then $N = 2m$ and $g = N\mu(\sigma_2) + 2 = m$. If m is even then $N = m = 2(g-1)$.

By Lemma 3.15 a pair (σ_3, N) is admissible if and only if $N = m$ is even and $m/2$ is odd. Every such pair is rigid by Proposition 3.17. We have $N = 2(g-1)$ and g is even.

By Lemma 3.11 a pair (σ_4, N) is admissible if and only if $N = \mathrm{lcm}(3, m)$ and m is even. We have $\mu(\sigma_2) = \frac{2}{3} - \frac{1}{m}$. If $3 \mid m$ then $N = m = \frac{3}{2}(g-1)$ and $g \equiv 1 \pmod 4$. In this case we have $c(\sigma_4, N) = \varphi(3) = 2$ by Proposition 3.12. If $3 \nmid m$ then $N = 3m$ and $g = 2m - 1$. We have $g \equiv 3$ or $g \equiv 7 \pmod{12}$. In this case (σ_4, N) is rigid by Proposition 3.12.

By Lemma 3.15 a pair (σ_5, N) is admissible if and only if $N = \mathrm{lcm}(3, m)$ and m is odd. For $m = 3$ we have $N = g = 3$, hence we assume $m > 3$. Every such pair is rigid by Proposition 3.16. If $3 \mid m$ then $N = m = \frac{3}{2}(g-1)$ and $g \equiv 3 \pmod 4$. If $3 \nmid m$ then $N = 3m$ and $g = 2m - 1$. We have $g \equiv 1$ or $g \equiv 9 \pmod{12}$.

By Lemma 3.11 a pair (σ_6, N) is admissible if and only if $N = \mathrm{lcm}(4, m)$. We have $\mu(\sigma_2) = \frac{3}{4} - \frac{1}{m}$. For $m = 4$ we have $N = g = 4$ which contradicts $N > g$. For $m \in \{7, 9, 11\}$ we have $N = 4m$ and $g = 3m - 2$. For $m \in \{6, 10\}$ we have $N = 2m$ and $g = \frac{3}{2}m$. For $m = 8$ we have $N = 8$ and $g = 7$. By Proposition 3.12 we have $c(\sigma_6, 8) = \varphi(4) = 2$ and $c(\sigma_6, m) = 1$ for $m \neq 8$.

It follows from Lemma 3.15 that (σ_7, N) is admissible if and only if $N = m$, $4 \mid m$ and $\frac{m}{4}$ is odd. This holds only for $m = 4$, but then $N = g = 4$ which contradicts $N > g$.

By Lemma 3.11 a pair (σ_8, N) is admissible only for $m = 6$, $N = 30$ and $g = 21$. By Proposition 3.12 this pair is rigid.

By Lemma 3.15 a pair (σ_9, N) is admissible only for $m = N = 5$ and $(m, N) = (7, 35)$. In the former case we have $g = 5$ which contradicts the assumption $N > g$. In the letter case we have $g = 25$ and the pair is rigid by Proposition 3.17.

By Lemma 3.7 for $i \in \{10, 11\}$ a pair (σ_i, N) is admissible if and only if N is even, and such pair is rigid by Proposition 3.9. We have $N = 2(g-2)$. Note that $N > g$ only for $g > 4$.

By Lemma 3.2 a pair (σ_{12}, N) is admissible if and only if $4 \mid N$ and $8 \nmid N$. Such pair is rigid by Proposition 3.4. We have $N = 2(g-2)$ and $4 \mid g$. □

Theorem 4.2. *Suppose that $g \geq 11$ is odd and $g \notin \{15, 19, 21, 25, 31\}$. If $N > \frac{3}{2}(g-2)$ and $C(g, N) > 0$ then*

$$N \in \left\{2g, 2(g-1), 2(g-2), \frac{3}{2}(g+1), \frac{3}{2}(g-1)\right\}.$$

Furthermore, $C(g,2g) = C\big(g,2(g-1)\big) = 1$*,* $C\big(g,2(g-2)\big) = 2$*, and*

$$C\big(g,\frac{3}{2}(g+1)\big) = \begin{cases} 1 & \text{for } (g \bmod 12) \in \{1,3,7,9\} \\ 0 & \text{for } (g \bmod 12) \in \{5,11\},\ g \neq 11 \end{cases}$$

$$C\big(g,\frac{3}{2}(g-1)\big) = \begin{cases} 2 & \text{for } g \equiv 1 \pmod 4 \\ 1 & \text{for } g \equiv 3 \pmod 4. \end{cases}$$

Proof. The assumptions about g guarantee that the numbers $2g, 2(g-1), 2(g-2), \frac{3}{2}(g+1)$ and $\frac{3}{2}(g-1)$ are all different (the only exception is $g = 11$, for which $2(g-2) = \frac{3}{2}(g+1)$), and also that there are no admissible pairs (σ_i, N) for $i \in \{6,8,9\}$ with such g in Table 1. Thus, the only possible values for N are those given in the theorem. For each of these values we calculate $C(g,N)$ by adding up $c(\sigma,N)$ for all admissible pairs (σ,N) from Table 1.

$$\begin{aligned} C(g,2g) &= c(\sigma_2(g),2g) = 1 \\ C(g,2(g-1)) &= c(\sigma_2(2(g-1)),2(g-1)) = 1 \\ C(g,2(g-2)) &= c(\sigma_{10},2(g-2)) + c(\sigma_{11},2(g-2)) = 2 \end{aligned}$$

For $N = 3(g+1)/2$ we have

$$\begin{aligned} C(g,N) &= c(\sigma_4(N/3),N) = 1 \quad \text{for } (g \bmod 12) \in \{3,7\} \\ C(g,N) &= c(\sigma_5(N/3),N) = 1 \quad \text{for } (g \bmod 12) \in \{1,9\} \end{aligned}$$

For $N = 3(g-1)/2$ we have

$$\begin{aligned} C(g,N) &= c(\sigma_4(N),N) = 2 \quad \text{for } g \equiv 1 \pmod 4 \\ C(g,N) &= c(\sigma_5(N),N) = 1 \quad \text{for } g \equiv 3 \pmod 4 \end{aligned}$$ □

Theorem 4.3. *Suppose that* $g \geq 4$ *is even. If* $N > \frac{3}{2}(g-2)$*,* $N > g$ *and* $C(g,N) > 0$ *then either* $N = 2(g-1)$ *or* $N = 2(g-2)$*. Furthermore,* $C\big(g,2(g-1)\big) = 2$ *and for* $g > 4$

$$C\big(g,2(g-2)\big) = \begin{cases} 3 & \text{if } 4 \mid g \\ 2 & \text{if } 4 \nmid g \end{cases}$$

Proof. As in the proof of Theorem 4.2, we use Table 1 to compute $C(g,N)$. For $N = 2(g-1)$ we heve

$$C\big(g,N\big) = c\big(\sigma_2(N),N\big) + c\big(\sigma_3(N),N\big) = 2$$

and for $N = 2(g-2)$ and $g > 4$

$$\begin{aligned} C\big(g,N\big) &= c\big(\sigma_{10},N\big) + c\big(\sigma_{11},N\big) = 2 \quad \text{if } 4 \nmid g \\ C\big(g,N\big) &= c\big(\sigma_{10},N\big) + c\big(\sigma_{11},N\big) + c\big(\sigma_{12},N\big) = 3 \quad \text{if } 4 \mid g \end{aligned}$$ □

Theorem 4.4. *The following is the complete list of values of* $C(g,N)$ *such that* $N > g \geq 3$*,* $N > \frac{3}{2}(g-2)$ *and* $C(g,N)$ *is not as in Theorem 4.2.*

$$\begin{array}{ccccc} C(5,6)=4 & C(7,12)=2 & C(7,8)=2 & C(9,12)=3 & C(15,20)=1 \\ C(19,28)=1 & C(21,30)=3 & C(25,35)=1 & C(25,36)=3 & C(31,44)=1 \end{array}$$

Proof. We consider odd genera excluded by the assumption of Theorem 4.2. For $g = 3$ we have only two admissible pairs with $N > g$, namely $(\sigma_2(3), 6)$ and $(\sigma_2(4), 4)$. Since each of them is rigid by Proposition 3.12, we have $C(3, 4) = C(3, 6) = 1$, which agrees with Theorem 4.2. For $g = 5$ we have $2(g-2) = \frac{3}{2}(g-1) = 6$ and $C(5, 6) = c(\sigma_{10}, 6)+c(\sigma_{11}, 6)+c(\sigma_4(6), 6) = 1 + 1 + 2 = 4$. For $g = 7$ we have $2(g - 1) = \frac{3}{2}(g + 1) = 12$ and $8 < \frac{3}{2}(g - 1) < 2(g - 2)$; $C(7, 12) = c(\sigma_2(12), 12) + c(\sigma_4(4), 12) = 2$ and $C(7, 8) = c(\sigma_6(8), 8) = 2$. For $g = 9$ we have $12 = \frac{3}{2}(g - 1)$ and $C(9, 12) = c(\sigma_4(12), 12) + c(\sigma_6(6), 12) = 2 + 1 = 3$ For $g = 15$ we have $20 < \frac{3}{2}(g - 1)$ and $C(15, 20) = c(\sigma_6(10), 20) = 1$. For $g = 19$ we have $\frac{3}{2}(g - 1) < 28 < \frac{3}{2}(g + 1)$ and $C(19, 28) = c(\sigma_6(7), 28) = 1$. For $g = 21$ we have $30 = \frac{3}{2}(g - 1)$ and $C(21, 30) = c(\sigma_4(30), 30) + c(\sigma_8(6), 30) = 2 + 1 = 3$. For $g = 25$ we have $36 = \frac{3}{2}(g - 1)$ and $C(25, 36) = c(\sigma_4(36), 36) + c(\sigma_6(9), 36) = 3$ and $C(25, 35) = c(\sigma_9(7), 35) = 1$. For $g = 31$ we have $44 < \frac{3}{2}(g - 1)$ and $C(31, 44) = c(\sigma_6(11), 44) = 1$, □

Corollary 4.5. *Suppose that* $g \geq 5$ *is odd and* $g \neq 5 \pmod{12}$*. Let* N *be the maximum odd integer such that* $C(g, N) > 0$*. Then*

$$N = \begin{cases} \frac{3}{2}(g+1) & \text{for } g \equiv \quad \pmod 4, \\ \frac{3}{2}(g-1) & \text{for } g \equiv 3 \pmod 4. \end{cases}$$

Furthermore, $C(g, N) = 1$.

References

[1] C. Bagiński, M. Carvacho, G. Gromadzki, R. Hidalgo. On periodic self-homeomorphisms of closed orientable surfaces determined by their orders, Preprint 2014, submited.

[2] E. Bujalance. Cyclic groups of automorphisms of compact nonorientable Klein surfaces without boundary. *Pacific J. Math.* **109** (1983), 279–289.

[3] E. Bujalance, F.J. Cirre, M.D.E. Conder. Extensions of finite cyclic group actions on non-orientable surfaces. Trans. Amer. Math. Soc. **365** (2013), no. 8, 4209–4227.

[4] E. Bujalance, F. J. Cirre, M. D. E. Conder, B. Szepietowski. Finite group actions on bordered surfaces of small genus. *J. Pure Appl. Algebra* **214** (2010), 2165–2185.

[5] E. Bujalance, J.J. Etayo, J. M. Gamboa, G. Gromadzki. *Automorphisms Groups of Compact Bordered Klein Surfaces. A combinatorial approach* Lecture Notes in Mathematics **1439** (Springer-Verlag 1990).

[6] K. Fujiwara. On the outer automorphism group of a hyperbolic group. *Israel J. Math.* **131** (2002), 277–84.

[7] F. J. González-Acuña and J. M. Márquez-Bobadilla. On the homeotopy group of the non orientable surface of genus three. *Rev. Colombiana Mat.* **40** (2006), 75–79.

[8] L. Greenberg. Maximal Fuchsian Groups. *Bull. Amer. Math. Soc.* **69** (1963), 569–573.

[9] G. Gromadzki, W. Marzantowicz. On conformal dynamics on orientable and nonorientable Riemann surfaces. *Fundamenta Mathematicae* **213** (2011), 169–190.

[10] W. J. Harvey. On branch loci in Teichmüller space. *Trans. Amer. Math. Soc.* **153** (1971), 387–399.

[11] W. J. Harvey, C. Maclachlan. On mapping-class groups and Teichmüller spaces, *Proc. Lond. Math. Soc.* **30** (1975), 496–512.

[12] S. Hirose. On periodic maps over surfaces with large periods. *Tohoku Math. J.* **62** (1) (2010), 45–53.

[13] M. Korkmaz. Mapping class groups of nonorientable surfaces. *Geom. Dedicata* **89** (2002), 109–103.

[14] J. Nielsen. Die Struktur periodischer Transformationen von Flächen, *Math. -fys. Medd. Danske Vid. Selsk* **15** (1) (1937), English transl. in Jakob Nielsen collected works, Vol. **2**, 65–102.

[15] D. Singerman. Symmetries of Riemann surfaces with large automorphism group. *Math. Ann.* **210** (1974), 17–32.

[16] D. Singerman. Finitely maximal Fuchsian groups, *J. London Math. Soc.* (2) 6 (1972), 29–38.
[17] M. Stukow. Dehn twists on nonorientable surfaces. *Fund. Math.* **189** (2006), 117–147.
[18] S. Wang. Maximum orders of periodic maps on closed surfaces, *Topol. Appl.* **41** (1991) 255–262.

Both Authors: Institute of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland

E-mail address: grom@mat.ug.edu.pl, blaszep@mat.ug.edu.pl