REAL LINEAR CHARACTERS OF THETA GROUPS

AND MODULAR IMBEDDINGS FOR REAL QUADRATIC FIELDS

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1. Introduction

Let n be a positive integer and $n \geq 2$. In this paper we consider the theta subgroup Θ_n of Siegel modular group Γ_n of degree n , which acts on the generalized Siegel upper half space H_n by the usual way. In [3], Endres investigated the multiplier systems of Θ_n nnd proved that

$$\Theta_n / \Theta_n' \cong \begin{cases} Z/4Z & \text{if } n = 2 , \\ Z/4Z \times Z/2Z & \text{if } n \geq 3 , \end{cases}$$

where G' means the commutator subgroup of group G . From this, we see that Θ_2 has three subgroups of index 2 and Θ_n has only one subgroup of index 2 if $n \geq 3$. The main purpose of this paper is to determine all subgroups of Θ_n of index 2, which is equivalent to describe all linear characters of Θ_n . Now we define the standard theta series θ_n of degree n by

$$\theta_n(Z) = \Sigma_{x \in Z^n} \exp(2\pi i\, {}^t xZx) \qquad (Z \in H_n) .$$

Then we have the character λ_n of Θ_n of order 2 given by

$$\lambda_n(\sigma) = \left(\frac{\theta_n(\sigma(Z))}{\theta_n(Z)}\right)^4 j(\sigma,Z)^2 \qquad (\sigma \in \Theta_n),$$

where $j(\sigma,Z)$ is the standard automorphic factor of Γ_n on H_n . Therefore $\ker(\lambda_n)$ is the unique subgroup of Θ_n of index 2 if $n \geq 3$. In the case $n=2$, we remember the fact that Γ_2 / Γ_2' $\cong Z/2Z$ (see Reiner [8]). Hence there exists uniquely the character of Γ_2 of order 2. We denote it by $\tilde{\Lambda}$. Since $\Theta_2 \not\subset \Gamma_2'$, the restriction Λ of $\tilde{\Lambda}$ on Θ_2 is different from λ_2 . This implies that $\{ H \mid H$ is a subgroup of Θ_2 and $(\Theta_2:H) = 2 \} =$ $\{ \ker(\lambda_n), \ker(\Lambda), \ker(\lambda_2\Lambda) \}$. Thus, our problem is stated as follows. We should give explicit formulas of the values of λ_n , Λ and $\lambda_2\Lambda$ for each element of Θ_n , a priori, which have already given for each generator of Θ_n (see Endres [3] and Kirchheimer [5]. We shall anser this problem in Section 2 for λ_n and in

Section 4 for $\lambda_2\Lambda$. As its application, we can see the connection with modular imbeddings over real quadratic fields, which is introduced by Hammond [4]. We shall show in Section 5 that Hilbert modular groups over certain real quadratic fields can be imbeded into $\ker(\lambda_2\Lambda)$, which is more sharp than the result of [6], where we did into a conjugate group of Θ_2 .

Notations. For a commutative ring R with identity element, we denote by $M(n,R)$ the ring of all $n \times n$-matrices with entries in R and by $GL(n,R)$ the group of all invertible elements. The identity element and the zero element of $M(n,R)$ are denoted by 1_n and 0_n , respectively. For each element σ of $M(2n,R)$, we write $A_\sigma = A$, $B_\sigma = B$, $C_\sigma = C$ and $D_\sigma = D$ if $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $A,B,C,D \in M(n,R)$. We denote by ${}^tA$ the transposed matrix of A for $A \in M(n,R)$, and put $A^* = {}^tA^{-1}$ for $A \in GL(n,R)$. We also put $J^{(n)} = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$, $S_p(2n,R) = \{A \in M(2n,R) \mid {}^tAJ^{(n)}A = J^{(n)}\}$, and $SM^0(n,R) = \{ A \in M(n,R) \mid {}^tA = A$ and all diagonal elements of A are even $\}$. $\mathrm{diag}(a_1,a_2,\cdots,a_n)$ denotes the diagonal matrix whose (i,i)-component is equal to a_i for each i . For a matrix

A , we denote by tr(A) and det(A) the trace and determinant, respectively, of A . Finally, we use the notations Z , Q and R for the ring of rational integers, the field of rational numbers and the field of real numbers, respectively.

2. Linear characters given by θ_n

Let n be a positive integer and $n \geq 2$. As usual, we define Γ_n, Θ_n, H_n, and θ_n by

$$\Gamma_n = S_p(2n,Z),$$

$$\Theta_n = \{ \sigma \in \Gamma_n \mid {}^tA_\sigma C_\sigma,\ {}^tB_\sigma D_\sigma \in SM^0(n,Z) \},$$

$$H_n = \{ X+iY \mid X,Y \in M(n,R) \text{ and } Y \text{ is positive definite} \},$$

$$\theta_n = \Sigma_{x\in Z^n} \exp(2\pi i\, {}^t xZx) \qquad (Z \in H_n).$$

We let every element σ of Γ_n, hence Θ_n, act on H_n by

$$\sigma(Z) = (A_\sigma Z + B_\sigma)(C_\sigma Z + D_\sigma)^{-1} \qquad (Z \in H_n).$$

Then we have the theta multiplier system ν_n of degree n given by

$$\nu_n(\sigma) = \frac{\theta_n(\sigma(Z))}{\theta_n(Z)} \sqrt{\det(C_\sigma Z + D_\sigma)} \qquad (\sigma \in \Theta_n).$$

It is well known that $\nu_n(\sigma)^8 = 1$ for all elements $\sigma \in \Theta_n$. Thus, we get two characters λ_n and κ_n of Θ_n of order 2 and 4, respectively, if we put for each element σ of Θ_n

$$\lambda_n(\sigma) = \nu_n(\sigma)^4, \qquad \kappa_n(\sigma) = \nu_n(\sigma)^2.$$

Theorem 1 $\lambda_n(\sigma) = (-1)^{tr({}^tC_\sigma B_\sigma)}$ for all elements $\sigma \in \Theta_n$.

Proof. Put

$$\chi(\sigma) = (-1)^{tr({}^tC_\sigma B_\sigma)}$$

for each element σ of Θ_n . Then we see that χ is a character of Θ_n . In fact, since for $\sigma, \tau \in \Theta_n$

$$B_{\sigma\tau} = A_\sigma B_\tau + B_\sigma D_\tau \ , \quad C_{\sigma\tau} = C_\sigma A_\tau + D_\sigma C_\tau \ ,$$

we have

$$\begin{aligned} tr({}^tC_{\sigma\tau}B_{\sigma\tau}) &= tr({}^tA_\tau {}^tC_\sigma A_\sigma B_\tau) + tr({}^tA_\tau {}^tC_\sigma B_\sigma D_\tau) \\ &\quad + tr({}^tA_\tau {}^tC_\sigma A_\sigma B_\tau) + tr({}^tC_\tau {}^tD_\sigma B_\sigma D_\tau) \\ &= tr(B_\tau {}^tA_\tau {}^tC_\sigma A_\sigma) + tr(D_\tau {}^tA_\tau {}^tC_\sigma B_\sigma) \\ &\quad + tr(B_\tau {}^tC_\tau {}^tD_\sigma A_\sigma) + tr(D_\tau {}^tC_\tau {}^tD_\sigma B_\sigma) \end{aligned}$$

Since $B_\tau {}^tA_\tau$, ${}^tC_\sigma A_\sigma$, $D_\tau {}^tC_\tau$, ${}^tD_\sigma B_\sigma \in SM^0(n,Z)$, we see that

$$tr({}^tC_{\sigma\tau}B_{\sigma\tau}) \equiv tr(D_\tau {}^tA_\tau {}^tC_\sigma B_\sigma) + tr(B_\tau {}^tC_\tau {}^tD_\sigma A_\sigma) \pmod 2,$$

because $tr(XY) \equiv 0 \pmod 2$ for $X, Y \in SM^0(n,Z)$. Thus, we obtain

$$tr({}^tC_{\sigma\tau}B_{\sigma\tau}) \equiv tr({}^tC_\sigma B_\sigma) + tr({}^tC_\tau B_\tau) \ .$$

Hence, χ is a character of Θ_n .

Now we know that Θ_n is generated by the following elements (see Eichler [2]) :

$$U_V = \begin{pmatrix} V & 0_n \\ 0_n & V^* \end{pmatrix} \qquad (V \in GL(n,Z)),$$

$$T_S = \begin{pmatrix} 1_n & S \\ 0_n & 1_n \end{pmatrix} \qquad (S \in SM^0(n,Z)),$$

$$J_k^{(n)} = \begin{pmatrix} E_k^{(n)} & 1_n - E_k^{(n)} \\ E_k^{(n)} - 1_n & E_k^{(n)} \end{pmatrix} \qquad (k=0,1,2,\cdots,n-1),$$

where $E_k^{(n)} = \mathrm{diag}(1,\cdots,1,0,\cdots,0)$ with k-times 1 . It is easily shown by the definition of χ that

$$\chi(U_V) = \chi(T_S) = 1 \quad (V \in GL(n,Z),\ S \in SM^0(n,Z)),$$

$$\chi(J_k^{(n)}) = (-1)^{n-k} \quad (k = 0,1,2,\cdots,n-1).$$

On the other hand, from Endres [3]

$$\lambda(U_V) = \lambda(T_S) = 1 \quad (V \in GL(n,Z),\ S \in SM^0(n,Z)),$$

$$\lambda(J_k^{(n)}) = (-1)^{n-k} \quad (k = 0,1,2,\cdots,n-1).$$

Therefore $\chi = \lambda$.

We denote by Θ_n^+ , Θ_n' and $\Gamma_n(2)$ the kernel of λ_n , the commutator subgroup of Θ_n and the principal congruence subgroup of level 2 which is defined by

$$\Gamma_n(2) = \{ \sigma \in \Gamma_n \mid A_\sigma - 1_n \equiv B_\sigma \equiv C_\sigma \equiv D_\sigma - 1_n \equiv 0_n \pmod 2 \}.$$

It is easily seen that these three groups are normal subgroups of Θ_n . We have the following diagram :

$$\Theta_n \supsetneqq \Theta_n^+ \supset \Gamma_n(2)\Theta_n' \supsetneqq \Theta_n' ,$$

because $\Theta_n^+ \supset \Gamma_n(2)$ by Theorem 1 , $\Theta_n^+ \supset \Theta_n'$ in general and Θ_n' is a congruence subgroup of level 4 (see Endres [3, Lemma 2.2]).

Corollary If $n \geq 3$, then $\Theta_n^+ = \Gamma_n(2)\Theta_n'$.

Proof. Since $(\Theta_n : \Theta_n') = 4$ as stated in Section 1 , we see that $(\Theta_n^+ : \Theta_n') = 2$. Hence $(\Theta_n^+ : \Gamma_n(2)\Theta_n') = 1$ by the reason why $(\Gamma_n(2)\Theta_n' : \Theta_n') > 1$.

Remark. If $n = 2$, then we see that $(\Theta_2^+ : \Gamma_2(2)\Theta_2') = 2$. We shall discuss the character of Θ_2^+ of order 2 whose kernel coincides with $\Gamma_2(2)\Theta_2'$ in Section 3.

3. Restriction of $\tilde{\Lambda}$ on Θ_2^+

In this section we shall construct a real character of Θ_2^+, which coincides with the restriction of $\tilde{\Lambda}$ on Θ_2^+. For this purpose, we introduce several definitions and notations as follows. For each matrix M of $M(2,Z)$, we define $\overline{M}$ and $\psi(M)$ by

$$\overline{M} = M \bmod 2 ,$$

$$\psi(\overline{M}) = \begin{cases} 1 & \text{if } \overline{M} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Put

$$\Psi(\sigma) = \Psi(A_\sigma) + \Psi(B_\sigma) + \Psi(C_\sigma) + \Psi(D_\sigma)$$

for $\sigma \in M(4,Z)$. It is obvious that $0 \leq \Psi(\sigma) \leq 2$ for $\sigma \in GL(4,Z)$. So, we define $\Theta(\ell)$ by

$$\Theta(\ell) = \{ \sigma \in \Theta_2 \mid \Psi(\sigma) = \ell \} \qquad (\ell=0,1,2).$$

We write

$$\sigma_{11} = A_\sigma , \ \sigma_{12} = B_\sigma , \ \sigma_{21} = C_\sigma , \ \sigma_{22} = D_\sigma$$

and put

$$\Theta^{i,j} = \{ \sigma \in \Theta(1) \mid \overline{\sigma}_{ij} = 0 \}.$$

Then it is clear that

$$\Theta_2 = \cup_{\ell=0}^{2} \Theta(\ell) , \qquad \Theta(1) = \cup_{i=1}^{2} \cup_{j=1}^{2} \Theta^{i,j}$$

and these sums are disjoint. For a subset X of Θ_2, put

$$\overline{X} = X\Gamma_2(2)/\Gamma_2(2) \ .$$

We define U by

$$U = \{ U_V \mid V \in GL(2,Z) \}$$

and put

$$J = J^{(2)}, \quad W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_1 = J_1^{(2)} \ .$$

Lemma 1 (1) $\Theta(2) = U\Gamma_2(2) \cup JU\Gamma_2(2)$,

(2) $\Theta(1) = J\Theta(2) \cup JT_W\Theta(2) \cup T_WJ\Theta(2) \cup JT_WJ\Theta(2)$,

$\Theta(0) = J_1\Theta(1) \cup J_1\Theta(2)$

and these sums are disjoint.

(2) $|\overline{\Theta(0)}| = 36$, $|\overline{\Theta(1)}| = 24$, $|\overline{\Theta(2)}| = 12$.

Proof. (1) If $\sigma \in \Theta(2)$, then it must be that

$$\overline{B}_\sigma = \overline{C}_\sigma = 0 \quad \text{or} \quad \overline{A}_\sigma = \overline{D}_\sigma = 0 \ .$$

Hence we have $\Theta(2) = U\Gamma_2(2) \cup JU\Gamma_2(2)$. It is easily seen that

$$\Theta^{1,1} = JT_W\Theta(2) \ , \qquad \Theta^{1,2} = JT_WJ\Theta(2) \ ,$$

$$\Theta^{2,1} = T_W\Theta(2) \ , \qquad \Theta^{2,2} = T_WJ\Theta(2) \ .$$

This implies second equality. Since $J_1 \in \Theta(0)$ and $J_1\sigma \in \Theta(0)$ for all $\sigma \in \Theta(1) \cup \Theta(2)$, we get $J_1\Theta(1) \cup J_1\Theta(2) \subset \Theta(0)$. On the other hand, we see that $|\overline{\Theta(2)}| = 12$ since $|\overline{U}| = 6$. Hence $|\overline{\Theta(1)}| = 24$ and $|\overline{J_1\Theta(1) \cup J_1\Theta(2)}| = 36$. It is well known that $(\Gamma_2 : \Theta_2) = 10$ and $(\Gamma_2 : \Gamma_2(2)) = 720$. Therefore $|\overline{\Theta}_2| = 72$. Thus we have $|\overline{\Theta(0)}| = 36$. Hence $\overline{\Theta(0)} = \overline{J_1\Theta(1) \cup J_1\Theta(2)}$. This implies third equality. (2) We have shown in the proof of (1).

Proposition 1 (1) $\Theta_2^+ = \Theta(1) \cup \Theta(2)$.

(2) Θ_2^+ is generated by $\Gamma_2(2) \cup \{ J, T_W \} \cup U$.

Proof. (1) As stated in the proof of Lemma 1, $|\overline{\Theta}_2| = 72$. Hence $|\overline{\Theta_2^+}| = 36$. On the other hand, from Theorem 1, we see that $\Theta(1) \cup \Theta(2) \subset \Theta_2^+$. Therefore, by Lemma 1-(2), we get our assertion. (2) It is obtained by the fact that $\{ T_S | S \in SM^0(2,Z)\} \subset \{W\} \cup \Gamma_2(2)$.

For each matrix M of $M(2,Z)$, put

$$\eta(M) = (-1)^{A_M B_M + B_M C_M + C_M D_M} .$$

We note that η gives a character on $GL(2,Z)$ and $\eta(M) = \eta(M^*)$ for $M \in GL(2,Z)$. Moreover η gives a homomorphism as monoid of $\{ 0_2 \} \cup \{ M \in M(2,Z) | \det(M) \equiv 1 \pmod 2 \}$ to $\{1,-1\}$. Now we define η_1, η_2 and μ^+ by

$$\eta_1(\sigma) = \eta(A_\sigma)\eta(B_\sigma)\eta(C_\sigma)\eta(D_\sigma) \qquad (\ \sigma \in \Theta(1)\ ),$$

$$\eta_2(\sigma) = \eta(A_\sigma)\eta(B_\sigma) \qquad (\ \sigma \in \Theta(2)\ ),$$

and put

$$\Theta^\varepsilon(\ell) = \{\ \sigma \in \Theta(\ell)\ |\ \eta_\ell(\sigma) = \varepsilon 1\ \}$$

for $\varepsilon \in \{+,-\}$.

Theorem 2 μ^+ is a character of Θ_2^+ of order 2 and $\ker(\mu^+) = \Theta_2^+(1) \cup \Theta_2^+(2)$.

Proof. Let us start to define a subset H of Θ^+ by

$$H = \{\ \tau \in \Theta_2^+\ |\ \mu^+(\tau\sigma) = \mu^+(\tau)\mu^+(\sigma) \text{ for all } \sigma\in\Theta^+\}.$$

Then H becomes a subgroup of Θ^+ (see Bass-Milnor-Serre [1, Lemma 9.1] . In order to prove $H = \Theta^+$, it is enough to show that $\Gamma_2(2) \cup \{\ J, T_W\ \} \cup U \subset H$ from Proposition 1-(2). We note that

$$\mu^+(U_V) = \eta(V) \qquad (\ V \in GL(2,Z))$$

$$\mu^+(J) = 1\ ,$$

$$\mu^+(T_W) = -1\ ,$$

$$\mu^+(\sigma) = 1 \qquad\qquad (\ \sigma \in \Gamma_2(2)\ ),$$

$$\eta(W) = -1\ .$$

It is clear that $\Gamma_2(2) \subset H$. Let $\sigma \in \Theta(\ell)$. Then $J\sigma$, $U_V \in \Theta(\ell)$.

In fact, it is obvious from that

$$J\sigma = \begin{pmatrix} C_\sigma & D_\sigma \\ -A_\sigma & B_\sigma \end{pmatrix} \quad , \qquad U_V = \begin{pmatrix} VA_\sigma & VB_\sigma \\ V^*C_\sigma & V^*D_\sigma \end{pmatrix} \quad .$$

At the same time, we have

$$\mu^+(J\sigma) = \mu^+(\sigma)\ , \qquad \mu^+(U_V\sigma) = \eta(V)\mu^+(\sigma)$$

by the property $\eta(V^*) = \eta(V)$. Since $\mu^+(J) = 1$ and $\mu^+(U_V) = \eta(V)$

mentioned above, we obtain

$$\mu^+(J\sigma) = \mu^+(J)\mu\ (\sigma)\ , \quad \mu^+(U_V\sigma) = \mu^+(\sigma)$$

for all $\sigma \in \Theta_2^+$. Next we shall show that $T_W \in H$. If $\sigma \in \Theta(2)$,

then we see that $T_W \in \Theta(1)$. Therefore there exist $V \in GL(2,Z)$

such that

$$\{\ \overline{A}_\sigma\ ,\ \overline{B}_\sigma\ ,\ \overline{C}_\sigma\ ,\ \overline{D}_\sigma\ \} = \{\ 0_2\ ,\ 0_2\ ,\ \overline{V}\ ,\ \overline{V}^*\ \}\ ,$$

$$\{\ \overline{A}_{T_W\sigma}, \overline{B}_{T_W\sigma}, \overline{C}_{T_W\sigma}, \overline{D}_{T_W\sigma}\ \} = \{\ 0_2, \overline{V}, \overline{V}^*, \overline{WV}\ \}\ .$$

Hence $\mu^+(\sigma) = \eta(V)$ and $\mu^+(T_W\ ) = \eta(WV) = \eta(W)\eta(V) = -\eta(V)$.

This implies that $\mu^+(T_W\sigma) = \mu^+(T_W)\mu^+(\sigma)$ for all $\sigma \in \Theta(2)$. If

$\sigma \in \Theta(1)$, then there exists $V \in GL(2,Z)$ such that

$$\{ \overline{A}_\sigma , \overline{B}_\sigma , \overline{C}_\sigma , \overline{D}_\sigma \} = \{ 0_2 , \overline{V} , \overline{V}^* , \overline{WV} \} .$$

Hence $\mu^+(\sigma) = \eta(WV) = \eta(W)\eta(V) = -\eta(V)$. Assume that $\sigma \in \Theta^{i,j}$.

Then we have

$$\{ \overline{A}_{T_W\sigma}, \overline{B}_{T_W\sigma}, \overline{C}_{T_W\sigma}, \overline{D}_{T_W\sigma} \} = \begin{cases} \{ 0_2, 0_2, \overline{V}, \overline{V}^* \} \in \Theta(2) & (i=j) , \\ \{ 0_2, \overline{V}, \overline{WV}, \overline{WV}^* \} \in \Theta(1) & (i \neq j) . \end{cases}$$

This implies that $\mu^+(T_W\sigma) = \eta(V)$. Therefore $\mu^+(T_W\sigma) = -\mu^+(\sigma)$. Thus we obtain that $\mu^+(T_W\sigma) = \mu^+(T_W)\mu^+(\sigma)$ for all $\sigma \in \Theta(1)$, since $\mu^+(T_W) = -1$. Therefore we can get that $\mu^+(T_W\sigma) = \mu^+(T_W)\mu^+(\sigma)$ for all $\sigma \in \Theta_2^+$. Hence $T_W \in H$. Thus $H = \Theta_2^+$. It is obvious by the definition of $\Theta^\varepsilon(\ell)$ that $\ker(\mu^+) = \Theta^+(1) \cup \Theta^+(2)$.

Corollary 1 Let Λ^+ be the restriction of $\tilde{\Lambda}$ on Θ_2^+ . Then, $\mu^+ = \Lambda^+$.

Proof. From Kirchheimer [5 , 2.9], we know that

$$\Lambda^+(J) = 1 ,$$

$$\Lambda^+(T_W) = -1 ,$$

$$\Lambda^+(U_V) = (-1)^{(1+A_V+D_V)(1+B_V+C_V)+A_VD_V} .$$

Since $(1+A_V+D_V)(1+B_V+C_V) + A_VD_V \equiv A_VB_V + B_VC_V + C_VD_V \pmod 2$ for $V \in GL(2,\mathbb{Z})$, we get our assertion by Theorem.

Corollary 2 $\ker(\mu^{+}) = \Gamma_2(2)\Theta_2'$.

Proof. From Theorem 2, we have that $\ker(\mu^{+}) \supset \Gamma_2(2)$ and by Corollary 1 we see that $\ker(\mu^{+}) \supset \Theta_2'$. Hence,

$$\Theta_2^{+} \supsetneqq \ker(\mu^{+}) \supset \Gamma_2(2)\Theta_2' .$$

On the other hand, from Remark in the end of Section 2, it holds that $(\Theta_2^{+} : \Gamma_2(2)\Theta_2') = 2$: Therefore we obtain $\ker(\mu^{+}) = \Gamma_2(2)\Theta_2'$.

4. Construction of another character of Θ_2

In this section we shall construct a character of Θ_2 of order 2, by means of μ^+ given as Section 3. For $\sigma \in \Theta_2$ and $M \in M(4,Z)$, put

$$\Phi(\sigma) = A_\sigma + B_\sigma + C_\sigma + D_\sigma \quad ,$$

$$\Phi_0(\sigma) = A_\sigma + C_\sigma \quad ,$$

$$\phi(M) = (-1)^{B_M + C_M + B_M C_M}$$

$$\mu(\sigma) = \begin{cases} \phi(\Phi_0(\sigma)) & \text{if } \overline{\Phi(\sigma)} = 0 \, , \\ \phi(\Phi(\sigma)) & \text{otherwise.} \end{cases}$$

Theorem 3 μ is a character of Θ of order 2 and $\mu \neq \lambda_2 , \Lambda$.

Proof. We first note that

$$\mu(J\sigma) = \mu(\sigma J) = \mu(J_1\sigma) = \mu(\sigma J_1) = \mu(\sigma)$$

for all $\sigma \in \Theta_2$. In fact,

$$J\sigma = \begin{pmatrix} C_\sigma & D_\sigma \\ -A_\sigma & -B_\sigma \end{pmatrix} \quad , \qquad \sigma J = \begin{pmatrix} -B_\sigma & A_\sigma \\ -D_\sigma & C_\sigma \end{pmatrix} \quad ,$$

$$J_1\sigma = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ -a_{21} & -a_{22} & -b_{21} & -b_{22} \end{pmatrix} \quad ,$$

$$\sigma J_1 = \begin{pmatrix} a_{11} & -b_{12} & b_{11} & a_{12} \\ a_{21} & -b_{22} & b_{12} & a_{22} \\ c_{11} & -d_{12} & d_{11} & c_{12} \\ c_{21} & -d_{22} & d_{12} & c_{22} \end{pmatrix} ,$$

where a_{ij} , b_{ij}, c_{ij} , d_{ij} denote (i,j)-component of A_σ , B_σ , C_σ , D_σ , respectively. Hence

$$\overline{\Phi(J\sigma)} = \overline{\Phi(\sigma J)} = \overline{\Phi(J_1\sigma)} = \overline{\Phi(\sigma J_1)} = \overline{\Phi(\sigma)},$$

$$\overline{\Phi_0(J\sigma)} = \overline{\Phi_0(J_1\sigma)} = \overline{\Phi_0(\sigma)}.$$

Therefore $\mu(J\sigma) = \mu(J_1\sigma) = \mu(\sigma)$ for all $\sigma \in \Theta_2$ and $\mu(\sigma J) = \mu(\sigma J_1) = \mu(\sigma)$ if $\sigma \in \Theta_2$ and $\overline{\Phi(\sigma)} \neq 0_2$. If $\overline{\Phi(\sigma)} = 0_2$, then $\overline{A}_\sigma + \overline{C}_\sigma = \overline{B}_\sigma + \overline{D}_\sigma$. Hence $\overline{\Phi_0(\sigma J)} = \overline{\Phi_0(\sigma)}$: Therefore $\mu(\sigma J) = \mu(\sigma)$ if $\overline{\Phi(\sigma)} = 0_2$, hence for all $\sigma \in \Theta_2$. We also see that

$$\Phi_0(J_1) = \begin{pmatrix} a_{11} + c_{11} & b_{12} + d_{12} \\ a_{21} + c_{21} & b_{22} + d_{22} \end{pmatrix} \quad \text{and} \quad a_{ij} + c_{ij} \equiv b_{ij} + d_{ij} \pmod 2 .$$

Therefore $\overline{\Phi_0(\sigma J_1)} = \overline{\Phi_0(\sigma)}$. Thus $\mu(\sigma J_1) = \mu(\sigma)$ if $\overline{\Phi(\sigma)} = 0_2$, hence for all $\sigma \in \Theta_2$. Next, we show that both functions of μ and μ^+ take the same values on Θ_2^+ . To prove this, it is sufficient to show for all elements of $U\Gamma_2(2) \cup \Theta^{2,1}$, because $\mu(J\sigma) = \mu(\sigma J) = \mu(\sigma)$. Let $\sigma \in \Theta^{2,1}$. Then $\overline{\Phi(\sigma)} = \overline{A}_\sigma + \overline{A}_\sigma{}^* + \overline{WA}_\sigma$.

Therefore

$$\overline{\Phi(\sigma)} = \begin{cases} \overline{W} & \text{if } \overline{A}_\sigma = 1_2 , \\ 1_2 & \text{if } \overline{A}_\sigma = \overline{W} , \\ e_{22} & \text{if } \overline{A}_\sigma = \overline{T} , \\ e_{11} & \text{if } \overline{A}_\sigma = {}^t\overline{T} , \\ e_{21} & \text{if } \overline{A}_\sigma = \overline{TW} , \\ e_{12} & \text{if } \overline{A}_\sigma = \overline{WT} , \end{cases}$$

where e_{ij} denotes the matrix of $M(2,F_2)$ whose (i,j)-component is equal to 1 and otherwise 0 , and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus, we have

$$\mu(\sigma) = \begin{cases} 1 & \text{if } \overline{A}_\sigma = \overline{W} , \overline{T} \text{ or } {}^t\overline{T} , \\ -1 & \text{if } \overline{A}_\sigma = 1_2, \overline{TW} \text{ or } \overline{WT} . \end{cases}$$

On the other hand ,

$$\mu^+(\sigma) = \eta(WA_\sigma) = -\eta(A_\sigma) ,$$

and

$$\eta(M) = \begin{cases} 1 & \text{if } \overline{M} = 1_2, \overline{TW} \text{ or } \overline{WT} , \\ -1 & \text{if } \overline{M} = \overline{W} , \overline{T} \text{ or } {}^t\overline{T} . \end{cases}$$

Therefore we obtain that $\mu(\sigma) = \mu^+(\sigma)$ for all $\sigma \in \Theta^{2,1}$. Let $\sigma \in U\Gamma_2(2)$. Then, $\overline{\Phi(\sigma)} = \overline{A}_\sigma + \overline{A}_\sigma^*$. We observe that $\overline{\Phi(\sigma)} = 0_2$ if and only if $\overline{A}_\sigma = 1_2$ or $\overline{W}$. Assume that $\overline{\Phi(\sigma)} \neq 0_2$. Then

$$\overline{\Phi(\sigma)} = \begin{cases} e_{21} & \text{if } \overline{A}_\sigma = \overline{T} \ , \\ e_{12} & \text{if } \overline{A}_\sigma = {}^t\overline{T} \ , \\ e_{11} & \text{if } \overline{A}_\sigma = \overline{TW} \ , \\ e_{22} & \text{if } \overline{A}_\sigma = \overline{WT} \ . \end{cases}$$

Thus, we have

$$\mu(\sigma) = \begin{cases} 1 & \text{if } \overline{A}_\sigma = \overline{TW} \text{ or } \overline{WT} \ , \\ -1 & \text{if } \overline{A} = \overline{T} \text{ or } {}^t\overline{T} \ . \end{cases}$$

On the other hand,

$$\mu^+(\sigma) = \begin{cases} 1 & \text{if } \overline{A}_\sigma = \overline{TW} \text{ or } \overline{WT} \ , \\ -1 & \text{if } \overline{A}_\sigma = \overline{T} \text{ or } {}^t\overline{T} \ , \end{cases}$$

since $\mu^+(\sigma) = \eta(A_\sigma)$. Assume $\overline{\Phi(\sigma)} = 0_2$. Then $\overline{\Phi_0(\sigma)} = \overline{A}_\sigma$.

Hence

$$\mu(\sigma) = \begin{cases} 1 & \text{if } \overline{A}_\sigma = 1_2 \ , \\ -1 & \text{if } \overline{A}_\sigma = \overline{W} \ . \end{cases}$$

Therefore $\mu(\sigma) = \mu^+(\sigma)$ since $\mu^+(\sigma) = \eta(A_\sigma)$. Thus, we can show that $\mu(\sigma) = \mu^+(\sigma)$ for all $\sigma \in \Theta_2^+$. So, we are going to prove that μ is a character of Θ_2 . Let $\sigma, \tau \in \Theta_2$. If $\sigma, \tau \in \Theta_2^+$, then it is clear by the fact just proved above. If $\sigma \notin \Theta_2^+$, $\tau \in \Theta_2^+$, then there exists an element ρ of Θ_2^+ such that $\sigma = J_1\rho$. Hence $\mu(\sigma\tau) = \mu(J_1\rho\tau) = \mu(\rho\tau) = \mu(\rho)\mu(\tau) =$

$\mu(J_1\rho)\mu(\tau) = \mu(\sigma)\mu(\tau)$. We also see that $\mu(\tau\sigma) = \mu(\tau J_1\rho) = \mu(J_1\tau J_1\rho) = \mu(J_1\tau J_1)\mu(\rho) = \mu(\tau)\mu(\rho) = \mu(\tau)\mu(J_1\rho) = \mu(\tau)\mu(\sigma)$. If $\sigma, \tau \notin \Theta_2^+$, then there exist two elements ρ, ζ of Θ_2^+ such that $\sigma = J_1\rho$ and $\tau = J_1\zeta$. Then, $\mu(\sigma\tau) = \mu(J_1\rho J_1\zeta) = \mu(J_1\rho J_1)\mu(\zeta) = \mu(J_1\rho)\mu(J_1\zeta) = \mu(\sigma)\mu(\tau)$. Thus, we verify that μ is a character of Θ_2^+. Finally, we see that $\mu(U_W) = -1$, because $\overline{\Phi(U_W)} = 0_2$ and $\overline{\Phi_0(U_W)} = \overline{W}$. Therefore, the order of μ is equal to 2. It is aiso seen that $\mu(J_1) = 1$, since $\overline{\Phi(J_1)} = 0_2$ and $\overline{\Phi_0(J_1)} = 1_2$. This shows that $\mu \neq \lambda_2, \Lambda^+$.

5. Application to modular imbeddings

In this section we investigate the relations between the characters of theta group Θ_2 and modular imbeddings for certain real quadratic fields, which Hammond has introduced in [4] . Let p be a rational prime number and $p \equiv 1 \pmod 4$. Then, we can find an odd integer u and an even integer v such that $u^2 + v^2 = p$. Now, we consider the Hilbert modular group, which acts on H_1^2 , over the real quadratic field $Q(\sqrt{p})$. Put $F = Q(\sqrt{p})$ and denote by o the ring of integers of F . We use the notation $SL(2,R)$ in stead of $S_p(2,R)$ for a ring R . Put:

$$\Delta = \begin{pmatrix} u & v \\ v & -u \end{pmatrix} .$$

For $\alpha \in F$ and $g \in SL(2,F)$, put

$$\xi(\alpha) = a1_2 + b\Delta \qquad (\alpha = a + b\sqrt{p}),$$

$$\Xi(g) = \begin{pmatrix} \xi(A_g) & \xi(B_g) \\ \xi(C_g) & \xi(D_g) \end{pmatrix} .$$

Then, from Hammond [4], there exists a holomorphic imbedding E of H_1^2 into H_2 such that (Ξ,E) becomes a modular imbedding for F , that is, Ξ is a homomorphism of $SL(2,F)$ into $S_p(4,Q)$ satisfyning

$$\Xi(SL(2,o)) \subset \Gamma_2 ,$$

$$\Xi(g)(E(z)) = E(g(z)) ,$$

$$j(\Xi(g),E(z)) = (C_g z_1 + D_g)(C_g{}' z_2 + D_g{}') ,$$

where $g \in SL(2,o)$, $z = (z_1,z_2) \in H_1{}^2$ and $\alpha' = a - b\sqrt{p}$ for $\alpha = a + b\sqrt{p} \in F$.

We consider the case when $p \equiv 1 \pmod 8$. Then, from [6] , we know that $\Xi(SL(2,o)) \subset \rho\Theta_2\rho^{-1}$, where $\rho = \begin{pmatrix} 1_2 & 1_2 \\ K^{-1} & K \end{pmatrix}$ with $K = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Remark. $\rho\Theta_2\rho^{-1}$ is the subgroup of Γ_2 consisting of all elements leaving the characteristic $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ of theta function invariant. In general, put

$$K_n = \begin{pmatrix} 1_n & 1_n \\ 1_n & 0_n \end{pmatrix} ,$$

$$\rho_n = \begin{pmatrix} 1_{2n} & 1_{2n} \\ K_n{}^{-1} & K_n \end{pmatrix} ,$$

$$m_0 = (0,\cdots,0) \in R^{4n} ,$$

$$m_1 = (\tfrac{1}{2},\cdots,\tfrac{1}{2}) \in R^{4n} .$$

Then $\rho_n \in \Gamma_{2n}$ and ρ_n maps m_0 to m_1 , under the transformation formular among the characteristics of theta functions of degree $2n$.

Theorem 4 Let μ be the real characteh of Θ_2 as in Section 4 . Then $\Xi(SL(2,o)) \subset \rho(\ker(\mu))\rho^{-1}$.

Proof. We can easily see the following :

$$(\#)\quad \begin{aligned}
A_{\rho^{-1}\Xi(g)\rho} &= K\xi(A_g) + K\xi(B_g)K^{-1} - \xi(C_g) - \xi(D_g)K^{-1} ,\\
B_{\rho^{-1}\Xi(g)\rho} &= K\xi(A_g) + K\xi(B_g)K - \xi(C_g) - \xi(D_g)K ,\\
C_{\rho^{-1}\Xi(g)\rho} &= - K^{-1}\xi(A_g) - K^{-1}\xi(B_g)K^{-1} + \xi(C_g) + \xi(D_g)K^{-1},\\
D_{\rho^{-1}\Xi(g)\rho} &= - K^{-1}\xi(A_g) - K^{-1}\xi(B_g)K + \xi(C_g) + \xi(D_g)K .
\end{aligned}$$

Hence by direct calculation we obtain that $\overline{\Phi(\rho^{-1}\Xi(g)\rho)} = \overline{\xi(B_g)}$ and $\overline{\Phi_0(\rho^{-1}\Xi(g)\rho)} = \overline{\xi(A_g)} + \overline{\xi(B_g)K^{-1}}$. Therefore $\overline{\Phi_0(\rho^{-1}\Xi(g)\rho)} = \overline{\xi(A_g)}$ if $\overline{\Phi(\rho^{-1}\Xi(g)\rho)} = 0_2$ .. Thus, we have that $\mu(\Xi(g)) = \phi(\xi(A_g))$ or $\phi(\xi(B_g))$. On the other hand, we observe that $\phi(\xi(\alpha)) = 1$ for all $\alpha \in o$, because $v \equiv 0 \pmod 4$ (see [7 , Lemma in §2]. This shows that $\mu(\Xi(g)) = 1$ for all $g \in SL(2,o)$. Then $\rho^{-1}\Xi(SL(2,o))\rho \subset \ker(\mu)$, hence our assertion is proved.

We can prove naturally the following fact.

Proposition 2 ([7, Proposition 2]) For all $g \in SL(2,o)$

$$\lambda_2(\rho^{-1}\Xi(g)\rho) = (-1)^{Tr_{F/Q}(A_gB_g+B_gC_g+C_gD_g)} ,$$

where $Tr_{F/Q}(\alpha)$ means the trace of an element α of F over Q.

Proof. BY (#) , we see that

$$tr({}^tC_{\rho^{-1}\Xi(g)\rho}B_{\rho^{-1}\Xi(g)\rho})$$

$$\equiv tr(\xi(A_g{}^2 + A_gC_g + C_g{}^2))$$

$$+ tr(\xi(B_g{}^2 + B_gC_g + D_g{}^2))$$

$$+ tr(\xi(A_gB_g + C_gD_g))$$

$$+ tr((K^{-1}\xi(B_g)K^{-1} + K\xi(B_g)K)\xi(C_g)) .$$

Then, by [7 , Lemma 1] , this is congruent to $tr(\xi(A_gB_g+B_gC_g+C_gD_g)$ modulo 2 .

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