AN ETALE APPROACH TO THE NOVIKOV CONJECTURE

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Abstract. We show that the Novikov conjecture for a group π of finite homological type follows from the mod 2 acyclicity of the Higson compactifcation of an $E\pi$. We then show that for groups of finite asymptotic dimensions the Higson compactification is mod p acyclic for all p, and deduce the integral Novikov conjecture for these groups.

§1 Introduction

Ten years ago, the most popular approach to the Novikov conjecture went via compactifications. If a compact aspherical manifold, say, had universal cover which suitably equivariantly compactifies, already Farrell and Hsiang [FH] proved that the Novikov conjecture follows and subsequent work by many authors weakened the various hypotheses and extended the idea to other settings. (See e.g. [CP1],[CP2], [FW], [Ro2].)

In more recent years other coarse methods have supplanted the compactification method (most notably the embedding method of [STY] in the C* algebra setting). The reason for this is that it seemed to have a better chance of applying generally while compactifications effective for the Novikov compactifications seem to require some special geometry for their construction. For a brief moment, it seemed that this could conceivably not be the case. Higson introduced a general compactification of metric spaces (somewhat reminiscent of the Stone-Cech compactification) that automatically has half of the properties necessary for application to the Novikov conjecture, and naturally suggested that perhaps the other condition automatically holds: that is, that perhaps the Higson compactification of a universal cover is automatically acyclic. (See [Ro1].)

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Unfortunately, it was soon realized by Keesling and others (see [Ke], [DF]) that the Higson compactification, even for manifolds as small as $\mathbf{R}$, has nontrivial rational cohomology. Thus, it was felt that general compactifications were not suitable for the problem one has to use geometric compactifications. However, recently Gromov has harpooned the embedding approach by constructing finitely generated groups which do not uniformly embed in Hilbert space [G2],[G3]. Moreover, a number of authors (e.g. [HLS]) have showed that Gromov's groups can be used to construct counterexamples to general forms of the Baum-Connes conjecture. However, there do not yet seem to be counterexamples to the Novikov conjecture on the horizon, nor, in the absence of embedding, a potential method.

Moreover, for reasons that are not entirely clear, the embedding method has never been translated into pure topology; the results on integral Novikov conjecture so obtained, from the L-theoretic viewpoint never account for the prime 2, and there do not seem to be many results in pure algebraic K-theory (or A-theory) provable by this method.

This paper, therefore, seeks to somewhat rehabilitate the Higson compactification approach. First for generalities. We will show that

Theorem A1. *If the Higson compactification of $E\pi$ is mod 2 acyclic, and $B\pi$ is of finite type, then the integral Novikov conjecture for π holds at the prime 2.*

At odd primes we do not know how to prove any corresponding statement, in general, for reasons related to the examples in [DFW]: The L-spectrum away from 2 is periodic K-theory, and cohomologically acyclic spaces may not be acyclic for periodic K-theory. So, for odd primes we prove this statement with the stipulation of finite dimensionality.

Theorem A2. *If the Higson compactification of $E\pi$ is mod p acyclic and finite dimensional, and $B\pi$ is of finite type, then the integral Novikov conjecture for π holds at the prime p.*

Since the above infinite dimensional phenomenon does not arise for groups of "finite asymptotic dimension", it is possible for such groups to prove an integral result.

Theorem B. *If $B\pi$ is a finite complex and π has finite asymptotic dimension, then if $\mathbf{R}$ is a ring such that $\mathbf{L}(\mathbf{R})$ is finitely generated, then the $\mathbf{L}(\mathbf{R})$ assembly map for π is injective.*

This is not a new theorem. For $K(C^*\pi)$, this is due to [Yu]. In algebraic Ktheory there are papers by [Ba], [CG2], which apply to L-theory as well (without the restrictions on R); moreover [CFY] has yet another approach to the stronger L-theory result. The novelty is the method of using finite primes to repair the acyclicity of the Higson compactification and then using the etale idea to get our Novikov result.[1]

To be completely clear,

Theorem C. *If π is as in Theorem B above, then the Higson compactification of $B\pi$ is mod p acyclic for every p.*

We note that the class of discrete groups Γ with $\operatorname{asdim}\Gamma < \infty$ is very large. It contains hyperbolic groups, polycyclic groups, Coxeter groups, arithmetic groups, etc. (see [G1],[Ro3], [BD1], [DJ], [Ji], [CG1]). In fact it is not easy to construct a finitely presented group with infinite asymptotic dimension. It is especially difficult in the class of groups with finite cohomological dimension. Only recently using expanders Gromov outlined a construction of asymptotically infinite dimensional groups with finite $B\Gamma$ [G2].

To perhaps motivate and clarify our results, we we close with a description of the Higson compactification and of the Keesling elements. If X is a metric space, and $f : X \to \mathbf{R}$ is a bounded continuous function, we say that f has *decaying variation* if

$$\lim \operatorname{diam}(f(B(x,r))) = 0$$

for any fixed r and any sequence of points x going to infinity in X. Higson's compactification is the smallest one containing X densely so that all functions with decaying variation extend. Let C_h be the aggregate of such functions. We can embed X in $Maps[C_h : R]$. Then take the closure.

Let us now think about the first cohomology of the compactification of a uniformly simply connected X. We represent this by maps into S^1. On X, though, we can lift and view our class as being given by $exp(2\pi i f)$, where f has decaying variation (but is not necessarily bounded!). Two such maps are homotopic iff f and g differ by a bounded function, by lifting a homotopy.

Note therefore, that $H^1(;Z)$ is now an $\mathbf{R}$-vector space – one can multiply the function f by a real number! In particular, this cohomology vanishes with mod p coefficients. In this regard, the cohomology of the Higson compactification seems to resemble the uniformly finite homology of a uniformly contractible n manifold, [BW], which is known to be an $\mathbf{R}$ vector space, aside from the class coming from the fundamental class of X.

Our paper is organized as follows. Section 2 is devoted to descent, i.e. deducing Novikov conjectures from metric results. The last section is devoted to verifying the mod p acyclicity result for the finite asymptotic dimension case. It is done using ideas of quantitative algebraic topology. The key idea is that when homotopy groups are finite, it is sometimes possible to get extra Lipschitz conditions on maps, a priori.

[1]Note that the celebrated paper of [BHM] on the algebraic K-theory analogue of the Novikov conjecture is a very nice illustration of a completely different etale idea.

§2 $\mathbf{L}$-theory and assembly map with $\mathbf{Z}_p$ coefficients

For every ring with involution R (actually any additive category with involution) Ranicki defines a 4-periodic spectrum $\mathbf{L}_*(R)$ with $\pi_i(\mathbf{L}_*(R)) = L_i(R)$ [Ran1]. We use the notation $\mathbf{L} = \mathbf{L}_*(\mathbf{Z})$.

Below R is a PID with involution. For a metric space (X, d) by $C_X(R)$ we denote the boundedly controlled Pedersen-Weibel category whose objects are locally finite direct sums $A = \oplus_{x\in X} A(x)$ of finite dimensional free R-modules and morphisms are given by matrices with bounded propagations. For a subset $V \subset X$ we denote by $A(V)$ the sum $\oplus_{x\in V} A(x)$. Ranicki [Ran2] defined *X-bounded quadratic L-groups* $L_*(C_X(R))$ and the corresponding spectrum $\mathbf{L}_*(C_X(R))$. We will use a notation $\mathbf{L}^{bdd}(X) = \mathbf{L}_*(C_X(\mathbf{Z}))$.

Suppose that $\bar{X}$ is a compactification of X with the corona $Y = \bar{X} \setminus X$. Then one can define continuously controlled category $B_{X,Y}(R)$ by taking the same objects as above and with morphisms $f : A \to B$ to be homomorphisms such that for every $y \in Y$ and every neighborhood $U \subset \bar{X}$ there is a smaller neighborhood $V \subset U$ such that $f(A(V)) \subset A(U)$. This category is additive and hence also admits the L-theory. The corresponding spectrum for $R = \mathbf{Z}$ we denote by $\mathbf{L}^{cc}(X) = \mathbf{L}_*(B_{X,Y}(\mathbf{Z}))$.

For a spectrum $\mathbf{E} = \{E_k \mid k \in \mathbf{Z}\}$ by $H_i(X; \mathbf{E})$ we denote the $\mathbf{E}$-homology of X. If X is a complex, then $H_i(X; \mathbf{E}) = \pi_i(X_+ \wedge \mathbf{E}) = lim_{k\to\infty} \pi_{i+k}(X_+ \wedge E_k)$. If X is a compact metric space, by $H_i(X; \mathbf{E})$ we denote the Steenrod $\mathbf{E}$-homology of X, i.e. $H_i(X; \mathbf{E}) = \pi_i(\mathrm{holim}\{N^i_+ \wedge \mathbf{E}\})$ where $X = \lim_{\leftarrow} N^i$ is the inverse limit of polyhedra [CP2], [EH].

The following theorem was proven in [P], [CP1], [CP2].

Theorem 2.1. *$L_i(B_{X,Y}(R)) = H_{i-1}(Y; \mathbf{L}(R))$ for all i where the homology is the Steenrod $\mathbf{L}_*(R)$-homology.*

Subsets $A, B \subset X$ of a metric space (X, d) are called *diverging* if

$$\lim_{r\to\infty} d(A \setminus B_r(x_0), B \setminus B_r(x_0)) = \infty$$

where $x_0 \in X$ is any point, $B_r(x_0)$ is the r-ball centered at x_0, and the distance $d(A', B')$ between sets is the infimum of distances between $d(a, b)$, $a \in A'$, $b \in B'$. We note that the minimal compactification of X with respect to the property that the closures of every diverging subsets in X have empty intersection in the corona is the *Higson compactification*. A compactification $\tilde{X}$ of X is called *Higson dominated* if the identity map $1_X : X \to X$ admits a continuous extension $\bar{X} \to \tilde{X}$. We note that for every Higson dominated compactification of X with the corona Y there is a forgetful functor $C_X(R) \to B_{X,Y}(R)$. This functor defines a forgetful map of spectra $\phi : \mathbf{L}^{bdd}(X) \to \mathbf{L}^{cc}(X)$.

If X is a universal covering of a finite complex with the fundamental group Γ, then Γ acts on $C_X(R)$ and hence on $\mathbf{L}_*(C_X(R))$ with the fixed set $\mathbf{L}_*(C_X(R))^\Gamma = L_*(R\Gamma)$ [CP]. We note that the base points of $\mathbf{L}_*(C_X(R))$ are fixed under Γ-action. If a compactification of X with the corona Y is equivariant, then Γ acts on $B_{X,Y}(R)$ and hence on $\mathbf{L}_*(B_{X,Y}(R))$.

We recall that the homotopy fixed set $X^{h\Gamma}$ of a pointed space X with a Γ action on it is defined as the space of equivariant maps $Map_\Gamma(E\Gamma_+, X)$.

For a general spectrum $\mathbf{E}$ and a torsion free group Γ it was proven [C], [CP1] that

$$\mathbf{H}_*(B\Gamma;\mathbf{E}) \cong \mathbf{H}^{lf}_*(E\Gamma;\mathbf{E})^\Gamma \cong \mathbf{H}^{lf}_*(E\Gamma;\mathbf{E})^{h\Gamma}.$$

Here we use notation $\mathbf{H}_*(X;\mathbf{E})$ for the spectrum $\mathbf{S}$ with $\pi_i(\mathbf{S}) = H_i(X;\mathbf{E})$ and $\mathbf{H}^{lf}_*(X;\mathbf{E})$ for the spectrum $\mathbf{S}'$ with $\pi_i(\mathbf{S}) = H^{lf}_i(X;\mathbf{E})$. Formally, $H^{lf}_i(X;\mathbf{E})$ is defined for a locally compact space X as the Steenrod $\mathbf{E}$-homology of the one point compactification αX. We note that for a complex N, $\mathbf{H}_*(N;\mathbf{E}) = N_+ \wedge \mathbf{E}$ and for a compact space Y, $\mathbf{H}_*(Y;\mathbf{E}) = \operatorname{holim}\{N^\alpha_+ \wedge \mathbf{E}\}$ where N^α runs over nerves of all finite open covers of Y. If Γ acts on a compact space Y, then it acts on the set of all finite open covers of Y and hence on the spectrum $\mathbf{H}_*(Y;\mathbf{E})$.

The following theorem is a combination of results of Carlsson-Pedersen [CP1],[CP2] and Ranicki [Ran3]. The first part of it in a more general setting is discussed in [Ros], Theorem 3.3.

Theorem 2.2.

For every group Γ with finite classifying complex $B\Gamma$ there are morphism of spectra called the bounded control assembly map

$$A^{bdd} : \mathbf{H}^{lf}_*(E\Gamma;\mathbf{L}) \to \mathbf{L}^{bdd}(E\Gamma)$$

which close a homotopy commutative diagram

$$\begin{array}{ccc}
\mathbf{H}_*(B\Gamma;\mathbf{L}) & \xrightarrow{A} & \mathbf{L}_*(\mathbf{Z}\Gamma) \\
\simeq\downarrow & & trf\downarrow \\
\mathbf{H}^{lf}_*(E\Gamma;\mathbf{L})^{h\Gamma} & \xrightarrow{A^{bdd,h\Gamma}} & \mathbf{L}^{bdd}(E\Gamma)^{h\Gamma}
\end{array}$$

where A is the standard assembly map, the vertical arrows are the natural maps from fixed sets to homotopy fixed sets.

If the universal covering space $E\Gamma$ admits a Higson dominated equivariant compactification X with the corona Y then the diagram can be extended

$$
\begin{array}{ccc}
\mathbf{H}_*(B\Gamma;\mathbf{L}) & \xrightarrow{A} & \mathbf{L}_*(\mathbf{Z}\Gamma) \\
\simeq\big\downarrow & & trf\big\downarrow \\
\mathbf{H}_*^{lf}(E\Gamma;\mathbf{L})^{h\Gamma} & \xrightarrow{A^{bdd,h\Gamma}} & \mathbf{L}^{bdd}(E\Gamma)^{h\Gamma} \\
=\big\downarrow & & \phi^{h\Gamma}\big\downarrow \\
\mathbf{H}_*^{lf}(E\Gamma;\mathbf{L})^{h\Gamma} & \xrightarrow{A^{cc,h\Gamma}} & \mathbf{L}^{cc}(E\Gamma)^{h\Gamma}
\end{array}
$$

where $A^{cc,h\Gamma} = \phi^{h\Gamma} \circ A^{bdd,h\Gamma}$ is the composition with the forgetful map. Moreover, A^{cc} coincides with the boundary map from the Steenrod $\mathbf{L}$-homology exact sequence of pair (X, Y):

$$A_p^{cc} = \partial : \mathbf{H}_*^{lf}(E\Gamma;\mathbf{L}) \to \mathbf{H}_{*-1}(Y;\mathbf{L}).$$

Let $M(p)$ denote the Moore spectrum for the group $\mathbf{Z}_p$.

Lemma 2.1. *Let X be a finite-dimensional compact metric space. Suppose that X is $\mathbf{Z}_p$-acyclic, i.e., $\tilde{H}^*(X;\mathbf{Z}_p) = 0$ for the reduced Čech cohomology. Then X is acyclic for the reduced Steenrod $\mathbf{L} \wedge M(p^k)$-homology for all k.*

Proof. First we note $\tilde{H}_*(X;\mathbf{Z}_p) = 0$ for the reduced Steenrod homology. Using coefficient exact sequence we can show that $\tilde{H}_*(X;\mathbf{Z}_{p^k}) = 0$ for all k. For every spectrum $\mathbf{S}$ there is the Steenrod homology Atiyah-Hirzebruch spectral sequence with

$$E_{i,j}^2 = \tilde{H}_i(X; H_j(S^0,\mathbf{S}))$$

which converges to $\tilde{H}_*(X;\mathbf{S})$ provided X is finite dimensional compact metric space [EH],[KS]. Since all groups $H_q(S^0, \mathbf{L} \wedge M(p^k))$ are $\mathbf{Z}_{p^s}$, we have $E_{i,j}^2 = 0$ for all i and j. $\square$

REMARK 1. The finite dimensionality condition is essential here. There are acyclic compacta that have nontrivial mod p complex K-theory [T]. Namely, by results of Adams and Toda there is a map $f : \Sigma^k M(\mathbf{Z}_p, m) \to M(\mathbf{Z}_p, m)$ to a Moore space of the suspension of the Moore space which induces an isomorphism in K-theory. Then the inverse limit of the suspensions of f is a an acyclic compactum, since all bonding maps are trivial in cohomology, and it has nontrivial K-theory and nontrivial mod p K-theory.

REMARK 2. The spectrum $\mathbf{L}$ localized at 2 is equivalent to the Eilenberg-MacLane 4-periodic spectrum generated by the space [MM]

$$\prod_{i=1}^{\infty} K(\mathbf{Z}_{(2)}, 4i) \times K(\mathbf{Z}_2, 4i-2).$$

Thus, $\mathbf{L} \wedge M(2) = \mathbf{L}_{(2)} \wedge M(2)$ is the Eilenberg-MacLane spectrum generated by the space

$$\prod_{i=1}^{\infty} K(\mathbf{Z}_2, 4i) \times K(\mathbf{Z}_2, 4i-1) \times K(\mathbf{Z}_2, 4i-2).$$

Now if compactum X is $\mathbf{Z}_2$-acyclic it is $L \wedge M(2)$-acyclic without assumption of finite dimensionality of X.

DEFINITION. The mod p assembly map A_p is by the definition

$$A \wedge 1_{M(p)} : \mathbf{H}_*(B\Gamma; \mathbf{L}) \wedge M(p) \to \mathbf{L}_*(\mathbf{Z}\Gamma) \wedge M(p).$$

Theorem 2.3. *Suppose that the universal cover X of a finite aspherical complex B satisfies the mod p Higson conjecture and let $\dim \nu X < \infty$. Then the mod p^k assembly map for $\Gamma = \pi_1(B)$ is a split monomorphism for every k.*

Proof. Let let $m = \dim \nu X$. Using Schepin's spectral theorem [Dr1] one can obtain a metrizable $\mathbf{Z}_p$-acyclic Γ-equivariant compactification $\bar{X}$ of X with the corona Y of dimension $\dim Y = m$ (see the proof of Lemma 8.3. in [Dr1]). We introduce coefficients to the second diagram of Theorem 2.2 by the smash product with $M(p^k)$.

$$\begin{array}{ccc}
\mathbf{H}_*(B; \mathbf{L}) \wedge M(p^k) & \xrightarrow{A_p} & \mathbf{L}_*(\mathbf{Z}\Gamma) \wedge M(p^k) \\
\simeq \Big\downarrow & & \Big\downarrow \phi^{h\Gamma} trf \wedge 1 \\
\mathbf{H}_*^{lf}(X; \mathbf{L})^{h\Gamma} \wedge M(p^k) & \xrightarrow{A^{cc,h\Gamma} \wedge 1} & \mathbf{L}^{cc}(X)^{h\Gamma} \wedge M(p^k)
\end{array}$$

If Γ acts on a based space W such that the base point is free, there is the formula $(W \wedge N)^{h\Gamma} = W^{h\Gamma} \wedge N$ for the trivial action on N. Therefore, we have the commutative diagram

$$\begin{array}{ccc}
\mathbf{H}_*(B; \mathbf{L}) \wedge M(p^k) & \xrightarrow{A_p} & \mathbf{L}_*(\mathbf{Z}\Gamma) \wedge M(p^k) \\
\simeq \Big\downarrow & & \Big\downarrow \phi^{h\Gamma} trf \wedge 1 \\
(\mathbf{H}_*^{lf}(X; \mathbf{L}) \wedge M(p^k))^{h\Gamma} & \xrightarrow{(A^{cc} \wedge 1)^{h\Gamma}} & (\mathbf{L}^{cc}(X) \wedge M(p^k))^{h\Gamma}.
\end{array}$$

We show that
$$A^{cc}\wedge 1:\mathbf{H}^{lf}_*(X;\mathbf{L})\wedge M(p^k)\to \mathbf{L}^{cc}(X)\wedge M(p^k)$$
is an isomorphism. This would imply that $(A^{cc}\wedge 1)^{h\Gamma}$ is an isomorphism and hence A_p is a split monomorphism.

In view of Theorems 2.1 and 2.2 we need to show that

$$\partial\wedge 1:\mathbf{H}^{lf}_*(X;\mathbf{L})\wedge M(p^k)\to \mathbf{H}_{*-1}(Y;\mathbf{L})\wedge M(p^k)$$

is isomorphism. We note that $\partial\wedge 1_{M(p^k)}$ is equivalent to the boundary homomorphism for the pair $(\bar{X},Y)$ in $\mathbf{L}\wedge M(p^k)$-homology. By Lemma 2.1 we obtain $\mathbf{H}_*(\bar{X};\mathbf{L}\wedge M(p^k))=\mathbf{H}_*(pt;\mathbf{L}\wedge M(p^k))$ and hence $\partial\wedge 1_{M(p^k)}$ is an isomorphism. □

It is known that the universal coefficient formula with $\mathbf{Z}_p$ coefficients (UCF) holds true for every generalized homology theory and it is natural with respect to morphisms of spectra.

Proposition 2.1. *For every morphism of spectra $A:\mathbf{E}_1\to\mathbf{E}_2$ and every p and i there is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_i(\mathbf{E}_1)\otimes\mathbf{Z}_p & \longrightarrow & \pi_i(\mathbf{E}_1\wedge M(p)) & \longrightarrow & \pi_{i-1}(\mathbf{E}_1)*\mathbf{Z}_p & \longrightarrow & 0\\
 & & {\scriptstyle A_*\otimes 1}\big\downarrow & & {\scriptstyle (A\wedge 1)_*}\big\downarrow & & {\scriptstyle (A_{*-1})*1}\big\downarrow & & \\
0 & \longrightarrow & \pi_i(\mathbf{E}_2)\otimes\mathbf{Z}_p & \longrightarrow & \pi_i(\mathbf{E}_2\wedge M(p)) & \longrightarrow & \pi_{i-1}(\mathbf{E}_2)*\mathbf{Z}_p & \longrightarrow & 0.
\end{array}$$

Proof. We apply the smash product with $\mathbf{E}_i$, $i=1,2$ to the cofibration of spectra $\mathbf{S}\to\mathbf{S}\to M(p)$. Then the result follows from the homotopy exact sequence of the resulting cofibrations of spectra and the induced morphism between them

$$\begin{array}{ccccc}
E_1 & \longrightarrow & E_1 & \longrightarrow & E_1\wedge M(p)\\
\big\downarrow & & \big\downarrow & & \big\downarrow\\
E_2 & \longrightarrow & E_2 & \longrightarrow & E_2\wedge M(p).
\end{array}$$

□

Theorem 2.4. *Suppose that the mod p Higson conjecture holds for the universal cover of finite complex $B\Gamma$ for all p and let $\dim\nu\Gamma<\infty$. Then the integral assembly map A is a monomorphism.*

Proof. In view of compactness of $B\Gamma$ we have $\nu\Gamma = \nu E\Gamma$. By Theorem 2.3 $A \otimes 1_G$ is a split monomorphism for every finite abelian group G. Since $B\Gamma$ is a finite complex, the standard induction argument on the number of cells show that the group $H_i(M; \mathbf{L})$ is finitely generated for every i. Hence for every $\alpha \in H_i(M; \mathbf{L})$ there is p such that $\alpha \otimes 1 \in H_i(M; \mathbf{L}) \otimes \mathbf{Z}_p$ is not zero. By the UCF there is a monomorphism $H_i(M; \mathbf{L}) \otimes \mathbf{Z}_p \to H_i(M; \mathbf{L}(\mathbf{Z}_p))$ which together with the assembly map produces a commutative diagram

$$\begin{array}{ccc} H_i(M; \mathbf{L}) \otimes \mathbf{Z}_p & \longrightarrow & H_i(M; \mathbf{L} \wedge M(p)) \\ {\scriptstyle A\otimes 1}\big\downarrow & & {\scriptstyle A_p}\big\downarrow \\ L_i(\mathbf{Z}\Gamma) \otimes \mathbf{Z}_p & \longrightarrow & \pi_i(\mathbf{L}(\mathbf{Z}\Gamma) \wedge M(p)). \end{array}$$

This diagram implies that $A(\alpha) \neq 0$. □

Theorem 2.5. *Suppose that a group Γ admits a finite complex for a classifying space $B\Gamma$ and the Mod 2 Higson conjecture holds for $E\Gamma$. Then the Novikov Higher Signatures conjecture holds for Γ.*

Proof. In view of Remark 2 the argument of Theorem 2.3 for $p = 2$ works without the assumption $\dim E\gamma < \infty$. Hence the mod 2^k assembly map

$$A \wedge 1_{M(2^k)} : \mathbf{H}_*(B\Gamma; \mathbf{L}) \wedge M(2^k) \to \mathbf{L}_*(\mathbf{Z}\Gamma) \wedge M(2^k)$$

is a split monomorphism. On the group level by the Universal Coefficient Formula we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_i(B\Gamma; \mathbf{L}) \otimes \mathbf{Z}_{2^k} & \longrightarrow & H_i(B\Gamma; \mathbf{L} \wedge M(2^k)) & \longrightarrow & H_{i-1}(B\Gamma; \mathbf{L}) * \mathbf{Z}_{2^k} & \longrightarrow & 0 \\ & & {\scriptstyle A_*\otimes 1}\big\downarrow & & {\scriptstyle (A\wedge 1)_*}\big\downarrow & & {\scriptstyle (A_{*-1})*1}\big\downarrow & & \\ 0 & \longrightarrow & \pi_i(\mathbf{L}_*(\mathbf{Z}\Gamma)) \otimes \mathbf{Z}_{2^k} & \longrightarrow & \pi_i(\mathbf{L}_*(\mathbf{Z}\Gamma) \wedge M(2^k)) & \longrightarrow & \pi_{i-1}(\mathbf{L}_*(\mathbf{Z}\Gamma) * \mathbf{Z}_{2^k} & \longrightarrow & 0. \end{array}$$

Since the group $H_i(B\Gamma; \mathbf{L})$ is finitely generated, it can be presented as $\oplus_{F_i}\mathbf{Z} \oplus Tor_i$. Since $A_* \otimes 1_{\mathbf{Z}_{2^k}}$ is a monomorphism, the kernel $ker(A_*)$ consists of 2^k divisible elements. Since k is arbitrary, $ker(A_*|_{\oplus\mathbf{Z}}) = 0$. Therefore, $A_* \otimes 1_{\mathbf{Q}}$ is a monomorphism. □

§3 Mod p Higson conjecture for asymptotically finite dimensional sapces

We recall that a map $f : X \to Y$ between metric spaces is λ-Lipschitz if $d_Y(f(x), f(x')) \leq \lambda d_X(x, x')$ for all $x, x' \in X$. Denote by

$$L(f) = \sup_{x \neq x'} \{\frac{d_Y(f(x), f(x'))}{d_X(x, x')}\}$$

the minimal Lipschitz constant of f.

Every simplicial complex K carries a metric where all simplexes are the standard euclidean of size one. We will call the maximal such metric *uniform* and usually we will denote corresponding metric space as K_U. Note that the metric space K_U is geodesic If it is not specified, we will assume that a finite complex always supplied with the uniform metrics.

The following Lemma is a partial case of Theorem A from [SW].

Lemma 3.1. *Let Y be a finite simplicial complex with $\pi_n(Y)$ finite. Then for every λ there is μ such that every map $f : B^n \to Y$ with $L(f|_{S^{n-1}}) \leq \lambda$ can be deformed to a μ-Lipschitz map $g : B^n \to Y$ by means of a homotopy $h_t : B^n \to Y$ with $h_t|_{S^{n-1}} = f|_{S^{n-1}}$.*

Lemma 3.2. *Let L be a finite dimensional complex and let K be a finite complex with finite homotopy groups $\pi_i(K)$ for $i \leq \dim L + 1$. Let $f, g : L \to K$ be homotopic Lipschitz maps. Then every homotopy between f and g can be deformed to a Lipschitz homotopy $H : L \times [0, 1] \to K$.*

Proof. Let $F : L \times I \to K$ be a homotopy between f and g. By induction on n, and using Lemma 3.1 we construct a μ_n-Lipschitz map $H_n : L^{(n)} \times I \cup L \times \{0, 1\} \to K$ which is a deformation of F restricted to the n-skeleton $L^{(n)}$ such that H_n extends H_{n-1}. Here the fact that L and hence $L \times I$ are geodesic is essential for the argument because this condition will guarantee that the union of λ-Lipschitz maps on $\Delta^n \times I$ is λ'-Lipschitz on $L^{(n)} \times I$ (see [Dr2] or [Dr3] for details). Then $H = H_m$ for $m = \dim L$. $\square$

Let $x_0 \in X$ be a base point in a metric space X. For $x \in X$ and $A \subset X$ we denote by $\|x\| = dist(x, x_0)$ and $\|A\| = \max\{\|z\| \mid z \in A\}$. By $B_r(x)$ we denote a closed r-ball in X centered in x. We use notation $B_r = B_r(x_0)$.

Let σ be an n-dimensional simplex spanned in a Euclidean space. By $s_\sigma : \sigma \to \Delta^n$ we denote a simplicial homeomorphism onto the standard n-simplex $\Delta^n = \{(x_i) \in \mathbf{R}^{n+1} \mid \sum x_i = 1, x_i \geq 0\}$. A locally finite simplicial complex L with a geodesic metric on it is called an *asymptotic polyhedron* if every simplex in L is isometric to a simplex σ spanned in a Euclidean space and $\lim_{\|\sigma\| \to \infty} L(s_\sigma) = 0$.

Proposition 3.3. *Let (K, d) be an asymptotic polyhedron and let d_U denote the uniform geodesic metric on K. Then the identity map $u : (K, d) \to (K, d_U)$ satisfies the condition (*).*

Proof. Let R and $\epsilon > 0$ be given. There is t such that $L(s_\sigma) < \epsilon/(2R)$ for σ with $\|\sigma\| \geq t$. Let x be with $\|x\| \geq t + 2R$. For every two points $z, z' \in B_R(x)$ a geodesic segment $J = [z, z']$ joining them does not intersect B_t. There is a partition of J: $z = z_0 < z_1 \cdots < z_m = z'$ such that every segment $J_i = [z_i, z_{i+1}]$ lies in a simplex σ_i.

Hence $L(\sigma_i) < \epsilon/(2R)$. Therefore $d_U(z_i, z_{i+1}) \leq \epsilon/(2R)d(z_i, z_{i+1})$ and hence $d_U(z, z') \leq \sum \epsilon/(2R)d(z_i, z_{i+1}) = \epsilon/(2R)d(z, z') \leq \epsilon$. $\square$

Gromov defined the *asymptotic dimension* [G1] $\operatorname{asdim} X$ of a metric space X as follows: $\operatorname{asdim} X \leq n$ if for every d there are $n+1$ d-disjoint uniformly bounded families $\mathcal{U}_i$, $i = 0, \dots, n$ of subsets of X such that the union $\mathcal{U} = \cup \mathcal{U}_i$ is a cover of X. A family of subsets $\mathcal{A}$ is said to be d-disjoint if for every sets A and A' we have $d(x, x') > d$ for all $x \in A$ and $x' \in A'$.

Gromov's definition can be equivalently reformulated as follows: $\operatorname{asdim} X \leq n$ if for any arbitrary large number d there is a uniformly bounded open cover $\mathcal{U}$ of X with multiplicity $\leq n+1$ and with the Lebesgue number $\geq d$ (see Assertion 1 in [BD2] for a proof).

We note that for every n-dimensional asymptotic polyhedron L, $\operatorname{asdim} L = n$.

Let $\mathcal{U}$ be an open cover of a metric space X. The canonical projection to the nerve $p : X \to Nerve(\mathcal{U})$ is defined by the partition of unity $\{\phi_U : X \to \mathbf{R}\}_{U \in \mathcal{U}}$, where $\phi_U(x) = d(x, X \setminus U)/\sum_{V \in \mathcal{U}} d(x, X \setminus V)$. The family $\{\phi_U : X \to \mathbf{R}\}_{U \in \mathcal{U}}$ defines a map p to the Hilbert space $l_2(\mathcal{U})$ of square summable functions on $\mathcal{U}$ with the Dirac functions δ_U, $U \in \mathcal{U}$ as the basis. The nerve $N(\mathcal{U})$ of the cover $\mathcal{U}$ is realized in $l_2(\mathcal{U})$ by taking every vertex U to δ_U. Clearly, the image of p lies in the nerve. Given a family of positive numbers $\bar{\lambda} = \{\lambda_U\}_{U \in \mathcal{U}}$ we can change the above imbedding of the nerve $N(\mathcal{U})$ into $l_2(\mathcal{U})$ by taking each vertex U to $\lambda_U \delta_U$. Then the projection $p_{\mathcal{U}}^{\bar{\lambda}} : X \to N(\mathcal{U})$ to modified realization is given by the formula $p_{\mathcal{U}}^{\bar{\lambda}}(x) = \{\lambda_U \phi_U(x)\}_{U \in \mathcal{U}}$.

For a subset $A \subset X$ we denote by $L(\mathcal{U}|_A)$ the Lebesgue number of $\mathcal{U}$ restricted to A. Thus, $L(\mathcal{U}|_A) = \inf_{y \in A} \max_{V \in \mathcal{U}} d(y, X \setminus V)$.

Proposition 3.4. *Let $\mathcal{U}$ be a locally finite cover of a geodesic metric space X with the multiplicity $\leq m$. Then the above projection to the nerve $p_{\mathcal{U}}^{\bar{\lambda}} : X \to N = N(\mathcal{U})$ for $\bar{\lambda} = \{\lambda_U\}$ with $\lambda_U = L(\mathcal{U}|_U)/(2m+1)^2$ is 1-Lipschitz where the nerve is taken with the intrinsic metric induced from $l_2(\mathcal{U})$.*

Proof. We show that the map $\bar{p} = p_{\mathcal{U}}^{\bar{\lambda}}$ is $1 - Lipschitz$ as a map to $l_2(\mathcal{U})$. The partition of N into simplices defines a locally finite partition on X such that $\bar{p}$ is 1-Lipschitz on every piece considered as the map to N with the intrinsic metric. Since X is geodesic metric space, this would imply that $\bar{p}$ is 1-Lipschitz. Let $x, y \in X$ and $U \in \mathcal{U}$. The triangle inequality implies

$$|d(x, X \setminus U) - d(y, X \setminus U)| \leq d(x, y).$$

It is easy to see that $\sum_{V \in \mathcal{U}} d(x, X \setminus V) \geq L(\mathcal{U}|_U) = (2m+1)^2 \lambda_U$ for $x \in U$. Then

$|\phi_U(x) - \phi_U(y)| \leq$

$$\frac{1}{\sum_{V\in\mathcal{U}} d(x, X\setminus V)} d(x,y) + d(y, X\setminus U)|\frac{1}{\sum_{V\in\mathcal{U}} d(x, X\setminus V)} - \frac{1}{\sum_{V\in\mathcal{U}} d(y, X\setminus V)}|$$

$$\leq \frac{1}{(2m+1)^2\lambda_U} d(x,y) + \frac{1}{(2m+1)^2\lambda_U}(\sum_{V\in\mathcal{U}} |d(x, X\setminus V) - d(y, X\setminus V)|) \leq \frac{1}{\sqrt{2m+1}\lambda_U} d(x,y).$$

Then $\|p(x) - p(y)\| =$

$$= (\sum_{U\in\mathcal{U}} \lambda_U^2(\phi_U(x) - \phi_U(y))^2)^{\frac{1}{2}} \leq ((2m)\frac{1}{2m+1} d(x,y)^2)^{\frac{1}{2}} \leq d(x,y).$$

□

Lemma 3.5. *Suppose that X is a geodesic metric space with* $\operatorname{asdim} X \leq n$ *and let $f : X \to \mathbf{R}_+$ be a given proper function. Then there are a compact set $C \subset X$ and a map $\phi : X \to N$ to a $2n+1$-dimensional asymptotic polyhedron with diam$(\phi^{-1}(\Delta)) \leq f(z)$ for all $z \in \phi^{-1}(\Delta) \setminus C$.*

Proof. We construct ϕ as the projection $p_{\mathcal{U}}^{\bar{\lambda}}$ from Proposition 3.4 to the nerve of a cover $\mathcal{U}$ of X.

Fix a monotone tending to infinity sequence of l_i. Since $\operatorname{asdim} X \leq n$, for every i there is a uniformly bounded cover $\mathcal{U}_i$ of multiplicity $n+1$ with the Lebesgue number $L(\mathcal{U}^i) > l_i$. Let m_i be an upper bound for the diameter of elements of $\mathcal{U}_i$. We define a sequence of r_i such that $f(X \setminus B_{r_i - m_i}) \geq 2m_{i+1}$. We take $\mathcal{U} = \cup_i \mathcal{U}_i'$ where

$$\mathcal{U}_i' = \{U \setminus B_{r_{i-1}} \mid U \in \mathcal{U}_i, U \cap B_{r_i} \neq \emptyset\}.$$

We take $C = B_{r_1}$ and check the condition of Lemma. By the definition the preimage $\phi^{-1}(\Delta)$ lies in the union $\cup U_J'$ of sets from $\mathcal{U}$ with nonempty intersection. Let $x \in \cap U_j'$. Each U_j' belongs to the cover $\mathcal{U}_i$ for some i. Let k be maximal among those is. Then $x \notin B_{r_{k-1}}$. Therefore $U_j' \subset X \setminus B_{r_{k-1} - m_{k-1}}$ for all j. Hence $f(z) \geq 2m_k \geq diam \cup_j U_j'$ for $z \in X \setminus B_{r_{k-1} - m_{k-1}}$ and $k > 1$.

Since $l_i \to \infty$ the nerve N realized in $l_2(\mathcal{U})$ as above with the intrinsic metric is an asymptotic polyhedron. Since the covers $\mathcal{U}_i'$ and $\mathcal{U}_j'$ do not intersect for $|i - j| > 1$, the multiplicity of $\mathcal{U}$ is $\leq 2n+2$. □

REMARK. By a standard dimension theoretic trick the polyhedron N can be chosen to be n-dimensional.

A metric space X is called *uniformly contractible* if there is a function $S : \mathbf{R}_+ \to \mathbf{R}_+$ such that every ball $B_r(x)$ is contractible to a point in the ball $B_{S(r)}(x)$.

Let $x_0 \in X$ be a base point. We denote $\|x\| = d_X(x, x_0)$.

Lemma 3.6. *Let X be a uniformly contractible proper metric space with* $\operatorname{asdim} X = n$. *Then given a proper function $g : X \to \mathbf{R}_+$ there are an n-dimensional asymptotic polyhedron N, a proper 1-Lipschitz map $\phi : X \to N$, and a proper homotopy inverse map $\gamma : N \to X$ with $d(x, \gamma\phi(x)) < g(x)$ for all $x \in X$. Moreover, there is a compact set $C \subset X$ such that $diam(f^{-1}(\Delta) \leq g(x)$ for all $x \in \Delta \setminus C$.*

Proof. We define by induction on i a lift γ on the i-skeleton $N^{(i)}$ of the nerve of a cover of X given by Lemma 3.5 for an appropriate choice of f. We take $\gamma(v) \in \phi^{-1}(v)$ for every vertex v. Then using the uniform contractibility of X we can extend γ with control over the 1-skeleton $N^{(1)}$ and so on. Without loss of generality we may assume that X is a polyhedron of the dimension n supplied with a triangulation of mesh ≤ 1. By induction on i we define a homotopy $H : X^{(i)} \times I \to X$ joining the identity map with $\gamma \circ \phi$. We consider a function $\psi(x) = \|x\| - \max\{d(x, y) \mid y \in H(x \times I)\}$. If ψ tends to infinity, then the map H is proper. Therefore it suffices to show that ψ tends to infinity for an appropriate choice of f. Let S be a contractibility function. We define $\rho(t) = S^{-1}(t/2)$ where S^{-1} is the inverse function for S. Then we take $f = \rho^{2n+1} \circ g$, the composition of g and $2n+1$ times iteration of ρ. Clearly $g \leq f$. We assume here that $g(x) \leq \|x\|/2$. Then it is easy to verify that $\psi(x) \geq \|x\|/2$. $\square$

A metric space X is called *proper* if every ball $B_r(x)$ in X is compact. We recall that the Higson compactification $\bar{X}$ of a proper metric space X can be defined as the maximal ideal space of the completion of the ring of bounded functions with the gradient tending to zero at infinity [Ro1]. The defining property of the Higson corona is the following:

(**) *A continuous map $f : X \to Z$ to a compact metric space is extendable to the Higson corona νX if and only if it satisfies the condition: For arbitrary large R*

$$\lim_{\|x\| \to \infty} diam(f(B_R(x)) = 0.$$

Note that a proper Lipschitz map $f : X \to Y$ induces a continuous mapping between the Higson coronas $\bar{f} : \nu X \to \nu Y$.

Theorem 3.7. *Let X be a uniformly contractible geodesic proper metric space with a finite asymptotic dimension and let $\bar{X}$ be the Higson compactification. Then $\check{H}^n(\bar{X}; \mathbf{Z}_p) = 0$ for all n and all p.*

Proof. We show that every map $\alpha : \bar{X} \to K(\mathbf{Z}_p, n)$ is null homotopic. Since $\bar{X}$ is compact, the image $\alpha(\bar{X})$ is contained in the k-skeleton $K = K(\mathbf{Z}_p, n)^{(k)}$, which is a finite complex. We fix a geodesic metric on K. Let ϵ_K be an injectivity radius in K, i.e. every two points within a distance ϵ_K can be joined by the unique geodesic (We may enlarge K to a manifold with boundary to speak freely about the injectivity radius). Since the map $\alpha|_X \to K$ is extendable over the Higson corona the function $R_\alpha(t) = L(\alpha|_{X\setminus B_t(x_0)})$ tends to zero at infinity. We apply Lemma 3.6 with $g(x) \le \min\{\epsilon_K/R_\alpha(\|x\|/2), \|x\|/4\}$ to obtain an asymptotic polyhedron N and maps $\phi : X \to N$ and $\gamma : N \to X$. We may assume that ϕ is surjective. Let $[u,v]$ be an edge in N, then $d_K(\alpha\gamma(u), \alpha\gamma(v)) \le R_\alpha(t_0)d_X(\gamma(u),\gamma(v))$ where $t_0 = \min\{\|\gamma(u)\|, \|\gamma(v)\|\}$. We may assume that there are $x, y \in X$ such that $\phi(x) = u$ and $\phi(y) = v$. Then

$$d_X(\gamma(u),\gamma(v)) \le d_X(x,y)+\epsilon_K/R_\alpha(\frac{1}{2}\|x\|)+\epsilon_K/R_\alpha(\frac{1}{2}\|y\|) \le diam\phi^{-1}[u,v]+2\epsilon_K/R_\alpha(\frac{1}{2}\|x\|)$$

provided $\|x\| \le \|y\|$. Because of the inequality $g(x) \le \|x\|/4$ we have that $R_\alpha(t_0) \le R_\alpha(\|x\|/2)$. We may assume that $f < g$ and then $diam(\phi^{-1}([u,v])) < g(x)$. Summarizing all this, we obtain the inequality $d_K(\alpha\gamma(u), \alpha\gamma(v)) \le 3\epsilon_K$. This means that the map $\alpha\circ\gamma$ is $3\epsilon_K$-Lipschitz for N taken with the uniform metric.

Since X is contractible, the map $\alpha\circ\gamma$ is null homotopic. Note that the homotopy groups $\pi_i(K)$ are finite for $i \le \dim N + 1$. We apply Lemma 3.2 to obtain a λ-Lipschitz homotopy $H : N_U \times I \to K$ of $\alpha\circ\gamma$ to a constant map. This homotopy defines a Lipschitz map $\tilde{H} : N_U \to K^I_\lambda$ to the space of λ-Lipschitz mappings of the unit interval I to K. We note that the space K^I_λ is compact. Then by Proposition 3.3 $\tilde{H}\circ u : N \to K^I_\lambda$ satisfies the Higson extendibility condition (**). Let $\bar{h} : \bar{N} \to K^I_\lambda$ be the extension over the Higson corona. This extension defines a map $\bar{H} : \bar{N}\times I \to K$. The map $\bar{H}$ is a homotopy between the extension $\xi = \overline{\alpha\circ\gamma}$ and a constant map. To complete the proof, we show that α is homotopic to $\xi\circ\bar{\phi}$ where $\bar{\phi}$ is the extension of the Lipschitz map ϕ to the Higson compactifications. Note that

$$d_K(\alpha(x), \alpha\gamma\phi(x)) \le R_\alpha(t_0)d(x,\gamma\phi(x)) \le R_\alpha(t_0)\epsilon_K/R_\alpha(\|x\|/2) \le \epsilon_K$$

where $t_0 = \min\{\|x\|, \|\gamma\phi(x)\|\} \ge \|x\|/2$. Then for every $x \in X$ we join the points $\alpha(x)$ and $\alpha\gamma\phi(x)$ by the unique geodesic $\psi_x : I \to K$. This defines a map $\tilde{\psi} : X \to K^I_\mu$. Since both α and $\alpha\circ\gamma\circ\phi$ satisfy the condition (*), the map $\tilde{\psi}$ has the property (*). Let $\bar{\psi} : \bar{X} \to K^I_\mu$ be the extension of $\tilde{\psi}$ to the Higson corona. The map $\bar{\psi}$ defines a homotopy $\Psi : \bar{X}\times I \to K$ between α an $\xi\circ\bar{\phi}$. □

Corollary 3.8. *Suppose that a group Γ has a finite asymptotic dimension and $B\Gamma$ is a finite complex. Then the integral assembly map $A : H_*(B\Gamma;\mathbf{L}) \to L_*(\mathbf{Z}\Gamma)$ is a monomorphism.*

Proof. In view of the inequality $\dim \nu\Gamma \le \operatorname{asdim}\Gamma$ [DKU] we can apply Theorem 2.4. □

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