Hodge structure of fibre integrals associated to the affine hypersurface in a torus

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ABSTRACT. *We calculate the fibre integrals of the affine hypersurface in a torus in the form of their Mellin transforms. Especially, our method works efficiently for an affine hypersurface defined by a so called "simpliciable" polynomial. The relations between poles of Mellin transforms of fibre integrals, the mixed Hodge structure of the cohomology of the hypersruface, the hypergeometric differential equation, and the Euler characteristic of fibres are clarified.*

0 Introduction

In this note we propose a simple method to calculate concretely fibre integrals associated to the affine hypersurface in a torus. We establish an expression of the position of poles of the Mellin transform with the aid of the mixed Hodge structure of an hypersurface Z_f define by a $\Delta-$regular polynomial explained by V. Batyrev [2]. The trial to relate the asymptotic behaviour of a fibre integral with the Hodge structure of the fibre variety goes back to [15] where Varchenko established the equivalence of the asymptotic Hodge structure and the mixed Hodge structure in the sense of Deligne-Steenbrink for the case of plane curves and (semi-)quasihomogneous singularities.

Later on, several authors ([1], [6], [10],[11],[12]) have pursued studies on the the asymptotic behaviour of fibre integrals in making use of the Mellin transforms. Their main idea consists in the fact, that it is possible to visualize the asymptotic behaviour (i.e. the filtration) of fibre integrals by means of the poles of Mellin transform. Especially in the case of complete intersection singularities, the advantage of this method is quite clear. Let us remark also that not only poles of the Mellin transform but its zeros play role in the calculus of the global monodromy of the fibre integrals (e.g. see Proposition 5.3). The relation between the poles of the Mellin transform and the mixed Hodge structure has been explained for examples of isolated complete intersections of space curve type in [13].

In this note, we illustrate the clarity of this approach in taking the example of a hypersurface in a torus defined by so called simpliciable polynomial (see Definition 2). It serves as an introduction to the author's main paper in preparation [14] where he establishes similar results for the fibre integrals of the "simpliciable" complete intersection singularities.

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1 Hodge structure of the cohomology group of a hypersurface in a torus

In this section we review fundamental notions on the Hodge structure of the cohomology group of a hypersurface in a torus after [2], [3].

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Let Δ be a convex $n-$dimensional convex polyhedron in $\mathbf{R}^n$ with all vertices in $\mathbf{Z}^n$. Let us define a ring $S_\Delta \subset \mathbf{C}[x_1^{\pm}, \cdots, x_n^{\pm}]$ of the Laurent polynomial ring as follows:

$$S_\Delta := \mathbf{C} \cup \bigcup_{\frac{\vec{\alpha}}{k} \in \Delta, \exists k \geq 1} \mathbf{C} \cdot x^{\vec{\alpha}}. \tag{1.1}$$

We denote by $\Delta(f)$ the convex hull of the set $\vec{\alpha} \in supp(f)$ and call it the Newton polyhedron of $f(x)$. We introduce the following Jacobi ideal:

$$J_{f,\Delta} = \langle x_1 \frac{\partial f}{\partial x_1}, \cdots, x_n \frac{\partial f}{\partial x_n} \rangle \cdot S_{\Delta(f)}. \tag{1.2}$$

Let τ be a $\ell-$dimensional face of $\Delta(f)$ and define

$$f^\tau(x) = \sum_{\vec{\alpha} \in \tau \cap supp(f)} a_{\vec{\alpha}} x^{\vec{\alpha}}, \tag{1.3}$$

where $f(x) = \sum_{\vec{\alpha} \in supp(f)} a_{\vec{\alpha}} x^{\vec{\alpha}}$. The Laurent polynomial $f(x)$ is called Δ regular, if $\Delta(f) = \Delta$ and for every $\ell-$dimensional face $\tau \subset \Delta(f)$ $(\ell > 0)$ the polynomial equations:

$$f^\tau(x) = x_1 \frac{\partial f^\tau}{\partial x_1} = \cdots = x_n \frac{\partial f^\tau}{\partial x_n} = 0,$$

have no common solutions in $\mathbf{T}^n = (\mathbf{C}^\times)^n$.

Proposition 1.1 *Let f be a Laurent polynomial such that $\Delta(f) = \Delta$. Then the following conditions are equivalent.*

(i) The elements $x_1 \frac{\partial f}{\partial x_1}, \cdots, x_n \frac{\partial f}{\partial x_n}$ gives rise to a regular sequence in $S_{\Delta(f)}$

(ii)

$$dim(\frac{S_\Delta}{J_{f,\Delta}}) = n! vol(\Delta).$$

(iii) f is $\Delta-$regular.

It is possible to introduce a filtration on S_Δ, namely $\vec{\alpha} \in S_k$ if and only if $\frac{\vec{\alpha}}{k} \in \Delta$. Consequently we have an increasing filtration;

$$\mathbf{C} \cong \{0\} = S_0 \subset S_1 \subset \cdots \subset S_n \subset \cdots,$$

that induces a filtration on $\frac{S_\Delta}{J_{f,\Delta}}$:

$$F^0(\frac{S_\Delta}{J_{f,\Delta}}) \subset F^1(\frac{S_\Delta}{J_{f,\Delta}}) \subset \cdots \subset F^n(\frac{S_\Delta}{J_{f,\Delta}}).$$

This is called the Hodge filtation of $\frac{S_\Delta}{J_{f,\Delta}}$. It is worthy to remark here that the Hodge filtration ends up with $n-$th term.

Let us remind us of the notion of Ehrhart polynomial:

Definition 1 *Let Δ be an $n-$dimensional convex polytope. Denote the Poincaré series of graded algebra S_Δ by*

$$P_\Delta(t) = \sum_{k \geq 0} \ell(k\Delta) t^k,$$

$$Q_\Delta(t) = \sum_{k \geq 0} \ell^*(k\Delta) t^k,$$

where $\ell(k\Delta)$ (resp.$\ell^*(k\Delta)$) represents the number of integer points in $k\Delta$. (resp. interior integer points in $k\Delta$.) Then

$$\Psi_\Delta(t) = \sum_{k=0}^{n} \psi_k(\Delta) t^k = (1-t)^{n+1} P_\Delta(t),$$

$$\Phi_\Delta(t) = \sum_{k=0}^{n} \varphi_k(\Delta) t^k = (1-t)^{n+1} Q_\Delta(t),$$

are called Erhart polynomials which satisfy

$$t^{n+1}\Psi_\Delta(t^{-1}) = \Phi_\Delta(t).$$

Further, the main object of our study will be the cohomology group of the hypersurface $Z_f := \{x \in \mathbf{T}^n; f(x) = 0\}$. We have an important isomorphism on the Hodge filtration of $PH^{n-1}(Z_f)$.

Theorem 1.2 *([2]) For the primitive part $PH^{n-1}(Z_f)$ of $H^{n-1}(Z_f)$, the following isomorphism holds;*

$$\frac{F^i PH^{n-1}(Z_f)}{F^{i+1}PH^{n-1}(Z_f)} \cong Gr_F^{n-i}\left(\frac{S_\Delta}{J_{f,\Delta}}\right) = \frac{F^i\left(\frac{S_\Delta}{J_{f,\Delta}}\right)}{F^{i+1}\left(\frac{S_\Delta}{J_{f,\Delta}}\right)}. \tag{1.4}$$

Furthermore

$$dim\ Gr_F^{n-i}\left(\frac{S_\Delta}{J_{f,\Delta}}\right) = \sum_{q \geq 0} h^{i,q}(PH^{n-1}(Z_f)) = \psi_{n-i}(\Delta),$$

for $i < n-1$ and dim $Gr_F^1\left(\frac{S_\Delta}{J_{f,\Delta}}\right) = \psi_1(\Delta) + n = \ell(\Delta) - 1$.

As for the weight filtration, we have the following characterization. We understand the notion of the stratum of the support of the algebra $\frac{S_\Delta}{J_{f,\Delta}}$ in identifying a polynomial $x^{\vec{\alpha}} \in S_\Delta$ with $\vec{\alpha} \in \mathbf{Z}^n$.

Theorem 1.3 *The weight filtration on $PH^{n-1}(Z_f)$ is defined as a decreasing filtration*

$$0 = W_{n-2} \subset W_{n-1} \subset \cdots \subset W_{2n-2} = PH^{n-1}(Z_f),$$

such that $W_{n+i-1} \cong \{$ the integer points located on the strata with dimension $\geq (n-i+1)$ of supp$\left(\frac{S_\Delta}{J_{f,\Delta}}\right)$ but not on the $(n-i)-$ dimensional stratum.$\}$ for $0 \leq i \leq n-2$. $W_{2n-2} \cong \{$ the integer points located on the strata with dimension ≥ 2 of supp$\left(\frac{S_\Delta}{J_{f,\Delta}}\right)$ and those on $1-$skeleton except one integer point $\}$.

2 Preliminary combinatorics

Let us consider a polynomial

$$f(x) = \sum_{1 \leq i \leq M} x^{\vec{\alpha}(i)} \tag{2.1}$$

with $M \geq N+1$. Here $\vec{\alpha}(i)$ denotes the multi-index

$$\vec{\alpha}(i) = (\alpha_1^i, \cdots, \alpha_N^i) \in \mathbf{Z}^N.$$

In the case when $M > N$ we associate to $f(x)$ another polynomial in $M-1$ variables $f^{\sigma}(x,x')$

$$f^{\sigma}(x,x') = \sum_{i=1}^{M-N-1} x'_i x_1^{\alpha(\sigma(i))} + \sum_{j=M-N}^{M} x_i^{\alpha(\sigma(j))} \tag{2.2}$$

with $\sigma \in \mathfrak{S}_M$, the permutation group of M elements. Here we used the notation of the multi-index:

$$\vec{\alpha}(\sigma(i)) = (\alpha_1^{\sigma(i)}, \cdots, \alpha_N^{\sigma(i)}) \in \mathbf{Z}^N.$$

In this situation, the expression $u(f^{\sigma}(x,x')+s)$ is a polynomial depending on $(M+1)$ variables $(x_1,\cdots,x_N,x'_1,\cdots,x'_{M-N-1},s,u)$. Further we shall assume

$$supp(f^{\sigma}) \cap int(\Delta(f^{\sigma})) = \emptyset. \tag{2.3}$$

Here $\Delta(f^{\sigma})$ denotes the Newton diagramme of $f^{\sigma}(x,x')$. Let us introduce new variables $T_1, \cdots T_{M+1}$:

$$T_1 = ux'_1 x_1^{\vec{\alpha}(\sigma(1))}, T_2 = ux'_2 x_2^{\vec{\alpha}(\sigma(2))}, \cdots \tag{2.4}$$

$$T_{M-N-1} = ux'_{M-N-2} x^{\vec{\alpha}(\sigma(M-N-1))}, T_{M-N} = ux^{\vec{\alpha}(\sigma(M-N))}, \cdots, T_{M+1} = us.$$

To express the situation in a compact form, we use the following notations:

$$\Xi :=^t (x_1,\cdots,x_N,x'_1,\cdots,x'_{M-N-1},u,s), \tag{2.5}$$

$$Log\ T :=^t (log\ T_1, \cdots, log\ T_{M+1}) =^t (\tau_1,\cdots,\tau_{M+1}), \tag{2.6}$$

$$Log\ \Xi :=^t (\log\ x_1, \cdots, \log\ x_N, \log\ x'_1, \cdots, \log\ x'_{M-N-1}, \log\ u,, \log\ s). \tag{2.7}$$

In making use of these notations , we have the relation

$$\tau_1 = log\ u + log\ x'_1 + < \vec{\alpha}(\sigma(1)), log\ x >, \cdots, \tag{2.8}$$

$$\tau_{M-N-1} = log\ u + log\ x'_{M-N-1} + < \vec{\alpha}(\sigma(M-N-1)), log\ x >,$$

$$\tau_{M-N} = log\ u + < \vec{\alpha}(\sigma(M-N)), log\ x >, \cdots, \tau_{M+1} = log\ u + log\ s.$$

We can rewrite the relation (2.8) with the aid of a matrix $\mathsf{L}^{\sigma} \in End(\mathbf{Z}^{M+1})$, as follows:

$$Log\ T = \mathsf{L}^{\sigma} \cdot Log\ X. \tag{2.9}$$

where

$$\mathsf{L}^{\sigma} = \begin{bmatrix} \alpha_1^{\sigma(1)} & \cdots & \alpha_N^{\sigma(1)} & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \alpha_1^{\sigma(2)} & \cdots & \alpha_N^{\sigma(2)} & 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ \vdots & \cdots & \vdots & 0 & 0 & 1 & \cdots & 0 & 0 & 1 \\ \alpha_1^{\sigma(M-N-1)} & \cdots & \alpha_N^{\sigma(M-N-1)} & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ \alpha_1^{\sigma(M-N)} & \cdots & \alpha_N^{\sigma(M-N)} & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \alpha_1^{\sigma(M)} & \cdots & \alpha_N^{\sigma(M)} & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}, \tag{2.10}$$

Further we shall assume that the determinant of the matrix L^σ is positive. This assumption is always satisfied without loss of generality, if we permute certain column vectors of the matrix, which evidently corresponds to the change of positions of variables x. We denote the determinant by $\gamma^\sigma = det(L^\sigma)$. The row vectors of L^σ will be denoted by $\vec{e}^\sigma_1, \cdots, \vec{e}^\sigma_{M+1}$. Later we will make use of the notation of variables $X := (X_1, \cdots X_{M-1}) := (x_1, \cdots, x_N, x'_1, \cdots, x'_{M-N-1})$ and that of the polynomial $f^\sigma(x, x') = f^\sigma(X)$.

Definition 2 *We call that polynomial $f(x)$ is simpliciable if for every $\sigma \in S_M$, $det(L^\sigma) = \gamma^\sigma \neq 0$.*

For $\tau \subset \Delta(f^\sigma)$ we denote by $\Sigma(\tau)$ a $(dim\ \tau + 1)-$ dimensional simplex consisting all segments connecting $\{0\}$ and a point of τ. Let us define a graded algebra

$$S_\tau := \bigcup_{\frac{\alpha}{k} \in \Sigma(\tau), \exists k \geq 1} \mathbf{C} X^\alpha. \tag{2.11}$$

and a polynomial

$$f^{\sigma,\tau}(X) := \sum_{\alpha \in supp(f^\sigma) \cap \tau} X^\alpha \tag{2.12}$$

Lemma 2.1 *If $f(x)$ is a simpliciable polynomial, then $f^\sigma(X)$ is $\Delta(f^\sigma)-$ regular.*

Proof *The condition $det(L^\sigma) = \gamma^\sigma \neq 0$ yields that $X_1 \frac{\partial f^{\sigma,\tau}}{\partial X_1}$, $X_2 \frac{\partial f^{\sigma,\tau}}{\partial X_2}$, $\cdots$ $X_{M-1} \frac{\partial f^{\sigma,\tau}}{\partial X_{M-1}}$, form a regular sequence in S_τ for any face $\tau \subset \Delta(f^\sigma)$.* **Q.E.D.**

3 Mellin transforms

In this section we proceed to the calculation of the Mellin transform of the fibre integrals associated to the hypersurface $Z_{f^\sigma+s} = \{X \in \mathbf{T}^{M-1}; f^\sigma(X)+s = 0\}$ defined by a simpliciable polynomial. First of all we consider the fibre integral taken along the fibre $\gamma(s) \in H_{M-2}(Z_{f^\sigma+s})$ as follows,

$$I^\sigma_{X^{\mathbf{J}},\partial\gamma}(s) := \int_{\gamma(s)} \frac{X^{\mathbf{J}-\mathbf{1}} dX}{df^\sigma(X)} = \int_{\partial\gamma(s)} \frac{X^{\mathbf{J}} dX}{(f^\sigma(X)+s)X^{\mathbf{1}}} \tag{3.1}$$

where $\partial\gamma(s) \in H_{M-1}(\mathbf{T}^{M-1} \setminus Z_{f^\sigma+s})$ is a cycle obtained after the application of ∂, Leray's coboundary operator. Here $X^{\mathbf{1}} = X_1 \cdots X_{M-1}$, $X^{\mathbf{J}} = X_1^{i_1} \cdots, X_{M-1}^{i_{M-1}}$. See [9] and [16] for the Leray's coboundary operator.

The Mellin transform of $I^\sigma_{X^{\mathbf{J}},\partial\gamma}(s)$ is defined by the following integral:

$$M^\sigma_{X^{\mathbf{J}}}(z) := \int_\Pi (-s)^z I^\sigma_{X^{\mathbf{J}},\partial\gamma}(s) \frac{ds}{s}. \tag{3.2}$$

Here Π stands for a cycle in $\mathbf{C}$ that avoids the poles of $I_{X^{\mathbf{J}},\partial\gamma}(s)$. We assume that on $\partial\gamma(s) \times \Pi$, $\Re(f^\sigma(X)+s) \to +\infty$. Let us deform the integral (3.2) in making use of the definition (3.1):

$$\begin{aligned} M^\sigma_{X^{\mathbf{J}}}(z) &= \int_{\partial\gamma \times \mathbf{R}_- \times \Pi} e^{u(f^\sigma(X)+s)} X^{\mathbf{J}} u(-s)^z \frac{dX}{X^{\mathbf{1}}} \wedge \frac{du}{u} \wedge \frac{ds}{s} \\ &= \frac{1}{\gamma^\sigma} \int_{\partial\gamma \times \mathbf{R}_- \times \Pi} e^{\Psi(T)} \prod_{q=1}^{M} T_q^{\mathcal{L}_q(\mathbf{J},z)} \prod_{q=1}^{M} \bigwedge \frac{dT_q}{T_q}. \end{aligned} \tag{3.3}$$

where

$$\Psi(T) = T_1(X,s,u) + \cdots + T_M(X,s,u) = u(f^\sigma(X)+s) \tag{3.4}$$

where each term $T_i(X,s,u)$ represents a monomial term of variables X,s,u of the polynomial (3.4). By virtue of the simple structure of the matrix L^σ (2.10), we can consider the simplex polyhedron $\tau^\sigma_q \in \mathbf{R}^{M-1}$ defined as $\langle \vec{e}^\sigma_1, \overset{q}{\vee}, \vec{e}^\sigma_{M+1}\rangle$. where we identify $\vec{e}^\sigma_i \in \mathbf{Z}^{M+1}$ with that of $\mathbf{Z}^{M-1}$ after ignoring the last two elements. It means that we identify $\vec{e}^\sigma_i$ with the $i-$th row vector of the matrix L^σ of which one removes the last two columns ${}^t(0,0,\cdots,0,1),{}^t(1,1,\cdots,1) \in \mathbf{Z}^{M+1}$. The chain $\partial\gamma \times \mathbf{R}_- \times \Pi$ can be deformed in $\mathbf{C}^M$ so far as it does not encounter the singularity of the integrand.

Proposition 3.1 *1) The Mellin transform $M^\sigma_{X\mathbf{J}}(z)$ of the fibre integral associated to the simpliciable polynomial $f^\sigma(X)$ has the following form.*

$$M^\sigma_{X\mathbf{J}}(z) = g(z)\prod_{q=1}^{M}\Gamma(\mathcal{L}_q(\mathbf{J},z)), 1 \leq q \leq M+1, \tag{3.5}$$

where $g(z)$ is a rational function in $e^{\frac{\pi i z}{\gamma^\sigma}}$ with $\gamma^\sigma = (M-1)!vol(\Delta(f^\sigma))$. The linear function

$$\mathcal{L}_q(\mathbf{J},z) = {}^t(\mathbf{J},z,1)\vec{w}^\sigma_q, \tag{3.6}$$

where $\vec{w}^\sigma_q$ is the $q-th$ column vector of the matrix $(\mathsf{L}^\sigma)^{-1}$.

2) The $M+1$ linear functions $\mathcal{L}_q(\mathbf{J},z)$ are classified into the following three groups.

$$\mathcal{L}_{M+1}(\mathbf{J},z) = \frac{\gamma^\sigma}{\gamma^\sigma}z = z. \tag*{$(3.7)_1$}$$

For q such that $\vec{w}^\sigma_q = B^\sigma_q(\vec{v}^\sigma_q,1,-1)$ for some $\vec{v}^\sigma_q \in \mathbf{Z}^{M-1}$,and $B^\sigma_q \neq 0$,

$$\mathcal{L}_q(\mathbf{J},z) = \frac{B^\sigma_q(< \vec{v}^\sigma_q, \mathbf{J} > +z-1)}{\gamma^\sigma},, \tag*{$(3.7)_2$}$$

For q such that $\vec{w}^\sigma_q = B^\sigma_q(\vec{v}^\sigma_q,0,0)$ for some $\vec{v}^\sigma_q \in \mathbf{Z}^{M-1}$, and $B^\sigma_q \neq 0$,

$$\mathcal{L}_q(\mathbf{J},z) = \frac{B^\sigma_q(< \vec{v}^\sigma_q, \mathbf{J} >)}{\gamma^\sigma}. \tag*{$(3.7)_3$}$$

Here the case $(3.7)_3$ corresponds to such q that dim $\tau^\sigma_q < M-1$.

3)

$$|B^\sigma_q| = (M-1)!vol(\tau^\sigma_q). \tag{3.8}$$

4) For $\mathbf{J} \in \tau^\sigma_q \cap \Delta(f^\sigma)$, with dim $\tau^\sigma_q = M-1$, $\tau^\sigma_q \neq \Delta(f^\sigma)$,

$$\langle \vec{v}^\sigma_q, \mathbf{J}\rangle = 1.$$

$$\langle \vec{v}^\sigma_{M+1}, \mathbf{J}\rangle = 0.$$

5) Suppose τ^σ_q has a face of $\Gamma(f^\sigma) := \{\vec{\alpha} \in \mathbf{R}^{M-1} : \vec{\alpha} \in (supp(f^\sigma) + \mathbf{R}^{\mathbf{M-1}}_+)\}$ then $B^\sigma_q > 0$. We will denote the set of such indices q with strictly positive B^σ_q by $I^+ \subset \{1,\cdots,M+1\}$.

6) We denote by $\chi(Z_{f^\sigma+1})$ the Euler-Poincaré characteristic of the hypersurface $Z_{f^\sigma+1} = \{X \in \mathbf{T}^{M-1}; f^\sigma(X)+1\}$ here under the constant 1 we understand a generic value for $f^\sigma(X)$. The following equality holds,

$$\sum_{q\in I^+} B^\sigma_q = (M-1)!vol_{M-1}\big(\Delta(f^\sigma(X)+1)\big) = (-1)^M\chi(Z_{f^\sigma+1}). \tag{3.9}$$

7) $\sum_{q=1}^{M+1} B^\sigma_q = 0$.

Proof 1) The definition of the $\Gamma-$ function sounds as follows;

$$\int_C e^{-T}T^\sigma \frac{dT}{T} = (1-e^{2\pi i\sigma})\Gamma(\sigma),$$

for the unique nontrivial cycle C turning around $T=0$ that begins and returns to $\Re T \to +\infty$. We apply it to the integral (3.3) and get (3.5). The points 2)- 5) are reduced to the linear algebra. For example 3) can be shown, if one remembers the definition of M minors of the matrix L^σ calculated in removing the $M-$th column. 4) If $\mathbf{J} \in \tau_q^\sigma$, the vector $\vec{e}_i^\sigma$ is orthogonal to $(\vec{v}_{M+1}^\sigma, 1, -1)$ for $i \neq q$ and $\langle \vec{e}_q^\sigma, B_q^\sigma(\vec{v}_{M+1}^\sigma, 1, -1)\rangle = \gamma^\sigma$. The result on the $M-$th and $(M+1)-$st element is explained by the fact that $\vec{e}_{M+1}^\sigma = (0,\cdots,0,1,1)$ is orthogonal to $(\vec{v}_{M+1}^\sigma, 1, -1)$ for $1 \leq q \leq M$. 6) Apply the Theorem by [4],[8] on the Euler characteristic. 7) The $(M+1)-$st column vector of L^σ is orthogonal to the $M-$th row vector of ${\mathsf{L}^\sigma}^{-1}$, $(B_1^\sigma, \cdots, B_{M+1}^\sigma)$. **Q.E.D.**

Let us introduce the set $\bar{q} \in I^- \subset \{1,\cdots,M+1\}$ defined as the set of such indices $\bar{q}$ that $B_{\bar{q}}^\sigma < 0$ and $q^0 \in I^0$ as $B_{q^0}^\sigma = 0$. then the above statement 7) entails the relation

$$\sum_{\bar{q}\in I^-} B_{\bar{q}}^\sigma = -(\sum_{q\in I^+} B_q^\sigma). \tag{3.10}$$

Corollary 3.2 *Under the above situation, the Mellin inverse of $M_{X^{\mathbf{J}},\gamma}^\sigma(s)$ with properly chosen periodic function $g(z)$ with period γ^σ :*

$$I_{X^{\mathbf{J}},\gamma'}^\sigma(s) = \int_{\check{\Pi}} g(z)\frac{\prod_{a\in I^+}\Gamma\big(\mathcal{L}_a(\mathbf{J},z)\big)}{\prod_{\bar{a}\in I^-}\Gamma\big(1-\mathcal{L}_{\bar{a}}(\mathbf{J},z)\big)} s^{-z}dz, \tag{3.11}$$

defines a convergent analytic function in $-\pi < arg\ s < \pi$.

Proof In applying the Stirling's formula

$$\Gamma(z+1) \sim (2\pi z)^{\frac{1}{2}} z^z e^{-z},\ \ \Re\, z \to +\infty,$$

to the integrand of (3.11), we take into account the relation (3.10). As for the choice of the rational function $g(z)$ one makes use of Nörlund's technique [7]. In this way we can choose such $g(z)$ that the integrand is of exponential decay on $\check{\Pi}$. **Q.E.D.**

Example Let us illustrate by a simple example the above procedures.

$$f(x) = x_1^5 + x_1^2x_2 + x_1x_2^2 + x_2^4. \tag{3.12}$$

We have 4 possibilities to add a new variable x_1' so that the polynomial (3.12) becomes a simplicial.

$$f^{\sigma_1}(x,x') = x_1'x_1^5 + x_1^2x_2 + x_1x_2^2 + x_2^4.$$
$$f^{\sigma_2}(x,x') = x_1^5 + x_1'x_1^2x_2 + x_1x_2^2 + x_2^4.$$
$$f^{\sigma_3}(x,x') = x_1^5 + x_1^2x_2 + x_1'x_1x_2^2 + x_2^4.$$
$$f^{\sigma_4}(x,x') = x_1^5 + x_1^2x_2 + x_1x_2^2 + x_1'x_2^4.$$

Let us calculate L^{σ_3} and $(\mathsf{L}^{\sigma_3})^{-1}$.

$$\mathsf{L}^{\sigma_3} = \begin{bmatrix} 5 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$(\mathsf{L}^{\sigma_3})^{-1} = \frac{1}{7}\begin{bmatrix} 3 & -4 & 0 & 1 & 0 \\ 2 & -5 & 0 & 3 & 0 \\ 1 & -6 & 7 & -2 & 0 \\ 8 & -20 & 0 & 5 & 7 \\ -8 & 20 & 0 & -5 & 0 \end{bmatrix}.$$

Let us denote by $\vec{e}_1 = (5,0,0)$,$\vec{e}_2 = (2,1,0)$,$\vec{e}_3 = (1,2,1)$, $\vec{e}_4 = (0,4,0)$,$\vec{e}_5 = (0,0,0)$. Then we have

$$vol(\tau_5) = 3! vol(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4) = 7.$$

Similarly $vol(\tau_4) = 5$, $vol(\tau_3) = 0$, $vol(\tau_2) = 20$, $vol(\tau_1) = 8$. Remark $\tau_1 + \tau_3 + \tau_4 + \tau_5 = \tau_2$ (a subdivision of simplex into three simplices) which yields $7 + 8 + 5 = 20$. The face not affected (see Definition 3 below) by σ_3 is that spanned by $\vec{e}_1, \vec{e}_2, \vec{e}_4$.

4 Hodge structure of the fibre integrals

Now we can state the relationship between the Hodge structure of the $PH^{M-2}(Z_{f^\sigma})$ and the poles of the Mellin transform after suitable period function multiplication $\frac{1}{g(z)}M^{\sigma}_{X^{\mathbf{J}},\gamma'}(z) = \frac{\prod_{a\in I^+}\Gamma\left(\mathcal{L}_a(\mathbf{J},z)\right)}{\prod_{\bar{a}\in I^-}\Gamma\left(1-\mathcal{L}_{\bar{a}}(\mathbf{J},z)\right)}$. We will misuse the expression "the poles of the Mellin transform" in meaning those of $\frac{\prod_{a\in I^+}\Gamma\left(\mathcal{L}_a(\mathbf{J},z)\right)}{\prod_{\bar{a}\in I^-}\Gamma\left(1-\mathcal{L}_{\bar{a}}(\mathbf{J},z)\right)}$.

Theorem 4.1 *1) For $X^{\mathbf{J}} \in Gr_F^p Gr^w_{M-2} PH^{M-2}(Z_{f^\sigma})$, $0 \le p \le M-1$, the following properties hold*

a)

$$M-2-p < \left\langle \vec{v}^{\sigma}_{q^0}, \mathbf{J}\right\rangle < M-1-p \;\; \textit{for at least one } q^0 \in I^+$$

$$0 < \left\langle \vec{v}^{\sigma}_{q}, \mathbf{J}\right\rangle < M-1-p \;\; \textit{for general } q \in I^+$$

$$0 < \left\langle \vec{v}^{\sigma}_{\bar{q}}, \mathbf{J}\right\rangle < M-p \;\; \textit{for } \bar{q} \in I^-$$

if not $\left\langle \vec{v}^{\sigma}_{q}, \mathbf{J}\right\rangle = 0$.

b) *The maximal pole of the Mellin transform satisfies;*

$$2+p-M < z < 3+p-M.$$

Here the pole is not necessarily a simple pole.

2) For $X^{\mathbf{J}} \in Gr_F^p Gr^w_{M-1} PH^{M-2}(Z_{f^\sigma})$, $0 \le p \le M-1$, the following properties hold

a) *There exists unique index $q \in I^+$ such that:*

$$\left\langle \vec{v}^{\sigma}_{q}, \mathbf{J}\right\rangle = M-1-p$$

b) *The maximal pole of the Mellin transform is the simple pole*

$$z = 2+p-M.$$

3) For $X^{\mathbf{J}} \in Gr_F^p Gr^w_{M-2+r} PH^{M-2}(Z_{f^\sigma})$, $1 \le r \le M-2$, $0 \le p \le M-1$, the following properties hold.

a) *There exist $r+1$ indices $q_1, \cdots q_{r+1}$ such that:*

$$\left\langle \vec{v}^{\sigma}_{q_1}, \mathbf{J}\right\rangle = \left\langle \vec{v}^{\sigma}_{q_2}, \mathbf{J}\right\rangle = \cdots = \left\langle \vec{v}^{\sigma}_{q_{r+1}}, \mathbf{J}\right\rangle = M-1-p$$

but no such $r+2$ pair of indices $q_1, \cdots, q_{r+2}$.

b) *The maximal pole of the Mellin transform satisfies;*

$$z = 2+p-M,$$

which is of order $r+1$.

Proof of the theorem can be achieved by a combination of Theorems 1.2, 1.3 and the Proposition 3.1, 1),2),4). We remember here that the $\Gamma(z)$ has simple poles at $z = 0, -1, -2, \cdots$.

The above theorem mentions about how the Hodge structure of $PH^{M-2}(Z_{f^\sigma})$ influences on the poles of the Mellin transform. How about the original Hodge structure $PH^{N-1}(Z_f)$? To state this relationship, we need to introduce the following notion.

Definition 3 *The face $\tau \in \Delta(f)$ is called "not affected by σ" $\in S_M$ if $\tau \in \Delta(f^\sigma)$ after the extension of $(i_1, \cdots, i_N) \in \tau \subset \mathbf{R}^N$ into $\mathbf{R}^M$ transforming it into the vector $(\mathbf{i}, 0) = (i_1, \cdots, i_N, 0, \cdots, 0, 0) \in \mathbf{R}^M$.*

The face not affected by σ for the polynomial (2.2) is a face (or its sub-face) spanned by the vertices

$$\sum_{j=M-N}^{M} x_i^{\alpha(\sigma(j))}$$

i.e. vertices free of x_i'.

Theorem 4.2 *1) For $x^{\mathbf{i}} \in Gr_F^p Gr_{N-1}^w PH^{N-1}(Z_f)$, $0 \le p \le N$, for which $(\mathbf{i}, 0)$ lies on a face τ not affected by σ, the following properties hold*

a)

$$N - p < \left\langle \vec{v}^\sigma_{q^0}, (\mathbf{i}, 0) \right\rangle < N - 1 - p \quad for\ q^0 \in I^+$$

$$0 < \left\langle \vec{v}^\sigma_{q}, (\mathbf{i}, 0) \right\rangle < N - 1 - p \quad for\ general\ q \in I^+$$

$$0 < \left\langle \vec{v}^\sigma_{\bar{q}}, (\mathbf{i}, 0) \right\rangle < N - p \quad for\ \bar{q} \in I^-$$

if not $\left\langle \vec{v}^\sigma_{q}, (\mathbf{i}, 0) \right\rangle = 0$.

b) *The maximal pole of the Mellin transform satisfies;*

$$-N + p < z < 1 - N + p.$$

Here the pole is not necessarily a simple pole.

The proof is based on the argument where one applies Theorem 4.1 to a face τ_q^σ not affected by σ containing the integer point$(\mathbf{i}, 0)$.

We remark the following simple combinatorial fact.

Proposition 4.3 *For every $x^{\mathbf{i}} \in Gr_F^p Gr_{N-1}^w PH^{N-1}(Z_f)$, there exists an element $\sigma \in S_M$ such that $x^{\mathbf{i}}$ is not affected by σ. That is to say there exists $\sigma \in S_M$ such that $x^{\mathbf{i}} \in S_{\Delta(f)} \cap S_{\Delta(f^\sigma)}$.*

5 Hypergeometric group associated to the fibre integrals

Let us introduce two differential operators of order $\Delta^\sigma := (M-1)!vol_{M-1}(\Delta(f^\sigma(X) + 1)) = |\chi(Z_{f^\sigma+1})| = |I^+| = |I^-|$;

$$P^\sigma_{\mathbf{J}}(\vartheta_s) = \prod_{q \in I^+} \prod_{j=0}^{B_q^\sigma - 1} \mathcal{L}_q(\mathbf{J}, -\vartheta_s + \frac{\gamma^\sigma j}{B_q^\sigma}) \tag{5.1}$$

$$Q^\sigma_{\mathbf{J}}(\vartheta_s) = \prod_{\bar{q} \in I^-} \prod_{j=1}^{-B_{\bar{q}}^\sigma} \big(1 - \mathcal{L}_{\bar{q}}(\mathbf{J}, -\vartheta_s - \gamma^\sigma(1 + \frac{j}{B_{\bar{q}}^\sigma}))\big), \tag{5.2}$$

where I^+, I^- are those sets of indices introduced in §3.

Theorem 5.1 *The fibre integral $I^{\sigma}_{X^{\mathbf{J}},\gamma}(s)$ is annihilated by the operator*

$$R^{\sigma}_{\mathbf{J}}(\vartheta_s) = P^{\sigma}_{\mathbf{J}}(\vartheta_s) - s^{\gamma^{\sigma}} Q^{\sigma}_{\mathbf{J}}(\vartheta_s), \tag{5.3}$$

that is to say

$$[P^{\sigma}_{\mathbf{J}}(\vartheta_s) - s^{\gamma^{\sigma}} Q^{\sigma}_{\mathbf{J}}(\vartheta_s)] I^{\sigma}_{X^{\mathbf{J}},\gamma}(s) = 0. \tag{5.4}$$

It is worthy to remark that the operator $R^{\sigma}_{\mathbf{J}}(\vartheta_s)$ is a push-forward of the Pochhammer hypergeometric operator of order Δ^{σ},

$$P^{\sigma}_{\mathbf{J}}(\gamma^{\sigma}\vartheta_t) - tQ^{\sigma}_{\mathbf{J}}(\gamma^{\sigma}\vartheta_t),$$

by the Kummer covering $t = s^{\gamma^{\sigma}}$.

We consider solutions $u_{\ell}(t), 1 \leq \ell \leq \Delta^{\sigma}$ to the equation

$$[P^{\sigma}_{\mathbf{J}}(\gamma^{\sigma}\vartheta_t) - tQ^{\sigma}_{\mathbf{J}}(\gamma^{\sigma}\vartheta_t)] u_{\ell}(t) = 0, \tag{5.5}$$

with the asymptotic behaviour

$$u_{\ell}(t) \cong t^{\rho^{\ell}_{\mathbf{J}}} \ (log\ t)^{\nu} A_{\ell}(t).$$

Here $0 \leq \nu \leq M-1$, $A_{\ell}(t)$ holomorphic in the neighbourhood of $t=0$. Similarly, we consider the asymptotic behaviour at $t = \infty$ of the solutions to (5.5)

$$v_{\ell}(t) \cong (\frac{1}{t})^{\bar{\rho}^{\ell}_{\mathbf{J}}} \ (log\ t)^{\mu} B_{\ell}(\frac{1}{t}).$$

Here $0 \leq \mu \leq M-1$, $B_{\ell}(\frac{1}{t})$ holomorphic in the neighbourhood of $\frac{1}{t} = 0$.

Under this situation, we define characteristic polynomials of the exponents of solutions to (5.5) at $t=0$

$$X_{0,\mathbf{J}}(t) = \prod_{\ell=1}^{\Delta^{\sigma}} (t - e^{2\pi \rho^{\ell}_{\mathbf{J}} \sqrt{-1}}),$$

and $t = \infty$

$$X_{\infty,\mathbf{J}}(t) = \prod_{\ell=1}^{\Delta^{\sigma}} (t - e^{2\pi \bar{\rho}^{\ell}_{\mathbf{J}} \sqrt{-1}}).$$

Corollary 5.2 *The characteristic polynomials defined above can be calculated in the following way.*

$$X_{0,\mathbf{J}}(t) = \prod_{q \in I^+} (t^{B^{\sigma}_q} - e^{-2\pi(1-\langle \vec{v}^{\sigma}_q, \mathbf{J}\rangle)\frac{B^{\sigma}_q}{\gamma^{\sigma}}\sqrt{-1}}), \tag{5.6}$$

$$X_{\infty,\mathbf{J}}(t) = \prod_{\bar{q} \in I^-} (t^{-B^{\sigma}_{\bar{q}}} - e^{-2\pi(1-\langle \vec{v}^{\sigma}_{\bar{q}}, \mathbf{J}\rangle)\frac{B^{\sigma}_{\bar{q}}}{\gamma^{\sigma}}\sqrt{-1}}).$$

We introduce two vectors $(A_1, A_2, \cdots, A_{\Delta^{\sigma}})$, $(B_1, B_2, \cdots, B_{\Delta^{\sigma}}) \in \mathbf{C}^{\Delta^{\sigma}}$, after the following relation:

$$X_{0,\mathbf{J}}(t) = t^{\Delta^{\sigma}} + A_1 t^{\Delta^{\sigma}-1} + A_2 t^{\Delta^{\sigma}-2} + \cdots + A_{\Delta^{\sigma}},$$

$$X_{\infty,\mathbf{J}}(t) = t^{\Delta^{\sigma}} + B_1 t^{\Delta^{\sigma}-1} + B_2 t^{\Delta^{\sigma}-2} + \cdots + B_{\Delta^{\sigma}}.$$

Let us denote by $\omega^i, i = 0, 1, 2, \cdots, \gamma^{\sigma} - 1$ the non-zero singular points of the equation (5.5) i.e. $\{s \in \mathbf{C}; \prod_{q \in I^+} B^{\sigma}_q - (\prod_{\bar{q} \in I^-} B^{\sigma}_{\bar{q}}) s^{\gamma^{\sigma}} = 0\}$.

Proposition 5.3 *A representation of the hypergeometric group (global monodromy group) of the solutions to* (5.5) *is given by*

$$M_0 = h_0^{\gamma^\sigma}, M_{\omega^0} = h_1 = (h_0 h_\infty)^{-1}, M_\infty = h_\infty^{\gamma^\sigma}, M_{\omega^i} = h_\infty^{-i} h_1 h_\infty^i (i = 1, 2, \cdots, \gamma^\sigma - 1), \tag{5.7}$$

for the matrices

$$h_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & -A_{\Delta^\sigma} \\ 1 & 0 & \ddots & 0 & -A_{\Delta^\sigma-1} \\ 0 & 1 & \ddots & 0 & -A_{\Delta^\sigma-2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -A_1 \end{pmatrix}, \tag{5.8}$$

$$(h_\infty)^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -B_\Delta^\sigma \\ 1 & 0 & \ddots & 0 & -B_{\Delta^\sigma-1} \\ 0 & 1 & \ddots & 0 & -B_{\Delta^\sigma-2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -B_1 \end{pmatrix}.$$

where M_{ω^i} denotes the monodromy action around the point $\omega^i \in \mathbf{CP}^1_s$.

proof The monodromies of the solutions annihilated by $P^\sigma_{\mathbf{J}}(\gamma^\sigma \vartheta_t) - tQ^\sigma_{\mathbf{J}}(\gamma^\sigma \vartheta_t)$ are given by h_0, (resp. h_1, h_∞) after [5]. at $t = 0$, (resp.$t = \frac{\prod_{q \in I^+} B_q^\sigma}{\prod_{\bar{q} \in I^-} B_{\bar{q}}^\sigma}, \infty$). Let us think of a γ^σ−leaf covering $\tilde{\mathbf{CP}}^1_t$ of $\mathbf{CP}^1_s$ that corresponds to the Kummer covering $s^{\gamma^\sigma} = t$. In lifting up the path around $t = 1$ the first leaf of $\tilde{\mathbf{CP}}^1_s$, the monodromy h_1 is sent to the conjugation with a path around $t = \infty$. That is to say we have $M_{\omega^1} = h_\infty^{-1} h_1 h_\infty$. For other leaves the argument is similar. **Q.E.D.**

In combining the above result with that of Theorem 4.1, 3), we get the following.

Corollary 5.4 *For $X^{\mathbf{J}} \in Gr_F^p Gr^w_{M-2+r} PH^{M-2}(Z_{f^\sigma})$, $1 \le r \le M-2$, $0 \le p \le M-1$ the size of a Jordan cell of the monodromies M_0 with unit eigenvalue is $r+1$.*

It is enough to remember the following relation for a cycle avoiding $z + \alpha = 0$:

$$(r+1)! \int_C \frac{s^{-z}}{(z+\alpha)^{r+1}} dz = \int_C s^{-z} [(\frac{d}{dz})^r \frac{1}{(z+\alpha)}] dz$$

$$= \int_C \frac{1}{(z+\alpha)} [(-\frac{d}{dz})^r s^{-z}] dz = \int_C \frac{1}{(z+\alpha)} s^{-z} (log\ s)^r dz = 2\pi\sqrt{-1} s^\alpha (log\ s)^r.$$

References

[1] D.BARLET et H.M.MAIRE, *Asymptotiques des intégrales fibres,* Ann. Inst.Fourier **43**,no.5, (1993), 1267-1299.

[2] V.V.BATYREV, *Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori* , Duke Math.J. **69**,no.2, (1993), 349-409.

[3] V.DANILOV, A.G.KHOVANSKI, *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers* , Math.USSR Izv. **29** (1987),No.2, pp.279-298.

[4] A.G.KHOVANSKI, *Newton polyhedra and the genus of complete intersections* , Funkts. Anal. Appl. **12** (1977), No.2, pp.51-61.

[5] A.H.M.LEVELT, *Hypergeometric functions,* Indagationes Math.**23**(1961), pp.361-403.

[6] F.LOESER, C.SABBAH, *Equations aux différences finies et déterminants d'intégrales de fonctions multiformes* , Comment.Math.Helvetici **66** (1991), pp. 458- 503.

[7] I.NÖRLUND, *Hypergeometric functions,* Acta Math. **94**, (1955/56), pp.289-349.

[8] M.OKA, *On the topology of the Newton boundary II,* J.Math. Soc. Japan **32**, (1980), pp.65-92.

[9] F.PHAM, *Introduction à l'étude topologique des singularités de Landau,* Gauthier-Villars, 1967.

[10] C.SABBAH, *Proximité évanescente, I. La structure polaire d'un $\mathcal{D}$-module -,* Compositio Math. **62** (1987), pp. 283- 328.

[11] C.SABBAH, *Proximité évanescente, II. Equations fonctionelles pour plusieurs fonctions analytiques* , Compositio Math. **64** (1987), pp. 213- 241.

[12] C.SABBAH, *Lieu des pôles d'un système holonome d'équations aux différences finies* , Bull.Soc.Math.France **120** (1992), pp. 371- 396.

[13] S.TANABÉ, *Transformée de Mellin des intégrales- fibres associées aux singularités isolées d'intersection complète quasihomogènes* , Compositio Math. **130** (2002), pp.119-160.

[14] S.TANABÉ, *Transformée de Mellin des intégrales- fibres associées aux singularités isolées d'intersection complète II* , in preparation.

[15] A.N.VARCHENKO, *Asymptotic Hodge structure in the vanishing cohomology* , Math.USSR Izvestiya **18** (1982),No.3, pp. 471- 512.

[16] V.A.VASSILIEV, *Ramified integrals, singularities and Lacunas,*Kluwer Academic Publishers, Dordrecht, 1995.

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