TILTING BUNDLES ON SOME FANO VARIETIES VIA THE FROBENIUS MORPHISM

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Abstract. Let X be a smooth algebraic variety over an algebraically closed field k of characteristic $p > 0$, and $\mathsf{F}\colon X \to X$ the absolute Frobenius morphism. The goal of this paper is to compute the cohomology groups $\mathrm{H}^i(\mathcal{E}nd(\mathsf{F}_*\mathcal{O}_X))$ for some X. We first prove some facts about the Frobenius pushforward $\mathsf{F}_*\mathcal{O}_X$ for projective bundles over smooth bases and blowups of surfaces. We then give several applications of these results and show that for $i > 0$ the above cohomology groups vanish on some toric Fano threefolds and on del Pezzo surfaces.

1. Introduction

This paper started as an attempt to describe derived categories of coherent sheaves on some algebraic varieties. Let X be a smooth proper algebraic variety over an algebraically closed field k. The description in question can be stated as an equivalence of categories:

$$\Phi\colon\ \mathrm{D}^b(X) \simeq \mathrm{D}^b(A-\mathrm{mod}), \tag{1}$$

where $\mathrm{D}^b(X)$ is the bounded derived category of coherent sheaves on X, and $\mathrm{D}^b(A-\mathrm{mod})$ is the bounded derived category of finitely generated left modules over a finite-dimensional associative algebra A. Equivalences as above are called tilting equivalences and can be obtained by constructing the so-called tilting bundles on X (see Section 2.4 for definitions). Recent works on the quantization of algebraic varieties in positive characteristic ([2],[9]) suggest to look at reducing our variety X modulo a prime number to construct tilting bundles.

We work over an algebraically closed field k of characteristic $p > 0$. Let X be a smooth proper variety over k, and F the Frobenius morphism. Our main interest is in computing groups $\mathrm{Ext}^i(\mathsf{F}_*\mathcal{O}_X, \mathsf{F}_*\mathcal{O}_X) = \mathrm{H}^i(\mathcal{E}nd(\mathsf{F}_*\mathcal{O}_X))$ (or, more generally, $\mathrm{Ext}^i(\mathsf{F}_*\mathcal{L}, \mathsf{F}_*\mathcal{L})$ for a line bundle $\mathcal{L}$ on X). The main result of the first part of the paper is a short exact sequence for a $\mathbb{P}^1$-bundle over a smooth base that connects the bundle $\mathsf{F}_*\mathcal{O}_X$, where X is the total space of $\mathbb{P}^1$-bundle, with the bundle $\mathsf{F}_*\mathcal{O}_S$ on the base space S. We then give several applications of these sequences: in particular, we give a different proof of cohomology vanishing of the sheaf of differential operators on the flag variety $\mathbf{SL}_3/\mathbf{B}$ ([5]) not using representation theory of algebraic groups in positive characteristic. Further applications to the D-affinity of flag varieties in positive characteristic will be discussed in a subsequent paper ([17]). We show that the derived Beilinson–Bernstein equivalence established in ([2]) combined with the above vanishing allows to conclude that the bundle $\mathsf{F}_*\mathcal{O}_X$ is tilting, where X is the flag variety of a semisimple simply connected algebraic group in type either $\mathbf{A}_2$ or $\mathbf{B}_2$, and the prime number p is greater than h, the Coxeter number of the corresponding group. This was shown recently by different methods in a series of papers ([7], [10]). We also compute the bundle $\mathsf{F}_*\mathcal{O}_X$ for a number of smooth toric Fano threefolds and prove that this bundle is tilting in some cases. At the end we consider rational surfaces that are obtained by blowing up a number of points on $\mathbb{P}^2$ in general position (e.g., del Pezzo surfaces) and prove that $\mathrm{E}xt^i(\mathsf{F}_*\mathcal{O}_X, \mathsf{F}_*\mathcal{O}_X) = 0$ for $i > 0$.

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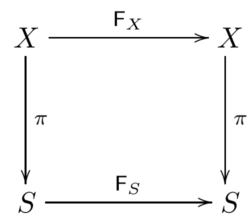
2. Preliminaries

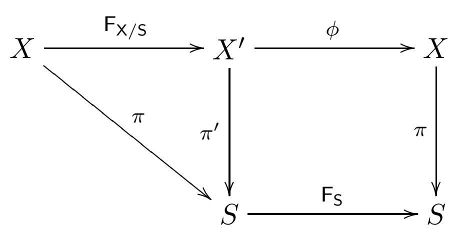
2.1. **The Frobenius morphism.** The material here is taken from ([8]). Let k be an algebraically closed field of characteristic $p > 0$, and X a scheme over k. The absolute Frobenius morphism F_X is an endomorphism of X that acts identically on the topological space of X and raises functions on X to the p-th power:

$$\mathsf{F}_X \colon X \to X, \qquad f \in \mathcal{O}_X \to f^p \in \mathcal{O}_X. \tag{2}$$

Let $\pi \colon X \to S$ be a morphism of k-schemes. Then there is a commutative square:



Denote X' the scheme $(S, \mathsf{F}_S) \times_S X$ obtained by the base change under F_S from X. The morphism F_X defines a unique S-morphism $\mathsf{F} = \mathsf{F}_{X/S} \colon X \to X'$, such that there is a commutative diagram:



The composition of upper arrows $\phi \circ \mathsf{F}$ is equal to F_X, and the square is cartesian. The morphism F is said to be the relative Frobenius morphism of X over S. The morphism F_X is not a morphism of S-schemes. On the contrary, the morphism $\mathsf{F}_{X/S}$ is a morphism of S-schemes.

Proposition 2.1. *Let S be a scheme over k, and $\pi \colon X \to S$ a smooth morphism of relative dimension n. Then the relative Frobenius morphism $\mathsf{F} \colon X \to X'$ is a finite flat morphism, and the $\mathcal{O}'_X$-algebra $\mathsf{F}_*\mathcal{O}_X$ is locally free of rank p^n.*

Let $\pi \colon X \to S$ be a smooth morphism as above, and $\mathcal{F}$ a coherent sheaf (a complex of coherent sheaves) on X. Proposition 2.1 implies:

$$\mathrm{R}^i\pi_*\mathsf{F}_{X*}\mathcal{F} = \mathsf{F}_{S*}\mathrm{R}^i\pi_*\mathcal{F}. \tag{3}$$

Indeed, it follows from the spectral sequence for the composition of two functors:

$$\mathrm{R}^i\pi_*\mathsf{F}_{X*}\mathcal{F} = \mathrm{R}^i(\pi\circ\mathsf{F}_X)_*\mathcal{F} = \mathrm{R}^i(\mathsf{F}_S\circ\pi)_*\mathcal{F} = \mathsf{F}_{S*}\mathrm{R}^i\pi_*\mathcal{F}. \tag{4}$$

Let $S = \mathrm{Spec}(k)$. In this case the schemes X and X' are isomorphic as abstract schemes (but not as k-schemes). By slightly abusing notations, we will skip the subscript at the absolute Frobenius morphism and denote it simply F, as the relative Frobenius morphism. More generally, for any $m\geq 1$ one defines m-th Frobenius twists $X^{(m)}$ and there is a morphism $\mathsf{F}_m\colon X\to X^{(m)}$, where $\mathsf{F}_m = \mathsf{F}\circ\dots\circ\mathsf{F}$ (m times). Let ω_X be the canonical invertible sheaf on X. Recall that the duality theory for finite flat morphisms ([6]) yields that a right adjoint functor $\mathsf{F}_m^!$ to F_{m*} is isomorphic to

$$\mathsf{F}_m^!(?) = \mathsf{F}_m^*(?)\otimes\omega_{X/X^{(m)}} = \mathsf{F}_m^*(?)\otimes\omega_X^{1-p^m}. \tag{5}$$

2.2. **Differential operators.** The material here is taken from ([4]) and ([5]). Let X be a smooth scheme over k. Consider the product $X\times X$ and the diagonal $\Delta\subset X\times X$. Let $\mathcal{J}_\Delta$ be the sheaf of ideals of Δ.

Definition 2.1. *An element $\phi\in\mathcal{E}nd_k(\mathcal{O}_X)$ is called a differential operator if there exists some integer $n\geq 0$ such that*

$$\mathcal{J}_\Delta^n\cdot\phi = 0. \tag{6}$$

One obtains a sheaf $\mathcal{D}_X$, the sheaf of differential operators on X. Denote $\mathcal{J}_\Delta^{(n)}$ the sheaf of ideals generated by elements a^n, where $a\in\mathcal{J}_\Delta$. There is a filtration on the sheaf $\mathcal{D}_X$ given by

$$\mathcal{D}_X^{(n)} = \{\phi\in\mathcal{E}nd_k(\mathcal{O}_X)\colon \mathcal{J}_\Delta^{(n)}\cdot\phi = 0\}. \tag{7}$$

Since k has characteristic p, one checks:

$$\mathcal{D}_X^{(p^n)} = \mathcal{E}nd_{\mathcal{O}_X^{p^n}}(\mathcal{O}_X). \tag{8}$$

Indeed, the sheaf $\mathcal{J}_\Delta$ is generated by elements $a\otimes 1 - 1\otimes a$, where $a\in\mathcal{J}_\Delta$, hence the sheaf $\mathcal{J}_\Delta^{(p^n)}$ is generated by elements $a^{p^n}\otimes 1 - 1\otimes a^{p^n}$. This implies (8). One also checks that this filtration exhausts the whole $\mathcal{D}_X$, so we can write:

$$\mathcal{D}_X = \bigcup_{n\geq 1}\mathcal{E}nd_{\mathcal{O}_X^{p^n}}(\mathcal{O}_X). \tag{9}$$

One can rewrite the isomorphism (8), using the Frobenius morphism:

$$\mathcal{D}_X^{(p^n)} = \mathcal{E}nd_{\mathcal{O}_X}(\mathsf{F}_{n*}\mathcal{O}_X). \tag{10}$$

Thus,

$$\mathcal{D}_X = \bigcup_{n\geq 1}\mathcal{E}nd_{\mathcal{O}_X}(\mathsf{F}_{n*}\mathcal{O}_X). \tag{11}$$

2.3. **Derived localization theorem.** We need to recall the derived Beilinson–Bernstein localization theorem ([2]). To this end, recall the definition of crystalline differential operators:

Definition 2.2. *Let $\mathcal{T}_X$ be the tangent sheaf to X. The sheaf D_X of crystalline differential operators (or PD-differential operators) is defined to be the enveloping algebra of the tangent Lie algebroid, that is it is generated by functions $f \in \mathcal{O}_X$ and vector fields $\partial \in \mathcal{T}_X$ with relations $[\partial, f] = \partial(f), \partial_1\partial_2 - \partial_2\partial_1 = [\partial_1, \partial_2]$.*

Let $\mathbf{G}$ be a semisimple algebraic group over k, $\mathbf{G}/\mathbf{B}$ the flag variety, and $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of the corresponding Lie algebra. Consider the category $\mathrm{D}_{\mathbf{G}/\mathbf{B}}$-mod of coherent $\mathrm{D}_{\mathbf{G}/\mathbf{B}}$-modules and the category $\mathcal{U}(\mathfrak{g})_0$-mod of finitely generated modules over $\mathcal{U}(\mathfrak{g})$ with the trivial action of the Harish–Chandra part of the center of $\mathcal{U}(\mathfrak{g})$ ([2]). The derived localization theorem (*loc.cit.*) states:

Theorem 2.1. *Let char $k = p > h$, where h is the Coxeter number of the group $\mathbf{G}$. Then there is an equivalence of derived categories:*

$$\mathrm{D}^b(\mathrm{D}_{\mathbf{G}/\mathbf{B}} - \mathrm{mod}) \simeq \mathrm{D}^b(\mathcal{U}(\mathfrak{g})_0 - \mathrm{mod}), \tag{12}$$

2.4. **Derived categories of coherent sheaves.** In this section we recall some facts about semiorthogonal decompositions in derived categories of coherent sheaves and tilting equivalences. We refer the reader to [14] for the definition of semiorthogonal decompositions in derived categories.

2.4.1. *Semiorthogonal decompositions.* The results of this section are taken from [14]. Let S be a smooth scheme over an algebraically closed field K, and $\mathcal{E}$ a vector bundle of rank n over S. Denote $X = \mathbb{P}_S(\mathcal{E})$ the projectivization of the bundle $\mathcal{E}$. Let $\pi \colon X \to S$ be the projection, and $\mathcal{O}_\pi(-1)$ the relative invertible sheaf on X. Finally, for a smooth scheme S denote $\mathrm{D}^b(S)$ the bounded derived category of coherent sheaves on S.

Theorem 2.2. *The category $\mathrm{D}^b(X)$ admits a semiorthogonal decomposition:*

$$\mathrm{D}^b(X) = \langle \pi^* \mathrm{D}^b(S) \otimes \mathcal{O}_\pi(-n+1), \pi^* \mathrm{D}^b(S) \otimes \mathcal{O}_\pi(-n+2), \dots, \pi^* \mathrm{D}^b(S) \rangle. \tag{13}$$

Further, we need a particular case of another Orlov's theorem (*loc.cit.*) Consider a smooth scheme X and a closed smooth subscheme $i \colon Y \subset X$ of codimension two. Let $\tilde{X}$ be the blowup of X along Y. There is a cartesian square:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\;j\;} & \tilde{X} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\;i\;} & X \end{array}$$

Here $\tilde{Y}$ is the exceptional divisor. If $\mathcal{N}_{Y/X}$ is normal bundle to Y in X then the projection p is the projectivization of the bundle $\mathcal{N}_{Y/X}$. Denote $\mathcal{O}_p(-1)$ be the relative invertible sheaf with respect to p.

Theorem 2.3. *The category* $\mathrm{D}^b(\tilde{X})$ *admits a semiorthogonal decomposition:*

$$\mathrm{D}^b(\tilde{X}) = \langle j_*(p^*\,\mathrm{D}^b(X)\otimes\mathcal{O}_p(-1)), \pi^*\,\mathrm{D}^b(X)\rangle. \tag{14}$$

2.4.2. *Tilting equivalences.*

Definition 2.3. *A coherent sheaf* $\mathcal{E}$ *on* X *is called a* tilting generator *of the bounded derived category* $\mathcal{D}^b(X)$ *of coherent sheaves on* X *if the following holds:*

(1) *The sheaf* $\mathcal{E}$ *is a tilting object in* $\mathcal{D}^b(X)$ *– that is, for any* $i \geq 1$ *we have* $\mathrm{Ext}^i(\mathcal{E},\mathcal{E}) = 0$
(2) *The sheaf* $\mathcal{E}$ *generates the derived category* $\mathcal{D}^b(X)$ *of complexes bounded from above – that is, if for some object* $\mathcal{F}\in\mathcal{D}^b(X)$ *we have* $\mathrm{R}Hom^\bullet(\mathcal{F},\mathcal{E}) = 0$*, then* $\mathcal{F} = 0$.

Tilting sheaves are a tool to construct derived equivalences. One has:

Lemma 2.1. *Let* X *be a smooth scheme,* $\mathcal{E}$ *a tilting generator of the derived category* $\mathcal{D}^b(X)$*, and denote* $R = \mathcal{E}nd(\mathcal{E})$*. Then the algebra* R *is left-Noetherian, and the correspondence* $\mathcal{F}\mapsto \mathrm{R}Hom^\bullet(\mathcal{E},\mathcal{F})$ *extends to an equivalence*

$$\mathcal{D}^b(X) \to \mathcal{D}^b(R\text{-mod}^{\mathrm{fg}}) \tag{15}$$

between the bounded derived category $\mathcal{D}^b(X)$ *of coherent sheaves on* X *and the bounded derived category* $\mathcal{D}^b(R\text{-mod}^{\mathrm{fg}})$ *of finitely generated left* R*-modules.*

The derived Beilinson–Bernstein equivalence (Theorem 2.1) implies the following:

Lemma 2.2. *Let* $\mathbf{G}$ *be a semisimple algebraic group over* k*, and* $X = \mathbf{G}/\mathbf{B}$ *the flag variety. Let char* $k = p > h$*, where* h *is the Coxeter number of the group* $\mathbf{G}$*. Then the bundle* $\mathsf{F}_*\mathcal{O}_X$ *satisfies the condition (2) of Definition 2.3.*

Proof. We need to show that for an object $\mathcal{F}\in\mathcal{D}^b(X)$ the equality $\mathrm{R}Hom^\bullet(\mathcal{F},\mathsf{F}_*\mathcal{O}_X) = 0$ implies $\mathcal{F} = 0$. By adjunction we get:

$$\mathbb{H}^\bullet(X,\mathsf{F}^*\mathcal{F}) = 0. \tag{16}$$

The object $\mathsf{F}^*\mathcal{F}$ is an object of the category $\mathrm{D}^b(\mathrm{D}_X-\mathrm{mod})$ (in fact, $\mathsf{F}^*\mathcal{F}$ is an object of the category $\mathrm{D}^b(\mathcal{E}nd(\mathsf{F}_*\mathcal{O}_X)-\mathrm{mod})$ – this is the so-called the Cartier descent, [12]). Now $\mathsf{F}^*\mathcal{F}$ is annihilated by the functor $\mathrm{R}\Gamma$. Under the assumption on the prime number p, this functor is an equivalence of categories by Theorem 2.1. Hence, $\mathsf{F}^*\mathcal{F} = 0$, and $\mathcal{F} = 0$ as well, q.e.d. □

3. A few lemmas

3.1. E*xt*-**groups.** Recall that for a variety X the m-th Frobenius twist of X is denoted $X^{(m)}$. One has a morphism of k-schemes $\mathsf{F}_m : X \to X^{(m)}$.

Let $\pi : Y \to X^{(m)}$ be an arbitrary morphism. Consider the cartesian square:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{p_2} & Y \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\mathsf{F}_m} & X^{(m)} \end{array}$$

Lemma 3.1. *The fibered product $\tilde{Y}$ is isomorphic to the left uppermost corner in the cartesian square:*

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & \Delta^{(m)} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_{\Delta^{(m)}}} \\ X \times Y & \xrightarrow{\mathsf{F}_m \times \pi} & X^{(m)} \times X^{(m)} \end{array}$$

where $\Delta^{(m)}$ is the diagonal in $X^{(m)} \times X^{(m)}$. If π is flat then one has an isomorphism of sheaves $i_*\mathcal{O}_{\tilde{Y}} = (\mathsf{F}_m \times \pi)^*(i_{\Delta^{(m)}}{}_*\mathcal{O}_{\Delta^{(m)}})$.

Proof. The isomorphism of two fibered products follows from the definition of fibered product. The isomorphism of sheaves follows from flatness of the Frobenius morphism and from flat base change. □

Definition 3.1. *Let $Y^{(m)} \overset{i}{\hookrightarrow} X^{(m)}$ be a closed subscheme. The fibered product $Y^{(m)} \times_{X^{(m)}} X$ as defined in the diagram:*

$$\begin{array}{ccc} Y' \times_{X^{(m)}} X & \longrightarrow & Y^{(m)} \\ \downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{\mathsf{F}_m} & X^{(m)} \end{array}$$

is called the m-th Frobenius neighbourhood of the subscheme $Y^{(m)}$ in X.

Consider the cartesian square:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi_2} & X \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \mathsf{F}_m} \\ X & \xrightarrow{\mathsf{F}_m} & X^{(m)} \end{array}$$

Lemma 3.2. *The fibered product $\tilde{X}$ is isomorphic to the m-th Frobenius neighbourhood of the diagonal $\Delta' \subset X' \times X'$.*

Proof. Apply Lemma 3.1 to $Y = X$ and $\pi = \mathsf{F}_m$. □

Recall that a right adjoint functor $\mathsf{F}^!_m(?)$ to $\mathsf{F}_{m*}(?)$ is isomorphic to $\mathsf{F}^*_m(?) \otimes \omega_X^{1-p^m}$. We get:

$$\mathrm{E}xt^k_X(\mathsf{F}_{m*}\mathcal{O}_X, \mathsf{F}_{m*}\mathcal{O}_X) = \mathrm{E}xt^k(\mathcal{O}_X, \mathsf{F}^!_m\mathsf{F}_{m*}\mathcal{O}_X) =$$
$$= \mathrm{E}xt^k(\mathcal{O}_X, \mathsf{F}^*_m\mathsf{F}_{m*}\mathcal{O}_X \otimes \omega_X^{1-p^m}) = \mathrm{H}^k(X, \mathsf{F}^*_m\mathsf{F}_{m*}\mathcal{O}_X \otimes \omega_X^{1-p^m}). \tag{17}$$

Lemma 3.3. *There is an isomorphism of cohomology groups:*

$$\mathrm{H}^k(X, \mathsf{F}^*_m\mathsf{F}_{m*}(\mathcal{O}_X) \otimes \omega_X^{1-p^m}) =$$
$$= \mathrm{H}^k(X \times X, (\mathsf{F}_m \times \mathsf{F}_m)^*({i_{\Delta^{(m)}}}_*\mathcal{O}_{\Delta^{(m)}}) \otimes ({\omega_X}^{1-p^m} \boxtimes \mathcal{O}_X)). \tag{18}$$

Proof. This lemma was proved in [15] (Lemma 2.3) for $m = 1$. For convenience of the reader, let us reproduce the proof. Consider the above cartesian square. By flat base change one gets an isomorphism of functors, the Frobenius morphism F_m being flat:

$$\mathsf{F}^*_m\mathsf{F}_{m*} = \pi_{1*}{\pi_2}^*. \tag{19}$$

Note that all the functors $\mathsf{F}_{m*}, \mathsf{F}^*_m, \pi_{1*}, {\pi_2}^*$ are exact, the Frobenius morphism F_m being affine. The isomorphism (19) implies an isomorphism of cohomology groups

$$\mathrm{H}^k(X, \mathsf{F}^*_m\mathsf{F}_{m*}(\mathcal{O}_X) \otimes \omega_X^{1-p^m}) = \mathrm{H}^k(X, \pi_{1*}{\pi_2}^*(\mathcal{O}_X) \otimes \omega_X^{1-p^m}). \tag{20}$$

By projection formula the right-hand side group in (20) is isomorphic to $\mathrm{H}^k(\tilde{X}, {\pi_2}^*\mathcal{O}_X \otimes {\pi_1}^*\omega_X^{1-p^m})$. Let p_1 and p_2 be the projections of $X \times X$ onto the first and the second component respectively, and let $\tilde{i}$ be the embedding $\tilde{X} \hookrightarrow X \times X$. One sees that $\pi_1 = p_1 \circ \tilde{i}$, $\pi_2 = p_2 \circ \tilde{i}$. Hence an isomorphism of sheaves

$${\pi_2}^*\mathcal{O}_X \otimes {\pi_1}^*\omega_X^{1-p^m} = \tilde{i}^*(p_2^*\mathcal{O}_X \otimes p_1^*\omega_X^{1-p^m}) = \tilde{i}^*(\omega_X^{1-p^m} \boxtimes \mathcal{O}_X). \tag{21}$$

From these isomorphisms and from the projection formula one gets

$$\mathrm{H}^k(\tilde{X}, {\pi_2}^*\mathcal{O}_X \otimes {\pi_1}^*\omega_X^{1-p^m}) = \mathrm{H}^k(\tilde{X}, \tilde{i}^*(\omega_X^{1-p^m} \boxtimes \mathcal{O}_X)) =$$
$$= \mathrm{H}^k(X \times X, \tilde{i}_*\mathcal{O}_{\tilde{X}} \otimes (\omega_X^{1-p^m} \boxtimes \mathcal{O}_X)). \tag{22}$$

By Lemma 3.2 the subscheme $\tilde{X}$ is isomorphic to the m-th Frobenius neighbourhood of the diagonal $\Delta^{(m)}$ in $X \times X$; thus

$$\tilde{i}_*\mathcal{O}_{\tilde{X}} = (\mathsf{F}_m \times \mathsf{F}_m)^*({i_{\Delta^{(m)}}}_*\mathcal{O}_{\Delta^{(m)}}). \tag{23}$$

Applying Lemma 3.1 to $\pi = \mathsf{F}_m$ finishes the proof. □

Corollary 3.1. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two vector bundles on X. There is an isomorphism of cohomology groups:*

$$\mathrm{E}xt^k(\mathsf{F}_*\mathcal{E}_1, \mathsf{F}_*\mathcal{E}_2) = \mathrm{H}^k(X, \mathsf{F}^!_m\mathsf{F}_{m*}(\mathcal{E}_2) \otimes \mathcal{E}_1^*) =$$
$$= \mathrm{H}^k(X \times X, (\mathsf{F}_m \times \mathsf{F}_m)^*({i_{\Delta^{(m)}}}_*\mathcal{O}_{\Delta^m}) \otimes ((\mathcal{E}_1^* \otimes \omega_X^{1-p^m}) \boxtimes \mathcal{E}_2)) =$$
$$= \mathrm{H}^k(X \times X, (\mathsf{F}_m \times \mathsf{F}_m)^*({i_{\Delta^{(m)}}}_*\mathcal{O}_{\Delta^m}) \otimes (\mathcal{E}_2 \boxtimes (\mathcal{E}_1^* \otimes \omega_X^{1-p^m})). \tag{24}$$

In particular,

$$\mathrm{H}^k(X, \mathsf{F}_m^*\mathsf{F}_{m*}(\mathcal{O}_X)\otimes\omega_X^{1-p^m}) =$$

$$= \mathrm{H}^k(X\times X, (\mathsf{F}_m\times\mathsf{F}_m)^*({i_{\Delta^{(m)}}}_*\mathcal{O}_{\Delta^m})\otimes(\mathcal{O}_X\boxtimes{\omega_X}^{1-p^m})). \tag{25}$$

Proof. Flat base change implies an isomorphism of functors

$$\mathsf{F}_m^*\mathsf{F}_{m*} = \pi_{2*}{\pi_1}^*. \tag{26}$$

Repeating verbatim the proof of Lemma 3.3 one gets the statement. □

3.2. $\mathbb{P}^1$ **- bundles.** Assume given a smooth variety S and a locally free sheaf $\mathcal{E}$ on S. Let $X = \mathbb{P}_S(\mathcal{E})$ be the projectivization of the bundle $\mathcal{E}$ and $\pi : X \to S$ the projection. Denote $\mathcal{O}_\pi(-1)$ the relative invertible sheaf.

Lemma 3.4. *There is a short exact sequence of vector bundles on X:*

$$0 \to \pi^*\mathsf{F}_*\mathcal{O}_S \to \mathsf{F}_*\mathcal{O}_X \to \pi^*(\mathsf{F}_*(\mathsf{S}^{p-2}\mathcal{E}\otimes\det\,\mathcal{E}))\otimes\det\,(\mathcal{E}^*))\otimes\mathcal{O}_\pi(-1)\to 0. \tag{27}$$

Proof. By Theorem 2.2, the category $\mathrm{D}^b(X)$ has a semiorthogonal decomposition:

$$\mathrm{D}^b(X) = \langle \mathrm{D}_{-1}, \mathrm{D}_0\rangle, \tag{28}$$

where D_i for $i = 0, -1$ is a full subcategory of $\mathrm{D}^b(X)$ that consists of objects $\pi^*(\mathcal{F})\otimes\mathcal{O}_\pi(i)$, for $\mathcal{F}\in\mathrm{D}^b(S)$. Decomposition (28) means that for any object $A\in\mathrm{D}^b(X)$ there is a distinguished triangle:

$$\dots\to\pi^*\mathrm{R}^{\cdot}\pi_*A\to A\to\pi^*(\tilde{A})\otimes\mathcal{O}_\pi(-1)\to\pi^*\mathrm{R}^{\cdot}\pi_*A[1]\to\dots. \tag{29}$$

The object $\tilde{A}$ can be found by twisting the triangle (29) by $\mathcal{O}_\pi(-1)$ and applying the direct image π_* to the obtained triangle. Given that $\mathrm{R}^{\cdot}\pi_*\mathcal{O}_\pi(-1) = 0$, we obtain an isomorphism:

$$\mathrm{R}^{\cdot}\pi_*(A\otimes\mathcal{O}_\pi(-1)) \simeq \tilde{A}\otimes\mathrm{R}^{\cdot}\pi_*\mathcal{O}_\pi(-2). \tag{30}$$

Now $\mathrm{R}^{\cdot}\pi_*\mathcal{O}_\pi(-2) = \det(\mathcal{E})[-1]$. Twisting both sides of the isomorphism (30) by $\det(\mathcal{E}^*)$, we get:

$$\tilde{A} = \mathrm{R}^{\cdot}\pi_*(A\otimes\mathcal{O}_\pi(-1))\otimes\det\,\mathcal{E}^*[1]. \tag{31}$$

Let now A be the vector bundle $\mathsf{F}_*\mathcal{O}_X$. The triangle (29) becomes in this case:

$$\dots\to\pi^*\mathrm{R}^{\cdot}\pi_*\mathsf{F}_*\mathcal{O}_X\to\mathsf{F}_*\mathcal{O}_X\to\pi^*(\tilde{A})\otimes\mathcal{O}_\pi(-1)\to\pi^*\mathrm{R}^{\cdot}\pi_*\mathsf{F}_*\mathcal{O}_X[1]\to\dots\quad. \tag{32}$$

where $\tilde{A} = \mathrm{R}^{\cdot}\pi_*(\mathsf{F}_*\mathcal{O}_X\otimes\mathcal{O}_\pi(-1))\otimes\det\,\mathcal{E}^*[1]$. Recall ((3), Subsection 2.1) that for a coherent sheaf $\mathcal{F}$ on X one has an isomorphism $\mathrm{R}^i\pi_*\mathsf{F}_*\mathcal{F} = \mathsf{F}_*\mathrm{R}^i\pi_*\mathcal{F}$. Therefore,

$$\mathrm{R}^{\cdot}\pi_*\mathsf{F}_*\mathcal{O}_X = \mathsf{F}_*\mathrm{R}^{\cdot}\pi_*\mathcal{O}_X = \mathsf{F}_*\mathcal{O}_S. \tag{33}$$

On the other hand, $\mathrm{R}^{\cdot}\pi_*(\mathsf{F}_*\mathcal{O}_X\otimes\mathcal{O}_\pi(-1)) = \mathrm{R}^{\cdot}\pi_*(\mathsf{F}_*\mathcal{O}_\pi(-p)) = \mathsf{F}_*\mathrm{R}^{\cdot}\pi_*\mathcal{O}_\pi(-p)$. The relative Serre duality for π gives an isomorphism:

$$\mathrm{R}^{\cdot}\pi_*\mathcal{O}_\pi(-p) = \mathsf{S}^{p-2}(\mathcal{E})\otimes\det\,(\mathcal{E})[-1]. \tag{34}$$

Let $\tilde{\mathcal{E}}$ be the vector bundle $\mathsf{S}^{p-2}(\mathcal{E})\otimes\det\,(\mathcal{E})$. Putting these isomorphisms together we see that the triangle (32) can be rewritten as follows:

$$\dots\to\pi^*\mathsf{F}_*\mathcal{O}_S\to\mathsf{F}_*\mathcal{O}_X\to\pi^*(\mathsf{F}_*(\tilde{\mathcal{E}})\otimes\det\,(\mathcal{E}^*))\otimes\mathcal{O}_\pi(-1)\xrightarrow{[1]}\dots, \tag{35}$$

and we see that the above distinguished triangle is in fact a short exact sequence of vector bundles on X:

$$0 \to \pi^*\mathsf{F}_*\mathcal{O}_S \to \mathsf{F}_*\mathcal{O}_X \to \pi^*(\mathsf{F}_*(\tilde{\mathcal{E}}) \otimes \det\,(\mathcal{E}^*)) \otimes \mathcal{O}_\pi(-1) \to 0. \tag{36}$$

□

Remark 3.1. Note that the sequence (36) could be obtained without resorting to the semiorthogonal decomposition (28). Indeed, as we have seen, one has a canonical morphism of locally free sheaves:

$$\pi^*\pi_*\mathsf{F}_*\mathcal{O}_X = \pi^*\mathsf{F}_*\mathcal{O}_S \to \mathsf{F}_*\mathcal{O}_X. \tag{37}$$

This morphism is an embedding of vector bundles (this can be established easily by a computation in local coordinates; it follows, in fact, that for any smooth morphism with connected rational fibers such a morphism is always an embedding of vector bundles). The cokernel of the canonical morphism is a locally free sheaf $\mathcal{F}$ such that $\mathrm{R}^{\cdot}\pi_*\mathcal{F} = 0$. Hence, $\mathcal{F} = \pi^*(\tilde{\mathcal{F}}) \otimes \mathcal{O}_\pi(-1)$ for some sheaf $\tilde{\mathcal{F}}$ on X. Indeed, using flat base change and the condition $\mathrm{R}^{\cdot}\pi_*\mathcal{F} = 0$ we see that the sheaf $\mathcal{F}$, up to the tensor product by pullback under π of a sheaf $\tilde{\mathcal{F}}$ on X, is isomorphic to $\mathcal{O}_\pi(-1)$. The explicit form of the sheaf $\tilde{\mathcal{F}}$ can be obtained by the same arguments as above.

Assume now that the vector bundle $\mathsf{F}_*\mathcal{O}_S$ is almost exceptional on S, that is

$$\mathrm{E}xt^i(\mathsf{F}_*\mathcal{O}_S, \mathsf{F}_*\mathcal{O}_S) = 0 \tag{38}$$

for $i > 0$. Applying the functor $\mathrm{H}om(?, \mathsf{F}_*\mathcal{O}_X)$ to the sequence (36), we get a long exact cohomology sequence:

$$\begin{aligned} 0 &\to \mathrm{H}om(\pi^*(\mathsf{F}_*(\tilde{\mathcal{E}}) \otimes \det\,(\mathcal{E}^*)) \otimes \mathcal{O}_\pi(-1), \mathsf{F}_*\mathcal{O}_X) \to \mathrm{H}om(\mathsf{F}_*\mathcal{O}_X, \mathsf{F}_*\mathcal{O}_X) \to \\ &\to \mathrm{H}om(\pi^*\mathsf{F}_*\mathcal{O}_S, \mathsf{F}_*\mathcal{O}_X) \to \mathrm{E}xt^1(\pi^*(\mathsf{F}_*(\tilde{\mathcal{E}}) \otimes \det\,(\mathcal{E}^*)) \otimes \mathcal{O}_\pi(-1), \mathsf{F}_*\mathcal{O}_X) \to \\ &\to \mathrm{E}xt^1(\mathsf{F}_*\mathcal{O}_X, \mathsf{F}_*\mathcal{O}_X) \to \mathrm{E}xt^1(\pi^*\mathsf{F}_*\mathcal{O}_S, \mathsf{F}_*\mathcal{O}_X) \to \dots \end{aligned} \tag{39}$$

From adjunction one has

$$\mathrm{E}xt^i(\pi^*\mathsf{F}_*\mathcal{O}_S, \mathsf{F}_*\mathcal{O}_X) = \mathrm{E}xt^i(\mathsf{F}_*\mathcal{O}_S, \mathsf{F}_*\mathcal{O}_S) = 0 \tag{40}$$

for $i > 0$ by our assumption. It follows from (39) that if the groups

$$\mathrm{E}xt^i(\pi^*(\mathsf{F}_*(\tilde{\mathcal{E}}) \otimes \det\,(\mathcal{E}^*)) \otimes \mathcal{O}_\pi(-1), \mathsf{F}_*\mathcal{O}_X), \tag{41}$$

are zero for $i > 0$ then

$$\mathrm{E}xt^i(\mathsf{F}_*\mathcal{O}_X, \mathsf{F}_*\mathcal{O}_X) = 0 \tag{42}$$

for $i > 0$. Using adjunction of functors once again, we obtain an isomorphism:

$$\mathrm{E}xt^i(\pi^*(\mathsf{F}_*(\tilde{\mathcal{E}}) \otimes \det\,(\mathcal{E}^*)) \otimes \mathcal{O}_\pi(-1), \mathsf{F}_*\mathcal{O}_X) = \mathrm{E}xt^i(\mathsf{F}_*(\tilde{\mathcal{E}}) \otimes \det(\mathcal{E}^*), \mathsf{F}_*\mathsf{S}^p\mathcal{E}^*). \tag{43}$$

This group can be computed using Corollary 3.1.

Remark 3.2. Twisting the short exact sequence (36) by $\mathcal{O}_\pi(1)$ and applying the direct image $\mathrm{R}^{\cdot}\pi_*$ to this sequence, we obtain:

$$0 \to \mathsf{F}_*\mathcal{O}_S \otimes \mathrm{R}^{\cdot}\pi_*\mathcal{O}_\pi(1) \to \mathrm{R}^{\cdot}\pi_*(\mathsf{F}_*\mathcal{O}_X \otimes \mathcal{O}_\pi(1)) \to \mathsf{F}_*(\tilde{\mathcal{E}}) \otimes \det\,(\mathcal{E}^*) \to 0 \tag{44}$$

or, rather

$$0 \to \mathsf{F}_*\mathcal{O}_S \otimes \mathcal{E}^* \to \mathsf{F}_*\mathsf{S}^p\mathcal{E}^* \to \mathsf{F}_*(\tilde{\mathcal{E}}) \otimes \det\,(\mathcal{E}^*) \to 0. \tag{45}$$

Twisting the above sequence by det $(\mathcal{E})$, we obtain:

$$0 \to \mathsf{F}_*\mathcal{O}_S \otimes \mathcal{E} \to \mathsf{F}_*\mathsf{S}^p\mathcal{E} \to \mathsf{F}_*\tilde{\mathcal{E}} \to 0. \tag{46}$$

For a rank two vector bundle there is a well-known short exact sequence (cf. [15]):

$$0 \to \mathsf{F}^*\mathcal{E} \to \mathsf{S}^p\mathcal{E} \to \tilde{\mathcal{E}} \to 0, \tag{47}$$

and we see that the sequence (46) is obtained by applying the functor F_* to the sequence (47).

Remark 3.3. Lemma 3.4 can be generalized for projective bundles of arbitrary rank. If $\mathcal{E}$ is a vector bundle of rank n over a scheme S and $X\colon = \mathbb{P}(\mathcal{E})$ is the projective bundle then there is a filtration on the bundle $\mathsf{F}_*\mathcal{O}_X$ with associated graded factors being of the form $\pi^*\mathcal{F}_i \otimes \mathcal{O}_\pi(-i)$, where $\mathcal{F}_i$ are some vector bundles over S, and $0 \leq i \leq n-1$. Let us work out an example of a vector bundle of rank 3.

Lemma 3.5. *Let $\mathcal{E}$ be a rank 3 vector bundle over a scheme S, and $X\colon = \mathbb{P}(\mathcal{E})$ the projective bundle. Then the bundle $\mathsf{F}_*\mathcal{O}_X$ has a three-step filtration with associated graded factors being:*

$$\pi^*\mathsf{F}_*\mathcal{O}_S,\ \pi^*\mathcal{G} \otimes \mathcal{O}_\pi(-1),\ \pi^*(\mathsf{F}_*(\mathsf{S}^{p-2}\mathcal{E} \otimes \det\,\mathcal{E}) \otimes \det\,\mathcal{E}^*) \otimes \mathcal{O}_\pi(-2), \tag{48}$$

where $\mathcal{G}$ is a vector bundle on S fitting in a short exact sequence:

$$0 \to \mathsf{F}_*\mathsf{F}^*\mathcal{E}^* \to \mathsf{F}_*\mathsf{S}^p\mathcal{E}^* \to \mathcal{G} \to 0. \tag{49}$$

Proof. Theorem 2.2 states that the category $\mathrm{D}^b(X)$ has a semiorthogonal decomposition of three pieces:

$$\mathrm{D}^b(X) = \langle \pi^*\,\mathrm{D}^b(S) \otimes \mathcal{O}_\pi(-2), \pi^*\,\mathrm{D}^b(S) \otimes \mathcal{O}_\pi(-1), \pi^*\,\mathrm{D}^b(S)\rangle. \tag{50}$$

This decomposition produces a distinguished triangle:

$$\cdots \to \pi^*\mathsf{F}_*\mathcal{O}_S \to \mathsf{F}_*\mathcal{O}_X \to \mathcal{A} \xrightarrow{[1]} \dots \quad , \tag{51}$$

where $\mathcal{A}$ is an object of $\mathrm{D}^b(X)$ that, in turn, fits into a distinguished triangle:

$$\dots \pi^*\mathcal{G} \otimes \mathcal{O}_\pi(-1) \to \mathcal{A} \to \pi^*\mathcal{F} \otimes \mathcal{O}_\pi(-2) \xrightarrow{[1]} \dots \quad . \tag{52}$$

Here $\mathcal{G}$ and $\mathcal{F}$ are objects of $\mathrm{D}^b(X)$. We can find these objects using the same arguments as above. Indeed, tensor the triangle (52) with $\mathcal{O}_\pi(-1)$ and apply the functor $\mathrm{R}^{\cdot}\pi_*$. We obtain an isomorphism:

$$\mathrm{R}^{\cdot}\pi_*(\mathcal{A} \otimes \mathcal{O}_\pi(-1)) = \mathcal{F} \otimes \mathrm{R}^{\cdot}\pi_*\mathcal{O}_\pi(-3). \tag{53}$$

Now $\mathrm{R}^{\cdot}\pi_*\mathcal{O}_\pi(-3) = \det\,\mathcal{E}[-2]$. Tensoring the triangle (51) with $\mathcal{O}_\pi(-1)$ and applying $\mathrm{R}^{\cdot}\pi_*$, we obtain an isomorphism:

$$\mathrm{R}^{\cdot}\pi_*(\mathcal{A} \otimes \mathcal{O}_\pi(-1)) = \mathrm{R}^{\cdot}\pi_*(\mathsf{F}_*\mathcal{O}_X \otimes \mathcal{O}_\pi(-1)) = \mathrm{R}^{\cdot}\pi_*(\mathsf{F}_*\mathcal{O}_\pi(-p)) = \mathsf{F}_*\mathrm{R}^{\cdot}\pi_*\mathcal{O}_\pi(-p), \tag{54}$$

and $\mathrm{R}^{\cdot}\pi_*\mathcal{O}_\pi(-p) = \mathsf{S}^{p-2}\mathcal{E} \otimes \det\mathcal{E}[-2]$ by the Serre duality. Hence,

$$\mathcal{F} = \mathsf{F}_*(\mathsf{S}^{p-2}\mathcal{E} \otimes \det\mathcal{E}) \otimes \det\mathcal{E}^*. \tag{55}$$

Similarly, twisting the triangle 52 with $\mathcal{O}_\pi(1)$ and applying $\mathrm{R}^{\cdot}\pi_*$ we obtain that the object $\mathcal{G}$ is in fact a vector bundle fitting in a short exact sequence:

$$0 \to \mathsf{F}_*\mathcal{O}_S \otimes \mathcal{E}^* \to \mathsf{F}_*\mathsf{S}^p\mathcal{E}^* \to \mathcal{G} \to 0. \tag{56}$$

This proves the statement. □

3.3. **Blowups of surfaces.** Let X be a smooth variety and Y its smooth subvariety of codimension two. Consider the blow-up of Y in X. Recall notations from Subsection 2.4.1: there is a cartesian diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\ j\ } & \tilde{X} \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\ i\ } & X \end{array}$$

Here $\tilde{Y}$ is the exceptional divisor. If $\mathcal{N}_{Y/X}$ is normal bundle to Y in X then the projection p is the projectivization of the bundle $\mathcal{N}_{Y/X}$. Let $\mathcal{O}_p(-1)$ be the relative invertible sheaf.

Lemma 3.6. *There is a short exact sequence:*

$$0 \to \pi^*\mathsf{F}_*\mathcal{O}_X \to \mathsf{F}_*\mathcal{O}_{\tilde{X}} \to j_*(\mathcal{O}_p(-1) \otimes p^*E) \to 0. \tag{57}$$

Here E is a coherent sheaf on Y which fits into a short exact sequence:

$$0 \to E \to i^*\pi_*\mathsf{F}_*\mathcal{O}_{\tilde{X}} \to \mathrm{R}^0 p_* j^*\mathsf{F}_*\mathcal{O}_{\tilde{X}} \to 0. \tag{58}$$

Proof. By Theorem 2.3, the category $\mathrm{D}^b(\tilde{X})$ admits a semiorthogonal decomposition:

$$\mathrm{D}^b(\tilde{X}) = \langle j_*(p^*\,\mathrm{D}^b(X) \otimes \mathcal{O}_p(-1)), \pi^*\,\mathrm{D}^b(X)\rangle. \tag{59}$$

This means that there is a distinguished triangle:

$$\cdots \to \pi^*\mathrm{R}^{\cdot}\pi_*\mathsf{F}_*\mathcal{O}_X \to \mathsf{F}_*\mathcal{O}_{\tilde{X}} \to j_*(\mathcal{O}_p(-1) \otimes p^*E) \xrightarrow{[1]} \dots \quad . \tag{60}$$

Consider the canonical morphism $\pi^*\mathrm{R}^{\cdot}\pi_*\mathsf{F}_*\mathcal{O}_{\tilde{X}} \to \mathsf{F}_*\mathcal{O}_{\tilde{X}}$. One has:

$$\mathrm{R}^{\cdot}\pi_*\mathsf{F}_*\mathcal{O}_{\tilde{X}} = \mathsf{F}_*\mathrm{R}^{\cdot}\pi_*\mathcal{O}_{\tilde{X}} = \mathsf{F}_*\pi_*\mathcal{O}_{\tilde{X}} = \mathsf{F}_*\mathcal{O}_X. \tag{61}$$

Indeed, $\mathrm{R}^{\cdot}\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$. The morphism $\pi^*\mathsf{F}_*\mathcal{O}_X \to \mathsf{F}_*\mathcal{O}_{\tilde{X}}$ is an injective map of coherent sheaves at the generic point of $\tilde{X}$. Therefore it is an embedding of coherent sheaves, the sheaves $\pi^*\mathsf{F}_*\mathcal{O}_X$ and $\mathsf{F}_*\mathcal{O}_{\tilde{X}}$ being locally free.

Taking sheaf cohomology $\mathcal{H}^{\cdot}$ of the sequence (60) we see that the object p^*E has cohomology only in degree zero, hence E is a coherent sheaf and the sequence (60) in fact becomes a short exact sequence (57). To determine the sheaf E apply the functor j^* to the sequence (57):

$$0 \to \mathsf{L}^1 j^* j_*(\mathcal{O}_p(-1) \otimes p^*E) \to j^*\pi^*\mathsf{F}_*\mathcal{O}_X \to j^*\mathsf{F}_*\mathcal{O}_{\tilde{X}} \to \mathcal{O}_p(-1) \otimes p^*E \to 0. \tag{62}$$

Recall that the normal bundle $\mathcal{N}_{\tilde{Y}/\tilde{X}}$ to $\tilde{Y}$ in $\tilde{X}$ is isomorphic to $\mathcal{O}_{\tilde{Y}}(\tilde{Y}) = \mathcal{O}_p(-1)$. Hence, the sheaf $\mathsf{L}^1 j^* j_*(\mathcal{O}_p(-1)\otimes p^*E)$ is isomorphic to $\mathcal{O}_p(-1)\otimes p^*E\otimes\mathcal{O}_{\tilde{Y}}(-\tilde{Y}) = p^*E$, and the sequence (62) becomes

$$0\to p^*E\to j^*\pi^*\mathsf{F}_*\mathcal{O}_X\to j^*\mathsf{F}_*\mathcal{O}_{\tilde{X}}\to\mathcal{O}_p(-1)\otimes p^*E\to 0. \tag{63}$$

Applying now to the sequence (63) the functor $\mathrm{R}^{\cdot}p_*$ and taking into account that $\mathrm{R}^{\cdot}p_*\mathcal{O}_p(-1)=0$ we get a short exact sequence

$$0\to E\to \mathrm{R}^0p_*j^*\pi^*\mathsf{F}_*\mathcal{O}_X\to\mathrm{R}^0p_*j^*\mathsf{F}_*\mathcal{O}_{\tilde{X}}\to 0. \tag{64}$$

Now $\mathrm{R}^0p_*j^*\pi^*\mathsf{F}_*\mathcal{O}_X = \mathrm{R}^0p_*p^*i^*\mathsf{F}_*\mathcal{O}_X = i^*\mathsf{F}_*\mathcal{O}_X = i^*\pi_*\mathsf{F}_*\mathcal{O}_{\tilde{X}}$, and the sequence (58) follows. □

Consider a particular case when X is a smooth surface and Y is a point $y\in X$. Let $\tilde{X}$ be the blown-up surface and l be the exceptional divisor, $l=\mathbb{P}^1$.

Corollary 3.2. *There is a short exact sequence:*

$$0\to\pi^*\mathsf{F}_*\mathcal{O}_X\to\mathsf{F}_*\mathcal{O}_{\tilde{X}}\to j_*\mathcal{O}_l(-1)^{\oplus\frac{p(p-1)}{2}}\to 0. \tag{65}$$

Proof. The category $\mathrm{D}^b(y)$ is equivalent to $\mathrm{D}^b(\mathrm{Vect}-k)$, since y is a point. Hence, we just need to compute the multiplicity of the sheaf $j_*\mathcal{O}_l(-1)$ in the sequence (65) or the rank of vector space E. This multiplicity is equal to the corank of the morphism of sheaves $\pi^*\mathsf{F}_*\mathcal{O}_X\to\mathsf{F}_*\mathcal{O}_{\tilde{X}}$ at the point y.

Proposition 3.1. *The corank is equal to* $\frac{p(p-1)}{2}$.

Proof. Choose the local coordinates x,y on $\tilde{X}$. Then x,xy are the local coordinates on X. The stalk of the sheaf $\pi^*\mathsf{F}_*\mathcal{O}_X$ at y is then $k[x,y]/(x^p,(xy)^p)$ whereas the stalk of the sheaf $\mathsf{F}_*\mathcal{O}_{\tilde{X}}$ at y is $k[x,y]/(x^p,y^p)$. We see now that the cokernel of the map $k[x,y]/(x^p,(xy)^p)\to k[x,y]/(x^p,y^p)$ consists of monomials x^ay^b such that $0\le a<b<p$, hence the statement. □

Corollary 3.2 is proven. □

Remark 3.4. Note that the sequence (65) could be obtained without resorting to Theorem 2.3. Indeed, there is an embedding of locally free sheaves $i\colon\pi^*\mathsf{F}_*\mathcal{O}_X\hookrightarrow\mathsf{F}_*\mathcal{O}_{\tilde{X}}$; the cokernel of the map i is a coherent sheaf $\mathcal{F}$ on $\tilde{X}$ supported on the exceptional divisor l. Clearly, $\mathrm{R}^{\cdot}\pi_*\mathcal{F}=0$. The sheaf $\mathcal{F}$ is the direct sum of its locally free part $\mathcal{F}^{\mathsf{lf}}$ and its torsion part $\mathcal{F}^{\mathsf{tors}}$. The sheaf $\mathcal{F}^{\mathsf{lf}}$ is the direct sum of line bundles $\mathcal{O}_l(m), m\in\mathbb{Z}$ by a Grothendieck theorem. From the condition $\mathrm{R}^{\cdot}\pi_*\mathcal{F}=0$ we conclude that the only possibility for the sheaf $\mathcal{F}$ is that it has no torsion and the only line bundles that can occur in the decomposition of $\mathcal{F}$ are those of the form $\mathcal{O}_l(-1)$, q.e.d. The sequences arising in Lemma 3.6 could obtained as well by using similar arguments.

4. A few examples

In this section we show some applications of the above results.

4.1. **Flag variety in type $\mathbf{A}_2$.** Consider the group $\mathbf{SL}_3$ over k and the flag variety $\mathbf{SL}_3/\mathbf{B}$. In [5] it was proved (Theorem 4.5.4 in *loc.cit.*) that the sheaf of differential operators on $\mathbf{SL}_3/\mathbf{B}$ has vanishing higher cohomology (recall that for flag varieties this implies the D-affinity, see ([5])). Below we give a different proof of this vanishing theorem.

Theorem 4.1. *Let X be the flag variety $\mathbf{SL}_3/\mathbf{B}$. Then* $\mathrm{E}xt^i(\mathsf{F}_{m*}\mathcal{O}_X, \mathsf{F}_{m*}\mathcal{O}_X) = 0$ *for $i > 0$ and $m \geq 1$.*

Corollary 4.1. $\mathrm{H}^i(X, \mathcal{D}_X) = 0$ *for $i > 0$.*

Proof. Recall (see Subsection 2.2) that the sheaf $\mathcal{D}_X$ is the direct limit of sheaves of matrix algebras:

$$\mathcal{D}_X = \bigcup_{n\geq 1} \mathcal{E}nd_{\mathcal{O}_X}(\mathsf{F}_{n*}\mathcal{O}_X). \tag{66}$$

Clearly, for some i and $n \geq 1$ the vanishing $\mathrm{E}xt^i(\mathsf{F}_{n*}\mathcal{O}_X, \mathsf{F}_{n*}\mathcal{O}_X) = \mathrm{H}^i(X, \mathcal{E}nd(\mathsf{F}_{n*}\mathcal{O}_X)) = 0$ implies $\mathrm{H}^i(X, \mathcal{D}_X) = 0$. □

Proof. The flag variety $\mathbf{SL}_3/\mathbf{B}$ is isomorphic to an incidence variety. For convenience of the reader, recall some facts about incidence varieties. Let V be a vector space of dimension n. The incidence variety X_n is the set of pairs $X_n: = (l \subset \mathrm{H} \subset V)$, where l and H are a line and a hyperplane in V, respectively. The variety X_n is fibered over $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$:

$$\begin{array}{ccccc} & & X_n & & \\ & \swarrow^{p} & & \searrow^{\pi} & \\ \mathbb{P}(V) & & & & \mathbb{P}(V^*) \end{array}$$

Let $0 \subset \mathcal{U}_1 \subset \mathcal{U}_{n-1} \subset V \otimes \mathcal{O}_X$ be the tautological flag on X_n. The projection π is projectivization of the bundle $\Omega^1(1)$ on $\mathbb{P}(V^*)$. Let $\mathcal{O}_p(-1)$ and $\mathcal{O}_\pi(-1)$ be the relative tautological line bundles with respect to projections p and π, respectively. Note that $\mathcal{U}_1 = p^*\mathcal{O}(-1) = \mathcal{O}_\pi(-1), \mathcal{U}_{n-1} = \pi^*\Omega^1(1)$. Let π/l be the quotient bundle:

$$0 \rightarrow p^*\mathcal{O}(-1) \rightarrow \pi^*\Omega^1(1) \rightarrow \pi/l \rightarrow 0. \tag{67}$$

Denote $\mathcal{O}(i,j)$ the line bundle $p^*\mathcal{O}(i) \otimes \pi^*\mathcal{O}(j)$. The canonical line bundle ω_X is isomorphic to $\mathcal{O}(-n,-n)$. To compute the $\mathrm{E}xt$-groups, let us apply Lemma 3.1. Recall that this lemma states an isomorphism of the following groups:

$$\mathrm{E}xt^i_X(\mathsf{F}_{m*}\mathcal{O}_X, \mathsf{F}_{m*}\mathcal{O}_X) = \mathrm{H}^k(X \times X, (\mathsf{F}_m \times \mathsf{F}_m)^*({i_{\Delta^{(m)}}}_*\mathcal{O}_{\Delta^m}) \otimes (\mathcal{O}_X \boxtimes {\omega_X}^{1-p^m})). \tag{68}$$

For incidence varieties, however, there is a nice resolution of the sheaf $i_*\mathcal{O}_\Delta$, the Koszul resolution ([11], Proposition 4.17). Recall briefly its construction. Consider the following double complex of sheaves $C^{\bullet,\bullet}$ on $X_n \times X_n$:

$$
\begin{array}{ccccc}
\cdots & \longrightarrow & \Psi_{1,0} \boxtimes \mathcal{O}(-1,0) & \longrightarrow & \mathcal{O}_{X_n \times X_n} \\
 & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & \Psi_{1,1} \boxtimes \mathcal{O}(-1,-1) & \longrightarrow & \Psi_{0,1} \boxtimes \mathcal{O}(0,-1) \\
 & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots
\end{array}
$$

The total complex of $C^{\bullet,\bullet}$ is a left resolution of the structure sheaf of the diagonal $\Delta \subset X_n \times X_n$. Truncate $C^{\bullet,\bullet}$, deleting all terms except those belonging to the intersection of the first n rows (from 0-th up to $(n-1)$-th) and the first $n-1$ columns, and consider the convolution of the remaining double complex. Denote $\tilde{C}^{\bullet}$ the convolution. The truncated complex has only two non-zero cohomology: $\mathcal{H}^0 = \mathcal{O}_{\Delta}$, and $\mathcal{H}^{-2(n-1)}$. The latter cohomology can be explicitly described:

$$\mathcal{H}^{-2(n-1)} = \bigoplus_{i=0}^{n-1} \wedge^i(\pi/l)(-1,0) \boxtimes \wedge^i(\pi/l)^*(-n+1,-n). \tag{69}$$

It follows that there is the following distinguished triangle ($\sigma_{\geq}$ stands for the stupid truncation):

$$\cdots \to \mathcal{H}^{-2n+2}[2n-2] \to \sigma_{\geq -2n+2}(\tilde{C}^{\bullet}) \to i_*\mathcal{O}_{\Delta} \to \mathcal{H}^{-2n+2}[2n-1] \to \dots \quad . \tag{70}$$

Let us come back to the case of $\mathbf{SL}_3/\mathbf{B} = X_4 = X$. Using the above triangle and Lemma 3.1, we can prove Theorem 4.1 almost immediately. In this case, the above triangle looks as follows:

$$\cdots \to \mathcal{H}^{-2}[2] \to \sigma_{\geq -2}(\tilde{C}^{\bullet}) \to i_*\mathcal{O}_{\Delta} \to \mathcal{H}^{-2}[3] \to \dots, \tag{71}$$

where the truncated complex $\sigma_{\geq -2}(\tilde{C}^{\bullet})$ is quasiisomorphic to:

$$0 \to \Psi_{1,1} \boxtimes \mathcal{O}(-1,-1) \to \Psi_{1,0} \boxtimes \mathcal{O}(-1,0) \oplus \Psi_{0,1} \boxtimes \mathcal{O}(0,-1) \to \mathcal{O}_{X \times X} \to 0, \tag{72}$$

and there is an isomorphism

$$\mathcal{H}^{-2} = \mathcal{O}(-1,0) \boxtimes \mathcal{O}(-1,-2) \oplus \pi/l \otimes \mathcal{O}(-1,0) \boxtimes (\pi/l)^* \otimes \mathcal{O}(-1,-2). \tag{73}$$

We need to compute the groups $\mathrm{H}^k(X \times X, (\mathsf{F}_m \times \mathsf{F}_m)^*({i_{\Delta^{(m)}}}_*\mathcal{O}_{\Delta^m}) \otimes (\mathcal{O}_X \boxtimes {\omega_X}^{1-p^m}))$. Apply the functor $(\mathsf{F}_m \times \mathsf{F}_m)^*$ to the triangle (71) and tensor it then with the sheaf $(\mathcal{O}_X \boxtimes {\omega_X}^{1-p^m}))$. Let us first prove that $\mathbb{H}^i(X \times X, (\mathsf{F}_m \times \mathsf{F}_m)^*(\sigma_{\geq -2}(\tilde{C}^{\bullet})) \boxtimes (\mathcal{O}_X \boxtimes {\omega_X}^{1-p^m})) = 0$ for $i > 0$. Recall that $\omega_X = \mathcal{O}(-2,-2)$. The sheaves $\Psi_{i,j}$ have right resolutions consisting of direct sums of ample line bundles and the sheaf $\mathcal{O}_X$ ([11]). This implies that $\mathrm{H}^i(X, \mathsf{F}_m^*(\Psi_{1,0})) = \mathrm{H}^i(X, \mathsf{F}_m^*(\Psi_{0,1})) = 0$ for $i > 1$ and $\mathrm{H}^i(X, \mathsf{F}_m^*(\Psi_{1,1})) = 0$ for $i > 2$. Along the second argument in (72) we get, after tensoring it with $\omega_X^{1-p^m}$, ample line bundles. Ample line bundles have no higher cohomology by the Kempf theorem ([13]). The spectral sequence then gives $\mathbb{H}^i(X\times X,(\mathsf{F}_m\times\mathsf{F}_m)^*(\sigma_{\geq -2}(\tilde{C}^\bullet))\boxtimes(\mathcal{O}_X\boxtimes{\omega_X}^{1-p^m}))$ $=0$ for $i>0$.

Note that π/l is a line bundle. One sees that $\pi/l=\mathcal{O}(1,-1)$, hence the sheaf $\mathcal{H}^{-2}$ is isomorphic to $\mathcal{O}(-1,0)\boxtimes\mathcal{O}(-1,-2)\oplus\mathcal{O}(0,-1)\boxtimes\mathcal{O}(-2,-1)$. Thus

$$(\mathsf{F}_m\times\mathsf{F}_m)^*\mathcal{H}^{-2}\otimes(\mathcal{O}_X\boxtimes{\omega_X}^{1-p^m})= \tag{74}$$
$$=\mathcal{O}(-p^m,0)\boxtimes\mathcal{O}(-p^m,-2p^m)\otimes\omega_X^{1-p^m}\oplus\mathcal{O}(0,-p^m)\boxtimes\mathcal{O}(-2p^m,-p^m)\otimes\omega_X^{1-p^m}.$$

Let us prove that the latter bundle has only one non-vanishing cohomology, namely H^3. The Serre duality gives:

$$\mathrm{H}^i(X,\mathsf{F}_m^*\mathcal{O}(-1,0))=\mathrm{H}^{3-i}(X,\mathsf{F}_m^*\mathcal{O}(-1,-2)\otimes\omega_X^{1-p^m}), \tag{75}$$

and

$$\mathrm{H}^i(X,\mathsf{F}_m^*\mathcal{O}(0,-1))=\mathrm{H}^{3-i}(X,\mathsf{F}_m^*\mathcal{O}(-2,-1)\otimes\omega_X^{1-p^m}). \tag{76}$$

It is therefore sufficient to show that the left-hand sides in both (75) and (76) are non-zero only for one value of i. Consider for example the cohomology group $\mathrm{H}^i(X,\mathsf{F}_m^*\mathcal{O}(-1,0))$. The line bundle $\mathcal{O}(-1,0)$ is isomorphic to $p^*\mathcal{O}(-1)$, hence $\mathrm{H}^i(X,\mathsf{F}_m^*\mathcal{O}(-1,0))=\mathrm{H}^i(\mathbb{P}^2,\mathsf{F}_m^*\mathcal{O}(-1))=\mathrm{H}^i(\mathbb{P}^2,\mathcal{O}(-p^m))$. The line bundle $\mathcal{O}(-p^m)$ on $\mathbb{P}^2$ is either acyclic (for $p=2$ and $m=1$) or has only one non-zero cohomology group in top degree. The Künneth formula now gives that $\mathrm{H}^i(X\times X,(\mathsf{F}_m\times\mathsf{F}_m)^*\mathcal{H}^{-2}\otimes(\mathcal{O}_X\boxtimes{\omega_X}^{1-p^m}))=0$ for $i\neq 3$. Remembering the distinguished triangle (71), we get the proof. □

Remark 4.1. A very similar argument was used in ([15]) where the case of 3-dimensional quadrics was studied. The vanishing of higher cohomology of the sheaf $\mathcal{D}_{X_n}$ on incidence varieties X_n for arbitrary n was proved in ([16]).

Corollary 4.2. *Assume $p>3$. Then $\mathsf{F}_*\mathcal{O}_X$ is a tilting bundle.*

Proof. The first condition of Definition 2.3 is satisfied by Theorem 4.1. The second one is satisfied by Lemma 2.2. Hence, there is an equivalence of categories:

$$\mathrm{D}^b(X)\simeq\mathrm{D}^b(\mathrm{End}(\mathsf{F}_*\mathcal{O}_X)-\mathrm{mod})), \tag{77}$$

and X being smooth, the algebra $\mathrm{End}(\mathsf{F}_*\mathcal{O}_X)$ has finite homological dimension (cf. [7]). □

4.2. **Toric Fano varieties.** Here we work out several examples of Fano toric varieties.

Lemma 4.1. *Let X be the projective bundle $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1))$ over $\mathbb{P}^n$. Then $\mathrm{Ext}^i(\mathsf{F}_*\mathcal{O}_X,\mathsf{F}_*\mathcal{O}_X)=0$ for $i>0$.*

Proof. We first consider the case $n>1$. Denote $\mathcal{E}=\mathcal{O}\oplus\mathcal{O}(1)$, and let $\pi\colon X\to\mathbb{P}^n$ be projection. Recall the short sequence (see Subsection 3.2):

$$0\to\pi^*\mathsf{F}_*\mathcal{O}_{\mathbb{P}^n}\to\mathsf{F}_*\mathcal{O}_X\to\pi^*(\mathsf{F}_*(\tilde{\mathcal{E}})\otimes\det\,(\mathcal{E}^*))\otimes\mathcal{O}_\pi(-1)\to 0. \tag{78}$$

Here $\tilde{\mathcal{E}}=\mathsf{S}^{p-2}\mathcal{E}\otimes\det\mathcal{E}$. We first observe that the sequence (78) splits, that is

$$\mathsf{F}_*\mathcal{O}_X=\pi^*\mathsf{F}_*\mathcal{O}_{\mathbb{P}^n}\oplus\pi^*(\mathsf{F}_*\tilde{\mathcal{E}}\otimes\det\mathcal{E}^*)\otimes\mathcal{O}_\pi(-1). \tag{79}$$

Indeed, by adjunction one obtains an isomorphism

$$\mathrm{E}xt^1(\pi^*(\mathsf{F}_*\tilde{\mathcal{E}}\otimes\det\mathcal{E}^*)\otimes\mathcal{O}_\pi(-1),\pi^*\mathsf{F}_*\mathcal{O}_{\mathbb{P}^n}) = \mathrm{E}xt^1(\mathsf{F}_*\tilde{\mathcal{E}}\otimes\det\mathcal{E}^*,\mathsf{F}_*\mathcal{O}_{\mathbb{P}^n}\otimes\pi_*\mathcal{O}_\pi(1)), \tag{80}$$

the group in the right hand side being isomorphic to

$$\mathrm{E}xt^1(\mathsf{F}_*\tilde{\mathcal{E}},\mathsf{F}_*\mathcal{O}_{\mathbb{P}^n}\otimes\mathcal{E}). \tag{81}$$

Recall that $\tilde{\mathcal{E}} = \mathsf{S}^{p-2}\mathcal{E}\otimes\det\mathcal{E} = \bigoplus_{k=1}^{k=p-1}\mathcal{O}(k)$. It is well known that the Frobenius push-forward of any line bundle on $\mathbb{P}^n$ splits into direct sum of line bundles (e.g.,[7]). More generally, the Frobenius push-forward of a line bundle on a smooth toric variety has the same property ([3]). Hence, both terms in the group (81) are direct sums of line bundles. However, a line bundle on $\mathbb{P}^n$ can have non-zero cohomology either in zero or in top degree. Thus, the group (81) is zero and the bundle $\mathsf{F}_*\mathcal{O}_X$ splits.

We need to prove that $\mathrm{E}xt^i(\pi^*(\mathsf{F}_*\tilde{\mathcal{E}}\otimes\det\mathcal{E}^*)\otimes\mathcal{O}_\pi(-1),\mathsf{F}_*\mathcal{O}_X) = 0$ for $i>0$. This reduces to showing that

$$\mathrm{E}xt^i(\mathsf{F}_*\tilde{\mathcal{E}},\mathsf{F}_*\tilde{\mathcal{E}}) = 0, \text{ and } \mathrm{E}xt^i(\mathsf{F}_*\tilde{\mathcal{E}},\mathsf{F}_*\mathcal{O}_{\mathbb{P}^n}\otimes\mathcal{E}) = 0 \tag{82}$$

for $i>0$.

Proposition 4.1. Let $1\leq k\leq p-1$. Then line bundles that occur in the decomposition of $\mathsf{F}_*\mathcal{O}(k)$ are isomorphic to $\mathcal{O}(l)$ for $-n\leq l\leq 0$.

Proof. Let $\mathsf{F}_*\mathcal{O}(k) = \oplus\ \mathcal{O}(a_i)$. Tensoring $\mathsf{F}_*\mathcal{O}(k)$ with $\mathcal{O}(-1)$, we obtain the bundle $\mathsf{F}_*\mathcal{O}(k)\otimes\mathcal{O}(-1) = \mathsf{F}_*\mathcal{O}(k-p)$, and the latter bundle has no global sections by the assumption. Hence, all $a_i\leq 0$. On the other hand, $\mathsf{F}_*\mathcal{O}(k)$ for such k has no higher cohomology. This gives $a_i\geq -n$. □

Lemma 4.1 implies that the groups in (82) are zero for $i>0$, hence the statement.

Now look at the case $n=1$. The toric variety – the projective bundle $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1))$ over $\mathbb{P}^1$ – is a ruled surface $\mathbb{F}_1$ and is isomorphic to the blowup of a point on $\mathbb{P}^2$. Blowups of $\mathbb{P}^2$ are treated in the next section. A straightforward check using Lemma 3.1, however, gives that the sequence (78) splits for $n=1$ as well, and the rest of the proof is the same as above.

In fact, the decomposition of the bundle $\mathsf{F}_*\mathcal{O}_X$ into a direct sum of line bundles allows to check when $\mathsf{F}_*\mathcal{O}_X$ generates the category $\mathrm{D}^b(X)$. Let us treat the simplest case of $\mathbb{P}^1$. Recall the decomposition:

$$\mathsf{F}_*\mathcal{O}_{\mathbb{F}_1} = \pi^*\mathsf{F}_*\mathcal{O}_{\mathbb{P}^1}\oplus\pi^*(\mathsf{F}_*\tilde{\mathcal{E}}\otimes\det\mathcal{E}^*)\otimes\mathcal{O}_\pi(-1). \tag{83}$$

One has $\mathsf{F}_*\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus p-1}$, and it can be easily verified (at least for $p>2$) that a similar decomposition holds for $\mathsf{F}_*\tilde{\mathcal{E}}\otimes\det\mathcal{E}^*$:

$$\mathsf{F}_*\tilde{\mathcal{E}}\otimes\det\mathcal{E}^* = \mathcal{O}_{\mathbb{P}^1}^{\oplus a}\oplus\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus b}, \tag{84}$$

where a and b are non-zero multiplicities. It follows from Theorem 2.2 that the set of line bundles $\mathcal{O}_{\mathbb{F}_1},\pi^*\mathcal{O}_{\mathbb{P}^1}(-1),\mathcal{O}_\pi(-1),\pi^*\mathcal{O}_{\mathbb{P}^1}(-1)\otimes\mathcal{O}_\pi(-1)$ generates the category $\mathrm{D}^b(\mathbb{F}_1)$, hence for $p>2$ the bundle $\mathsf{F}_*\mathcal{O}_{\mathbb{F}_1}$ generates $\mathrm{D}^b(\mathbb{F}_1)$, that is $\mathsf{F}_*\mathcal{O}_{\mathbb{F}_1}$ is a tilting bundle. □

Similarly, one checks the following:

Lemma 4.2. *Let X be the projective bundle $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(2))$ over $\mathbb{P}^n$, and $n>2$. Then the bundle $\mathsf{F}_*\mathcal{O}_X$ is almost exceptional (i.e. $\mathrm{E}xt^i(\mathsf{F}_*\mathcal{O}_X,\mathsf{F}_*\mathcal{O}_X)=0$ for $i>0$).*

Proof. The proof is completely analogous to that of the previous lemma. It turns out, however, that for $n=2$ there are non-vanishing $\mathrm{E}xt$-groups in top degree. A direct sum decomposition for $\mathsf{F}_*\mathcal{O}_X$ as in the previous lemma holds anyway. □

Recall that according to the classification of smooth toric Fano threefolds ([1]), there are 18 isomorphism classes of smooth Fano toric threefolds. Among these are the following projective bundles:

$$\mathbb{P}^3,\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(2)),\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(1)),\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)),$$
$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)),\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1)),\ \mathbb{P}(\mathcal{O}_{X_1}\oplus\mathcal{O}_{X_1}(l)), \tag{85}$$

The other varieties in the list are products of del Pezzo surfaces X_k (the blowups of $\mathbb{P}^2$ at k points) and $\mathbb{P}^1$:

$$\mathbb{P}^2\times\mathbb{P}^1,\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1,X_1\times\mathbb{P}^1,X_2\times\mathbb{P}^1,X_3\times\mathbb{P}^1, \tag{86}$$

and there are yet six varieties, of which four are isomorphic to del Pezzo fibrations over $\mathbb{P}^1$. The similar calculations as above give:

(i) Let X be the projective bundle $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1,1))$ over $\mathbb{P}^1\times\mathbb{P}^1$. Then $\mathrm{E}xt^i(\mathsf{F}_*\mathcal{O}_X,\mathsf{F}_*\mathcal{O}_X)=0$ for $i>0$.

(ii) Let X be the projective bundle $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1,-1))$ over $\mathbb{P}^1\times\mathbb{P}^1$. Then $\mathrm{E}xt^1(\mathsf{F}_*\mathcal{O}_X,\mathsf{F}_*\mathcal{O}_X)\neq 0$.

In the next section we will see that $\mathrm{E}xt^i(\mathsf{F}_*\mathcal{O}_{X_k},\mathsf{F}_*\mathcal{O}_{X_k})=0$ for a del Pezzo surface X_k. Thus, the $\mathrm{E}xt$-groups vanish for all varieties in (86). It should be possible to compute the bundle $\mathsf{F}_*\mathcal{O}_X$ for $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))$ and $\mathbb{P}(\mathcal{O}_{X_1}\oplus\mathcal{O}_{X_1}(l))$ using the above methods though we did not attempt to do this.

5. Some rational surfaces

Theorem 5.1. *Let X_k be a smooth surface that is obtained by blowing up of a set of k points on $\mathbb{P}^2$ in general position, $k\geq 1$. Then*

$$\mathrm{E}xt^i(\mathsf{F}_*\mathcal{O}_{X_k},\mathsf{F}_*\mathcal{O}_{X_k})=0 \tag{87}$$

for $i>0$.

Proof. We prove the theorem by induction. The case of $\mathbb{P}^2=:X_0$ is well known (e.g., [7]). Assume that

$$\mathrm{E}xt^i(\mathsf{F}_*\mathcal{O}_{X_k},\mathsf{F}_*\mathcal{O}_{X_k})=0 \tag{88}$$

for $i>0$ and $X_k=\tilde{\mathbb{P}}^2_{x_0,\dots,x_{k-1}}$, $k>1$. Take a point x_k in general position with respect to the points $x_0,\dots,x_{k-1}$ and consider the blow-up of X_k at x_k:

$$\begin{array}{ccc} l_k & \xrightarrow{\ i\ } & X_{k+1} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi_k} \\ x_k & \longrightarrow & X_k \end{array}$$

By Corollary 3.2 there is a short exact sequence:

(89) $$0 \to \pi_k^*\mathsf{F}_*\mathcal{O}_{X_k} \to \mathsf{F}_*\mathcal{O}_{X_{k+1}} \to i_*\mathcal{O}_{l_k}(-1)^{\oplus\frac{p(p-1)}{2}} \to 0.$$

Applying the functor $\mathrm{H}om(\mathsf{F}_*\mathcal{O}_{X_{k+1}}, ?)$ to the sequence (89), we get a long exact sequence:

(90) $$0 \to \mathrm{H}om(\mathsf{F}_*\mathcal{O}_{X_{k+1}}, \pi_k^*\mathsf{F}_*\mathcal{O}_{X_k}) \to \mathrm{H}om(\mathsf{F}_*\mathcal{O}_{X_{k+1}}, \mathsf{F}_*\mathcal{O}_{X_{k+1}}) \to$$
$$\to \mathrm{H}om(\mathsf{F}_*\mathcal{O}_{X_{k+1}}, i_*\mathcal{O}_l(-1)^{\oplus\frac{p(p-1)}{2}}) \to \mathrm{E}xt^1(\mathsf{F}_*\mathcal{O}_{X_{k+1}}, \pi_k^*\mathsf{F}_*\mathcal{O}_{X_k}) \to \dots \quad .$$

Let us first consider the groups $\mathrm{E}xt^m(\mathsf{F}_*\mathcal{O}_{X_{k+1}}, \pi_k^*\mathsf{F}_*\mathcal{O}_{X_k})$. By adjunction we have:

(91) $$\mathrm{E}xt^m(\mathsf{F}_*\mathcal{O}_{X_{k+1}}, \pi_k^*\mathsf{F}_*\mathcal{O}_{X_k}) = \mathrm{E}xt^m(\mathcal{O}_{X_{k+1}}, \mathsf{F}^*\pi_k^*\mathsf{F}_*\mathcal{O}_{X_k} \otimes \omega_{X_{k+1}}^{1-p}) =$$
$$= \mathrm{H}^m(X_{k+1}, \pi_k^*\mathsf{F}^*\mathsf{F}_*\mathcal{O}_{X_k} \otimes \omega_{X_{k+1}}^{1-p}).$$

Recall that the canonical sheaves are related by the formula:

(92) $$\omega_{X_{k+1}} = \pi_k^*\omega_{X_k} \otimes \mathcal{O}_{X_{k+1}}(l_k).$$

For any $n > 0$ there is the short exact sequence:

(93) $$0 \to \mathcal{O}_{X_{k+1}}(-nl_k) \to \mathcal{O}_{X_{k+1}} \to \mathcal{O}_{nl_k} \to 0.$$

The sheaf $\mathcal{O}_{nl_k}$ has a filtration with associated graded factors being $\mathcal{J}_{l_k}^m/\mathcal{J}_{l_k}^{m+1} = \mathcal{O}_{l_k}(m), 0 < m < n$. Hence, $\mathrm{R}^{\cdot}\pi_{k*}\mathcal{O}_{nl_k} = \mathcal{O}_{nx_k}$. Applying the direct image functor π_{k*} to the sequence (93), we get:

(94) $$0 \to \pi_{k*}\mathcal{O}_{X_{k+1}}(-nl_k) \to \mathcal{O}_{X_k} \to \mathcal{O}_{nx_k} \to 0.$$

Thus, $\mathrm{R}^{\cdot}\pi_{k*}\mathcal{O}_{X_{k+1}}(-nl_k) = \mathcal{J}_{x_k}^n$. Finally:

(95) $$\mathrm{R}^{\cdot}\pi_{k*}(\omega_{X_{k+1}}^{1-p}) = \omega_{X_k}^{1-p} \otimes \mathcal{J}_{x_k}^{p-1}$$

We obtain an isomorphism:

(96) $$\mathrm{H}^m(X_{k+1}, \pi_k^*\mathsf{F}^*\mathsf{F}_*\mathcal{O}_{X_k} \otimes \omega_{X_{k+1}}^{1-p}) = \mathrm{H}^m(X_k, \mathsf{F}^*\mathsf{F}_*\mathcal{O}_{X_k} \otimes \omega_{X_k}^{1-p} \otimes \mathcal{J}_{x_k}^{p-1}).$$

Consider a short exact sequence

(97) $$0 \to \mathcal{J}_{x_k}^{p-1} \to \mathcal{O}_{X_k} \to \mathcal{O}_{(p-1)x_k} \to 0,$$

and its tensor product with the vector bundle $\mathsf{F}^*\mathsf{F}_*\mathcal{O}_{X_k} \otimes \omega_{X_k}^{1-p} = \mathsf{F}^!\mathsf{F}_*\mathcal{O}_{X_k} :\ = \mathcal{E}_k$

(98) $$0 \to \mathcal{J}_{x_k}^{p-1} \otimes \mathcal{E}_k \to \mathcal{E}_k \to \mathcal{E}_k \otimes \mathcal{O}_{(p-1)x_k} \to 0.$$

By the induction assumption we have

(99) $$\mathrm{H}^i(X_k, \mathcal{E}_k) = \mathrm{E}xt^i(\mathsf{F}_*\mathcal{O}_{X_k}, \mathsf{F}_*\mathcal{O}_{X_k}) = 0$$

for $i > 0$.

Proposition 5.1. *The map* $\mathrm{H}^0(X_k, \mathcal{E}_k) \to \mathrm{H}^0(X_k, \mathcal{E}_k \otimes \mathcal{O}_{(p-1)x_k})$ *is surjective.*

Proof. We again proceed by induction. For $k = 1$ we have $X_1 = \mathbb{P}^2$ and $\mathcal{E}_0 = \mathsf{F}^*\mathsf{F}_*\mathcal{O}_{\mathbb{P}^2} \otimes \omega_{\mathbb{P}^2}^{1-p} = \mathcal{O}_{\mathbb{P}^2}(3p-3) \oplus \mathcal{O}_{\mathbb{P}^2}(2p-3)^{p_1} \oplus \mathcal{O}_{\mathbb{P}^2}(p-3)^{p_2}$, where p_1 and p_2 are the multplicities. In fact, these multplicities are equal to

$$p_1 = \frac{(p-1)(p+4)}{2}, \quad p_2 = \frac{(p-1)(p-2)}{2} \quad . \tag{100}$$

A dimension count gives in this case that there is a surjection $\mathrm{H}^0(\mathbb{P}^2, \mathcal{E}_0) \twoheadrightarrow \mathrm{H}^0(\mathbb{P}^2, \mathcal{E}_0 \otimes \mathcal{O}_{(p-1)x_1})$. Indeed, the dimension of the group $\mathrm{H}^0(\mathbb{P}^2, \mathcal{E}_0)$ is given by

$$\dim \mathrm{H}^0(\mathbb{P}^2, \mathcal{E}_0) = \binom{3p-1}{2} + \binom{2p-1}{2} \cdot p_1 + \binom{p-1}{2} \cdot p_2. \tag{101}$$

On the other hand, the space $\mathrm{H}^0(\mathbb{P}^2, \mathcal{E}_0 \otimes \mathcal{O}_{(p-1)x_1})$ imposes $\frac{(p-1)(p-2)}{2}(1 + p_1 + p_2)$ conditions and one sees that the dimension of this space is less than the right-hand side in (101). Moreover, the dimension count shows that there is even a surjection $\mathrm{H}^0(\mathbb{P}^2, \mathcal{E}_0) \twoheadrightarrow \mathrm{H}^0(\mathbb{P}^2, \mathcal{E}_0 \otimes \mathcal{O}_{\mathcal{Z}})$, where $\mathcal{Z}$ is an artinian subscheme of $\mathbb{P}^2$ of the length $\leq 2p$. Assume that for some $l \leq k$ we have a surjection $\mathrm{H}^0(X_l, \mathcal{E}_l) \to \mathrm{H}^0(X_l, \mathcal{E}_l \otimes \mathcal{O}_{\mathcal{Z}})$, where $\mathcal{Z}$ is an artinian subscheme of X_l of length $\leq 2p$; in particular, this inductive assumption implies a surjection $\mathrm{H}^0(X_k, \mathcal{E}_k) \to \mathrm{H}^0(X_k, \mathcal{E}_k \otimes \mathcal{O}_{(p-1)x_k})$, or, equivalently, $\mathrm{H}^1(X_k, \mathcal{J}_{x_k}^{p-1} \otimes \mathcal{E}_k) = 0$. Consider the sequence (89). This is a sequence of coherent sheaves on X_{k+1}. Applying to it the functor $\mathsf{F}^!$ on X_{k+1}, we obtain:

$$0 \to \mathsf{F}^!(\pi_k^*\mathsf{F}_*\mathcal{O}_{X_k}) \to \mathcal{E}_{k+1} \to \mathsf{F}^!(i_*\mathcal{O}_{l_k}(-1)^{\oplus \frac{p(p-1)}{2}}) \to 0. \tag{102}$$

One has

$$\mathsf{F}^!(\pi_k^*\mathsf{F}_*\mathcal{O}_{X_k}) = \mathsf{F}^*(\pi_k^*\mathsf{F}_*\mathcal{O}_{X_k}) \otimes \omega_{X_{k+1}}^{1-p} = \tag{103}$$

$$= \pi_k^*(\mathsf{F}^*\mathsf{F}_*\mathcal{O}_{X_k} \otimes \omega_{X_k}^{1-p}) \otimes \mathcal{O}_{X_{k+1}}(-(p-1)l_k) = \pi_k^*\mathcal{E}_k \otimes \mathcal{O}_{X_{k+1}}(-(p-1)l_k).$$

Denote the line bundle $\mathcal{O}_{X_{k+1}}(-(p-1)l_k)$ by $\mathcal{L}_k$ and the sheaf $\mathsf{F}^!(i_*\mathcal{O}_{l_k}(-1)^{\oplus \frac{p(p-1)}{2}})$ by $\mathcal{M}_k$. Tensor the sequence (102) with the sheaf $\mathcal{J}_{x_{k+1}}^{p-1}$:

$$0 \to \mathcal{T}or^1(\mathcal{M}_k, \mathcal{J}_{x_{k+1}}^{p-1}) \to \pi_k^*\mathcal{E}_k \otimes \mathcal{L}_k \otimes \mathcal{J}_{x_{k+1}}^{p-1} \to \mathcal{E}_{k+1} \otimes \mathcal{J}_{x_{k+1}}^{p-1} \to \mathcal{M}_k \otimes \mathcal{J}_{x_{k+1}}^{p-1} \to 0. \tag{104}$$

We need to prove that $\mathrm{H}^1(X_{k+1}, \mathcal{E}_{k+1} \otimes \mathcal{J}_{x_{k+1}}^{p-1}) = 0$. First observe that $\mathcal{M}_k \otimes \mathcal{J}_{x_{k+1}}^{p-1} = \mathcal{M}_k$ since the sheaf $\mathcal{J}_{x_{k+1}}$ is isomorphic to $\mathcal{O}_{X_{k+1}}$ in the neighbourhood of the support of the sheaf $\mathcal{M}_k$. Further, for any coherent sheaf $\mathcal{F}$ on X_{k+1} one has an exact sequence:

$$0 \to \mathrm{H}^1(X_k, \mathrm{R}^0\pi_{k*}\mathcal{F}) \to \mathrm{H}^1(X_{k+1}, \mathcal{F}) \to \mathrm{H}^0(X_k, \mathrm{R}^1\pi_{k*}\mathcal{F}) \to \dots \tag{105}$$

Applying to the short exact sequence

$$0 \to \mathcal{O}_{X_{k+1}} \to \mathcal{O}_{X_{k+1}}(l_k) \to i_*\mathcal{O}_{l_k}(-1) \to 0 \tag{106}$$

the functor $\mathsf{F}^!$, we get:

$$0 \to \omega_{X_{k+1}}^{1-p} \to \omega_{X_{k+1}}^{1-p} \otimes \mathcal{O}_{X_{k+1}}(pl_k) \to \mathsf{F}^!(i_*\mathcal{O}_{l_k}(-1)) \to 0. \tag{107}$$

Taking the direct image π_{k*} gives:

$$0 \to \omega_{X_k}^{1-p} \otimes \mathcal{J}_{x_k}^{p-1} \to \omega_{X_k}^{1-p} \to \pi_{k*}(\mathsf{F}^!(i_*\mathcal{O}_{l_k}(-1)) \to 0. \tag{108}$$

Therefore

$$\mathrm{R}^0\pi_{k*}(\mathsf{F}^!(i_*\mathcal{O}_{l_k}(-1)) = \omega_{X_k}^{1-p} \otimes \mathcal{O}_{(p-1)x_k}, \tag{109}$$

and

$$\mathrm{R}^1\pi_{k*}(\mathsf{F}^!(i_*\mathcal{O}_{l_k}(-1)) = 0. \tag{110}$$

We see that $\mathrm{R}^1\pi_{k*}(\mathcal{M}_k \otimes \mathcal{J}_{x_{k+1}}^{p-1}) = 0$ and $\mathrm{R}^0\pi_{k*}(\mathcal{M}_k \otimes \mathcal{J}_{x_{k+1}}^{p-1})$ is a skyscraper sheaf. By (105) we get:

$$\mathrm{H}^1(X_{k+1}, \mathcal{M}_k) = \mathrm{H}^1(X_{k+1}, \mathcal{M}_k \otimes \mathcal{J}_{x_{k+1}}^{p-1}) = 0. \tag{111}$$

The following two observations finish the proof: first, by the induction assumption, one has:

$$\mathrm{H}^1(X_{k+1}, \pi_k^*\mathcal{E}_k \otimes \mathcal{L}_k \otimes \mathcal{J}_{x_{k+1}}^{p-1}) = \mathrm{H}^1(X_k, \mathcal{E}_k \otimes \mathcal{J}_{x_k \cup \pi_k(x_{k+1})}^{p-1}) = 0. \tag{112}$$

Secondly, the sheaf $\mathcal{T}or^1(\mathcal{M}_k, \mathcal{J}_{x_{k+1}}^{p-1})$ is a torsion sheaf supported on the exceptional divisor l_k; hence, $\mathrm{H}^2(X_{k+1}, \mathcal{T}or^1(\mathcal{M}_k, \mathcal{J}_{x_{k+1}}^{p-1})) = 0$. Considering the spectral sequence associated to the sequence (104) we get $\mathrm{H}^1(X_{k+1}, \mathcal{E}_{k+1} \otimes \mathcal{J}_{x_{k+1}}^{p-1}) = 0$, q.e.d. □

Taking into account (99) and Proposition 5.1, from the long exact cohomology sequence associated to (98) we get:

$$\mathrm{H}^i(X_k, \mathcal{E}_k \otimes \mathcal{J}_x^{p-1}) = 0 \tag{113}$$

for $i > 0$. Hence, the left-hand side group in (91) is zero for $m > 0$. Now consider the groups $\mathrm{E}xt^m(\mathsf{F}_*\mathcal{O}_{X_{k+1}}, i_*\mathcal{O}_{l_k}(-1)^{\oplus\frac{p(p-1)}{2}}) = \mathrm{H}^m(X_{k+1}, \mathsf{F}^!(i_*\mathcal{O}_{l_k}(-1)^{\oplus\frac{p(p-1)}{2}}))$. From (109) and (110) one sees immediately that $\mathrm{H}^m(X_{k+1}, \mathsf{F}^!(i_*\mathcal{O}_{l_k}(-1)^{\oplus\frac{p(p-1)}{2}})) = 0$ for $m > 0$.

Finally, from (91) we get $\mathrm{E}xt^m(\mathsf{F}_*\mathcal{O}_{X_{k+1}}, \mathsf{F}_*\mathcal{O}_{X_{k+1}}) = 0$ for $m > 0$, and the theorem follows.

□

Another proof uses the Cartier isomorphism ([8]).

Under a bound on the prime number p, for del Pezzo surfaces (that is for surfaces X_k for $k \leq 9$) the bundle $\mathsf{F}_*\mathcal{O}_{X_k}$ should generate the whole category $\mathrm{D}^b(X_k)$ (see, e.g., an example of the ruled surface $\mathbb{F}_1 = X_2$ in the previous section). The proof will be given elsewhere.

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