2. Mathematische Arbeitstagung (Neue Serie)

14. - 20. Juni 1995

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

Germany

MPI / 95-50

Inhalt

Teilnehmerliste

Programm der Mathematischen Arbeitstagung 1995

Kurzfassungen der Vorträge

TEILNEHMERLISTE

D. Alekseevsky (Moskau/z.Zt.MPI)
P. Arias (Bonn)
N. Askitas (z.Zt.MPI)
R. Assenmacher (Bonn)
M. Avdispahić (Sarajevo/z.Zt.MPI)
D. Baeumer (MPI)
A. Baker (Glasgow)
W. Ballmann (Bonn)
H.-J. Baues (MPI)
P. Bayer (Barcelona)
K. Behrend (z.Zt.MPI)
K. Belabas (Bordeaux I)
R. Berndt (Hamburg)
Yu. Bilu (Ben Gurion U./z.Zt.MPI)
J-M. Bismut (Orsay)
S. Böcherer (Mannheim)
H. Boden (U. of Michigan/z.Zt.MPI)
C.-F. Bödigheimer (Bonn)
R. Borcherds (UC Berkeley)
A. Buium (Bukarest/z.Zt.MPI)
U. Bunke (Berlin)
D. Burghelea (Ohio State U./z.Zt.MPI)
C. Byrne (Springer-Verlag)
M. Cardenas (Binghamton/z.Zt.MPI)
C. Casacuberta (Barcelona)
Chai, Ching-Li (U. of Pennsylvania/z.Zt.MPI)
S. Chmutov (Pereslavl/z.Zt.MPI)
H. Cohen (Bordeaux I)
P.B. Cohen (CNRS Paris)
F. Connolly (U. of Notre Dame/z.Zt.MPI)
E. Dahlhaus
D. Dais (TU Berlin)
J.F. Davis (Indiana U.)
A. Degtyarev (POMI/z.Zt.MPI)
A. Deitmar (Heidelberg)
Th. Delzant (Strasbourg)
A. Dessai (Mainz)
A. Dzhumadildaev (U. München)
W. Ebeling (Hannover)
J. Eells (Warwick)
M.A. Efendiev (FU Berlin)
B. Elsner (Nizza)
D. Erle (Dortmund)
H. Esnault (Essen)
U. Everling (Eichstätt)
V. Ezhov (Adelaide)
C. Faber (U. Amsterdam)
G. Faltings (MPI)
B. Fantechi (Trento/z.Zt.Bonn & MPI)
D. Feldhusen (Bonn)
M. Fels (Bonn)
A. Fel'shtyn (Greifswald)
M. Fontaine (Bonn)
R. Friedrich (Bonn)
H. Gangl (MPI)
L. Geatti (Rom II)
G. van der Geer (U. Amsterdam)
E. Getzler (M.I.T.)
L. Göttsche (z.Zt.MPI)
D. Grigoriev (Pennsylvania St.U.)
F. Grunewald (Düsseldorf)
P. Guerzhoy (Haifa/z.Zt.MPI)
K. Haberland (Jena)
R. Hain (Duke U.)
I. Hambleton (McMaster U./z.Zt.MPI)
G. Harder (Bonn & MPI)
V. Heiermann (Humboldt-U. Berlin)
J. Heinze (Springer-Verlag)
Ch. Herald (UC Irvine/z.Zt.MPI)
F. Hirzebruch (MPI)
Hoda, Elsherbiny (Kairo)
W. Hoffmann (MPA Humboldt-U. Berlin)
G. Höhn (MPI)
R.-P. Holzapfel (Humboldt-U. Berlin)
Hsiang, Wu-yi (UC Berkeley)
J. Huebschmann (Lille/z.Zt.MPI)
D. Husemoller (Haverford College)
D. Huybrechts (MPI)
H.-C. Im Hof (Basel)
A. Jähde (MPI)
J. Jahnel (Göttingen)
V. Jeschonnek (Bonn)
J-M. Kantor (U. Paris VII)
N. Kapouleas (Brown U./z.Zt.Bonn & MPI)
M. Karpinski (Bonn)
Kato, Fumiharu (Kyoto)
R. Kaufmann (MPI)
I. Kausz (Köln)
R. Kellerhals (MPI)
V. Kharlamov (Strasbourg)
Kim, Jongsu (Seoul/z.Zt.MPI)
Th. Kleinjung (Bonn)
Ch. Klingenberg (Tübingen)
M. Klingholz (Düsseldorf)
H. Knörrer (ETH Zürich)
H. Koch (MPA Humboldt-U. Berlin)
P. Koepke (Bonn)
K. Köhler (Bonn)
M. Kontsevich (UC Berkeley/z.Zt.MPI)
Yu. Kordyukov (Ufa/z.Zt.MPI)
U. Koschorke (Siegen)
T. Kotnik (Tübingen)

O. Küchle (Bayreuth/z.Zt.MPI)
H. Kurke (Humboldt-U. Berlin)
J-P. Labesse (ENS Paris)
S. Lang (Yale U.)
F. Lárusson (Purdue U.)
Le, Hong Van (MPI)
I. Leary (z.Zt.MPI)
R. Lee (Hong Kong UST)
B. Leeb (Bonn)
Th. Lehmkuhl (Göttingen)
L. Lemaire (U.Libre de Bruxelles)
F. Leprévost (Paris VII/z.Zt.MPI)
M. Lönne (Hannover)
E. Looijenga (Utrecht)
Yu. Manin (MPI)
G. Mason (UC Santa Cruz)
G. Matic (U. of Georgian/z.Zt.MPI)
K.H. Mayer (Dortmund)
G. Mikhalkin (Michigan State U./z.Zt.MPI)
A. Miller (MPI)
V. Moncrief (Yale U.)
B. Moonen (Bonn)
J. Morgan (Columbia U.)
B.Z. Moroz (c/o MPI)
T. Mrowka (Harvard)
S. Müller-Stach (Essen)
J. Murakami (Osaka/z.Zt.MPI)
W. Nahm (Bonn)
Namikawa, Yoshinori (Sophia U./z.Zt.MPI)
F. Nicolae (Romanian Academy)
S. Ochanine (U. of Kentucky)
K. Ohba (Ochanomizu U./z.Zt.MPI)
F. Oort (Utrecht)
A. Panchishkin (Inst.Fourier, Grenoble)
E. Pedersen (SUNY Binghamton/z.Zt.MPI)
Phung Ho Hai (U. München)
V. Pilyugina (c/o U. Greifswald)
R. Pink (Mannheim)
T. Pirashvili (Tbilisi/z.Zt.MPI)
V. Platonov (Minsk/z.Zt.MPI)
M. Polyak (UC Berkeley/z.Zt.MPI)
F. Pop (Heidelberg)
V. Popov (Sydney/z.Zt.MPI)
P. Pragacz (Polish Acad.Sci./z.Zt.MPI)
Ch. Priplata (Düsseldorf)
D. Quillen (Oxford)
R. Rabi (London/z.Zt.MPI)
L. Ramero (Essen)
N. Reshetikhin (UC Berkeley)
M. Richartz (Bonn)
J.-E. Riedel (Bonn)
B. Roberts (U. of Chicago/z.Zt.MPI)
F. Rodriguez-Villegas (Princeton U./z.Zt.MPI)
M. Rosellen (Bonn)
B. Runge (MPA Humboldt-U. Berlin)
Saito, Masa-Hiko (Kyoto)
G. Salomonsen (Aarhus/z.Zt.MPI)
L. Saper (Duke U.)
A. Scharf-Knappe (Bonn)
P. Schenzel (MPA Humboldt-U. Berlin)
G. Schmalz (Bonn)
U. Schmickler-Hirzebruch (Vieweg Verlag)
A. Schmitz-Tewes (Bonn)
P. Schmutz (MPI)
A. Schönhage (Bonn)
R. Schoof (Rom II)
M. Schröder (Mannheim)
R. Schulze-Pillot (Köln/z.Zt.MPI)
L. Schwachhöfer (U. of Pennsylvania/z.Zt.MPI)
J. Schwermer (Eichstätt)
W.K. Seiler (Mannheim)
St. Shatz (U. of Pennsylvania)
I. Shimada (Hokkaido U./z.Zt.MPI)
O. Sick (Bonn)
A. Silverberg (Ohio State U.)
M. Smirnov (Princeton U./z.Zt.MPI)
A. Sofer (z.Zt.MPI)
B. Steinert (Bonn)
M. Stephanoudakis (z.Zt.MPI)
A. Stipsicz (UC Irvine)
M. Stoll (Bonn)
J. Strooker (Utrecht)
Sun, Xiaotao (Acad.Sinica/z.Zt.MPI)
H. Tamanoi (I.H.E.S.)
R. Taylor (Oxford)
M. Teicher (Bar Ilan U.)
A. Tonks (z.Zt.MPI)
Y. Tschinkel (Harvard)
P. Turner (Oxford/z.Zt.MPI)
K. Ueno (Kyoto)
J. Urbanowicz (Warschau/z.Zt.Bonn)
A. Vasiu (z.Zt.MPI)
B. Vettel (Bonn)
E. Viehweg (Essen)
M. Walgenbach (Dortmund)
Wang, Ron (z.Zt.MPI)
B. Wajnryb (Haifa/z.Zt.MPI)
Ch. Weber (McMaster U./z.Zt.MPI)
M. Weber (Bonn)
H. Wente (U. of Toledo/z.Zt.Bonn & MPI)
U. Weselmann (Bonn)
G. Wüstholz (ETH Zürich)
D. Zagier (MPI & Utrecht)
E. Zeidler (Leipzig)
E.-W. Zink (Humboldt-U. Berlin)

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

and

Mathematisches Institut
der Universität Bonn
Wegelerstraße 10
53115 Bonn

Program of the Mathematische Arbeitstagung 1995 (I)

Wednesday, June 14, 1995

3:30 – 4:15 p.m.	Opening and first program discussion
5:00 – 6:00 p.m.	M. KONTSEVICH (Berkeley and IHES) The moduli space of all Calabi-Yau threefolds
8:00 – 9:50 p.m.	Chamber Music Concert: Knopp-Melançon Duo Aula der Universität, Hauptgebäude (entrance from "Am Hof" street across from Bouvier bookstore)

Thursday, June 15, 1995

10:15 – 11:15 a.m.	F. POP (Heidelberg) Characterization of fields by their Galois groups
12:00 – 1:00 p.m.	M. SMIRNOV (Princeton U.) Radon transform and the 4th Hilbert problem
5:00 – 6:00 p.m.	T. MROWKA (Harvard) Seiberg-Witten invariants and 4-manifolds, I
8:00 – 11 (?) p.m.	Rector's Party Festsaal der Universität, Hauptgebäude (entrance from "Am Hof" street across from Bouvier bookstore)

Friday, June 16, 1995

10:00 – 10:20 a.m.	Program discussion for the Saturday and Sunday lectures
10:30 – 11:30 a.m.	J. MORGAN (Columbia U.) Seiberg-Witten invariants and 4-manifolds, II
1:00 – 8:00 p.m.	Boat trip to Unkel Departure with "Carmen Sylva" near the Kennedy Bridge

All lectures will take place in the "Großer Hörsaal," Wegelerstraße 10.

There will be *tea breaks* on Wednesday from 4:15 to 5 p.m. and on other days (except Friday) from 11:15 to 12 a.m. and 4:15 to 5 p.m. At this time also *mail* will be distributed and you will have the opporunity to pay your *Tagungsbeitrag* of 20 marks.

Lists of participants and other information will be available at the entrance of the lecture room. All participants are requested to put their name on the list!

All Arbeitstagung participants and those accompanying them are warmly invited to the chamber music program on Wednesday evening and to the party on Thursday.

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

and

Mathematisches Institut
der Universität Bonn
Wegelerstraße 10
53115 Bonn

Program of the Mathematische Arbeitstagung 1995 (II)

Saturday, June 17, 1995

10:15 – 11:15 a.m.	R. BORCHERDS (Berkeley) Product expansions of automorphic forms
12:00 – 13:00 p.m.	F. OORT (Utrecht) Resolution of singularities by alterations (a report on work of A.J. de Jong)
5:00 – 6:00 p.m.	A. BUIUM (Romanian Acad.Sci. Bucarest) p-adic jets, Manin maps and related questions

Sunday, June 18, 1995

10:00 – 10:15 a.m.	Program discussion for the Monday and Tuesday lectures
10:15 – 11:15 a.m.	C-L. CHAI (U. of Pennsylvania) Density of ordinary abelian varieties in the moduli space
12:00 – 13:00 p.m.	R. TAYLOR (Oxford) Galois representations and modular forms, I
5:00 – 6:00 p.m.	E. LOOIJENGA (Utrecht) On the tautological ring of M_g

All lectures will take place in the "Großer Hörsaal," Wegelerstraße 10.

There are daily *tea breaks* from 11:15 to 12 a.m. and 4:15 to 5 p.m. At this time also *mail* will be distributed and you will have the opporunity to pay your *Tagungsbeitrag* of 20 marks.

Lists of participants and other information will be available at the entrance of the lecture room. All participants are requested to put their name on the list!

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

and

Mathematisches Institut
der Universität Bonn
Wegelerstraße 10
53115 Bonn

Program of the Mathematische Arbeitstagung 1995 (III)

Monday, June 19, 1995

10:15 – 11:15 a.m.	G. FALTINGS (MPI) Tate cycles
12:00 – 13:00 p.m.	R. PINK (Mannheim) Mumford-Tate conjecture
5:00 – 6:00 p.m.	B. LEEB (U. Bonn) Rigiditiy of spaces of non-positive curvature

Tuesday, June 20, 1995

10:15 – 11:15 a.m.	U. BUNKE (Berlin) Spectral theory on symmetric spaces: group cohomology and ζ-functions
12:00 – 13:00 p.m.	I. HAMBLETON (McMaster U.) On topological similarities of representations of finite groups
5:00 – 6:00 p.m.	R. TAYLOR (Oxford) Galois representations and modular forms, II

All lectures will take place in the "Großer Hörsaal," Wegelerstraße 10.

There are daily *tea breaks* from 11:15 to 12 a.m. and 4:15 to 5 p.m. At this time also *mail* will be distributed and you will have the opporunity to pay your *Tagungsbeitrag* of 20 marks.

Lists of participants and other information will be available at the entrance of the lecture room. All participants are requested to put their name on the list!

Titel: The moduli space of all Calabi-Yau threefolds

Autor: M. Kontsevich
Adresse: UC Berkeley and IHES

This talk is about an old idea of M. Reid (in complex geometry) and a recent discovery of A. Strominger (in string theory)
+ many speculations ...

<u>STRING THEORY</u>: The space-time for heterotic strings compactified to 4 dimensions is of the form $\mathbb{R}^4$ (Minkowski space-time) $\times X$ where X is a Calabi-Yau 3-fold, i.e. a compact Riemannian manifold with holonomy $\subset SU(3)$.

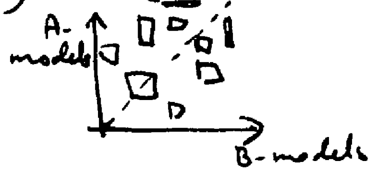
<u>Theorem</u> (Yau): $\{$CY manifolds$\} \overset{1:1}{\longleftrightarrow} \{$compact complex 3-fold X with $K_X = 0$, X has a holomorphic volume element $vol \in H^0(X, \Lambda^3 T^*_X)$, $vol \neq 0$ everywhere + a Kähler class $[\omega] \in H^2(X; \mathbb{R})$ $\}$.

Additional parameter in string theory:
B-field $\in H^2(X; i\mathbb{R})$

Locally the moduli space of string theories
$= \left(\begin{array}{c}\text{moduli of complex structures}\\ \mathcal{M}od(X)\end{array}\right) \times \underbrace{\left(\begin{array}{c}\text{moduli of } [\omega]\\ \text{and B-field}\end{array}\right)}_{\text{domain in } H^2(X;\mathbb{C})}$

Globally each component is a product of two Kähler manifolds
$\left(\begin{array}{c}\text{moduli of B-model}\\ = \mathcal{M}od(X)\end{array}\right) \times \left(\text{moduli of A-model}\right)$

Total moduli space:



each box = product of two algebraic varieties$/\Gamma$

Now $\sim$8.000 families are known.
Picture is symmetric (Mirror Symmetry).

<u>Unification of strings</u>: A. Strominger (April 95) added a "glue" (black holes) and the total space is connected.
B-projection is a part of a beautiful moduli space $\mathcal{M}$

MILES REID'S MODULI SPACE (8 years ago)

Pre-definition: $\mathcal{M} :=$ the set of birational types of compact complex 3-folds X with $K_X = 0$, $\pi_1 X = 0$.
X is not Kähler in general, no string theory...

Basic construction (H. Clemens, R. Friedman)

Consider rational curves C in X.
A good situation: all C are smoothly embedded $C \simeq \mathbb{CP}^1 \hookrightarrow X$, do not intersect each other and $N_{X/C} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.
One expects that it holds generically.

Step 1 Contract finitely many $(-1,-1)$-curves $\{C_i\}$ to points. We get a singular space X' with quadratic singularities $\sum_1^4 x_j^2 = 0$.

Step 2 Flat deformations X'' of X' $\left(\sum_1^4 x_j^2 = \varepsilon_i\right)$

$$0 \to H^1(X, T_X) \to \underset{\text{1st order deformations}}{T^1} \to \mathbb{C}^{\{C_i\}} \xrightarrow{C_i \mapsto [C_i]} H_2(X; \mathbb{C}) \to \underset{\text{obstructions}}{T^2} \to 0$$

If we have enough curves, $[C_i]$ span $H_2(X; \mathbb{Z})$ then $T^2 = 0$ and moduli space $Mod(X')$ is smooth.
$Mod(X')$ is stratified by singularities of X''.

Theorem As a stratified space $Mod(X') \sim$ domain in $H^1(X, T_X) \times$ domain in $\mathrm{Ker}\left(\mathbb{C}^{\{C_i\}} \twoheadrightarrow H_2(X; \mathbb{C})\right)$ with the stratifications coming from coordinates in $\mathbb{C}^{\{C_i\}}$.

Deformed manifold X'' from the open stratum has $\pi_1 X'' = 0$, $K_{X''} = 0$ and $H^2(X'', \mathbb{Z}) = 0$.
Such manifolds are diffeomorphic to $(S^3 \times S^3) \# (S^3 \times S^3) \# \ldots \# (S^3 \times S^3)$ (C.T.C. Wall)

Conjecture (M. Reid) $\mathcal{M}$ is connected.
For a large group of examples it was checked by physicists, but the physical meaning was not clear

MIRROR SYMMETRY AND GEOMETRY OF $\mathcal{M}$

We construct for each CY 3-fold X two Lagrangean analytic cones.

<u>B-model</u>: pick a symplectic base in $H^3(X;\mathbb{C}) \sim \mathbb{C}^{2b+2}$

$H^{3,0}(X) \subset \mathbb{C}^{2b+2}$ is a 1-dim subspace depending on $b = \operatorname{rk} H^1(X, T_X) = h^{2,1}(X)$ parameters.

$$\mathcal{L}_B(X) := \bigcup_{\tilde{X}:\ \text{deformed } X} H^{3,0}(\tilde{X}) \subset \mathbb{C}^{2b+2}$$ is a Lagrangean cone.

<u>A-model</u>: define a function in a domain in $H^2(X;\mathbb{C})$:

$$F([\omega]) = \int_X \frac{\omega^3}{3!} + \sum_{C:\ \text{rational curves}} \mathrm{Li}_3\left(\exp \int_C \omega\right)$$

$$\mathrm{Li}_3(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^3}$$

$$F^{(2)}([\omega], t) := t^2 F^{(1)}\left(\frac{[\omega]}{t}\right)$$ homogeneity degree $=2$.

t: additional variable

$\mathcal{L}_A(X) :=$ graph of differential of $F^{(2)}$

$\mathcal{L}_A(X) \subset \mathbb{C}^{2a+2}$, $a := \operatorname{rk} H^2(X)$, again a Lagrangean cone

<u>Mirror Symmetry</u>: for dual CY 3-folds X, Y

$$\mathcal{L}_A(X) \sim \mathcal{L}_B(Y), \quad \mathcal{L}_B(X) \sim \mathcal{L}_A(Y)$$

(after symplectic linear transformations).

Solutions of Picard-Fuchs equations give predictions for numbers of rational curves.

Consider "universal" cone $\mathcal{L}_B$ for Reid's moduli space $\mathcal{M}$. It has strata $\mathcal{L}_B(X)$ singularities in normal directions are governed by linear relations between fundamental classes of rational curves.

MS is a duality between strata in $\mathcal{L}_B$:

interchanging

Shape and singularity

It is analogous to the Fourier transform.

MASTER EQUATION (a very wild guess)

Let $F^{(2)}$ be a function of homogeneity degree $=2$. The master equation is:

Fourier transform $(\exp F^{(2)}) = \exp(\text{Legendre transform of } F^{(2)})$

Guess Mirror Symmetry gives a solution of it in ∞ many variables.

Examples of finite-dimensional solutions (D. Kazhdan)

0) $F^{(2)}$ = quadratic form

1) $F^{(2)}(x_i, t) = t^{-1} Q^{(3)}(x_i)$, $i=1,\dots,n$, $Q^{(3)}$ is a cubic form over a local field K

Replace exp by a character $K \to U(1)$.

$n=3$ $\quad Q = x_1 x_2 x_3$ or a norm in cubic field

$n=9$ $\quad Q = \det(x_{ij})_{3\times 3}$ + twisted forms

$n=27$ $\quad Q$ is E_6-invariant

It is the complete list of algebraic cones coming from variations of Hodge structures in MS.

2) $F = \dfrac{x_1 \cdots x_{n+2}}{y_1 \cdots y_n}$ (over local fields)

PHYSICS OF TRANSITIONS BETWEEN CY.

A. Strominger: in 10-dimensional string theory there is a conserved charge α

α is a closed self-dual 5-form. $d\alpha = 0$, $*\alpha = \alpha$

Reduction to dim $= 4$ $\quad \alpha \rightsquigarrow \int_{Z^3} \alpha \in \Omega^2(\mathbb{R}^4)$

where Z^3 is a 3-cycle in X. Closed 2-form gives a charge for point-like objects in our space $\mathbb{R}^3$ ("black holes").

It leads to new physical phenomena.

Cycles Z^3 should be "special Lagrangian" submanifolds in X.

Choose a holomorphic volume element $vol \in H^0(X, K_X)$ with $|vol| = 1$

<u>Definition</u> $Z \subset X$ is <u>special lagrangian</u> iff $\exists \varphi \in \mathbb{R}$, $Re(e^{i\varphi} vol)|_Z = 0$ + Z is lagrangian.

Such cycle Z has minimal volume in its homology class (Harvey-Lawson calibrated geometries)

$Z \to$ black hole with mass $= \dfrac{vol(Z)}{vol(X)}$

Degeneration of complex structure on X to conical singularities give new massless particles and saves physics from discontinuities. (Z is a vanishing cycle).

<u>conjecture</u> (C. Vafa) zeta-regularized

$$\prod_{\text{black holes}} \text{masses} = \prod \det (\underbrace{\Delta^{ij}}_{\text{Laplacian on } \Omega^{ij}(X)})^{\text{some powers.}}$$

References

H. Clemens, Adv. Math. 47 (1983), 107-230

R. Friedman, Proc. Symp. Pure Math. vol 53 (1991) 103-134

M. Reid, Math. Ann. 278 (1987), 329-334

P. Candelas, A. Dale, C. Lütken, R. Schimmrigk, Nucl. Phys. B298 (1988) 493-525

P. Green, T. Hübsch, Comm. Math. Phys. 119 (1988), 431-441

G. Tian, in Essays on Mirror Manifolds, 1992 (Yau, ed.)

next 3 preprints one can get sending e-mail To: hep-th@xxx.lanl.gov Subject: get ******* with empty message. (Subject: help for the help!)

A. Strominger, hep-th/9504090

B. Green, D. Morrison, A. Strominger hep-th/9504145

C. Vafa, hep-th/9505023

D. Kazhdan's calculations are not published yet.

Let κ be an arbitrary field, $\overline{\kappa}$ an algebraic closure of κ, and let $G_\kappa = \mathrm{Aut}\,(\overline{\kappa}|\kappa)$ denote the absolute Galois group of κ. The aim of my talk was to present the following result and to give the idea of its proof:

Main Theorem. *Let K and L be finitely generated fields of characteristic zero. Then $G_K \cong G_L$ implies $L \cong K$. More precisely, there exists an functorial bijection*

$$\mathrm{Isom}\,(L,K) \longrightarrow \mathrm{Out}\,(G_K, G_L),$$

where Out *denotes outer isomorphisms of profinite groups. Equivalently, for every isomorphism of profinite groups $\Phi : G_K \to G_L$ there exists a unique isomorphism of fields $\phi : \overline{L} \to \overline{K}$ such that*

$$\Phi(g) = \phi^{-1}\, g\, \phi \quad (g \in G_K),$$

hence in particular, ϕ maps L isomorphically onto K.

Questions relating the two basic objects *fields* K on the one side and *absolute Galois groups* G_K on the other side go back to the well known Artin-Schreier Theorem which mainly asserts that $G_K \cong G_{\mathbf{R}}$ implies that K is real closed. The corresponding assertion for the p-adics is much more difficult to prove, the beginning step being made by Neukirch for the case $K \subset \overline{\mathbf{Q}}$. This result of Neukirch was the starting point for the proof of the above Main Theorem in the case dim $= 1$, i.e., in the case K and L are number fields: This is the Theorem of Neukirch–Ikeda–Iwasawa–Uchida. The next step was made by the author by solving the dim $= 2$ case, and a new different proof for this case was recently given by M. Spiess.

Outline of the proof of the Main Theorem

Let $\Phi : G_K \to G_L$ be as in the Main Theorem.

A) Let $\mathcal{D}^1(K)$ denote the space of all Zariski prime divisors of K, and denote $\mathcal{D}^1(L)$ correspondingly. Then Φ gives rise to a functorial bijection $\varphi : \mathcal{D}^1(K) \to \mathcal{D}^1(L)$ which we call the *local correspondence.* From the construction of φ it follows that for given $v \in \mathcal{D}^1(K)$ and the corresponding $w \in \mathcal{D}^1(L)$ there exist "local" isomorphisms $\Phi_v : G_{Kv} \to G_{Lw}$. Thus there exist functorial isomorphisms $\phi_v : \overline{Lw} \to \overline{Kv}$ defined by Φ_v. The idea is to "interpolate" the local isomorphism ϕ_v in order to get ϕ with the desired properties.

B) We will say that a subset $D \subset \mathcal{D}^1(K)$ is geometric if it is defined by the set of all Weil prime divisors of some quasi-projective, normal model $X \to \mathbf{Q}$ of K over $\mathbf{Q}$. The next step in the proof is to show that the local correspondence φ is *geometrically continuous,* which means that φ maps every geometric set D of prime divisors of K onto a geometric set of prime divisors of L. In this proof we use in an essential way the fact that K has complete, regular models. Hence this step in the proof does not work in general.[1)]

C) Using again the local correspondence one shows that Φ is compatible with the cyclotomic characters. Thus using Kummer Theory one shows that for every prime number ℓ Φ gives rise to an isomorphism $\hat{\phi} : \hat{L} \to \hat{K}$, where $\hat{}$ denotes the ℓ-adic completion of the multiplicative groups of the fields. The main problem now is to show that $\hat{\phi}$ maps the image of $L^\times$ in $\hat{L}$ isomorphically onto the image of $K^\times$ in $\hat{L}$. For doing this one uses the induction hypothesis as mentioned at A) and secondly, the hilbertianity of the fields in discussion.

D) Next one shows that $\hat{\phi}$ is induced by a field isomorphism $\phi_K : L \to K$, which moreover, is functorial. To finish the proof of the Main Theorem one replaces K by finite extentions, say M, and L by the corresponding finite extentions, say N. One gets isomorphisms $\phi_M : N \to M$, which by the functorially, all are compatible with each other. Hence we obtain a field isomorphism $\phi : \overline{L} \to \overline{K}$ which defines Φ.

Final comments

We finally want to remark that the question/answers presented above fit into the "anabelian geometry" of Grothendieck, which roughly speaking means that certain categories of S-schemes, called "anabelian", should be mapped via the fundamental group to full subcategories of the category of all profinite groups with augmentation to $\pi_1 S$ and outer isomorphisms. Equivalently, if X and

---

[1)] On the meeting A. de Jong announced a desingularization result by "alterations", which nevertheless would suffice for our purposes. Thus the Main Theorem seems to hold in all characterisitcs.

Y are objects of some anabelain category, there there should exist a functorial bijection

$$\mathrm{Isom}_S(X, Y) \longrightarrow \mathrm{Out}_{\pi_1 S}(\pi_1 X, \pi_1 Y).$$

The Main Theorem above gives a positive unswer to the *Fundamental conjecture of the birational anabelian geometry* in characteristic zero, which asserts that via the fundamental group the category of all finitely generated fields of characteristic zero is equivalent to a full subcategory of the profinite groups and outer isomorphisms of such groups.

There is already some progress in the study of anabelian curves, but here one is far away from the "anabelian dream" of Grothendieck. One should mention here the results of Nakamura which essentially use deep results of Deligne and Ihara on the arithmetic Galois action of the ground field on the geometric fundamental group of curves.

Hilberts 4-th Problem, Radon Transform and Analogs of Differential Forms

Mikhail Smirnov

Princeton University and Max-Planck-Institut für Mathematik, Bonn*)

1. Introduction.

In his talk at the International Congress of Mathematicians in Paris in 1900 D.Hilbert formulated his famous 23 problems. The 4-th problem was entitled "the problem of lines as shortest curves between two points. " This problem was formulated quite broadly but it inspired a lot of works in geometry. The motivation for this problem came from the Hilbert's own work on foundations of geometry.

The first progress on this problem was made by H.Hamel in 1903 [Ha]. It was followed by the work of P.Funk in 1930 [Fu], J.Douglas [D] in 1942 and finally by a series of works by H.Busemann [Bu] on the geometry of geodesics. Later appeared a book by A.V.Pogorelov devoted to the Hilbert's 4-th problem.

The Hilbert's problem and its generalization that we deal with can be formulated as follows. To describe all Finsler metrics [1]) or more general Lagrangians whose extremals are straight lines. More generally this problem can be formulated in this form: to describe all k-dimensional Lagrangians in $\mathbf{R}^n$ such that all k-dimensional planes are their extremals (of course they have other extremals too).

The solution that is given here uses the Radon transform connected with a pair of Grassmannians (section 6). The results on the Radon transform are described in details in the previous works [GS1], [GS2]. The solution that uses symplectic geometry will appear in the forthcoming paper of Alvarez, Gelfand and Smirnov [AGS].

First, we describe Finsler metrics in $\mathbf{R}^n$ whose extremals are straight lines. They are solutions of a system of PDE's and they can also be described as the image under the Radon transform of positive measures on the space of hyperplanes in $\mathbf{R}^n$.

Second, we describe all k-dimensional Lagrangians in $\mathbf{R}^n$ that have all k-dimensional planes among their extremals. These Lagrangians are thr solutions of a system of PDE's and this system coincides with the lower order terms of the Euler-Lagrange equations.

On the other hand, these Lagrangians are the images under the Radon transform of the measures on the space of $(n-k)$-planes in $\mathbf{R}^n$. This Radon transform, first defined in the previous work [GS1] is different from the classical Radon transform, but the system of PDE's for k-Lagrangians can be interpreted as the standard F.John's system of PDE's describing the image of this special Radon transform.

The k-dimensional Lagrangians introduced here are analogues of closed differential forms. Other analogues of differential forms and De Rham complex were studied by M. Baranov and A.S.Shvarts [BS]. Although their approach to the approach presented here have similarities they are somewhat different.

2. Even and Odd Densities and the Crofton Formula.

---

*)From September 1, 1995 Columbia University, New York

[1]) Let us remind the definition of a Finsler metric. It is a function $F(x,v)$ of a point x in the domain in $\mathbf{R}^n$ and a tangent vector $v \in T_x$ that is smooth in x, v, that is positive homogeneous in v: $F(x, \lambda v) = \lambda F(x,v)$ for all $\lambda \in \mathbf{R}$, and such that the indicatrix of F defined as $Ind_x F = \{v \in T_x | F(x,v) = 1\}$ is a twice differentiable closed *convex!* hypersurface in T_x. The Riemannian metric is a special example of a Finsler metric when $F(x,v) = \sqrt{quadratic\ form\ in\ v}$

Consider in $\mathbf{R}^n$ a k-dimensional manifold M^k. Suppose M^k has the following parametrization: $M^k = \{x(s_1,\dots,s_k)\in\mathbf{R}^n,\ \text{where } s\in\Omega^k\in\mathbf{R}^k\}$ We want a functional

$$S[M^k]=\int_{\Omega^k}L(x(s),\tfrac{\partial x}{\partial s_1},\dots,\tfrac{\partial x}{\partial s_k})ds_1\dots ds_k$$

to be independent on the parametrization of M^k. Then after a change of parametrization $s\to t$, L must change in the following way

$$L(x(t),\tfrac{\partial x}{\partial t_1},\dots,\tfrac{\partial x}{\partial t_k})=\det(\tfrac{\partial s_i}{\partial t_j})L(x(s),\tfrac{\partial x}{\partial s_1},\dots,\tfrac{\partial x}{\partial s_k})\ \text{ or}$$

$$L(x(t),\tfrac{\partial x}{\partial t_1},\dots,\tfrac{\partial x}{\partial t_k})=\left|\det(\tfrac{\partial s_i}{\partial t_j})\right|L(x(s),\tfrac{\partial x}{\partial s_1},\dots,\tfrac{\partial x}{\partial s_k}).$$

In the first case L is called an odd k-density, in the second case L is called an even k-density. The main example of odd k-densities are differential k-forms. An example of an even k density is a k-dimensional volume element.

Now let us explain what is a Crofton density or Crofton Lagrangian. It is an even density that has some additional properties. Let us replace a manifold M^k by its Crofton function. It is a function on the set $H_{n,n-k}$ of $(n-k)$-planes in $\mathbf{R}^n$. The value of the function $\mathrm{Crof}_{M^k}(\xi)$ on the $(n-k)$-plane ξ is equal to the number of intersection points of M^k and ξ. Crofton functions carry almost all information about the original manifold M^k.

Crofton Lagrangians (or Crofton densities) are such densities for which the Crofton formula is valid: i.e. there exists a measure $\mu(\xi)d\xi$ on the set $H_{n,n-k}$ of $(n-k)$-planes such that for every manifold $M^k\in\mathbf{R}^n$

$$\int_{M^k}L=\int_{H_{n,k}}\mathrm{Crof}_{M^k}(\xi)\mu(\xi)d\xi.$$

For example, according to the classical Crofton formula of the integral geometry an element of k-dimensional nonoriented volume in $\mathbf{R}^n$ is an even Crofton k-density. A Crofton density depends on the measure $\mu(\xi)d\xi$ on $H_{n,n-k}$. We shall write an explicit formula for a Crofton density by means of a special kind of $(n(n-k))$-dimensional Radon transform of the measure μ.

Example. Consider an even 1-density L in $\mathbf{R}^2$ given by the formula

$$L(x_1,x_2;v_1,v_2)=\frac{2(x_1v_1+x_2v_2)^2-(v_1^2+v_2^2)(x_1^2+x_2^2+1)}{(x_1^2+x_2^2+1)^2((v_1^2+v_2^2)(x_1^2+x_2^2+1)-(x_1v_1+x_2v_2)^2)^{\frac{1}{2}}}.$$

Let us parametrize lines in $\mathbf{R}^2$ by the equation $x_2=ax_1+b$. In coordinates a,b the corresponding dual measure $\mu(a,b)dadb$ is

$$\mu(a,b)=\frac{1}{2}|v_1|\frac{\partial^2}{\partial v_2^2}L(x_1,\ x_2;\ v_1,v_2)\Bigg|_{\substack{v_2=av_1\\ x_2=ax_1+b}}=\frac{-1-a^2+2b^2}{2(1+a^2+b^2)^{\frac{5}{2}}}.$$

The extremals of the problem $\int L(x_1(t),x_2(t),\dot{x}_1(t),\dot{x}_2(t))dt$ are straight lines.

3. System of PDE for Lagrangians solving Hilbert's Problem.

Theorem 1. *If a function $L(x;\ \mathbf{v}_1,\dots,\mathbf{v}_k)$ of a point $x\in\mathbf{R}^n$ and k-tangent vectors $\mathbf{v}_i=(v_i^1,\dots,v_i^n)\in T_x\mathbf{R}^n$ is an even Crofton density then it satisfies for all x and all $\mathbf{v}_1,\dots,\mathbf{v}_k$: $|\mathbf{v}_1|+\dots+|\mathbf{v}_k|\neq 0$ the following* *equations:*

$$\left(\sum_{s=1}^{n} v_i^s \frac{\partial}{\partial v_j^s}\right) L = \delta_{ij} L, \qquad i,j = 1,\ldots,k, \tag{1}$$

$$\text{and} \quad \left(\sum_{p=1}^{k}\sum_{i=1}^{n} v_p^i \frac{\partial^2}{\partial v_p^l \partial x^i}\right) L = \frac{\partial}{\partial x^l} L, \qquad l = 1,\ldots,k. \tag{2}$$

If L is a function of a point $x \in \mathbf{R}^n$ and k-tangent vectors $\mathbf{v}_i = (v_i^1, \ldots, v_i^n) \in T_x\mathbf{R}^n$ which is even in $\mathbf{v}_i$ and satisfies (1) *and* (2) *then it is a Crofton even k-density.*

The density L defines a functional

$$S[M^k] = \int_{\Omega^k} L(x(t), \tfrac{\partial x}{\partial t_1}, \ldots, \tfrac{\partial x}{\partial t_k}) dt_1 \ldots dt_k.$$

We want to find its extremal k dimensional surfaces. Let us write general Euler-Lagrange equations for k-dimensional extremals $x^m(t^1, \ldots, t^k)$ of the functional S. They are

$$\left(-\frac{\partial}{\partial x^l} + \sum_{p=1}^{k}\sum_{i=1}^{n} \frac{\partial x^i(t)}{\partial t^p}\frac{\partial^2}{\partial v_p^l \partial x^i} + \sum_{p=1}^{k}\sum_{j=1}^{k}\sum_{i=1}^{n} \frac{\partial^2 x^m(t)}{\partial t^j \partial t^p}\frac{\partial^2}{\partial v_p^s \partial v_j^i}\right) L(x(t), \tfrac{\partial x}{\partial t_1}, \ldots\ldots, \tfrac{\partial x}{\partial t_k}) = 0,$$

where $s = 1, \ldots, k,\ m = 1, \ldots, n$.

Theorem 2. *Differential operator* (2) *in the equations for a Crofton k-density coincide with the terms of Euler-Lagrange operator for L which do not contain second derivatives in v:*

$$-\frac{\partial}{\partial x^l} + \sum_{p=1}^{k}\sum_{i=1}^{n} v_p^i \frac{\partial^2}{\partial v_p^l \partial x^i} \tag{3}$$

From the last theorem it is possible to deduce

Theorem 3. *An even k-density satisfies the Crofton formula if and only if all k-planes in $\mathbf{R}^n$ are contained among its extremals.*

4. Finsler metrics in $\mathbf{R}^n$ whose extremals are sraight lines.

We consider 1-Lagrangians $L(x, \dot{x})$ that are even densities. If the indicatrix $Ind_x L = \{v \in T_x : L(x,v) = 1\}$ is convex (and twice differentiable) for all x then this Lagrangian is a *Finsler metric.*

Consider an even 1-density for which the Crofton formula is valid. It depends on a measure on the space of hyperplanes in $\mathbf{R}^n$ and can be expressed through this measure by the Radon transform (see section 5 and [GS1]).

Proposition 5. *The indicatrix of the Crofton 1-density with the positive dual function is strictly convex.*

So such density is a Finsler metric whose geodesics are straight lines. For 1-densities the equations for Lagrangians whose geodesics are straight lines can be written in a particularly simple form. One can find them already in the work of Hamel [Ha] and Funk [Fu]. strangely enough, P.Funk who along with Radon is the "father of the Radon transform" did not notice how the Radon transform can be applied to the Hilbert's 4-th problem, although he himself worked on that problem. H.Busemann [Bu2] noticed the connection between the Hilbert's 4-th problem and the classical integral geometry in the sense of Buffon, Crofton, Poincaré and Chern, but not to the Radon transform.

Now let us write the equations for Crofton 1-density.

Theorem 4. *An even 1-density $L(x,v)$ satisfies the Crofton formula if and only if*

$$\frac{\partial^2 L}{\partial x_i \partial v_j} - \frac{\partial^2 L}{\partial x_j \partial v_i} = 0, \quad \text{for} \quad all \quad i,j = 1,...,n, \quad and \; v \neq 0.$$

This is equivalent to the property that all extremals of l are straight lines.

Remark. These equations can be deduced from equations (2) of the theorem 1 using the homogeneity equations (1).

5. Radon transform for a pair of Grassmannians $G_{n+1,n}$ and $G_{n+1,k}$ and k-Lagrangians.

The Radon transform in the general situation maps a geometric object on the source space (function, differential form, connection) into integrals of this object over some family of submanifolds in the source space. So it is a mapping from geometric objects on the source space into geometric objects on a target space that is the family of submanifolds in the source space.

Let us define now the Radon transform that is used here. Let $\rho(\zeta)$ be a function on a manifold $H_{n,n-k}$ of $(n-k)$-planes in $\mathbf{R}^n$. To each $x \in \mathbf{R}^n$ we associate a variety $H_x = \{\zeta \in H_{n,n-k} | \zeta \text{ passes through } x\}$. To every pair (x,ℓ) where $x \in \mathbf{R}^n$, and ℓ is a k-subspace of a tangent space T_x we associate a pair: a variety H_x and a measure σ_ℓ on H_x which is a $k(n-k)$-dimensional subvariety in $H_{n,n-k}$.

The even Radon transform is given by

$$\phi(x,\ell) = \int\limits_{H_x} \rho(\zeta) d\sigma_\ell.$$

We can write now an explicit formula expressing a Crofton density L in terms of μ using the even Radon transform connected with a pair of Grassmann manifolds $G_{n+1,n}$ and $G_{n+1,k}$. This transform will be defined below. We shall do it in several steps.

1. We regard $\mathbf{R}^n_x$ as an affine part of $G_{n+1,n}$.

2. We go from the measure μ on $H_{n,n-k}$ to the function $\tilde{\mu}$ on a frame manifold $E_{n+1,k}$.

3. For every $x \in \mathbf{R}^n$ and k-tangent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in the point x we construct a differential $k(n-k)$-form Ω on the frame manifold $E_{n,k}$. Actually this form is constructed using not x but vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ which spans n-space from $G_{n+1,n}$ corresponding to the point x.

4. The $k(n-k)$-form Ω can be pulled down from the frame manifold $E_{n,k}$ which is a bundle over $G_{n,k}$ to a Grassmann manifold $G_{n,k}$ and

$$L(x,v) = \int_{G_{n,k}} \Omega_*.$$

Let us repeat this construction in details.

1. Compactification of $\mathbf{R}^n$ and $H_{n,n-k}$ Let us consider $\mathbf{R}^n_x$ as an affine part of $G_{n+1,n}$. Then the set of all $(n-k)$ planes in $\mathbf{R}^n_x$, which we have denoted by $H_{n,n-k}$, can be compactified to the Grassmannian $G_{n+1,n-k+1}$, which is canonically isomorphic to $G_{n+1,k}$. Consider two manifolds of frames: $E_{n+1,k}$, the manifold of k-frames in $\mathbf{R}^{n+1}$ and $E_{n+1,n}$, the manifold of n-frames in $\mathbf{R}^{n+1}$. They are bundles over $G_{n+1,k}$ and $G_{n+1,n}$ correspondingly.

2. Construction of $\tilde{\mu}$. To every measure $\mu(\zeta)d\zeta$ on $H_{n,n-k} \subset G_{n+1,n-k+1}$ we correspond in a canonical way a function $\tilde{\mu}(\mathbf{w})$ of a k-frame $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_k) \in E_{n+1,k}$ that has the property

$$\tilde{\mu}(A\mathbf{w}) = |\mathrm{det} A|^{-n-1} \tilde{\mu}(\mathbf{w}), \text{ for } \quad A \in GL(k, \mathbf{R}). \tag{7}$$

Because $G_{n+1,n-k+1}$ and $G_{n+1,k}$ are canonically isomorphic, every measure μ on $G_{n+1,n-k+1}$ is isomorphic to a measure μ' on $G_{n+1,k}$. We now consider a local coordinate system on $G_{n+1,k}$ given by

$$\begin{pmatrix} c_1^1 & \dots & c_q^1 & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ c_1^k & \dots & c_q^k & 0 & \dots & 1 \end{pmatrix},$$

where $q = n-k+1$. In local coordinates $||c_j^i||$ the measure μ' can be written as $\mu'(c_1^1, \dots, c_k^q)\ dc_1^1 \dots dc_k^q$, where $\mu'(c_1^1, \dots, c_k^q)$ is a function of $||c_j^i||$. For a k-frame $\mathbf{w}' = (\mathbf{w}'_1, \dots, \mathbf{w}'_k) \in E_{n+1,k}$ with $\mathbf{w}'_1 = (c_1^1, ..., c_q^1, 1, \dots, 0)$, $\dots$, $\mathbf{w}'_k = (c_1^k, ..., c_q^k, 0, \dots, 1)$ we define $\tilde{\mu}(\mathbf{w}')$ by the equality

$$\tilde{\mu}(\mathbf{w}') = \mu'\ (c_1^1, \dots, c_k^q)$$

and we extend $\tilde{\mu}$ for arbitrary $\mathbf{w} \in E_{n+1,k}$ using (7).

3. Construction of a form Ω. Consider now a point $x \in \mathbf{R}^n \subset G_{n+1,n}$. We can look at x as at the n-subspace $L_x \subset \mathbf{R}^{n+1}$. Let $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ be a basis of L_x. Then any k-frame $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_k)$ in L_x can be written in the unique way as

$$(\mathbf{w}_1, \dots, \mathbf{w}_k) = (\mathbf{u}_1, \dots, \mathbf{u}_n) \begin{pmatrix} t_1^1 & \dots & t_k^1 \\ \vdots & & \vdots \\ t_1^n & \dots & t_k^n \end{pmatrix}.$$

So when we fix a basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ of L_x the components of the matrix $T = ||t_i^j||$ form the local coordinate system on the manifold $E_{n,k}$ of k-frames in L_x. We define differential forms $\sigma_1, \dots, \sigma_k$ on $E_{n,k}$ as

$$\sigma_i = \sum_{(p_1 \dots p_n)} (-1)^s t_1^{p_1} t_2^{p_2} \dots t_k^{p_k} dt_i^{p_{k+1}} \dots dt_i^{p_n}, \quad i = 1, ..., k$$

where s is the sign of the permutation $(p_1, p_2, \dots, p_n)$. Symbolically σ_i can be written as

$$\sigma_i = \det \begin{pmatrix} t_1^1 & \dots & t_k^1 & dt_i^1 & \dots & dt_i^1 \\ \vdots & & \vdots & \vdots & & \vdots \\ t_1^n & \dots & t_k^n & dt_i^n & \dots & dt_n^i \end{pmatrix},$$

where we multiply differentials dt_i^j by the exterior product.

For a basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ of $L_x \in G_{n+1,n}$ and k tangent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in T_x\mathbf{R}^n$ we define a differential form Ω on $E_{n,k}$ as

$$\Omega(\mathbf{u}_1, \dots, \mathbf{u}_n;\ \mathbf{v}_1, ..., \mathbf{v}_k) =$$

$$= \tilde{\mu}((\mathbf{u}_1 \dots \mathbf{u}_n)T\,) \left| \sum_{i_1 < \dots < i_k} p_{i_1 \dots i_k} T_{i_1 \dots i_k} \right| \sigma_1 \wedge \dots \wedge \sigma_k,$$

$$\text{where} \quad p_{i_1 \dots i_k} = \det \begin{pmatrix} v_1^{i_1} & \dots & v_k^{i_1} \\ \vdots & & \vdots \\ v_1^{i_k} & \dots & v_{i_k}^k \end{pmatrix}, \ \text{and} \ T_{i_1 \dots i_k} = \det \begin{pmatrix} t_1^{i_1} & \dots & t_k^{i_1} \\ \vdots & & \vdots \\ t_1^{i_k} & \dots & t_k^{i_k} \end{pmatrix},$$

$1 \le i_1 < \dots < i_k \le n$.

4. Pull down of Ω to $G_{n,k}$. We can see that when we evaluate form Ω on $k(n-k)$ tangent vectors to $E_{n,k}$, components of these tangent vectors which are tangent to the fibers of the bundle $E_{n,k} \to G_{n,k}$ does not play any role. So Ω can be pulled down to $G_{n,k}$.

Theorem 5. *Every Crofton k-density $L(x;\ \mathbf{v}_1, ..., \mathbf{v}_k)$ in $\mathbf{R}^n$ can be represented as*

$$L(x;\ \mathbf{v}_1, ..., \mathbf{v}_k) = \int_{G_{n,k}} \Omega_*(\mathbf{u}_1, \dots, \mathbf{u}_n;\ \mathbf{v}_1, ..., \mathbf{v}_k),$$

where Ω_* is the form Ω pulled down to $G_{n,k}$ and $\mathbf{u}_1 ... \mathbf{u}_n$ is any n-frame spanning the space $L_x \in G_{n+1,n}$ corresponding to the point $x \in \mathbf{R}^n$.

6. Nonlocal differentials and the Hilbert transform for 1-Lagrangians.

We shall explain now in what sense even Crofton 1-densities can be viewed as "differentials" of functions. Let f be a function in $\mathbf{R}^n$. The usual differential df, which depends on a point x from $\mathbf{R}^n$ and a tangent vector v from $T_x\mathbf{R}^n$ can be written as $(df)(x,v) = \int_{-\infty}^{+\infty} \delta'(t) f(x - vt)dt$. The *nonlocal (or even) differential* of a function f is

$$(d^o f)(x,v) = \int_{-\infty}^{+\infty} \frac{1}{t^2} f(x - vt)dt = \int_0^{+\infty} \frac{f(x+vt) + f(x-vt) - 2f(x)}{t^2} dt.$$

We suppose that f is such that this integral converges at infinity.

Theorem 6. *Let $f(x)$ be a rapidly decreasing function in the Schwartz space $S(\mathbf{R}^n)$. Let f be the Radon transform of a function $F(\alpha)$: $f(x_1, \dots, x_n) = \int_{-\infty}^{+\infty} F(\alpha_1, \dots, \alpha_{n-1},\ x_n - (\alpha_1 x_1 + \dots + \alpha_{n-1}x_{n-1}))d\alpha_1 \dots d\alpha_{n-1}$. Let $L(x,v) = d^0 F$ be an even differential of the function F. Then*

(1) *$L(x,v)$ is a Crofton density in $\mathbf{R}^n$.*

(2) *Let $D^o_{\alpha_n} F(\alpha_1, \dots, \alpha_n)$, be the "even partial derivative" of F defined as $D^o_{\alpha_n} F(\alpha_1, \dots, \alpha_n) = \int \frac{1}{t^2} F(\alpha_1, \dots, \alpha_n - t)dt$. Then the the dual measure for L is $\mu(\alpha)d\alpha$ where $\mu(\alpha_1, \dots, \alpha_n) = D^o_{\alpha_n} F(\alpha_1, \dots, \alpha_n)$.*

Let us remind that an *even 1-density* is a function $L(x,v)$ of a point $x \in \mathbf{R}^n$ and a tangent vector v at the point x such that $L(x, \lambda v) = |\lambda| L(x,v)$ for every $\lambda \in \mathbf{R}$. An *odd 1-density* is a function $\theta(x,v)$ of a point $x \in \mathbf{R}^n$ and a tangent vector v such that $\theta(x, \lambda v) = \lambda \theta(x,v)$ for every $\lambda \in \mathbf{R}$. Differential 1-forms give a specific example of odd 1-densities, they are linear in v.

The 1-density $(HL)(x,v)$

$$(HL)(x,v) = P.V. \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{L(x - vt, v)}{t} dt = \frac{1}{\pi} \int_0^{+\infty} \frac{L(x+vt,v) - L(x-vt,v)}{t} dt$$

is called the *Hilbert transform* of a 1-density L. We suppose that L satisfy some growth conditions such that its Hilbert transform exists.

The Hilbert transform of an even 1-density is an odd 1-density. The Hilbert transform of an odd 1-density is an even 1-density.

Theorem 7. *The Hilbert transform of a closed 1-form with rapidly decreasing coefficients is the Crofton 1-density. The Hilbert transform of a rapidly decreasing Crofton 1-density L is a closed 1-form. ("Rapid decreasing of L" means that $L(x - vt, v)$ is a rapidly decreasing function of t for any fixed v and x). Let f be a rapidly decreasing function, $\omega = df$ is its differential and $L = \frac{1}{\pi} d^o f$ is its nonlocal differential (which is* *the Crofton 1-density). Then $(HL)(x,v) = \omega(x,v)$ and $(H\omega)(x,v) = L(x,v)$. So the Hilbert transform maps even differentials into odd ones and vice versa.*

We can summarize properties of Crofton 1-densities on the following diagram:

$$
\begin{tikzcd}
F \arrow[r, "\text{Radon transf.}"] \arrow[d, "D^{\circ}_{a_n}"'] & f \arrow[r, equal] \arrow[d, "d^{\circ}"'] & f \arrow[d, "d"'] & F \arrow[l, "\text{Radon transf.}"'] \arrow[d, "D_{a_n}"] \\
\mu \arrow[r, "\text{Even Radon tr.}"] & d^{\circ}f \arrow[r, leftrightarrow, "\text{Hilbert tr.}"] \arrow[d, equal] & df \arrow[d, equal] & n \arrow[l, "\text{Odd Radon tr.}"'] \\
 & \varphi & \omega & \\
 & \text{Crofton 1-density} & \text{Closed 1-form} &
\end{tikzcd}
$$

7. Crofton 1-Lagrangians on the hyperbolic plane.

Consider the hyperbolic plane as an upper half-plane $\{(x,y) \mid y > 0\}$ with the metric $\dfrac{dx^2+dy^2}{y^2}$. Take $z = x + iy$. The upper half-plane is a model of the Lobachevsky geometry, where Lobachevsky lines are vertical rays $\{(x,y) \mid x = x_0, y > 0\}$ and half circles $\{(x,y) \mid (x-x_0)^2 + y^2 = R^2, y > 0\}$.

Let us parametrize "lines" on the hyperbolic plane. To each half circle $\{(x,y) \mid (x-x_0)^2+y^2 = R^2, y > 0\}$ we correspond a point $\zeta = \zeta_1 + i\zeta_2$ where $\zeta_1 = x_0$ and $\zeta_2 = R$. Thus we parametrize almost all "lines" except vertical rays.

The set of all lines which come through the point (x,y) forms a branch of a hyperbola $(x-\zeta_1)^2+y^2 = \zeta_2^2$ on the dual plane ζ_1, ζ_2.

Let us as usual define even and odd 1-densities and also define Crofton densities with respect to Lobachevsky lines. For example the length element $\dfrac{dx^2+dy^2}{y^2} = \dfrac{v_1^2+v_2^2}{y^2}$ will be a Crofton 1-density.

Theorem 8. *Even Crofton densities on the Lobachevsky plane satisfy the following equation*

$$\left(\frac{\partial^2}{\partial x \partial v_2} - \frac{\partial^2}{\partial y \partial v_1} - \frac{(v_1^2+v_2^2)}{y v_1}\frac{\partial^2}{\partial v_2 \partial v_2}\right) L = 0, \quad \text{for} \qquad v \neq 0.$$

Crofton densities have the following integral representation through their dual measures on the set of Lobachevskian lines:

$$L(x,y;v_1,v_2) = \int_{-\infty}^{\infty} \mu(\zeta_1, ((x-\zeta_1)^2+y^2)^{1/2}) \frac{|yv_2 + (x-\zeta_1)v_1|}{((x-\zeta_1)^2+y^2)^{1/2}} d\zeta_1.$$

Extremals of arbitrary Crofton density are Lobachevsky lines.

8. Crofton Lagrangians as analogs of closed differential k-forms.

In the class of odd densities, densities for which the oriented analog of the Crofton formula is valid are exactly closed differential k-forms [GS1]. They have analogous representation through Radon transform. The PDE's for the image of this Radon transform are exactly conditions of closedness of the form.

Crofton k-densities give a good example of the integro-geometrictex prepr functional. There were several attempts recently to use integro-geometric functionals other then area in the functional integral in QFT. For example, Ambartsumian-Savvidi [S1] used for this Steiner functional of the classical integral geometry.

References

[AGS] Alvarez, J.C., Gelfand, I. M., Smirnov M.M., *Hilbert's 4-th Problem, Radon transform and symplectic geometry* to appear

[Am] Ambartzumian, R.V., Combinatorial Integral Geometry, Wiley NY 1982

[Bu1] Busemann, H., The Geometry of Geodesics, NY 1955

[Bu1] Busemann, H., Problem IV, In math. devel. arising from Hilbert problems, Proc. Symp. Pure math No. 28, Ed. F.Browder, AMS

[Cr] Crofton, M. W., *On the theory of local probability, Phil. Trans. Roy. Soc. London,* **158**, (1868), 181-199.

[Da] Darboux, *Lecons sur la théorie générale des surfaces,* Bd.3, s.54, Paris 1894.

[D] Douglas, J. *Solution of the inverse problem of the calculus of variations,* Trans. A.M.S. **50** (1941) 71-128.

[F] Funk, P., *Über die Geometrien, in denen die Geraden die Kürzesten sind , Monats. math. Phys.,* **37**, (1930), 152-158.

[GS1] Gelfand, I. M., Smirnov M.M., *Lagrangians satisfying Crofton formulas, Radon transforms and nonlocal differentials,* Advances in Mathematics, **109**, (1994) 188-227 .

[GS2] Gelfand, I. M., Smirnov M.M., *Crofton formulas and nonlocal differentials. in* The Gelfand Mathematical Seminars 1990-92, L.Corwin, I.Gelfand, J.Lepowsky eds., Birkhäuser, Boston 1993, 35-50.

[GG] Gelfand, I. M., Gindikin S.G., *Nonlocal inversion formulas in real integral geometry, Funct. Analysis and its Applications,* **11**, (1977) 12-19.

[H] Hamel, H., *Über die Geometrien, in denen die Geraden die Kürzesten sind , Math. Ann.,* **57**, (1903), 231-264.

[S] Hilbert, D., Gesammelte Abhandlungen, Bd. iii, s. 302 Springer

[S] Santalo, L. S., *"Integral Geometry and Geometric Probability",* Addison-Wesley, London 1976.

[H] Helgason, S., *"Groups and Geometric analysis",* Academic Press, N.Y. 1984.

[E] Ehrenpreis, L., *" Radon Transform"* (to appear).

[PS] Phong, D.H., Stein, E.M., *Hilbert integrals, singular integrals and Radon transform, Acta Math.* **157**, (1986) 99-157

[P] Pogorelov, A.V., Hilbert's Fourth Problem, Winston & Wiley NY 1982

[GS1] Guillemin, V., Sternberg, S., *Some problems in integral geometry and some related problems of microlocal analysis, Amer. J. Math.* **101**, (1979) 915-955.

[GS1] Guillemin, V., Sternberg, S., *"Geometric Asymptotics"* A.M.S. 1979

[Gr] Griffiths, P.A., *"Exterior differential systems and calculus of variations",* Boston, Birkhäuser, 1982.

[BCG] Bryant, R., Chern, S.-S., Gardner, R., Goldshmidt, H., Griffiths, P.A., *"Exterior differential systems",* N.Y., Springer, 1991.

[C] Chern, S.-S., *On the fundamental kinematic formula in the Euclidean space of n dimensions, American Journal of Mathematics* **74** (1952) 227-236.

[BS] Baranov, M. A., Schwartz, A. S. Cohomology of supermanifolds, *Funct. Analysis and its Applications,* **18**, (1984) 236-238 .

[BCG] Bryant, R., V.Guillemin, S.Helgason, R.Wells, eds. Integral Geometry, AMS 1987.

[S1] R.V. Ambartsumian, G.S. Sukiasian, G.K. Savvidi, K.G. Savvidy, Alternative model of random surfaces, Phys. Lett. B275 (1992) 99-102,

[S2] G.K. Savvidi, K.G. Savvidy, Interaction hierarchy, Phys. Lett. B337 (1994) 333-339.

Titel: Seiberg-Witten Invariants & 4-manifold topology I.

Autor: Tom Mrowka

Adresse: Dept of Mathematics
Caltech 253-37
Pasadena, CA 91125

This talk described recent progress in 4-manifold topology arising from the Seiberg-Witten Monopole Equation. In particular relations between scalar curvature and differential topology of 4-manifolds were stressed.

Many interesting problems about 4-manifolds center on the minimal genus function. Let X denote a compact, closed, oriented, connected 4-mfld. Any two dimensional homology class $\mathcal{S} \in H_2(X:\mathbb{Z})$ is represented by a smoothly embedded connected 2-manifold $\Sigma^2 \subset X^4$. Define

$$g : H_2(X:\mathbb{Z}) \to \{0,1,2,\dots\},$$

to be the minimal genus of a smoothly embedded connected 2-manifold representing the given homology class. g is clearly a differentiable invariant of the 4-manifold X.

Over the past four years Kronheimer and the author have made significant progress in understanding many properties of this function. In fact it turns out to be better to study the function

$$j : H_2(X;\mathbb{Z}) \to \mathbb{Z}$$

$$j(S) \equiv 2g(S) - 2 - S \cdot S.$$

An elementary property of j is that is j sublinear along a ray in the homology corresponding to a homology class of positive square.

Lemma: $\forall$ $S \in H_2(X;\mathbb{Z})$ with $S \cdot S > 0$

$$j(dS) \leq d\, j(S).$$

Then we have

Theorem (Kronheimer-M 1991) If X has non trivial Donaldson invariants then $\forall$ $S \in H_2(X;\mathbb{Z})$ with $S \cdot S \geq 0$ we have

$$j(S) \geq 0.$$

In 1993 we proved

Theorem: If X non trivial Donaldson invariants and $b_1(X)=0$ then $\exists$ cohomology classes $K_1, K_2, \ldots, K_s \in H^2(X;\mathbb{Z})$, uniquely determined by the Donaldson invariants, so that $\forall\ S \in H_2(X;\mathbb{Z})$ with $S\cdot S \geq 0$ we have

$$j(S) \geq \max_i |K_i(S)|,$$

and $K_i \equiv w_2(TX) \mod (2) \quad \forall i.$

The cohomology classes arising in this theorem are called the <u>basic classes</u> of X.

In October of 1994 Witten proposed a new and drastically simpler way of getting at the basic classes. Motivated by recent work with Seiberg he suggested that the following equation ought to relavant to four-manifold topology.

Recall that a $Spin_c$-structure on a 4-manifold consists of the following data. $W_\pm \to X$, a pair of Hermitian 2-plane bundles, and $\quad c: T^*X \to \mathrm{Hom}(W_+, W_-)$

satisfying $\quad c(\xi)^* c(\xi) = |\xi|^2 \mathbb{1}_{W^+}$ &

$$c(\xi)\, c^*(\xi) = |\xi|^2 \mathbb{1}_{W^-}.$$

The Seiberg-Witten monopole equation is then an equation for a pair (A, Φ) where A is a connection in L, the complex determinant line of W_+, and Φ in a section of W_+.

$$(M) \quad \begin{aligned} F_A^+ &= \sigma(\Phi) \\ \not{\partial}_A \Phi &= 0. \end{aligned}$$

Here $\sigma: W_+^- \to \Lambda^2_+(T^*X)$ is a certain quadratic bundle map obtained from the cone on the Hopf map $h: S^3 \to S^2$.

These equations are the minima of the following energy function

$$\mathcal{E}(A,\Phi) = \frac{1}{2}\int_X |D_A\Phi|^2 + \frac{1}{4}\int_X \left|\frac{s}{2} + |\Phi|^2\right|^2 + \frac{1}{4}\int_X |F_A^-|^2.$$

in fact the Weitzenbock formula for $\partial\!\!\!/_A \Rightarrow$

$$\mathcal{E}(A,\Phi) = \frac{1}{2}\int |\partial\!\!\!/_A \Phi|^2 + \frac{1}{4}\int |F_A^+ - \sigma(\Phi)|^2$$

$$+ \frac{1}{16}\int_X s^2 - 2\pi^2 c_1^2(L).$$

where s denotes the scalar curvature.

Following the lead of Donaldson's invariants one uses the solution spaces to these equations to define an invariant of smooth 4-manifolds with $b_+(X) > 1$. These invariants are a function

$$SW : \{\text{Spin}_{\mathbb{C}}\text{-structures}\} \to \mathbb{Z}.$$

Witten made a beautiful conjecture regarding the relation of SW to the Donaldson invariants of manifolds of simple type, and he and Seiberg gave a path integral arguement suggesting the validity of the relation.

The formal dimension of the solution space to (M) is

$$d = d(L) = \frac{1}{4}\left(L^2 - (2\chi(X) + 3\sigma(X)\right)$$

One of the greatest simplifications in this theory over the anti-self-dual case is the compactness of the moduli space. In fact one can use the Weitzenbock formula for $\not{D}_A$ to prove the following apriori C^0-bound for solutions to (M).

Lemma: If (A,Φ) solves (M) then

$$\max_{x\in X} |\Phi|^2(x) \leq \max_{x\in S} \{-s^2(x), 0\}$$

where s is the scaler curvature.

Proof. The Weitzenbock Formula implies

$$\langle \Phi, \not{D}_A^2 \Phi\rangle = \langle \Phi, \nabla_A^2 \Phi\rangle + \frac{s}{4}|\Phi|^2 + \frac{1}{2}\langle iF_A^+ \cdot \Phi, \Phi\rangle$$

and for a solution (A,Φ) this magically simplifies to

$$0 = \langle \Phi, \nabla_A^2 \Phi\rangle + \frac{s}{4}|\Phi|^2 + \frac{1}{4}|\Phi|^4.$$

or

$$0 = \frac{1}{2}\Delta|\Phi|^2 + |\nabla_A \Phi|^2 + \frac{s}{4}|\Phi|^2 + \frac{1}{4}|\Phi|^4.$$

At a maximum of $|\Phi|^2$ we have

$0 \geq \frac{s}{4}|\Phi|^2 + \frac{1}{4}|\Phi|^4$ and the result follows.

The determinant lines of $Spin_{\mathbb{C}}$-structures with non-trivial Seiberg-Witten invariants are called Seiberg-Witten basic classes and play the same role regarding the minimal genus problem as the basic classes.

Theorem (1994, Kronheimer-Mrowka & Morgan-Szabo-Taubes, Fintushel-Stern). If X is a four-manifold with $b_+(X) > 1$ and (W_+, W_-, c) is a $Spin_{\mathbb{C}}$-structure with $SW(W_+, W_-, c) \neq 0$ then for all $S \in H_2(X, \mathbb{Z})$ with $S \cdot S \geqslant 0$ we have

$$j(S) \geqslant |c_1(\det_{\mathbb{C}}(W_+)) \cdot S| .$$

One of the proofs exploits the fact that using the existance of the embedded surface, Σ, one can coustruct a family of metrics on X with a geometric limit $\mathbb{R} \times S^1 \times \Sigma$. Using a constant scalar curvature metric on this cylinder and the C^0-bound one arrives at the result.

In addition one can handle the $b^+(X) = 1$ case.

Theorem (1994, Kronheimer-Mrowka, Morgan-Szabo-Taubes, Fintushel-Stern). The minimum genus of a smoothly embedded surface in $\mathbb{C}P^2$ representing d ($\neq 0$) times the hyperplane class is.

$$\frac{(|d|-1)(|d|-2)}{2}$$

i.e. the genus of a smooth complex curve representing that class if $d>0$.

One can also prove the smooth invariance of the canonical class of a minimal complex surface using these invariants. In fact for surfaces of general type the is a slick proof using the existence of a Kähler Einstein metric and two forms of energy function.

The most striking result obtained to date with these new invariants is due to Taubes. He proves that for a symplectic 4-manifold the Seiberg-Witten invariants and the Gromov invariants are equivalent.

Titel: SEIBERG-WITTEN INVARIANTS AND 4-MANIFOLDS, PART II

Autor: JOHN W. MORGAN

Adresse: COLUMBIA UNIVERSITY
NEW YORK, NY 10027 USA

Let X be a closed, oriented riemannian 4-manifold. Let $\tilde{P} \to X$ be a Spin^c structure for X, i.e. a lifting of the principal $SO(4)$-bundle associated to the tangent bundle of X to a principal $\mathrm{Spin}^c(4)$-bundle. Then there are unitary $\mathbb{C}^2$-bundles $S^{\pm}(\tilde{P}) \to X$, the bundles of plus and minus spinors. Clifford multiplication defines an action of $\Omega^*(X;\mathbb{C})$ on the space of sections of $S^+(\tilde{P}) \oplus S^-(\tilde{P})$. In particular $\Omega^2_+(X; i\mathbb{R})$ acts as the space of symmetric, trace-free endomorphisms of sections of $S^+(\tilde{P})$.

The Seiberg-Witten equations are equations for a pair (A, ψ) where A is a unitary connection on the determinant line bundle $\mathcal{L}$ of $S^+(\tilde{P})$ and ψ is a section of $S^+(\tilde{P})$. They are:

$$\begin{cases} F_A^+ = \sigma(\psi) = \psi \otimes \psi^* - \dfrac{|\psi|^2}{2}\,\mathrm{Id} \\ \not{\partial}_A(\psi) = 0 \end{cases}$$

Autor: John W. Morgan Seite: 2

Modulo the action of the group $(S^1)^M$ of automorphisms of $\tilde{P}$ over $P_{SO(4)}(X)$, the solutions to these equations form a compact space of formal dimension

$$d = \frac{c_1(\mathcal{L})^2 - (2\chi(X) + 3\sigma(X))}{4}$$

For a generic perturbation, replacing the curvature equation by

$$F_A^+ = \sigma(\psi) + i h$$

for generic $h \in \Omega^2_+(X;\mathbb{R})$, the solutions form a compact, smooth manifold $M(P,h)$ inside the space $\mathcal{B}^*(P)$ of all pairs $\{(A,\psi) \mid \psi \not\equiv 0\}/(S^1)^M$. The space $\mathcal{B}^*(P)$ is a Hilbert manifold homotopy equivalent to $\mathbb{CP}^\infty \times K(H^1(M;\mathbb{Z}),1)$. The Seiberg-Witten invariant of $\tilde{P}$ is the homology class of $M(P,h)$ projected into the $\mathbb{CP}^\infty$ factor. (This class is zero if d is odd, and is an integer multiple of the generator when d is even.)

Theorem If $b_2^+(X) > 1$, then the Seiberg-Witten invariant of X is independent of the metric on X and on the perturbation h. Thus we have a differentiable invariant

$$SW: \{\text{Spin}^c\text{-structures on } X \text{ up to isomorphism}\} \to \mathbb{Z}.$$

This function is zero except at a finite set of Spin^c-structures.

Associated to any Spin^c-structure $\tilde{P}$ is $c_1(\mathcal{L}) \in H^2(X;\mathbb{Z})$. The function from Spin^c-structures to $H^2(X;\mathbb{Z})$ is onto the elements congruent to $w_2(X)$ modulo 2 and has finite fibers (of cardinality equal to the order of the 2-torsion subgroup of $H^2(X;\mathbb{Z})$). One often interprets the Seiberg-Witten invariant as a function

$$SW: H^2(X;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

by defining the value on a cohomology class to be the sum over all Spin^c-structures associated to that class of the invariants of the Spin^c-structure. This function has finite support contained in the subset of characteristic classes.

Computations: If X is a Kähler manifold and $\tilde{\mathfrak{L}}$ is a Spin^c-structure with determinant $\mathcal{L}$, then $S^+(\tilde{\mathfrak{L}}) = \mathcal{L}_0 \oplus \Lambda^{0,2}(X;\mathcal{L}_0)$ and $S^-(\tilde{\mathfrak{L}}) = \Lambda^{0,1}(X;\mathcal{L}_0)$ where $\mathcal{L}_0 = \sqrt{K_X \otimes \mathcal{L}}$.
Furthermore, the Dirac operator is identified with
$\bar\partial_{A_0} + \bar\partial^*_{A_0}$ for an appropriate connection A_0 on $\mathcal{L}_0$.
The Seibert-Witten equations then become.

$$\begin{cases} \bar\partial_{A_0}(\alpha) + \bar\partial^*_{A_0}(\beta) = 0. \\ F_A^{0,2} = \bar\alpha\beta \\ (F_A^{1,1})^+ = \frac{i}{2}(|\alpha|^2 - |\beta|^2)\,\omega \end{cases}$$

where ω is the Kähler form, $\alpha \in \Omega^0(X;\mathcal{L}_0)$, $\beta \in \Omega^{0,2}(X;\mathcal{L}_0)$
A is a connection on $\mathcal{L}$ and A_0 is the corresponding connection on $\mathcal{L}_0 = \sqrt{K_X \otimes \mathcal{L}}$.

Thm (Witten) Let X be a Kähler manifold with $b_2^+(X) > 1$. Suppose $\tilde{\mathfrak{L}}$ is a Spin^c-structure with $\det \tilde{\mathfrak{L}} = \mathcal{L}$ of negative degree. Then for any solution to the SW equations we have $\beta = 0$, A and A_0 are holomorphic and α is a non-trivial holomorphic section of $\mathcal{L}_0$. This yields and isomorphism between the moduli space of all solutions to the SW equations and the moduli space of effective divisors in X whose dual cohomology class is $c_1(\mathcal{L}_0) = \frac{K_X + c_1(\mathcal{L})}{2}$.
If $\deg \mathcal{L} > 0$, then $\alpha = 0$ and $\bar\beta$ is a holomorphic section of $K_X \otimes \mathcal{L}_0^{-1}$. In this case the moduli space of solutions to the SW equations is identified with the space of effective divisors cohomologous

to $K_X \otimes \mathcal{L}_0^{-1} = \frac{K_X - c_1(\mathcal{L})}{2}$.

Corollary: $SW(K_X) = \pm 1$, $SW(K_X^{-1}) = \pm 1$. If $SW(\mathcal{L}) \neq 0$, then $-\deg K_X \leq \deg \mathcal{L} \leq \deg K_X$.

Symplectic Manifolds: As in the Kähler case, for X symplectic $S^+(\tilde{P}) \cong \mathcal{L}_0 \oplus \Lambda^{0,2}(X; \mathcal{L}_0)$ and $S^-(\tilde{P}) = \Lambda^{0,1}(X; \mathcal{L}_0)$. Furthermore, for an appropriate connection on K_X we have $\not{\partial} = \bar{\partial} + \bar{\partial}^*$. Thus, again one writes a plus Spinor field as (α, β) with $\alpha \in \Omega^0(X; \mathcal{L}_0)$ and $\beta \in \Omega^{0,2}(X; \mathcal{L}_0)$. It is no longer true in general that either $\alpha = 0$ or $\beta = 0$ for any solution. Nevertheless, Taubes has introduced a 1-parameter family of perturbations so that in the limit this is the case. This leads to

Thm (Taubes) Let X be a symplectic manifold with $b_2^+(X) > 1$. Then $SW(K_X) = \pm 1$, $SW(K_X^{-1}) = \pm 1$, If $SW(\mathcal{L}) \neq 0$, then $|\deg \mathcal{L}| \leq \deg K_X$. Furthermore, if $\deg \mathcal{L} < 0$, then in the limit of the Taubes perturbation $\beta \to 0$ and α goes to a section vanishing on a pseudo-holomorphic curve. This sets up a correspondence between the Seiberg-Witten moduli space and the space of pseudo holomorphic curves cohomologous to $(K_X + c_1(\mathcal{L}))/2$. In particular, $SW(\mathcal{L}) =$ Gromov Inv. $((K_X + c_1(\mathcal{L}))/2)$. There is an analogous statement in the case when $\deg \mathcal{L} > 0$.

Titel: Infinite Product expansions of Automorphic Forms
Autor: Richard Borcherds Seite: 1
Adresse: Dept of Math, Evans Hall #3840, U.C. Berkeley,
CA 94720-3840, USA.
email: reb@

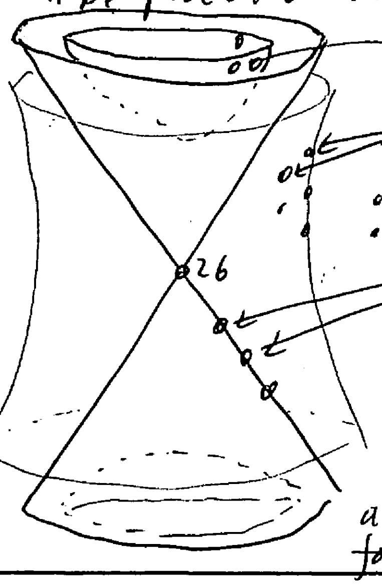
The Dedekind function $\Delta(\tau) = q\prod_{n>0}(1-q^n)^{24}$ $(q = e^{2\pi i\tau})$ satifies the functional equations $\Delta(\tau) = \Delta(\tau+1)$, $\Delta(\frac{-1}{\tau}) = \tau^{12}\Delta(\tau)$, so Δ is a modular form (i.e. $\Delta(\frac{a\tau+b}{c\tau+d}) = \tau^{12}\Delta(\tau)$) with a "nice" infinite product expansion. We will construct some other functions with similar properties.

The easiest example is the denominator function of the fake monster Lie algebra M, so we will first describe 4 properties of M:

(1) M is a generalized Kac-Moody algebra (GKM), which is graded by the even 26-dimensional unimodular Lorentzian lattice $II_{25,1}$ (its "root lattice"), and the multiplicity of the root $r \in II_{25,1}$ is $p_{24}(1-r^2/2)$ if $r \neq 0$, 26 if $r=0$, where $\sum_n p_{24}(1+n)q^n = \frac{1}{\Delta(\tau)} = q^{-1} + 24 + 324q + 3200q^2 + \cdots$

A GKM is roughly one with most properties of finite dimensional simple Lie algebras, so it has simple roots, a Weyl group, a Weyl character and denominator formula, and so on.

M can be pictured roughly as



M can be constructed using vertex algebras as the space of chiral strings on a 26-dim torus, and most of the properties above follow from the no-ghost theorem.

Autor: Richard Borcherds

(2) The denominator function of M is

$$\Phi(v) = e^{-2\pi i(\rho,v)} \prod_{\alpha>0}\left(1-e^{-2\pi i(\alpha,v)}\right)^{p_{24}(1-\alpha^2/2)} = \sum_{w\in W} \det(w) \sum_{n>0} \tau(n)\, e^{-2\pi i n(w(v),\rho)}$$

"Product over positive roots" α

sum over Weyl group W = reflection group of $II_{25,1}$

Ramanujan's function

vector of $II_{25,1}\otimes\mathbb{C}$ with $Im(v)\in$ positive cone.

ρ = Weyl vector $=(0,0,1)$ if $II_{25,1}$ is written as $\Lambda \oplus \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$, Λ = Leech lattice.

The simple roots are the vectors r with $(r,\rho) = -r^2/2$, i.e. $(\lambda, 1, \lambda^2/2-1)$, $\lambda\in\Lambda$, and $(0,0,n)$, $n>0$, with multiplicity 24.

(The simple roots are not linearly independent, so the existence of a Weyl vector is not trivial. $II_{25,1}$ seems to be the only indefinite lattice for which the corresponding Lie algebra has a Weyl vector.)

(3) The function $\Phi(v)$ is an automorphic form for the group $Aut(II_{26,2})^+$. More precisely it satisfies the equations

(a) $\Phi(v) = \Phi(v+r)$, $r\in II_{25,1}$

(b) $\Phi(v) = \pm\Phi(w(v))$, $w\in Aut(II_{25,1})^+$

(c) $\Phi\left(\frac{2v}{(v,v)}\right) = -\left(\frac{(v,v)}{2}\right)^{12}\Phi(v)$.

These 3 sorts of transformations generate a group isomorphic to $Aut(II_{26,2})$ acting on the Hermitean symmetric space $\{v\in II_{25,1}\otimes\mathbb{C} \mid Im(v)\in C\}$ in the same way that $\tau\to\tau+1$ and $\tau\to -1/\tau$ generate a group isomorphic to $SL_2(\mathbb{Z})$ acting on $\{\tau\in\mathbb{C} \mid Im(\tau)>0\}$. Equations (a) and (b) are trivial to prove, but (c) is more difficult. It can be proved by showing that both $\Phi(v)$ and $-\Phi\left(\frac{2v}{(v,v)}\right)\left(\frac{(v,v)}{2}\right)^{-12}$ satisfy the same differential equation and have the same boundary conditions.

(4) The zeros of $\Phi(v)$ are the rational quadratic divisors $\{v \mid a\frac{(v,v)}{2} + (b,v) + c = 0\}$ for $a, c \in \mathbb{Z}$, $b \in II_{25,1}$, $(b,b) - 2ac = 2$. Not all these zeros are zeros of factors of the infinite product of Φ, because this infinite product does not converge everywhere. The proof of this result follows by looking at the analytic continuation of $\log \Phi(v)$, whose fourier series ~~are dep~~ coefficients depend on $p_{24}(1+n)$. The singularities of any function are related to the asymptotic expansion of its Fourier series, so the singularities of $\log \Phi(v)$, and hence the zeros of $\Phi(v)$, can be read off from the Hardy-Ramanujan-Rademacher series for $p_{24}(1+n)$. For example, the zero $\{v \mid (v,v) = 2\}$ of Φ corresponds to the dominant term in the H-R-R series, and other zeros of Φ correspond to smaller terms in the series.

We can find an infinite series of functions like Φ as follows. Suppose $f(\tau) = \sum_n c(n) q^n$ is a level 1 modular form of weight $-s/2 < 0$ all of whose poles are at ~~cusps~~ cusps. Then for some $\rho \in II_{s+1,1}$, the function

$$\Phi(v) = e^{-2\pi i(\rho,v)} \prod_{\substack{\alpha \in II_{s+1,1} \\ \alpha > 0}} \left(1 - e^{-2\pi i(\alpha,v)}\right)^{c(-(\alpha,\alpha)/2)}$$

can be analytically continued to a meromorphic automorphic form for the group $\mathrm{Aut}(II_{s+2,2})^+$. The weight of Φ is $c(0)/2$, and all zeros of Φ lie on divisors of the form $\{v \mid \frac{a(v,v)}{2} + (b,v) + c = 0\}$ for $a, c \in \mathbb{Z}$, $b \in II_{s+1,1}$, and the multiplicities of the zeros and poles along these divisors are given by the coefficients $c(n)$ for $n < 0$.

We now give some applications of these forms.

Application 1: A mysterious connection with elliptic cohomology. As a consequence of conjectures about a geometric realization of elliptic cohomology, M. Hopkins conjectured that the constant term of $\frac{1}{24}\frac{\theta_K(\tau)}{\Delta(\tau)^{\dim(K)/24}}$ was integral for any even unimodular lattice K of dimension divisible by 24. If we take $s = \dim K + 1$, $II_{s+1,1} = K \oplus \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$, $f(\tau) = \frac{1}{\Delta(\tau)^{\dim(K)/24}}$, and look at the Weyl vector ρ of the corresponding automorphic form Φ, then this constant term is one of the coefficients of ρ, and must therefore be integral as $\rho \in II_{s+1,1}$. (There does not seem to be any obvious way to prove this congruence using ordinary properties of the theta function of K.)

Application 2: Moduli space of Enriques surfaces (over $\mathbb{C}$). The moduli space of Enriques surfaces is

$$\frac{\{v \in L\otimes\mathbb{C} \mid \operatorname{Im}(v)\in C\}}{\operatorname{Aut}\left(L\oplus\begin{pmatrix}0 & -2\\ -2 & 0\end{pmatrix}\right)^+} - (\text{irreducible divisor } D_0)$$

where $L = 2E_8 \oplus \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$ and C is the positive cone in $L\otimes\mathbb{R}$.

If M is the superalgebra of superstrings on a 10-dimensional torus, then M has a denominator function Φ similar to the one for the fake monster Lie algebra, and Φ is an automorphic form on $\{v\in L\otimes\mathbb{C} \mid \operatorname{Im}(v)\in C\}$ with respect to the group $\operatorname{Aut}\left(L\oplus\begin{pmatrix}0 & -2\\ -2 & 0\end{pmatrix}\right)^+$. Moreover M vanishes exactly on the ~~on~~ divisor D_0 and its conjugates and nowhere else. Hence the denominator function of the algebra of superstrings on this 10-dim torus is a section of a line bundle over the moduli space of Enriques surfaces with no ~~zeros~~. One consequence is that this moduli space is quasi affine, as the line bundle is known to be ample, and if an ample line bundle over a quasiprojective variety V is trivial then V is quasi affine.

Autor: Richard Borcherds Seite: 5

Application 3: Heegner points. Suppose V is the quotient of $\{v \in II_{s+1,1} \otimes \mathbb{C} \mid \mathrm{Im}(v) \in C\}$ by the group $\mathrm{Aut}(II_{s+2,2})^+$, so that V is a quasiprojective variety. The divisors $\{v \mid av^2 + (b,v) + c = 0\}$ correspond to points on $\mathrm{Pic}(V)$, which can be called Heegner points as they are (more or less) generalizations of Heegner points on modular curves. The automorphic forms Φ constructed above then give many relations between these Heegner points, as their zeros are all Heegner divisors of known multiplicities. For example, if $s = 24n$ then the group of Heegner points is generated by at most n elements, and if $n = 1$ or 2 it is free abelian with n generators.

Application 4: If we restrict the automorphic forms Φ to one dimensional spaces we get classical modular forms which can be written explicitly as infinite products. For example if $f(\tau)$ is the unique weight $1/2$ level 4 modular form with poles only at cusps of the form $3q^{-3} + \sum_{\substack{n \geq 1 \\ n \equiv 0,1 \bmod 4}} c_n q^n$, (so that $f(\tau) = 3q^{-3} - 744q + 80256q^2 - 257985q^5 + 5121792q^8 - 12288744q^9 \ldots$) then the elliptic modular function $j(\tau) = q^{-1} + 744 + 196884q + \ldots$ is equal to $q^{-1}\prod_{n \geq 1}(1-q^n)^{c(n^2)} = q^{-1}(1-q)^{-744}(1-q^2)^{80256}(1-q^3)^{-12288744} \ldots$

More generally any level 1 modular form with integral coefficients all of whose zeros are imaginary quadratic irrationals can be written as a similar infinite product.

Application 5: Many other automorphic forms (of higher level) ~~which~~ that can be written as infinite products are closely related to Lie algebras or hyperbolic reflection groups. (One example is the Φ of the fake monster Lie algebra, and the reflection group of $II_{25,1}$.) These Lie algebras are often acted on naturally by sporadic groups. Gritsenko and Nikulin have some recent preprints giving some more examples which are genus 2 Siegel modular forms.

Titel: Resolution of singularities by alterations (a report on work by A.J. de Jong)

Autor: Frans OORT Seite: 1
Adresse: A.J. de Jong & FO: Math. Inst., Budapestlaan 6
NL-3508 TA UTRECHT, The Netherlands
dejong@math.ruu.nl oort@math.ruu.nl

Def.: Let V and W be integral Noetherian schemes. A modification V of W is a proper birational morphism $\sigma: V \longrightarrow W$.

Def.: let W be integral Noetherian, an alteration of W is an integral Noetherian scheme V and a dominant, proper morphism $\varphi: V \longrightarrow W$ such that for an open set $U \subset W$, $U \neq \emptyset$, the induced morphism $\varphi^{-1}U \longrightarrow U$ is finite.

Problem: Resolution of singularities (by a modification): Let k be a field, and X a variety over k, does there exist a modification $\tilde{X} \xrightarrow{\varphi} X$ such that $\tilde{X}$ is regular?

This is solved by Hironaka in case $\mathrm{char}(k) = 0$.
In other cases there are partial results (Hironaka, Abhyankar, Lipman, ... etc.!).
A result of this kind is useful! many people are seriously looking for a solution (e.g. in case $\mathrm{char}(k) = p > 0$, or for a scheme in mixed characteristics).
The result by Johan de Jong does not answer the original problem, but solves a slightly weaker one. Very useful!

We report on (part of):

A. J. de Jong – Smoothness, semi-stability and alterations. Utrecht Preprint 916, June 1995.

A first approximation to one of his results:

Theorem: For any field k, for any variety X over k, $\exists$ alteration $\tilde{X} \xrightarrow{\varphi} X$ such that $\tilde{X}$ is regular.

Def. (sncd): Let S be a Noetherian scheme, $D \subset S$ a divisor, and $D_i \subset D$, $i \in I$, its irreducible components. We say that D is a strict normal crossings divisor in S if:

a) $\forall s \in D$, $\mathcal{O}_{S,s}$ is regular

b) D is reduced, i.e. $D = \bigcup_{i \in I} D_i$ as scheme,

c) $\forall J \subset I$, $D_J := \bigcap_{j \in J} D_j$ is a regular scheme, and $\mathrm{codim}(D_J \subset S) = \# J$.

(e.g. a union of coordinate hyperplanes in $\mathbb{A}^n$ is a sncd).

N.B. We say X is a variety over a field k, if $X \to \mathrm{Spec}(k)$ is separated, of finite type over k, and X is reduced and irreducible.

(⚠ $X \otimes \bar{k}$ can be reducible, need not be reduced.)

Here is the first result (Th. 3.1):

<u>Theorem</u> (A. J. de Jong): Let X be a variety over a field k, and let $Z \subsetneq X$ be a closed subset. There is an <u>alteration</u>

$$\varphi : \tilde{X} \longrightarrow X$$

and an open immersion $j : \tilde{X} \hookrightarrow \overline{\tilde{X}}$ such that:

a) $\overline{\tilde{X}}$ is a projective and <u>regular</u> variety, and

b) the closed subset $D := (j\varphi^{-1}Z) \cup (\overline{\tilde{X}} \setminus j\tilde{X})$ is a sncd in $\overline{\tilde{X}}$.

Moreover, if k is perfect, the alteration φ can be chosen to be generically étale.

$$\begin{tikzcd}
R(\tilde X) & \tilde X \arrow[rr, "j"] \arrow[dd, "\varphi"'] & & \overline{\tilde X} \arrow[r, hook] & \mathbb{P}^{?} \\
& & \tilde X \cap D \arrow[ul, hook'] \arrow[r, hook] \arrow[d] & D \arrow[u, hook] & \\
R(X) \arrow[uu, hook] & X \arrow[r, phantom, "\supset"] & Z & &
\end{tikzcd}$$

(proof by induction on $\dim X$, or if $\dim X = 1$)

[Here, $R(X)$ is the function field of X.]

We give a short (very incomplete) survey of the proof. By "*" we indicate a step where an extension of the function field $R(X)$ might be involved.

<u>Step 1</u>: <u>Improve the situation</u>. Assume $k = \bar{k}$, X is projective and normal, $\exists$ divisor $D \subset X$ such that $Z = \mathrm{Supp}(D)$.

<u>Step 2</u>: <u>choose a fibration</u> $f : X \to Y$ (with certain properties ..., especially $f|_Z : Z \to Y$ finite & generic étale).

$$
\begin{tikzcd}
 & & X \arrow[r, hook] \arrow[dr, "\pi"] & \mathbb{P}^N \supset L^{N-d-1} \arrow[d, dashed, "\pi_L"] \\
 & X' \arrow[ur, "\sigma"] \arrow[dl, "f"', "\text{stein}"] \arrow[dr] & & \mathbb{P}^d \ni P \arrow[dll, dashed, "\pi_P"] \\
Y' \arrow[rr, "\sim"] & & \mathbb{P}^{d-1} &
\end{tikzcd}
$$

dim $X = d$

$\left.\begin{matrix} <\infty + \\ \text{étale} \\ \therefore \cong \end{matrix}\right\}$

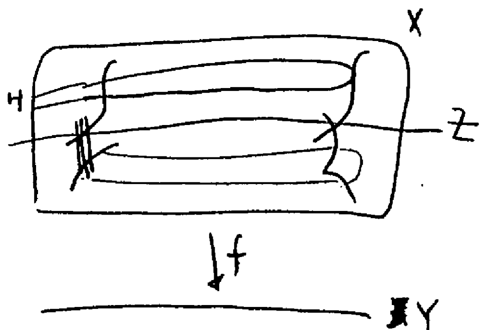
$P \notin L$ in general position

$\mathbb{P}^{d-1} = \{\ell \mid \ell \subset \mathbb{P}^d \text{ line}, P \in \ell\}$

$X' = \{(x,\ell) \in X \times \mathbb{P}^{d-1} \mid \pi(x) \in \ell\}$

σ: blowing up in $\pi^{-1}(\{P\})$.

* Step 3: Choose sections.
Construct $H \subset X$, $H \to Y$ finite and generically étale, $Z' := Z \cup H$, after base extension $Y' \to Y$ (separable) we can write $Z' = \cup\, \sigma_i Y$ (3 condition: H should intersect every geometric component of a fibre of f at least 3 times).



* Step 4: Use stable n-pointed curves. Over a non-empty open set, $U \subset Y$, $X|_U \to U$ and $\sigma_i|_U$ form a stable n-pointed curve over U. Choose $\ell \in \mathbb{Z}_{\geq 3}$, char$(k) \nmid \ell$, use $\overline{M}_{g,\ell}^{(n)}$: compactification of the moduli space of curves of genus g, level ℓ, n-pointed,

this gives:

$$
\begin{tikzcd}
\mathcal{C}' \arrow[drr, bend right] & \mathcal{C}'|_{U'} \arrow[l, hook'] \arrow[r, "\beta", "\sim"'] \arrow[dr] & X|_{U'} \arrow[r, hook] \arrow[d] & X' \arrow[dl, bend left, "f"] \\
 & & U' \subset Y' \arrow[ull, bend left, "\tau_i"] \arrow[ur, bend right, "\sigma_i"'] &
\end{tikzcd}
$$

$\beta \cdot (\tau_i|_U) = \sigma_i|_U$

Step (5): Extend $\beta : \mathcal{C}|_{U} \to X|_{U}$ to $\beta' : \mathcal{C}' \to X'$.
Consider $T := (\Gamma_\beta)^c \subset \mathcal{C} \times_Y X$, c: Zariski closure. Using Raynaud - Gruson "platification" (Invent. Math. 13 (1971), 1-89): after blowing up Y, and perform base change - strict transformations we achieve $X' \to Y'$, $T' \to Y$ flat, normalise Y', again base change and conclude:

$$\mathcal{C}' \xleftarrow{\sim} T' \longrightarrow X',$$

replace X' by $\mathcal{C}'$. Now $X' = \mathcal{C}' \to Y'$ has stable fibres!

*Step (6): Induction hypothesis. Apply the theorem to Y, pull back X, blow up in all components of $\mathrm{Sing}(X')$ of codim $= 2$, and arrive at:

(7) We have a projective variety X, $\dim X = d$, $\forall x \in X$ either
i) $x \in \mathrm{Reg}(X)$, (and Z is sncd divisor at x, or $x \notin Z$), or
ii) $x \in \mathrm{Sing}(X)$, $\exists\, s,t \in \mathbb{Z}$, $2 \le s \le t \le d-1$,

$$\widehat{\mathcal{O}}_{X,x} \cong k[[u,v,t_1,\dots,t_{d-1}]]/(uv - t_1 \times \dots \times t_s)$$

and locally on X at x, Z is given as $\mathcal{Z}(t_1 \times \dots \times t_r)$.

From (7), finish the proof of the theorem: each time you blow up in a component of $\mathrm{Sing}(X)$, the number of components goes down by one, after a finite number of steps we arrive at: X is regular, again applying a finite number of modifications: also Z is sncd. $\square$

The second half of the paper treats the case of $X \to \mathrm{Spec}(\text{complete DVR})$, arriving by an alteration at a situation of a "strict semi-stable pair" (generic fibre smooth, special fibre stable, Z sncd).

$p-$adic jets, Manin maps and related questions

Alexandru Buium

Let p be an odd prime integer and $n \in \mathbf{Z}$ any integer number. We define the "derivative" of n with respect to p by the formula

$$\frac{\partial n}{\partial p} := \frac{n - n^p}{p}$$

The map $\delta = \partial/\partial p : \mathbf{Z} \to \mathbf{Z}$ is not additive and does not satisfy the Leibniz rule; in a higher sense, however, this map plays the role of a derivation and may be used to build a theory of algebraic differential equations whose solutions are integers in numbers fields, and more generally in local fields. Then one can prove [B] arithmetic analogues of some remarkable classical results about algebraic differential equations on varieties defined over function fields, especially an arithmetic analogue of the Manin-Chai Theorem of the Kernel and an arithmetic analogue of Cassidy's theorem on subgroups of simple algebraic groups.

Here are the relevant definitions. We start by replacing $\mathbf{Z}$ by the larger ring $R = \hat{\mathbf{Z}}_p^{ur}$, the completion of the maximal unramified extension of $\mathbf{Z}_p$; we also extend the operator δ to R by the formula $\delta x = (\phi(x) - x^p)/p$. where $\phi : R \to R$ is the unique lifting of the Frobenius automorphism of the residue field k of R. For any integers $n \geq 0$, $N \geq 1$ let $R[T, T', ..., T^{(n)}]\hat{}$ denote the $p-$adic completion of the ring of polynomials $R[T, T', ..., T^{(n)}]$, each of the letters $T, T', ..., T^{(n)}$ standing for an $N-$tuple of indeterminates. Let X/R be a scheme of finite type. An $R-$valued function $\varphi : X(R) \to R$ will be called a *$\delta-$formal function of order* $\leq n$ (intuitively a "non linear differential operator") if for any point in X there exist an affine Zariski open neighbourhood $U \subset X$, an integer $N \geq 1$, a closed embedding $t : U \to \mathbf{A}^N$ and an element $\Phi \in R[T, T', ..., T^{(n)}]\hat{}$ such that

$$\varphi(P) = \Phi(t(P), t(P)', t(P)'', ..., t(P)^{(n)}), \quad P \in U(R)$$

(Note that for $P \in U(R)$ we have $t(P) \in R^N$ so it makes sense to consider $t(P)', t(P)'', ... \in R^N$). We denote by $\mathcal{O}^n(X)$ the ring of $\delta-$formal functions of order $\leq n$ on $X(R)$. If A/R is a smooth commutative group scheme of finite type , by a $\delta-$character of $A(R)$ we understand a $\delta-$formal function $\psi : A(R) \to R = \mathbf{G}_a(R)$ (of some order $n \in \mathbf{N}$) which is also an (additive) group homomorphism. Consider the intersection of all kernels of $\delta-$characters:

$$A^{\sharp}(R) := \bigcap Ker\ \psi \subset A(R)$$

Clearly $A^\sharp(R)$ contains the intersection

$$p^\infty A(R) := \bigcap_n p^n A(R) \subset A(R)$$

Here is our analogue of the Theorem of the Kernel [Man] [Ch]:

Theorem 1. *Assume A/R is an abelian scheme of relative dimension g with ordinary closed fibre A_0/k. Let $q_{ij}(A) \in 1+pR$, $1 \leq i,j \leq g$ be the Serre-Tate parameters. Assume $det((q_{ij}(A)-1)/p) \in R^\times$. Then $A^\sharp(R) = p^\infty A(R)$.*

Note that the condition $det((q_{ij}(A)-1)/p) \in R^\times$ is satisfied for a generic choice of the Serre-Tate parameters and is actually a condition on $q_{ij}(A)$ modulo p^2. So this condition may be viewed as saying that A/R is "sufficiently general", and is an arithmetic analogue of the trace zero condition in the classical Theorem of the Kernel.

To formulate the next result assume G/R is a smooth affine group scheme of finite type. Let $\mathcal{J} = (J_n)_n$ be a sequence of Hopf ideals $J_n \subset \mathcal{O}^n(G)$, $n \geq 0$, such that $J_{n+1} \cap \mathcal{O}^n(G) = J_n$. Intuitively such a sequence should be viewed as a system of "global algebraic differential equations" (with respect to our operator δ) that is "compatible with the group law". For such a sequence $\mathcal{J}$ set

$$\mathcal{J}^{sol} := \{P \in G(R);\ \varphi(P) = 0 \quad for \quad all \quad \varphi \in J_n,\ n \geq 0\}$$

Intuitively $\mathcal{J}^{sol}$ represents the "set of solutions" of $\mathcal{J}$. Conditions (*) imply $\mathcal{J}^{sol}$ is a subgroup of $G(R)$; such a subgroup will be called a *$\delta-$subgroup.* A subgroup $\Gamma \subset G(R)$ will be said to be *Zariski dense modulo p* if the image of the reduction modulo p homomorphism $\Gamma \subset G(R) \to G_0(k)$ is Zariski dense in $G_0(k)$ (where G_0 denotes as usual the closed fibre of G/R.) Finally, by a simple group over R we will understand, in this paper, a smooth affine group scheme of finite type G/R for which G_0/k is a simple algebraic group; we will say that G/R is *good at p* if either $p \neq 2,3$ in case G_0 is not of type A_n or p does not divide $n+1$ in case G_0 is of type A_n. Our analogue of Cassidy's Theorem [Ca] is the following

Theorem 2. *Let G/R be a simple group, good at p, and let $\Gamma \subset G(R)$ be a $\delta-$subgroup. Assume Γ is Zariski dense modulo p. Then $\Gamma = G(R)$.*

[B] A.Buium, Differential characters of abelian varieties over $p-$adic fields, Invent. Math., to appear.

[Ca] P.Cassidy, The classification of the semisimple differential algebraic groups and the linear semi-simple differential algebraic Lie algebras, J.Algebra 121, 1 (1989), 169-238.

[Ch] Ching-Li Chai, A note on Manin's Theorem of the Kernel, Amer. J. Math., 113 (1991), 387-389.

[Man] Yu.I.Manin, Algebraic curves over fields with differentiation, Izv. Akad. Nauk SSSR, Ser. Mat. 22 (1958), 737-756 = AMS translations Series 2, 37 (1964), 59-78.

DENSITY OF ORDINARY ISOGENY CLASSES IN THE MODULI SPACE

CHING-LI CHAI

Throughout this note k will denote an algebraically closed field of characteristic $p > 0$. For every abelian variety A over k one associates to it its p-divisible group $A[p^\infty] = \varinjlim A[p^n]$. We owe to Dieudonné-Manin the classification of p-divisible groups. Especially the p-divisible group $A[p^\infty]$ of an abelian variety A of dimension g over k is determined up to isogeny by a collection of $2g$ rational numbers $\lambda_1, \ldots, \lambda_{2g}$, called the slopes of $A[p^\infty]$, such that

$$0 \leq \lambda_1 \leq \cdots \leq \lambda_{2g} \leq 1 \quad \text{and} \quad \lambda_i + \lambda_{2g-i} = 1 \ \forall i .$$

The two extreme cases are
(a) (A is ordinary) $\lambda_1 = \cdots = \lambda_g = 0$, and $\lambda_{g+1} = \cdots = \lambda_{2g} = 1$.
(b) (A is supersingular) $\lambda_1 = \cdots = \lambda_{2g} = \frac{1}{2}$.
Experience has taught us that the ordinary abelian varieties exhibit behavior close to abelian varieties in characteristic 0, while supersingular abelian varieties have very different properties. Among the nice properties of ordinary abelian varieties, we mention only that the deformation space of ordinary abelian varieties have nice toroidal coordinates, also called Serre-Tate coordinates, such that the deformation space has a canonical group structure.

For fixed integers $g, d \geq 1$, denote by $\mathcal{A}_{g,d}$ the moduli space of abelian varieties A over k with a polarization of degree d^2. When $d = 1$, $\mathcal{A}_{g,1}$ is abbreviated to $\mathcal{A}_g$. Study of the fine structures of these moduli spaces was begun by Mumford in the late 1960's. In the late 1970's Norman and Oort proved that the ordinary locus $\mathcal{A}_{g,d}^{\mathrm{ord}}$ in $\mathcal{A}_{g,d}$ is open and dense. By the middle of 1980's much more information about these moduli spaces are known, due to Faltings' method of constructing arithmetic toroidal compactification of $\mathcal{A}_g$.

The theme of this talk is to study the geometry of Hecke correspondences on $\mathcal{A}_g$. Since we are in characteristic p, we will consider only the prime-to-p Hecke correspondences. It is well-known that over the complex numbers the moduli space of principally polarized abelian varieties has a uniformization as the quotient of the Siegel upper half space $\mathfrak{H}_g$ of genus g by the Siegel modular group $\mathrm{Sp}_{2g}(\mathbb{Z})$. Moreover it has Hecke correspondences coming from $\mathrm{GSp}_{2g}(\mathbb{Q})$. The prime-to-$p$ Hecke correspondences are those coming from $\mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)})$, where $\mathbb{A}_f^{(p)}$ denotes the finite prime-to-p adeles for $\mathbb{Q}$. We shall focus on one problem about the orbits of these Hecke correspondences in characteristic p. First we give a geometric description of the orbit under the prime-to-p Hecke correspondences.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Suppose that $x \in \mathcal{A}_g(k)$ corresponds to an abelian variety A_x. The prime-to-p Hecke orbit of x, denoted by $\mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)})(x)$, is the countable subset of $\mathcal{A}_g(k)$ consisting of all points $y \in \mathcal{A}_g(k)$ such that there exists an isogeny $\phi : A_y \to A_x$ with $\phi^*(\mathrm{pol}_{A_x}) = m \cdot \mathrm{pol}_{A_y}$ for some positive integer m, $(m,p) = 1$. We would like to know when the prime-to-p Hecke orbit $\mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)})(x)$ is Zariski dense in $\mathcal{A}_g$?

Clearly a necessary condition for this question is that x has to be ordinary. Otherwise $\mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)})(x)$ will be contained in a divisor in $\mathcal{A}_g$, namely the zero locus of the determinant of the Hasse-Witt matrix.

Theorem. *If x corresponds to an ordinary principally polarized abelian variety, then the prime-to-p Hecke orbit $\mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)})(x)$ of x is Zariski dense in $\mathcal{A}_g$.*

The method is based on the following trivial principle: Suppose that a group G operates on a scheme X, and Z is a closed subscheme of X stable under G. Then for any point $x \in Z$, the formal completion $Z_x^\wedge$ of Z at x is stable under the natural action of the stabilizer subgroup G_x on the formal completion Z_x of Z at x. This principle also holds for algebraic correspondences, for instance the prime-to-p Hecke correspondences on $\mathcal{A}_g$. In this case the stabilizer subgroup of a point $x = [A]$ comes from the ring of endomorphisms of the A, equipped with the Rosati involution coming from the principal polarization of A. We shall give an example of a principally polarized abelian variety whose Hecke orbit is dense using this principle.

Example. (M. Larsen) Assume that $k = \overline{\mathbb{F}_p}$ and let E be an ordinary elliptic curve over k, with its natural principal polarization λ_E. Let $(A, \lambda) = (E, \lambda_E)^{\oplus g}$, $g \geq 2$, and let $x \in \mathcal{A}_g(k)$ be the corresponding point. Then $\mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)})(x)$ is Zariski dense in $\mathcal{A}_g$.

Proof of Example. First of all, we show that x is a smooth point of the Zariski closure Z of $\mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)})(x)$. The usual argument which shows that any orbit for a connected algebraic group acting on an algebraic variety is smooth applies in this case: The smooth locus of Z is a nonempty open subscheme Z_{sm} of Z, which is stable under prime-to-p Hecke correspondences and contains points of $\mathcal{G}_\ell(x)$ by definition. Hence $x \in Z_{\ell,sm}$.

Let $\mathcal{O}$ be the endomorphism ring of E. It is well-known that $\mathcal{O}$ is an order in an imaginary quadratic number field, and $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_p$. The semigroup

$$S = \mathrm{GL}_g(\mathcal{O}_{(p)}) \cap \mathrm{M}_{g\times g}(\mathcal{O}) \cap \mathrm{GSp}_{2g}(\mathbb{Q})$$

operates on $(A, \lambda) = (E, \lambda_E)^{\oplus g}$ as endomorphisms, which are ℓ-power isogenies and preserve the polarization up to ℓ-power multiples. Also the elements in the semigroup S induce ℓ-power Hecke correspondences on $\mathcal{A}_g/k$. Consequently the formal completion $Z_x^\wedge \subseteq \mathcal{A}_{g,x}^\wedge$ of Z at x is stable under the natural action of S on the formal completion $\mathcal{A}_{g,x}^\wedge$ of $\mathcal{A}_g/k$ at x. Since $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $\mathrm{GL}_g(\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p) \cap \mathrm{GSp}_{2g}(\mathbb{Q}_p)$ is isomorphic to $\mathrm{GL}_g(\mathbb{Z}_p)$, and S is dense in $\mathrm{GL}_g(\mathbb{Z}_p)$ under this isomorphism. Therefore the formal subscheme $Z_x^\wedge$ of $\mathcal{A}_{g,x}^\wedge$ is stable under the action of $\mathrm{GL}_g(\mathbb{Z}_p)$.

We will only use a tiny part of the above information at the level of the tangent space, namely that the tangent space $\mathrm{T}_x Z$ of Z at x is stable under the natural action of

$\mathrm{GL}_g(\mathbb{Z}_p)$ on $\mathrm{T}_x(\mathcal{A}_g/k)$. But we know that the action of $\mathrm{GL}_g(\mathbb{Z}_p)$ on $\mathrm{T}_x(\mathcal{A}_g/k)$ factorizes through the natural surjection $\mathrm{GL}(\mathbb{Z}_p) \twoheadrightarrow \mathrm{GL}_g(\mathbb{F}_p)$. Moreover as a representation of $\mathrm{GL}_g(\mathbb{F}_p)$ the tangent space $T_x(\mathcal{A}_g/k)$ is isomorphic to the subspace of symmetric elements in $k^{\oplus g} \otimes_k k^{\oplus g}$. Now it is well known that the second symmetric product of the standard representation of $\mathrm{GL}_g(\mathbb{F}_p)$ is absolutely irreducible if $p > 2$. If $p > 2$, the invariant subspace $\mathrm{T}_x Z$ being nonzero, it has to be equal to $\mathrm{T}_x(\mathcal{A}_g/k)$, and we conclude that $Z = \mathcal{A}_g/k$ when $p > 2$.

When $p = 2$, one can still conclude that $Z = \mathcal{A}_g/k$. For instance one can use the stronger information that Z is stable under the action of $\mathrm{GL}_g(\mathbb{Z}_p)$, and use Serre-Tate coordinates on $\mathcal{A}_{g,x}$ to perform computation with higher order deformations. We omit the details here. $\square$

Naturally we would like to understand the stabilizer subgroup action on the deformation space of supersingular abelian varieties as well. Unfortunately our understanding is very limited in this case. This forces one to pursue a different method. We only outline the strategy.

- (Step 1) We may assume that the Zariski closure Z of the prime-to-p Hecke orbit of an ordinary point x is defined over $\overline{\mathbb{F}_p}$. For any ordinary $\overline{\mathbb{F}_p}$-rational point of Z we can find a Shimura subvariety M containing x, which is essentially a product of Hilbert-Blumenthal moduli varieties. Then one is reduced to a similar question for the density of the prime-to-p Hecke correspondences for the Shimura variety M. Of course the Hecke correspondences come from a smaller group now. The Zariski closure of the Hecke orbit in M will be denoted by Z_1 now.
- (Step 2) A calculation using the Serre-Tate coordinates shows that there are only a finite number of possibilities for Z_1.
- (Step 3) Combining a result of Ekedahl and Oort and a calculation at the boundary, one deduce that Z_1 contains a supersingular point.
- (Step 4) Inspect the stabilizer subgroup action at the supersingular point. Even though we have only very poor understanding of this action, we still can rule out other possibilities and conclude that Z_1 is indeed equal to M. Since M contains points of the type discussed in the Example, one conclude that Z is indeed equal to $\mathcal{A}_g$.

The following is a VERY INCOMPLETE list of references. Of course consulting them will yield more information on the literature.

References

[Ch] C.-L. Chai, *Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli*, to appear in Inv. Math.

[Ek] T. Ekedahl, *On supersingular curves and abelian varieties*, Math. Scand **60** (1987), 151-178.

[EO] T. Ekedahl and F. Oort, *Connected subsets of a moduli space of abelian varieties*, preprint (1994).

[F] G. Faltings, *Arithmetische Kompaktifizierung des Modulraums der abelschen Varietäten*, Lecture Notes in Math. 1111, Springer-Verlag, 1985, pp. 321-383.

[FC] G. Faltings and C.-L. Chai, *Degeneration of Abelian Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 22, Springer-Verlag, 1990.

[GH] B. H. Gross and M. J. Hopkins, *Equivariant vector bundles on the Lubin-Tate moduli space* (to appear).

[Ha] M. Hazewinkel, *Formal groups and applications*, Academic Press, 1978.

[dJ] A. J. de Jong, *The moduli space of polarized abelian varieties*, Math. Ann. **295** (1993), 485-503.

[Ka] ______, *Serre-Tate local moduli*, Séminaire de Géométrie Algébrique d'Orsay 1976-78, Expposé Vbis, Surface Algébriques, Lecture Notes in Math. 868, Springer-Verlag, 1981, pp. 138-202.

[LT2] J. Lubin and J. Tate, *Formal moduli for one-parameter formal Lie groups*, Bull. Soc. Math. France **94** (1966), 49-60.

[Me] W. Messing, *The Crystals Associated to Barsotti-Tate Groups: with Applications to Abelian Schemes*, Lecture Notes in Math. 370, Springer-Verlag, 1972.

[Mu] D. Mumford, *Abelian Varieties*, Tata Inst., Studies in Math. 5, Oxford University Press, 1974.

[N] P. Norman, *An algorithm for computing local moduli of abelian varieties*, Ann. Math. **101** (1975), 499-509.

[NO] P. Norman and F. Oort, *Moduli of abelian varieties*, Annals of Math. **112** (1980), 413-439.

[OO] T. Oda and F. Oort, *Supersingular abelian varieties*, Proc., Kyoto Univ., Kyoto, 1977, Kinokuniya Book Store, Tokyo, 1978, pp. 595-621.

[Ta] J. Tate, *Classes d'isogeny de variétés abéliennes sur un corps fini (d'apès T. Honda)*, Sém. Bourbaki Exp. 352 (1968/69), Lecture Notes in Math. 179, Springer Verlag, 1971.

[Z] T. Zink, *Cartiertheorie kommutativer formaler Gruppen*, Teubner-Texte Math. 68, Teubner, Leipzig, 1984.

Autor: RICHARD TAYLOR

Adresse: DPMMS
CAMBRIDGE UNIVERSITY
16 MILL LANE
CAMBRIDGE CB2 1SB U.K.

(1) The basic problem we wish to consider is to describe the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ together with certain additional structure. Namely the special element c = complex conjugation; and for each prime number p the decomposition group G_p, the inertia group I_p and the Frobenius element $\mathrm{Frob}_p \in G_p/I_p$. All this additional data is only really defined up to conjugation in $G_{\mathbb{Q}}$. Recall that $G_{\mathbb{Q}} \supset G_p \trianglerighteq I_p$, that $G_p \cong G_{\mathbb{Q}_p}$ and that $G_p/I_p \cong G_{\mathbb{F}_p} = \langle \mathrm{Frob}_p \rangle$.

The Cebotarev Density theorem tells us that Frobenius elements are dense in $G_{\mathbb{Q}}$ (or more precisely conjugacy classes of lifts of Frobenius elements are). Thus if we keep track of these elements in our description of $G_{\mathbb{Q}}$ we will "tie things down".

As an example class field theory tells us that the abelianisation $G_{\mathbb{Q}}^{ab}$ of $G_{\mathbb{Q}}$ can be described completely by

$$G_{\mathbb{Q}}^{ab} \cong \prod_p \mathbb{Z}_p^\times$$

$$I_p \leftrightarrow \mathbb{Z}_p^\times$$

$$\mathrm{Frob}_p \leftrightarrow (p, p, p, \ldots, p, \overset{p^{th}\text{ place}}{1}, p, \ldots).$$

Attention has focused on describing the representations of $G_{\mathbb{Q}}$ rather than $G_{\mathbb{Q}}$ itself. Class field theory handles 1 dim^l representations. In general d-dim^l representations are supposed to be connected with automorphic forms on GL_d — an extraordinary relationship between algebraic objects ($G_{\mathbb{Q}}$) and analytic/geometric ones (automorphic forms). In this talk we discussed 2-dim' representations and classical modular forms.

② If $N \in \mathbb{Z}_{\geq 1}$ set

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv d-1 \equiv 0 \ (N) \right\}$$

$$\mathfrak{h} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$$

$$Y_1(N) = \Gamma_1(N) \backslash \mathfrak{h}, \text{ a } \underline{\text{modular curve}}.$$

$H^1(Y_1(N), \mathbb{C})$ is a complex vector space whose dimension is easy to calculate. For each prime p one can define <u>Hecke operators</u> T_p (and S_p), which are endomorphisms of $H^1(Y_1(N), \mathbb{C})$ induced by simple correspondences on $Y_1(N) \times Y_1(N)$. The <u>Hecke algebra</u> $\mathbb{T}_1(N)$, i.e. the $\mathbb{Z}$-algebra generated by T_p and S_p $\forall p \nmid N$, plays a fundamental role in our story. It is a ~~[illegible]~~ reduced commutative $\mathbb{Z}$-algebra, which is finite and free as a $\mathbb{Z}$-module. For example it might be $\mathbb{Z}$, $\mathbb{Z}[i]$ or $\{(a,b) \in \mathbb{Z}^2 \mid a \equiv b \ (6)\}$. It turns out to encode a remarkable amount of subtle information about $G_{\mathbb{Q}}$.

The first hint of this is given by the following deep theorem of Eichler and Shimura from the '50's and '60's.

<u>THM</u> 1) Suppose $\overline{\theta} : \mathbb{T}_1(N) \to \overline{\mathbb{F}}_L$ is a ring homomorphism. Then $\exists!$ semi-simple representation (continuous)

$$\overline{\rho} = \overline{\rho}_{\overline{\theta}} : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{F}}_L)$$

such that a) $\det \overline{\rho}(c) = -1$, i.e. $\overline{\rho}$ is <u>odd</u>

b) if $p \nmid NL$ then $\overline{\rho}$ is <u>unramified</u> at p (i.e. $\overline{\rho}(I_p) = \{1\}$) and

$$\operatorname{tr} \overline{\rho}(\mathrm{Frob}_p) = \overline{\theta}(T_p) \quad ; \quad \det \overline{\rho}(\mathrm{Frob}_p) = p\, \overline{\theta}(S_p)$$

2) Suppose $\theta : \mathbb{T}_1(N) \to \overline{\mathbb{Q}}_L$ is a ring homomorphism. Then $\exists!$ irreducible representation (cts)

$$\rho = \rho_\theta : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{Q}}_L)$$

such that a) $\det \rho(c) = -1$

b) if $p \nmid NL$ then ρ is unramified at p and

$$\operatorname{tr} \rho(\mathrm{Frob}_p) = \theta(T_p) \quad ; \quad \det \rho(\mathrm{Frob}_p) = p\, \theta(S_p)$$

c) $\rho|_{I_L}$ is "potentially semi-stable".

We remark that the conditions b) enormously overdetermine $\overline{\rho}$ (resp. ρ) and the theorem says something very strong about the existence of number fields in which primes split in ways determined by $\overline{\theta}$ (resp. θ).

We should explain the term "potentially semi-stable". Potentially means $\rho|_{G_K}$ is semi-stable for some <u>finite</u> extension $K/\mathbb{Q}_L$. For us semi-stable will mean that either $\rho|_{I_K} \sim \begin{pmatrix} \epsilon & * \\ 0 & 1 \end{pmatrix}$, where ϵ is the cyclotomic character; or that $\rho|_{I_K}$ is "flat", i.e. $\det \rho|_{I_K} = \epsilon$ and there exists $A/\mathcal{O}_K$ an l-divisible group whose Tate module gives rise to the representation ρ. We warn the reader that this terminology is not completely standard.

③ Two important conjectures predict that this Eichler-Shimura construction exhausts a large class of 2-dim'l representations of $G_{\mathbb{Q}}$.

<u>CONJ</u> (SERRE) If $\bar{\rho}: G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{F}}_L)$ is continuous odd and absolutely irreducible then $\bar{\rho} = \bar{\rho}_{\bar{\theta}}$ for some $\bar{\theta}: \mathbb{T}_1(N) \to \overline{\mathbb{F}}_L$. (We say that $\bar{\rho}$ is <u>modular</u>.) Moreover one can take $N = \text{conductor}(\bar{\rho})\, L^{\delta_{\bar{\rho}}}$, where $\delta_{\bar{\rho}} = 0, 1$ or 2 is calculable from $\bar{\rho}|_{I_L}$ only.

<u>CONJ</u> (FONTAINE-MAZUR) If $\rho: G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{Q}}_L)$ is a continuous, odd and irreducible representation which is unramified at all but finitely many primes and whose restriction to the inertia group I_L is "potentially semi-stable"; then $\exists\ \theta: \mathbb{T}_1(N) \longrightarrow \overline{\mathbb{Q}}_L$ st $\rho \cong \rho_\theta$ (i.e. ρ is <u>modular</u>). Moreover one can take $N = \text{conductor}(\rho) . L^{\delta_\rho}$ where $\delta_\rho \in \mathbb{Z}_{\geq 0}$ depends only on $\rho|_{I_L}$.

Let us summarise what is known about these conjectures.

Serre's Conjecture:

a) If $\ell \geq 3$ and $\bar{\rho}$ is modular then $\bar{\rho}$ is modular with $N = N(\bar{\rho})\ell^{\delta_{\bar{\rho}}}$.

(This is the culmination of work of a large number of people over the last 10 years. They include Mazur, Ribet, Carayol, Gross, Coleman, Voloch, Edixhoven and Diamond.)

b) About the 1st half of the conjecture very little is known.

It is true for i) $\bar{\rho}: G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_2)$ (Hecke.....)

ii) $\bar{\rho}: G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_3)$ (Langlands–Tunnell)

iii) $\bar{\rho}: G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_5)$

$\det \bar{\rho} = \epsilon$, $\# \bar{\rho}(I_3) \mid 10$ (Wiles....)

ii) is based on Langlands' theory of base change and uses analytic methods (trace formula, L-functions). It also relies on two coincidences, namely that $GL_2(\mathbb{F}_3)$ is soluble and that there is a lift $GL_2(\mathbb{F}_3) \hookrightarrow GL_2(\mathbb{Z}[\sqrt{-2}])$.

Fontaine–Mazur Conjecture:

a) Carayol showed (before the conj was made) that if ρ is modular then one may take $N = \text{conductor}(\rho) \cdot (\text{power of } \ell)$.

b) Wiles' recent breakthrough was in the direction Serre's conj $\Rightarrow$ Fontaine-Mazur Conjecture.

THM (WILES + TW + D) Suppose $\ell \geq 3$ and that $\rho: G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_\ell)$ is a cts, irreducible representation which is unramified at all but finitely many primes. Suppose also that

1) the reduction $\bar{\rho}$ of ρ modulo ℓ is modular
2) $\bar{\rho}|_{\mathbb{Q}(\sqrt{(-1)^{\frac{\ell-1}{2}}\ell})}$ is absolutely irreducible
3) ρ is semi-stable.

Then ρ is modular.

(4) This result has many applications. Perhaps most importantly is that combined with Faltings' isogeny theorem it shows that all elliptic curves which are semi-stable at 3 and 5 are modular. This in turn has important implications for the study of elliptic curves.

Perhaps most amusing is:

<u>THM</u> (WILES) If $a, b, c, n \in \mathbb{Z}$; if $abc \neq 0$ and if $n \geq 3$ then $a^n + b^n \neq c^n$.

This can be deduced using an extraordinary trick of Frey. One may assume that $n \geq 5$, n is prime and $b \equiv 0 \ (2)$, $a \equiv -1 \ (4)$. Then Frey considers the elliptic curve

$$E: \ y^2 = x(x - a^n)(x + b^n).$$

If $\bar{\rho}_{E,n}$ denotes the representation of $G_{\mathbb{Q}}$ on $E[n]$ then

a) $\bar{\rho}_{E,n}$ is odd (easy)

b) $\bar{\rho}_{E,n}$ is irreducible (Mazur)

c) $N(\bar{\rho}_{E,n}) = 2$, $\delta_{\bar{\rho}_{E,n}} = 0$: by the theory of the Tate curve — as disc(E) is nearly an n^{th} power.

Thus

Serre's conjecture $\Rightarrow$ $\bar{\rho}_{E,n}$ modular on $\Gamma_1(2)$ $\xRightarrow{\text{explicit calculation}}$ #

$\Uparrow$ Ribet

$\bar{\rho}_{E,n}$ modular

$\Uparrow$ easy

$\rho_{E,l}$ modular for some prime l

$\Uparrow$ Wiles

$\bar{\rho}_{E,l}$ modular for some prime l + sufficiently irreducible

$\rho_{E,l}$ = rep on l-adic Tate module of E

Taking $l = 3$ and 5 and using Langlands–Tunnell and a trick finishes the proof.

(5) We now turn to the proof of Wiles' theorem. The argument we present is taken from the papers of Wiles and Taylor-Wiles, but includes simplifications found by Faltings. In the hope of increasing clarity we will only sketch the proof in a special case.

Fix $l \geq 3$ a prime and $\bar{\rho}: G_{\mathbb{Q}} \to GL_2(\mathbb{F}_l)$ a cts representation. We will suppose that:

- $\bar{\rho}|_{\mathbb{Q}(\sqrt{(-1)^{\frac{l-1}{2}} l})}$ is abs. irreducible
- $\bar{\rho}$ is modular
- $\det \bar{\rho} = \epsilon$
- if $p \neq l$ then $\bar{\rho}|_{I_p} \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ ($* = 0$ possible)
- $\bar{\rho}|_{I_l}$ is "semi-stable" (we leave the defn to the reader)

<u>THM A</u> Suppose $\rho: G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_l)$ satisfies $\rho \bmod l$ is modular, $\det \rho = \epsilon$, ρ ramifies at only finitely many primes and $\rho|_{I_l}$ is semi-stable. Then ρ is modular.

We will in fact prove a strengthening of this result — which seems to be easier to attack. To describe this let S denote a finite set of primes and A a complete noetherian local ring with residue field $\mathbb{F}_l$ (eg. $\mathbb{F}_l$, $\mathbb{Z}/l^2$, $\mathbb{Z}_l$, $\mathbb{Z}_l[[T]]$, $\{(a,b) \in \mathbb{Z}_l^2 \mid a \equiv b \ (l)\}$).

<u>Definition</u> $\rho: G_{\mathbb{Q}} \to GL_2(A)$ is a <u>lifting of $\bar{\rho}$ of type S</u> if

a) $\rho \bmod \mathfrak{m}_A = \bar{\rho}$

b) $\det \rho = \epsilon$

c) $\rho|_{I_l}$ is "semi-stable" (meaning left to reader)

d) if $p \notin S$ then $\rho|_{I_p}$ is minimally ramified, i.e.

- $\bar{\rho}$ unramified at $p \Rightarrow \rho$ unramified at p

- $\bar\rho|_{I_p} \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \Rightarrow \rho|_{I_p} \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$
- $\bar\rho|_{I_l}$ "flat" $\Rightarrow \rho|_{I_l}$ "flat".

What we will in fact prove is that for all S any lifting of type S is modular in an appropriate sense.

If $\theta : \mathbb{T}_1(N) \to \overline{\mathbb{Q}}_l$ and ρ_θ is of type S then we may replace N by $N_S = N(\bar\rho) l \prod_{p \in S} p^2$. There is a homomorphism $\bar\theta : \mathbb{T}_1(N_S) \to \mathbb{F}_l$ such that $\bar\rho_{\bar\theta} = \bar\rho$. let $\mathfrak{m} = \ker\bar\theta$. We set $\mathbb{T}_S = \mathbb{T}_1(N_S)_{\mathfrak{m}} / \bigcap \ker\theta$, where the intersection is over all θ so that ρ_θ is a lift of $\bar\rho$ of type θ. $\mathbb{T}_S$ is finite over $\mathbb{Z}_l$ and reduced. Moreover one can deduce from the Eichler-Shimura theorem that there is a unique representation
$$\rho_S^{mod} : G_{\mathbb{Q}} \to GL_2(\mathbb{T}_S)$$
with $\operatorname{tr} \rho_S^{mod}(\mathrm{Frob}\, p) = T_p$ for almost all p. Moreover ρ_S^{mod} is a lifting of $\bar\rho$ of type S.

<u>THM B</u> Suppose $\rho : G_{\mathbb{Q}} \to GL_2(A)$ is a lifting of $\bar\rho$ of type S then $\exists!\ \theta : \mathbb{T}_S \to A$ such that $\rho \sim \theta \circ \rho_S^{mod}$.

This is the strengthening of <u>THM A</u> we shall actually prove. It is a theorem of Mazur that there is some ring with this universal property. Namely:

<u>Thm</u> (Mazur) $\exists\ \rho_S^{univ} : G_{\mathbb{Q}} \to GL_2(R_S)$ a lifting of $\bar\rho$ of type S such that if $\rho : G_{\mathbb{Q}} \to GL_2(A)$ is any lifting of $\bar\rho$ of type S then $\exists!\ \theta : R_S \to A$ such that $\rho \sim \theta \circ \rho_S^{univ}$.

<u>Cor</u> $R_S \twoheadrightarrow \mathbb{T}_S$, naturally.

It suffices to show that this map is an isomorphism. Again Wiles[8] looks to prove a crucial strengthening of this result. Namely:

<u>THM C</u> $R_S \twoheadrightarrow \mathbb{T}_S$ is an isomorphism and these rings are complete intersections (i.e. of the form $\mathbb{Z}_l[[T_1, \ldots, T_r]]/(f_1, \ldots, f_r)$ for some $r, f_1, \ldots, f_r$).

Wiles deduces theorem C from the special case:

<u>THM D</u> $R_\phi \twoheadrightarrow \mathbb{T}_\phi$ is an isomorphism and these rings are complete intersections.

To deduce THM C from THM D Wiles uses an ingenious criterion from commutative algebra. This criterion can also be used to show that THM C is equivalent to a statement that is very close to the Bloch–Kato conjecture for the ~~adjoint~~ trace zero part of the adjoint of the l-adic representation of any character $\mathbb{T}_S \to \mathbb{Z}_l$.

(6) We now turn to the proof of theorem D. We will need to consider finite sets Q of primes q which have the properties $q \equiv 1 \ (l)$ and $\bar{\rho}$ is unramified at q and $\bar{\rho}(\mathrm{Frob}_q)$ has distinct eigenvalues. For such a set Q it is not hard to show that for $q \in Q$, $\rho_Q^{univ}|_{I_q}$ has the special form $\begin{pmatrix} \chi & 0 \\ 0 & \chi^{-1} \end{pmatrix}$ where

$$\chi : I_q \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_q^{nr}(\zeta_q)/\mathbb{Q}_q^{nr}) \cong (\mathbb{Z}/q\mathbb{Z})^\times \twoheadrightarrow \Delta_q$$

and where Δ_q denotes the maximal quotient of $(\mathbb{Z}/q\mathbb{Z})^\times$ of l-power order.

We will let $\Delta_Q = \prod_{q \in Q} \Delta_q$ and $\mathfrak{a}_Q$ denote the augmentation ideal of $\mathbb{Z}_l[\Delta_Q]$. If $Q = \{q_1, \ldots, q_r\}$

and $l^{\alpha_i} \| q_i - 1$ then note that

$$\mathbb{Z}_l[\Delta_Q] \cong \mathbb{Z}_l[[S_1, \ldots, S_r]] \big/ \left((1+S_1)^{l^{\alpha_1}}-1, \ldots, (1+S_r)^{l^{\alpha_r}}-1\right)$$

and $\mathfrak{a}_Q \leftrightarrow (S_1, \ldots, S_r)$.

We have the following 3 important observations:

ⓐ $R_\phi = R_Q / \mathfrak{a}_Q$ (easy)

ⓑ $\mathbb{T}_\phi \otimes \mathbb{Q}_l = (\mathbb{T}_Q / \mathfrak{a}_Q) \otimes \mathbb{Q}_l$ (easy)

ⓒ $\mathbb{T}_Q$ is a free $\mathbb{Z}_l[\Delta_Q]$-module and hence

$$\mathbb{T}_\phi = \mathbb{T}_Q / \mathfrak{a}_Q.$$

To prove ⓒ one finds a covering $Y \to X$ of open modular curves with Galois group Δ_Q such that one has an isomorphism $H^1(Y, \mathbb{Z}_l)^-_{\mathfrak{m}} \cong \mathbb{T}_Q$. Here $-$ refers to the action of c and $\mathfrak{m}$ is some max'l ideal of a Hecke algebra. Such isomorphisms go back to Mazur and rely on the q-expansion principle. Then one uses topological arguments to show that $H^1(Y, \mathbb{Z}_l)^-$ is free over $\mathbb{Z}_l[\Delta_Q]$ (following de Shalit).

One next has to choose the sets Q carefully.

Thm $\exists\, r \in \mathbb{Z}_{\geq 0}$ such that $\forall n \in \mathbb{Z}_{\geq 1}$ we can find a set Q_n as above such that

- $q \in Q_n \Rightarrow q \equiv 1 \ (l^n)$
- $\# Q_n = r$
- R_{Q_n} can be topologically generated over $\mathbb{Z}_l$ by r elements.

The proof of this theorem relies on Poitou-Tate duality in Galois cohomology and on a Cebotarev argument. Note that R_{Q_n} can be top. gen. by $\dim \mathfrak{m}_{Q_n}/(l, \mathfrak{m}_{Q_n}^2)$ elements and that $\mathfrak{m}_{Q_n}/(l, \mathfrak{m}_{Q_n}^2)$ can be interpreted in terms of Galois cohomology.

To explain the argument we shall suppose that we may take $n=\infty$ (which is clearly absurd). We may correct this argument by using a limiting process. In our ideal ($n=\infty$) situation we would have a diagram:

$$\begin{tikzcd} & \mathbb{Z}_l[[S_1,\dots,S_r]] \arrow[d] & \\ \mathbb{Z}_l[[X_1,\dots,X_r]] \arrow[r, two heads] & R_{Q_\infty} \arrow[r, two heads] \arrow[d] & R_\emptyset \arrow[d] \\ & \mathbb{T}_{Q_\infty} \arrow[r, two heads] & \mathbb{T}_\emptyset \end{tikzcd}$$

such that a) $R_\emptyset = R_{Q_\infty}/(S_1,\dots,S_r)$

$\mathbb{T}_\emptyset = \mathbb{T}_{Q_\infty}/(S_1,\dots,S_r)$

b) $\mathbb{T}_{Q_\infty}$ is a finite free $\mathbb{Z}_l[[S_1,\dots,S_r]]$-module.

Thus $\mathbb{T}_{Q_\infty}$ has Krull dimension $r+1$ and so the map $\mathbb{Z}_l[[X_1,\dots,X_r]] \twoheadrightarrow \mathbb{T}_{Q_\infty}$ must be an injective. Thus we have isomorphisms:

$$\mathbb{Z}_l[[X_1,\dots,X_r]] \xrightarrow{\sim} R_{Q_\infty} \xrightarrow{\sim} \mathbb{T}_{Q_\infty}.$$

Dividing by $(S_1,\dots,S_r)$ we get

$$\mathbb{Z}_l[[X_1,\dots,X_r]]/(S_1,\dots,S_r) \xrightarrow{\sim} R_\emptyset \xrightarrow{\sim} \mathbb{T}_\emptyset$$

as desired.

Titel: The tautological ring of M_g

Autor: Eduard Looijenga

Adresse: Mathematics, University of Utrecht
P.O. Box 80.010
NL-3508 TA Utrecht

email: looijen@math.ruu.nl

Consider systems $(C; x_1,\dots,x_n)$ where C is a nonsingular complex projective curve and $x_1,\dots,x_n$ are (not nec. distinct) points of C. Such systems are parametrized by a coarse moduli space $\mathcal{C}_g^n$ (for $n=0$ it is more traditional to denote this space by $\mathcal{M}_g$). Notice that there is a forgetful morphism $\pi^n: \mathcal{C}_g^n \to \mathcal{M}_g$.

Assigning to $(C; x_1,\dots,x_n)$ the complex line $T^*_{x_i}C$ defines a line bundle on $\mathcal{C}_g^n$ (in the orbifold sense); we denote its first Chern class by $K_i \in A^1(\mathcal{C}_g^n)$. Here $A^\cdot$ stands for the Chow ring modulo rational equivalence, tensorized with $\mathbb{Q}$. The basic tautological classes

$$\kappa_d := (\pi^1)_* K_1^{d+1} \in A^d(\mathcal{M}_g) \quad (d=1,2,\dots)$$

were introduced and studied by Mumford [M] and — with A^d replaced by $H^{2d}(\ ;\mathbb{Q})$ — by Morita and Miller. The subalgebra $R^\cdot(M_g)$ of $A^\cdot(M_g)$ generated by these classes is called the tautological ring of M_g. No one seemed to have any idea what this ring would look like until Carel Faber made a very precise conjecture about it (after ~~doing~~ numerous computations for low g).

Faber's conjecture ('93): $R^\cdot(\mathcal{M}_g)$ is a graded Gorenstein ring with socle in degree $g-2$. Moreover, κ_1 has the

Lefschetz property: if $k \in \{0,\dots,g-2\}$ has the same parity as g, then
$$\kappa_1^k : R^{\frac{g-2-k}{2}}(M_g) \to R^{\frac{g-2+k}{2}}(M_g)$$
is an isom.

Furthermore, Faber conjectures that $R^\cdot(M_g)$ is already generated by the κ_d with $d \leq \frac{g-1}{3}$ (Mumford [M] proved this for $d \leq g-2$) and predicts $\dim R^k(M_g)$.

Our main result supports this conjecture (but is ~~still~~ far from proving it).

<u>Thm</u>. The subalgebra of $A^\cdot(\mathcal{C}_g^n)$ generated by $K_1,\dots,K_n$ is zero in degree $> n+g-2$, whereas all its elements of degree $n+g-2$ are proportional to the hyperelliptic locus
$$\mathcal{H}_g^n := \{(C; x_1,\dots,x_n) : C \text{ hyperell.}, x_1 = \dots = x_n \text{ a Weierstr. pt.}\}.$$

Since the direct image of $K_1^{d_1+1} \cdots K_n^{d_n+1}$ under $\pi^n : \mathcal{C}_g^n \to M_g$ is $\kappa_{d_1} \cdots \kappa_{d_n}$ we find:

<u>Cor.</u> $R^k(M_g) = \begin{cases} 0 & \text{for } k > g-2 \\ \mathbb{Q}[\mathcal{H}_g^0] & k = g-2 \end{cases}$

(We also used that $[\mathcal{H}_g^0] \in R^{g-2}(M_g)$; it is however not known whether $[\mathcal{H}_g^0] \neq 0$.)

In particular, κ_1^{g-1} is numerically zero. Since κ_1 is an ample class we thus recover a theorem of Diaz [D] that says that M_g does not contain a complete subvariety of dim $>$ $g-2$. (As Oort [O] pointed out, these properties of κ_1 are independent of the characteristic, so it follows that Diaz's theorem is also valid in positive characteristic.)

The most natural definition of the tautological ring of $\mathcal{C}_g^n$ is ~~the subalgebra~~ to let it be the subalgebra $R^\cdot(\mathcal{C}_g^n)$ of $A^\cdot(\mathcal{C}_g^n)$ generated by $K_1, \ldots, K_n$, $\pi^* \kappa_d$ $(d=1,2,\ldots)$ and the classes of the diagonal divisors $(x_i = x_j)$ $(1 \le i < j \le n)$. It is not hard to show that the theorem still holds for this subalgebra.

The proof of the theorem is a mixture of topology and algebraic geometry. The statement on the degree $n+g-2$ classes is proved using the Mukai-Beauville transform applied to the universal Jacobian of genus g.

References:

[D] S. Diaz: A bound on the dimensions of complete subvarieties of M_g, Duke Math. J. 51 (1984), 405-408.

[L] E. Looijenga: On the tautological ring of M_g, to appear in Invent. Math.

[M] D. Mumford: Towards an enumerative geometry of the moduli space of curves, Arithmetic and Geometry II (M. Artin and J. Tate eds.) Birkhaüser Verlag 1983, 271-328.

[O] F. Oort: Complete varieties of moduli spaces. Abelian Varieties (W. Barth, K. Hulek, H. Lange eds.) de Gruyter Verlag 1995, 225-235.

Titel: Tate-cycles
Autor: Gerd Faltings
Seite: 1
Adresse: MPI f. Mathematik, Gottfried-Claren-Str. 26, 53227 Bonn

Suppose we are given a perfect field k of characteristic $p>0$, $V_0 = W(k)$ its ring of Witt-vectors, $K_0 = V_0[\frac{1}{p}]$ the fraction-field, K a finite totally ramified extension with integers V. For an abelian variety A over V Tate-cycles in $H^1_{et}(A\otimes\bar{K}, \mathbb{Z}_p)^{\otimes 2r}(r)$ $(r\geq 0)$ correspond to crystalline Tate-cycles, at least after inverting p (this was proved by Fontaine). We show that for $r\leq p-2$ this correspondence preserves integrality.

As an application one can study good models for Shimura-varieties of abelian or Hodge type. Namely points in the special fibre of their closure (in $\mathcal{A}_g$) can be obtained from abelian varieties with Tate-cycles as above. As a consequence Adrian Vasiu has shown in many cases that these closures are smooth.

Autor: Richard Pink
Adresse: Universität Mannheim (Germany)
D-68131 Mannheim
e-mail: pink@math.uni-mannheim.de

Part 1: Abelian Varieties.

$K \subset \mathbb{C}$ number field
A abelian variety over K
$A(\mathbb{C}) \cong \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ with complex structure on $\mathbb{R}^{2g}$ given by $h: \mathbb{C}^* \longrightarrow Gl_{2g}(\mathbb{R})$ (choose isomorphism).

Def.: Mumford-Tate group $G \subset Gl_{2g,\mathbb{Q}}$ is the smallest algebraic subgroup defined over $\mathbb{Q}$ such that h factors through $G(\mathbb{R})$.

ℓ rational prime $\Longrightarrow$

$$V_\ell A := \mathrm{Hom}(\mathbb{Q}_\ell, A[\ell^{-\infty}]) \overset{\text{(via above isom)}}{\cong} \mathbb{Q}_\ell^{2g}$$

has action of Galois:

$$\rho_\ell : \mathrm{Gal}(\overline{K}/K) \longrightarrow Gl_{2g}(\mathbb{Q}_\ell).$$

Def.: algebraic ℓ-adic monodromy group

$G_\ell \subset Gl_{2g,\mathbb{Q}_\ell}$:= Zariski closure of $\mathrm{im}(\rho_\ell)$.

Conjecture (M.T.): $\underset{\text{(identity component)}}{G_\ell^0} = G \times \mathbb{Q}_\ell$

Known: (a) $G_\ell^0 \subset G \times \mathbb{Q}_\ell$ (Deligne)
(b) Tate conjecture for endomorphisms
(c) G_ℓ^0 is reductive $\rightarrow$ (Faltings)

(d) Some special cases (Serre ...)

Theorem: $\mathrm{End}(A(\mathbb{C})) = \mathbb{Z}$

$\Rightarrow$ $\exists$ connected reductive group defined over $\mathbb{Q}$: $H \subset GL_{2g,\mathbb{Q}}$ such that for all primes ℓ in a set of Dirichlet density 1 we have:

$$G_\ell^\circ \simeq H \times \mathbb{Q}_\ell$$

conjugate under $GL_{2g}(\mathbb{Q}_\ell)$.

Moreover: H^{der} is $\mathbb{Q}$-simple, and H is itself the Mumford-Tate group of some abelian variety.

(joint work M. Larsen – R. P.)

Part 2: Drinfeld Modules

F = function field of smooth connected projective algebraic curve X over $\mathbb{F}_p$

$\infty \in X$ closed point

$A := \Gamma(X \setminus \{\infty\}, \mathcal{O}_X)$

$\mathbb{C}_p :=$ completion of algebraic closure of F_∞

$\Lambda \subset \mathbb{C}_p$ an A-submodule of finite type, rank $= r > 0$

$e_\Lambda(z) := z \cdot \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right)$ converges

$\Rightarrow \forall a \in A \ \exists \varphi(a) \in \left\{ \begin{array}{l} \text{additive polynomials} \\ = u_0 z + u_1 z^p + u_2 z^{p^2} + \ldots \end{array} \right\}$

with

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}_p & \xrightarrow{e_\Lambda} & \mathbb{C}_p & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow a & & \downarrow \varphi(a) & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}_p & \xrightarrow{e_\Lambda} & \mathbb{C}_p & \longrightarrow & 0 \end{array}$$

commutative.

Def.: The ring homomorphism

$$\varphi : A \longrightarrow \mathrm{End}_{\mathbb{F}_p}(\mathbb{G}_{a,\mathbb{C}_p}), \quad a \mapsto \varphi(a)$$

is called "Drinfeld Module of rank r"

Autor: Richard Pink	Seite: 4

$\wp \in X$ closed point $\neq \infty$

$\Rightarrow V_\wp \varphi := \mathrm{Hom}_A(F_\wp, \underbrace{\varphi[\wp^{-\infty}]}_{:= \{z \in \mathbb{C}_p \mid \exists n \geq 0 \; \varphi(\wp^n)(z)=0\}}) \cong F_\wp^r$

"Tate module"

Analog of Mumford-Tate group?

"Hodge Filtration" on $\Lambda \otimes \mathbb{C}_p$ suggests GL_r, if there are no nonobvious endomorphisms.

Suppose: φ defined over K = finite extension of F

that is: the coefficients of all $\varphi(a)$ lie in K.

Then $\mathrm{Gal}(\overline{K}/K)$ acts on $\varphi[\wp^{-\infty}]$

$\Rightarrow \rho_\wp : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Aut}(V_\wp \varphi) \cong GL_r(F_\wp)$

Theorem: Assume: (a) φ defined over K, K/F finite.

(b) $\mathrm{End}_{\overline{K}}(\varphi) = A$.

Then: $\mathrm{image}(\rho_\wp) \subset GL_r(F_\wp)$ open.

Among the ingredients are:

(1) semisimplicity, Tate conjecture (Taguchi 田口, Tamagawa 玉川 (≤ 1994)).

implies: Zariski closure of im(ρ_φ) is reductive and acts absolutely irreducibly.

(2) <u>Proposition</u>: The set of places of K where φ has ordinary reduction has Dirichlet density 1.

(3) <u>Abstract group theoretical result</u>:

<u>Theorem</u>: L local field,

G simple simply connected semisimple group over L

$\Gamma \subset G(L)$ <u>Zariski dense compact subgroup</u>

$\Longrightarrow$ $\exists$ subfield $L' \subset L$ which is closed and with L/L' finite

$\exists$ model G' of G over L'

such that $[\Gamma, \Gamma]$:= commutator subgroup

is an <u>open</u> compact subgroup of $G'(L')$.

<u>except</u> (possibly) if G has roots of different lengths with length ratio $\|\alpha\|^2/\|\beta\|^2 = p = \mathrm{char}(L)$

<u>SL_n: OKAY!</u>

Titel: Group cohomology and Zeta Functions

Autor: Ulrich Bunke and Martin Olbrich

Seite: 1

Adresse: Berlin

Let G be a real semi-simple Lie group of real rank one, $K \subset G$ be its maximal compact subgroup and $G = KAN$ be an Iwasawa decomposition. By $\mathcal{HC}(g, K)$ we denote the Harish-Chandra modules. For $(\pi, V_{\pi,K}) \in \mathcal{HC}(g, K)$ we have the canonical globalizations of $V_{\pi,K}$:

$$(V_{\pi,\omega} \subset V_{\pi,\infty} \subset V_{\pi,-\infty} \subset V_{\pi,-\omega} .$$

Let n be the Lie-algebra of N. Then $H^*(n, V_{\pi,K})$ is a finite-dimensional MA-module, where $M := Z_K(A)$.

Theorem 1: Let $(\pi, V_{\pi,K}) \in \mathcal{HC}(g, K)$. Then the inclusion $V_{\pi,K} \hookrightarrow V_{\pi,*-}$, $* = \infty, \omega$, induces an isomorphism of MA-modules:

$$H^p(n, V_{\pi,K}) \xrightarrow{\sim} H^p(n, V_{\pi,*}), \ \forall p \geq 0 .$$

Moreover, if $(\tilde{\pi}, V_{\tilde{\pi},K}) \in \mathcal{HC}(g, K)$ is the dual of $(\pi, V_{\pi,K})$, then we have the isomorphism of MA-modules

$$H^p(n, V_{\pi,-*})^{\sim} \cong H^{\dim(n)-p}(n, V_{\tilde{\pi},*}) \otimes \Lambda^{\dim(n)} n .$$

Let $\Gamma \subset G$ be a torsion-free, discrete, cocompact subgroup.

<u>Theorem 2</u>: Let $(\pi, V_\pi, K) \in \mathcal{HC}(\mathfrak{g}, K)$. Then for $* = \omega, \infty$ we have

i) $\dim H^p(\Gamma, V_{\pi, \pm *}) < \infty$

ii) $H^p(\Gamma, V_{\pi, -*})^\sim \simeq H^{\dim(G/K)-p}(\Gamma, V_{\tilde{\pi}, *})$.

The proof of the Theorems 1, 2 is obtained using a Γ- and $\mathfrak{n}$-acyclic resolution of $V_{\pi, -*}$ of the form:

$$0 \to V_{\pi, -*} \longrightarrow \mathcal{E}_0 \xrightarrow{D_0} \mathcal{E}_1 \xrightarrow{D_1} \mathcal{E}_2 \xrightarrow{D_2} \cdots, \qquad (1)$$

where $\mathcal{E}_i$ is the space of sections of a finite-dimensional vector bundle over G/K (satisfying growth conditions in case $* = \infty$).

Let $\mathfrak{a}$ be the Lie-algebra of A and $\rho \in \mathfrak{a}^\sim$ defined by $\rho(H) := \frac{1}{2} \operatorname{tr} \operatorname{ad}(H)_{|\mathfrak{n}}$, $\forall H \in \mathfrak{a}$. For $\sigma \in \hat{M}$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ we let

$$H^{\sigma,\lambda} = \{ f : G \to V_\sigma \mid f(gman) = a^{\lambda-\rho} \sigma(m)^{-1} f(g) \ \forall\, man \in MAN,\ f_{|K} \in L^2 \}$$

$\pi^{\sigma,\lambda}$ - left-regular action of G on $H^{\sigma,\lambda}$

be the principal series representation of G.

The same parameters enter into the Selberg zeta function

$$Z_\Gamma(s,\sigma) := \prod_{\substack{1\neq [g] \text{ primitive} \\ \text{conjugacy class} \\ \text{in } \Gamma}} \prod_{k=0}^{\infty} \det\left(1 - a_g^{-\rho-s} S^k(\mathrm{Ad}(m_g a_g)^{-1}_{|\mathfrak{n}}) \otimes \sigma(m_g)\right)$$

· notation: · S^k - k'th symmetric power

· $g \underset{\text{in } G}{\overset{\text{conj.}}{\sim}} m_g a_g \in MA$

· product converges for $\mathrm{Re}(s) > \rho$, has a meromorphic continuation.

The following Theorem was conjectured by S. Patterson

Theorem 3: For $* = \omega, \infty$, $\lambda \neq 0$, we have

i) $\sum_{p=0}^{\infty} (-1)^p \dim H^p(\Gamma, H_{-*}^{\sigma,\lambda}) = 0$

ii) $\mathrm{ord}_{s=\lambda} Z_\Gamma(s,\sigma) = -\sum_{p=0}^{\infty} (-1)^p p \dim H^p(\Gamma, H_{-*}^{\sigma,\lambda})$.

For $\lambda = 0$ one has to replace $H_{-*}^{\sigma,0}$ by $\hat{H}_{-*}^{\sigma,0}$ satisfying

$$0 \to H_{-*}^{\sigma,0} \to \hat{H}_{-*}^{\sigma,0} \to H_{-*}^{\sigma,0} \to 0 .$$

If $\Gamma < G$ is torsion-free, discrete and $vol(\Gamma \backslash G) < \infty$, then we have:

<u>Proposition 4:</u> Let $(\pi, V_{\pi,K}) \in \mathcal{HC}(\mathfrak{g}, K)$. Then

$$\dim H^p(\Gamma, V_{\pi,-\infty}) < \infty .$$

A computation of the cohomology can be approached by specifying (1). Thus let $\Gamma \subset SL(2,\mathbb{R})$ be a Fuchsian group of the first kind with r cusps. We abbreviate: $H^{1,\lambda} = H^\lambda$, $1 \in \hat{M}$ being the trivial representation. Let g be the genus of the surface $\Gamma \backslash \underline{\underline{H}}^2 = Y$. Then.

<u>Proposition 5:</u> i) $H^p(\Gamma, H^\lambda_{-\infty}) = 0 \quad \forall\, p \geq 2.$

ii) For $\lambda \neq -1/2, -3/2, -5/2, \ldots$

$$H^0(\Gamma, H^\lambda_{-\infty}) = \left\{ \begin{array}{l} \text{eigenfunctions of } \Delta_Y - 1/4 \\ \text{to the eigenvalue } -\lambda^2 \text{ of} \\ \text{moderate growth} \end{array} \right\}$$

$$H^1(\Gamma, H^\lambda_{-\infty}) = \{ \text{cusp forms in } Ker(\Delta_Y - 1/4 + \lambda^2) \}$$

$\Delta_Y : C^\infty(Y) \longrightarrow C^\infty(Y)$ - Laplace-Beltrami

iii) $H^0(\Gamma, H^{-1/2}_{-\infty}) = 2g + 2r - 1$

$H^1(\Gamma, H^{-1/2}_{-\infty}) = 2g + r - 1$

$H^0(\Gamma, H^{-k/2}_{-\infty}) = k(2g-2) + (k+1)r$

$H^1(\Gamma, H^{-k/2}_{-\infty}) = k(2g + r - 2)$

$k = 3, 5, 7, \dots$

The method can also be applied to Γ acting convex-cocompactly on G/K. We again consider the case $\Gamma \subset SL(2, \mathbb{R})$. Let r be the number of ends of $Y = \Gamma \backslash H^2$ and g be the genus. Let S^1 be the geodesic boundary of H^2. Then we have a limit set $\Lambda \subset S^1$ of Γ. Identifying

$$H^\lambda_{-\omega} = C^{-\omega}(S^1, (TS^1)^{\lambda - 1/2})$$

we define the Γ-invariant subspace $H^\lambda_{-\omega,\Lambda} \subset H^\lambda_{-\omega}$ by

$$H^\lambda_{-\omega,\Lambda} = \{ f \in H^\lambda_{-\omega} \mid \operatorname{supp} f \subset \Lambda \}$$

<u>Theorem 6</u>: For $\lambda \neq 0$ we have:

i) $H^*(\Gamma, H^{\lambda}_{-\omega,\lambda})$ is finite-dimensional

ii) $\dim H^0(\Gamma, H^{\lambda}_{-\omega,\lambda}) = \dim H^1(\Gamma, H^{\lambda}_{-\omega,\lambda})$

$H^p(\Gamma, H^{\lambda}_{-\omega,\lambda}) = 0 \quad , \forall p \geq 2$

More explicitly:

- $Re(\lambda) > 0$:

$$H^0(\Gamma, H^{\lambda}_{-\omega,\lambda}) = Ker_{L^2}(\Delta_Y - 1/4 + \lambda^2)$$

- $Re(\lambda) = 0, \ \lambda \neq 0$

$$H^0(\Gamma, H^{\lambda}_{-\omega,\lambda}) = 0$$

- $Re(\lambda) < 0, \ \lambda \neq -1/2, -3/2, -5/2, \dots$

$H^0(\Gamma, H^{\lambda}_{-\omega,\lambda}) \neq 0 \implies \lambda$ is a resonance (pole of scattering matrix)

- $\dim H^0(\Gamma, H^{-k/2}_{-\omega,\lambda}) = \begin{cases} \underline{\Gamma \text{ non-abelian}} & \\ (2g-2+r)k & , k \geq 2, g \neq 0 \\ (2g-2+r)k+1 & , k > 2, g = 0 \\ 2g+r & , k = 1 \\ \underline{\Gamma\text{-abelian}} & \\ 2 & \end{cases}$

$k = -1, 3, 5, \dots$

Titel: TOPOLOGICAL SIMILARITY OF REPRESENTATIONS

Autor: I. HAMBLETON Seite: 1

Adresse: MAX PLANCK INSTITUT FÜR MATHEMATIK, BONN
(MCMASTER UNIVERSITY, HAMILTON, ONT., CANADA)

(JOINT WORK WITH E.K. PEDERSEN)

Let G be a finite group and V, V' finite dimensional real orthogonal representations of G. Then V is said to be *topologically similar* to V' (notation $V \sim_t V'$) if there exists a homeomorphism $h: V \to V'$ which is G-equivariant.

The topological classification of G-representations was studied by de Rham [6]. He proved that if the topological similarity $h: V \to V'$ preserves the unit spheres and restricts to a diffeomorphism between $S(V)$ and $S(V')$, then V and V' are linearly isomorphic. Kuiper and Robbin [4] conjectured in 1973 that topological similarity implies linear equivalence, but this was disproved in 1981 by Cappell and Shaneson [1]. They constructed the first examples of non-linear similarities for $G = \mathbb{Z}/8$ or $G = \mathbb{Z}/4q$, q odd. Other examples were given in [2].

On the other hand, Hsiang and Pardon [3] and Madsen and Rothenberg [5] proved the Kuiper-Robbin conjecture for all groups G with $|G| = \text{odd}$ or $|G| = 2 \cdot \text{odd}$.

Now let G be a finite cyclic group, and consider possible topological similarities $V_1 \oplus W \sim_t V_2 \oplus W$ where V_1, V_2 are free G-representations (i.e. G acts freely on V_i away from 0).

Obvious Necessary Conditions

$V_1 \oplus W \sim_t V_2 \oplus W \Rightarrow \dim_{\mathbb{R}} V_i \geq 4$ and $S(V_1) \simeq_G S(V_2)$, i.e. the unit spheres are G-homotopy equivalent.

Our first result generalizes the Odd Order Theorem [3][5] to "first-time" similarities:

Theorem 1: Suppose that $V_1 \oplus W \sim_t V_2 \oplus W$ and $\mathrm{Res}_H V_1 \cong \mathrm{Res}_H V_2$ for all $H \subsetneqq G$. Then either $V_1 \cong V_2$ or $|G| \equiv 0 \pmod 4$, $\mathbb{R}_- \subseteq W$ and $V_1 \oplus \mathbb{R}_+ \oplus \mathbb{R}_- \sim_t V_2 \oplus \mathbb{R}_+ \oplus \mathbb{R}_-$.

($\mathbb{R}_{+/-}$ = trivial/non-trivial 1-dimensional G-rep.)

as an application of this result, we reduce the topological similarity problem for $G = \mathbb{Z}/4q$, q odd, to number theory. Let t denote the 2-dimensional G-rep on $\mathbb{R}^2 = \mathbb{C}$, with a generator of G acting by multiplication by $e^{2\pi i/|G|} := \zeta$. Any free representation has the form $V = t^{a_1} + \dots + t^{a_k}$, where $a_1, \dots, a_k$ are integers prime to $|G|$. The Reidemeister torsion invariant is

$$\Delta(V) = \prod_{i=1}^{k} (t^{r_i} - 1), \bmod \pm t^{\ell}, \text{ in } \mathbb{Q}G/(\Sigma_G).$$

If V_1, V_2 are free with $S(V_1) \simeq_G S(V_2)$ then $\Delta(V_1)/\Delta(V_2) \in (\mathbb{Z}G)^{\times}$ gives an element of $Wh(\mathbb{Z}G)$. The involution $g \mapsto w(g)g^{-1}$, where $w: G \to \mathbb{Z}/2$ is the non-trivial homomorphism, induces a $\mathbb{Z}/2$ action on $Wh(\mathbb{Z}G)$ and on $\tilde{K}_0(\mathbb{Z}G)$. We define the double coboundary

$$\delta^2: H^0(\mathbb{Z}/2; \tilde{K}_0(\mathbb{Z}G^-)) \longrightarrow H^1(\mathbb{Z}/2, Wh(\mathbb{Z}G^-))$$

from the arithmetic sequence in K-theory:

$$0 \to Wh(\mathbb{Z}G) \to Wh(\hat{\mathbb{Z}}G) \oplus Wh(\mathbb{Q}G) \to Wh(\hat{\mathbb{Q}}G) \to \tilde{K}_0(\mathbb{Z}G) \to 0$$

Autor: I. Hambleton Seite: 4

Theorem 2: Let $G = \mathbb{Z}/4q$, q odd, and $H = \mathbb{Z}/2q$. Then $V_1 \oplus W \sim_t V_2 \oplus W$ if and only if $S(V_1) \simeq_G S(V_2)$, $\mathrm{Res}_H V_1 \cong \mathrm{Res}_H V_2$ and $\{\Delta(V_1)/\Delta(V_2)\} \in \mathrm{Image}(\delta^2)$.

The computation of the double coboundary is difficult in general since the domain of δ^2 depends on class numbers of cyclotomic fields.

Theorem 3: Let $G = \mathbb{Z}/4q$, q odd, and suppose that the fields $\mathbb{Q}(\zeta_{4d})$, $d \mid q$, $(\forall d)$ have odd class number. Then G has no 6-dimensional non-linear similarities.

References

[1] S.E. Cappell + J.L. Shaneson, "Non-linear similarity", Ann. Math. 113 (1981), 315-355.

[2] ———————— + M. Steinberger + J. West, "Non-linear similarity begins in dimension six", Amer. J. Math. 111 (1989), 717-752

[3] W-C. Hsiang and W. Pardon, "When are topologically equivalent representations linearly equivalent?" Invent. math. 68 (1982) 275-316

Autor: I. Hambleton Seite: 5

[4] N. Kuiper + J.W. Robbin, "Topological classification of linear endomorphisms", Invent. math. 19 (1973), 83–106

[5] I. Madsen + M. Rothenberg, "On the classification of G-spheres, I: equivariant transversality" Acta Math. 160 (1988), 65-104

[6] G. de Rham, "Reidemeister's torsion invariant and rotations of S^n". Diff. Analysis Oxford U. Press, 1964, 27-36.