THE NONARCHIMEDEAN THETA CORRESPONDENCE FOR GSp(2) AND GO(4)

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ABSTRACT. In this paper we consider the theta correspondence between Irr(GSp(2)) and Irr(GO(X)) when k is a nonarchimedean local field and $\dim_k X = 4$. When $\det(X) = 1$, we determine all the elements of Irr(GO(X)) that occur in the correspondence, and when $\det(X) \neq 1$, we find all the infinite dimensional elements of Irr(GO(X)) that occur in the correspondence. We apply this result to prove a case of a conjecture of S.S. Kudla concerning the first occurance of a representation in the theta correspondence, and to construct series of supercuspidal representations of GSp$(2, k)$.

Suppose k is a nonarchimedean local field of characteristic zero and odd residual characteristic, X is an even dimensional nondegenerate symmetric bilinear space over k and n is a nonnegative integer. Let ω be the Weil representation of $\mathrm{Sp}(n, k) \times \mathrm{O}(X)$ corresponding to a fixed choice of nontrivial additive character of k, and let $\mathcal{R}_X(\mathrm{Sp}(n, k))$ be the set of elements of $\mathrm{Irr}(\mathrm{Sp}(n, k))$ that are nonzero quotients of ω; similarly define $\mathcal{R}_n(\mathrm{O}(X))$. By [W], the condition that $\pi \otimes_{\mathbf{C}} \sigma$ be a nonzero quotient of ω for π in $\mathcal{R}_X(\mathrm{Sp}(n, k))$ and σ in $\mathcal{R}_n(\mathrm{O}(X))$ defines a bijection between $\mathcal{R}_X(\mathrm{Sp}(n, k))$ and $\mathcal{R}_n(\mathrm{O}(X))$. By [R], the extension of ω to the subgroup R of $\mathrm{GSp}(n, k) \times \mathrm{GO}(X)$ consisting of pairs whose entries have the same similitude factor also defines a well behaved correspondence between $\mathrm{Irr}(\mathrm{GSp}(n, k)^+)$ and $\mathrm{Irr}(\mathrm{GO}(X))$. Here, $\mathrm{GSp}(n, k)^+$ is the subgroup of elements of $\mathrm{GSp}(n, k)$ having similitude factors equal to the similitude factor of some element of $\mathrm{GO}(X)$. More precisely, let $\mathcal{R}_X(\mathrm{GSp}(n, k)^+)$ be the set of elements of $\mathrm{Irr}(\mathrm{GSp}(n, k)^+)$ whose restrictions to $\mathrm{Sp}(n, k)$ are multiplicity free and have a constituent in $\mathcal{R}_X(\mathrm{Sp}(n, k))$; similarly define $\mathcal{R}_n(\mathrm{GO}(X))$. Then the condition

$$\mathrm{Hom}_R(\omega, \pi \otimes_{\mathbf{C}} \sigma) \neq 0$$

defines a bijection between $\mathcal{R}_X(\mathrm{GSp}(n, k)^+)$ and $\mathcal{R}_n(\mathrm{GO}(X))$. In this paper we consider $\mathcal{R}_2(\mathrm{GO}(X))$ when $\dim_k X = 4$. In this case, all elements of $\mathrm{Irr}(\mathrm{GO}(X))$ have multiplicity free restrictions to $\mathrm{O}(X)$, and by the theta dichotomy conjecture, $\mathrm{GSp}(2, k)^+$ can be replaced with $\mathrm{GSp}(2, k)$. If $\det(X) = 1$ then we determine

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$\mathcal{R}_2(\mathrm{GO}(X))$ completely, and if $\det(X) \neq 1$, then we find all the infinite dimensional elements of $\mathcal{R}_2(\mathrm{GO}(X))$. We also give two applications of this result. The first is a proof of a case of a new conjecture of S.S. Kudla concerning the first appearance of a representation in the theta correspondence. The second is the construction of series of supercuspidal representations of $\mathrm{GSp}(2,k)$. A summary of previous work on this example appears near the end of this introduction.

To state the main theorem we need some more terminology. Assume $\dim_k X = 4$. Let π be contained in $\mathrm{Irr}(\mathrm{GSO}(X))$. If π induces irreducibly to $\mathrm{GO}(X)$ we say that π is regular; otherwise, we say that π is invariant. Suppose that π is invariant. If y in X is anisotropic, then the stabilizer in $\mathrm{SO}(X)$ of y can be identified with $\mathrm{SO}(Y)$, where Y is the orthogonal complement to x. Suppose that π is invariant. We say that π is distinguished if π is invariant and there is a y such that

$$\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) \neq 0,$$

and, if $\det(X) \neq 1$, then Y is isotropic. Suppose that π is distinguished. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) = 1.$$

Since π is invariant, π extends to two different elements of $\mathrm{Irr}(\mathrm{GO}(X))$. Each provides an action of the nontrivial element of $\mathrm{O}(Y)/\mathrm{SO}(Y)$ on the above homomorphism space, and since the space is one dimensional, the actions are multiplication by ± 1. We denote by π^+ the extension inducing multiplication by 1, and by π^- the extension inducing multiplication by -1. The definitions of π^+ and π^- do not depend on the choice of y.

Theorem 6.3 (Main Theorem). *Assume* $\dim_k X = 4$. *Let* σ *be in* $\mathrm{Irr}(\mathrm{GO}(X))$. *In the case* $\det(X) \neq 1$, *assume* σ *is infinite dimensional. Then* σ *is in* $\mathcal{R}_2(\mathrm{GO}(X))$ *if and only if* σ *is not of the form* π^- *for some distinguished* π *in* $\mathrm{Irr}(\mathrm{GSO}(X))$.

This result is entirely analogous to the case $\dim_k X = 2n = 2$ considered by Hecke, Weil, Jacquet, Langlands and others. In this case, the role of $\mathrm{SO}(Y)$ is played by $\mathrm{SO}(X)$. For a description of this case, see section 7. Of course, we expect the theorem to hold for all σ in the case $\det(X) \neq 1$.

To describe the proof and make the theorem concrete, we characterize of $\mathrm{GSO}(X)$ in terms of units of quaternion algebras. If $\det(X) = 1$, either there is an exact sequence

$$1 \to k^\times \to \mathrm{Gl}(2,k) \times \mathrm{Gl}(2,k) \to \mathrm{GSO}(X) \to 1$$

or an exact sequence

$$1 \to k^\times \to D^\times \times D^\times \to \mathrm{GSO}(X) \to 1,$$

depending on the Hasse invariant of X. In the first case X is isotropic; in the second case, X is anisotropic. Here, D is the division quaternion algebra over k. If $\det(X) \neq 1$ then there is an exact sequence

$$1 \to K^\times \to k^\times \times \mathrm{Gl}(2,K) \to \mathrm{GSO}(X) \to 1.$$

Here, $K = k(\sqrt{\det(X)})$. If $\det(X) = 1$, the exact sequence gives a bijection between $\mathrm{Irr}(\mathrm{GSO}(X))$ and the subset of $\tau \otimes_{\mathbf{C}} \tau'$ in $\mathrm{Irr}(\mathrm{Gl}(2,k) \times \mathrm{Gl}(2,k))$ or $\mathrm{Irr}(D^\times \times D^\times)$ such that $\omega_\tau = \omega_{\tau'}$. If $\det(X) \neq 1$, the exact sequence gives a two to one map from $\mathrm{Irr}(\mathrm{GSO}(X))$ onto the subset of τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$ such that ω_τ factors through N_k^K; the two representations lying over τ correspond to the characters through which ω_τ factors.

Using these identifications, regular, invariant and distinguished have the following meanings for an element π of $\mathrm{Irr}(\mathrm{GSO}(X))$. Suppose $\det(X) = 1$, and let $\tau \otimes_{\mathbf{C}} \tau'$ correspond to π. Then π is regular if and only if $\tau \not\cong \tau'$, and if π is invariant, then π is distinguished. Suppose $\det(X) \neq 1$, and let π correspond to τ and the quasi-character χ of $k^\times$. In this case, π is regular if and only if τ is not Galois invariant. In contrast to the case $\det(X) = 1$, not all invariant representations are distinguished. Indeed, if π is invariant, so that τ is Galois invariant, then π is distinguished if and only if

$$\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi \circ \det) \neq 0.$$

In the nontrivial case when τ is infinite dimensional, nonvanishing is given by the following theorem. We claim no originality for this result. The proof is a straightfoward generalization of arguments from [H] and [F], along with some observations from [HST] on [T]. In the global case, this theorem goes back to [HLR].

Theorem 5.3 (Hakim-Flicker). *Let τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$ be infinite dimensional and Galois invariant. Let $\omega_\tau = \chi \circ \mathrm{N}_k^K$. Then the following are equivalent:*

(1) $\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi \circ \det) \neq 0$;

(2) *For every quasi-character ζ of $K^\times$ extending χ,*

$$\epsilon(\tau \otimes_{\mathbf{C}} \zeta^{-1}, 1/2, \psi_K) = \chi(-1);$$

(3) *τ is the base change of an element of $\mathrm{Irr}(\mathrm{Gl}(2,k))$ with central character $\chi\omega_{K/k}$.*

With these interpretations, we can explain the proof of the main theorem. We need to show that under the hypotheses of the theorem that every element of the form π^- is not in $\mathcal{R}_2(\mathrm{GO}(X))$, and that every element of $\mathrm{Irr}(\mathrm{GO}(X))$ not of the form π^- is in $\mathcal{R}_2(\mathrm{GO}(X))$. The first statement follows by an argument analogous to one in [HK]. This proof depends on a lemma that follows directly from a result of D. Prasad [P]: every distribution on $\mathcal{S}(X^2)$ invariant under $\mathrm{SO}(Y)$ is invariant under $\mathrm{O}(Y)$, for any Y as above.

To prove the second statement, we use the local analogue of the global method of computing a Fourier coefficient. Let σ in $\mathrm{Irr}(\mathrm{GO}(X))$ not be of the form π^-. Let z be in X^2. If $\det(z,z) \neq 0$, we will say that z is nondegenerate. As above, if z is nondegenerate, then the components of z generate a nondegenerate subspace, and the stabilizer in $\mathrm{O}(X)$ is isomorphic to $\mathrm{O}(Z)$, where Z is the orthogonal complement of the subspace. By Frobenius reciprocity, to show that σ is in $\mathcal{R}_2(\mathrm{GO}(X))$ it suffices to show that

$$\mathrm{Hom}_{\mathrm{O}(Z)}(\sigma^\vee, \mathbf{1}) \neq 0,$$

for some nondegenerate z. See section 6. First consider the case when σ is not induced from a regular element of $\mathrm{Irr}(\mathrm{GSO}(X))$ or is not of the form π^+. Then $\det(X) \neq 1$, and σ is the extension of an element of $\mathrm{Irr}(\mathrm{GSO}(X))$ corresponding to a τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$ and a quasi-character χ of $k^\times$ such that

$$\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi \circ \det) = 0.$$

With a proper choice of z and quasi-character ζ of $K^\times$ extending χ, using the Kirillov model of $\tau^\vee$, we show

$$L(f) = Z(\zeta^{-1}, f, 1/2)$$

is the required linear functional. Here, $Z(\zeta^{-1}, f, s)$ is the zeta function associated to f in $\tau^\vee$ and ζ. In particular, the invariance of L follows from the functional equation for $Z(\zeta^{-1}, f, s)$. When σ is induced from a regular element π of $\mathrm{Irr}(\mathrm{GSO}(X))$ or is of the form π^+ there is a simplification. In this case, by Theorem 4.4, it suffices to show that

$$\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi^\vee, 1) \neq 0$$

for some nondegenerate z. When X is isotropic we accomplish this by some Kirillov model constructions, in part analogous to those of the previous paragraph, and when X is anisotropic, we use Tunnell's work [T].

In combination with some other results, the main theorem can be used to prove a case of a conjecture of S.S. Kudla. To state the conjecture, suppose for the moment that $\dim_k X$ is arbitrary. For σ in $\mathrm{Irr}(\mathrm{GO}(X))$, let $n(\sigma)$ be the smallest integer n such that σ occurs in the theta correspondence with $\mathrm{GSp}(n,k)^+$.

Conjecture 7.1 (S.S. Kudla). *If σ is in $\mathrm{Irr}(\mathrm{GO}(X))$ then*

$$n(\sigma) + n(\sigma \otimes_{\mathbb{C}} \mathrm{sign}) = \dim_k X.$$

Actually, S.S. Kudla made his conjecture for the correspondence for isometries, but this is equivalent to the conjecture stated here. This conjecture is known to be true when $\dim_k X = 0$ or 2, but is open for all other cases. There is another conjecture of S.S. Kudla for representations of $\mathrm{GSp}(n,k)$. See section 7. We prove the following theorem. In the theorem, in the case $\det(X) \neq 1$, $\mathrm{Irr}_{\mathrm{BC\ inf}}(\mathrm{GO}(X))$ is a certain large set of elements of $\mathrm{Irr}(\mathrm{GO}(X))$ that includes all the supercuspidals. See section 7 for the definition.

Theorem 7.5. *Let $\dim_k X = 4$ and σ be in $\mathrm{Irr}(\mathrm{GO}(X))$. In the case $\det(X) \neq 1$ assume σ is infinite dimensional, σ is in $\mathrm{Irr}_{\mathrm{BC\ inf}}(\mathrm{GO}(X))$, and Conjecture 7.2 (S.S. Kudla) for $n = 1$. Then*

$$n(\sigma) + n(\sigma \otimes_{\mathbb{C}} \mathrm{sign}) = 4.$$

To prove the theorem, we characterize $\mathcal{R}_1(\mathrm{GO}(X))$ and $\mathcal{R}_3(\mathrm{GO}(X))$. That is, under the same assumptions as in the theorem, we specify $n(\sigma)$. To do so, we make use of the main theorem and [S] and [Co]. For a presentation of the information, see the tables in section 7.

The main theorem also can be used to construct series of supercuspidal representations of $\mathrm{GSp}(2,k)$ parametrized by series of representations of $\mathrm{Gl}(2,k)$, $D^\times$ and $\mathrm{Gl}(2,K)$. In the following theorem, if σ is in $\mathcal{R}_2(\mathrm{GO}(X))$, then let $\theta(\sigma)$ be the corresponding element of $\mathrm{Irr}(\mathrm{GSp}(2,k))$.

Theorem 8.2. *Assume X is as above and $\det(X) = 1$.*

(1) *(regular series 1_a) Suppose X is isotropic. If τ, τ' in $\mathrm{Irr}(\mathrm{Gl}(2,k))$ are supercuspidal, distinct and have the same central character, and if π is the element of $\mathrm{Irr}(\mathrm{GSO}(X))$ lying over $\tau \otimes_{\mathbb{C}} \tau'$, then*

$$\theta(\mathrm{Ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)} \pi)$$

in $\mathrm{Irr}(\mathrm{GSp}(2,k))$ is supercuspidal.

(2) *(regular series 1_b) Suppose X is anisotropic. If τ, τ' in $\mathrm{Irr}(D^\times)$ are distinct and have the same central character, and if π is the element of $\mathrm{Irr}(\mathrm{GSO}(X))$ lying over $\tau \otimes_{\mathbb{C}} \tau'$, then*

$$\theta(\mathrm{Ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)} \pi)$$

in $\mathrm{Irr}(\mathrm{GSp}(2,k))$ is supercuspidal.

If $\det(X) \neq 1$ then we also construct two series of supercuspidal representations of $\mathrm{GSp}(2,k)$. However, in contrast to the $\det(X) = 1$ case, the two series do not correspond to the two four dimensional symmetric bilinear spaces with the same determinant different from 1. In fact, conjecturally, these two symmetric bilinear spaces together give one correspondence. See section 1; in the statement of the following theorem we assume this discussion.

Theorem 8.3. *Assume X is as above and $d = \det(X) \neq 1$. Assume further that Conjecture 1.3 (theta dichotomy) with $\dim_k X = 4$ and $n = 2$, and Conjecture 7.2 with $n = 1$ hold.*

(1) *(regular series d) If τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$ is supercuspidal, not Galois invariant, but has Galois invariant central character, and if π and π' in $\mathrm{Irr}(\mathrm{GSO}(X))$ lie over τ, then*

$$\theta(\mathrm{Ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)} \pi), \qquad \theta(\mathrm{Ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)} \pi')$$

in $\mathrm{Irr}(\mathrm{GSp}(2,k))$ are supercuspidal.

(2) *(Invariant series d) Let τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$ be supercuspidal and Galois invariant. Exactly one π in $\mathrm{Irr}(\mathrm{GSO}(X))$ lying over τ is not distinguished, and if σ_1 and σ_2 are the two extensions of π to $\mathrm{GO}(X)$, then $\theta(\sigma_1)$ and $\theta(\sigma_2)$ in $\mathrm{Irr}(\mathrm{GSp}(2,k))$ are supercuspidal.*

In this paper we do not consider applications to functorality and the theory of L-packets. For some discussion of these topics see [V] and [HST]. In the appendix, we give tables showing the passage and bifurcation of the appropriate representations of $D^\times \times D^\times$, $\mathrm{Gl}(2,k)\times\mathrm{Gl}(2,k)$ and $\mathrm{Gl}(2,K)$ from these groups to $\mathrm{GSO}(X)$, $\mathrm{GO}(X)$ and finally $\mathrm{GSp}(2,k)$. The tables may be useful to the reader interested in these topics.

We will now make some remarks about previous work on the Weil representation and theta correspondence for similitudes when $\dim_k X = 4$ and $n = 2$. In [PSS] and [So1], in the case $\det(X) = 1$ and X isotropic, the induced Weil representation [R] is used to lift elements of $\mathrm{Irr}(\mathrm{GSO}(X))$ to representations of $\mathrm{GSp}(2,k)$. This construction is an analogue of the global definition of theta lifts, and uses elements of Whittaker models in place of automorphic forms. The problem of whether these representations of $\mathrm{GSp}(2,k)$ are irreducible is not resolved in [PSS] or [So1]. The work [HPS] in part investigates the case $\det(X) = 1$ and X anisotropic. In this case, as a consequence of Theorem 9.1 of [HPS], every element of $\mathrm{Irr}(\mathrm{GSO}(X))$ is an $\mathrm{SO}(X)$ quotient of ω. Using this result, one could prove the main theorem in this case using Theorems 4.3 and 4.4. Using the induced Weil representation, results from the previously mentioned papers, and the strong multiplicity one theorem for $\mathrm{GSp}(2)$ of [So2], a global argument in [V] lifts elements of $\mathrm{Irr}(\mathrm{GSO}(X))$ that are the local components of cuspidal, not invariant, automorphic representations of $\mathrm{GSO}(X)$ to $\mathrm{Irr}(\mathrm{GSp}(2,k))$. Included in these representations are the supercuspidal representations. Since it uses Whittaker models, in the case $\det(X) \neq 1$, this method fails to construct the representations that correspond to one of the extensions to $\mathrm{GO}(X)$ of the invariant but not distinguished elements of $\mathrm{Irr}(\mathrm{GSO}(X))$. Finally, [HST] makes many remarks and observations about the cases when X is isotropic, though it is mainly concerned with a certain global theta lifting, and its application to another problem. In particular, after the computation of the Fourier coefficient of the global theta lift it makes a conjecture essentially equivalent to the main theorem in the case X is isotropic; see the guess on page 399. However, instead of using the concept of distinguished representations, the guess is phrased in terms of ϵ factors. Even so, we rely heavily on the understanding of these ϵ factors from Lemma 14 of [HST].

In the first section we recall the theory of the theta correspondence for similitudes from [R]. In the second section we characterize $\mathrm{GO}(X)$ in terms of the units of quaternion algebras. Using this account, in the third section we parameterize $\mathrm{Irr}(\mathrm{GO}(X))$. In the fourth section we define the concept of being distinguished, and relate it to the theta correspondence. Distinguished representations for $\det(X) \neq 1$ are investigated in the fifth section. The main theorem is proven in the sixth section. In the remaining two sections we make the applications to S.S. Kudla's conjecture and the construction of supercuspidals.

I would like to thank S.S. Kudla for many useful comments, and especially for telling me about his conjectures. Thanks are also due to J. Hakim for some helpful conversations concerning his theorem.

We use the following notation. Let J be a group of td-type, as in [C]. Then $\mathrm{Irr}(J)$ is the set of equivalence classes of smooth admissible irreducible representations of J. If π is in $\mathrm{Irr}(J)$ then $\pi^\vee$ in $\mathrm{Irr}(J)$ is the contragredient representation of π, and ω_π is the central character of π. A quasi-character of J is a continuous homomorphism from J to $\mathbb{C}^\times$, and a unitary character of J is a continuous homomorphism from J to the group of complex numbers of absolute value 1. The trivial representation of

J on $\mathbb{C}$ will be denoted by $\mathbf{1}$. We will also use the notation of [GK] for restriction theory. Throughout the paper k is a nonarchimedean local field of characteristic zero and odd residual characteristic. Let D be the division quaternion algebra over k, with canonical involution $*$ and reduced norm N defined by $N(x) = xx^* = x^*x$. The canonical involution of the quaternion algebra $\mathrm{M}_2(k)$ will also be denoted by $*$; in this case the reduced norm is det. Let $(\ , \ )_k$ denote the Hilbert symbol of k. If K is a quadratic extension of k, then $\omega_{K/k}$ is the nontrivial character of $k^\times / \mathrm{N}_k^K(K^\times)$. For $d \in k^\times/k^{\times 2}$ we let $\epsilon(d) = (-1, -d)_k$.

1. The theta correspondence for similitudes. In this section we recall some results and definitions from [R]. Suppose that $(X, (\ , \ ))$ is a nondegenerate symmetric bilinear space over k of even dimension m, and let n be a nonnegative integer. Let $\mathrm{GO}(X)$ be the set of k linear automorphisms h of X such that there exists λ in $k^\times$ such that $(h(x), h(y)) = \lambda(x, y)$ for x and y in X. If h is in $\mathrm{GO}(X)$, then such a λ is unique, and will be denoted by $\lambda(h)$. Let $\mathrm{O}(X)$ be the subgroup of all h in $\mathrm{GO}(X)$ such that $\lambda(h) = 1$. Let sign : $\mathrm{GO}(X) \to \{\pm 1\}$ be the unitary character defined by $\mathrm{sign}(h) = \det(h)/\lambda(h)^{m/2}$. We let $\mathrm{GSO}(X) = \ker(\mathrm{sign})$. We will often describe $\mathrm{GO}(X)$ in terms of $\mathrm{GSO}(X)$ and an extra element of $\mathrm{GO}(X)$. Let h_0 in $\mathrm{GO}(X)$ be such that $h_0^2 = 1$ and h_0 is not in $\mathrm{GSO}(X)$. There is an action of the group $\{1, h_0\}$ on $\mathrm{GSO}(X)$ given by $h_0 \cdot h = h_0 h h_0$, and an isomorphism

$$\mathrm{GSO}(X) \rtimes \{1, h_0\} \cong \mathrm{GO}(X)$$

that takes (h, δ) to $h\delta$. Next, let $\mathrm{GSp}(n, k)$ be the group of all g in $\mathrm{Gl}(2n, k)$ such that for some λ in $k^\times$,

$${}^t g \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} g = \lambda \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

Again, if g is in $\mathrm{GSp}(2n, k)$, then such a λ is unique and will be denoted by $\lambda(g)$. Let $\mathrm{Sp}(n, k)$ be the subgroup of all g in $\mathrm{GSp}(n, k)$ such that $\lambda(g) = 1$. Let $\mathrm{GSp}(n, k)^+$ be the subgroup of all g in $\mathrm{GSp}(n, k)$ such that there exists h in $\mathrm{GO}(X)$ such that $\lambda(g) = \lambda(h)$. The group $\mathrm{GSp}(n, k)^+$ is a proper subgroup of $\mathrm{GSp}(n, k)$ if and only if $\det(X) \neq 1$. If $\det(X) \neq 1$, then $[\mathrm{GSp}(n, k) : \mathrm{GSp}(n, k)^+] = 2$. Fix a nontrivial additive character ψ of k.

To ψ, X and n, there is associated the **Weil representation** ω of $\mathrm{Sp}(n, k) \times \mathrm{O}(X)$ on $\mathcal{S}(X^n)$. In this paper we only will need to know the action of $\omega(1, h)$ for h in $\mathrm{O}(X)$, which is given by left translation:

$$\omega(1, h) \cdot \varphi(x) = L(h)\varphi(x) = \varphi(h^{-1}x).$$

There exists of an extension of ω to a representation of the larger group

$$R = \{(g, h) \in \mathrm{GSp}(n, k) \times \mathrm{GO}(X) : \lambda(g) = \lambda(h)\}.$$

This extension, called the **extended Weil representation** , will also be denoted by ω, and is very simply defined by

$$\omega(g, h)\varphi = \omega(g \begin{pmatrix} 1 & 0 \\ 0 & \lambda(g)^{-1} \end{pmatrix}, 1)L(h)\varphi.$$

The Weil representation defines a correspondence between $\mathrm{Irr}(\mathrm{Sp}(n, k))$ and $\mathrm{Irr}(\mathrm{O}(X))$. Let $\mathcal{R}_X(\mathrm{Sp}(n, k))$ be the set of all elements of $\mathrm{Irr}(\mathrm{Sp}(n, k))$ that are nonzero quotients of ω, and similarly define $\mathcal{R}_n(\mathrm{O}(X))$. As a consequence of a more general theorem of [W], we have

Theorem 1.1 (Howe-Waldspurger). *The set*

$$\{(\pi, \sigma) \in \mathcal{R}_X(\mathrm{Sp}(n,k)) \times \mathcal{R}_n(\mathrm{O}(X)) : \mathrm{Hom}_{\mathrm{Sp}(n,k)\times \mathrm{O}(X)}(\omega, \pi \otimes_{\mathbf{C}} \sigma) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}_X(\mathrm{Sp}(n,k))$ *and* $\mathcal{R}_n(\mathrm{O}(X))$.

A correspondence for similitudes is defined by the extended Weil representation. Let $\mathcal{R}_X(\mathrm{GSp}(n,k)^+)$ be the set of σ in $\mathrm{Irr}(\mathrm{GSp}(n,k)^+)$ such that $\sigma|_{\mathrm{Sp}(n,k)}$ is multiplicity free and has a constituent in $\mathcal{R}_X(\mathrm{Sp}(n,k))$. Similarly define $\mathcal{R}_n(\mathrm{GO}(X))$. From [R], section 4, we have:

Theorem 1.2. *The set*

$$\{(\pi, \sigma) \in \mathcal{R}_X(\mathrm{GSp}(n,k)^+) \times \mathcal{R}_n(\mathrm{GO}(X)) : \mathrm{Hom}_R(\omega, \pi \otimes_{\mathbf{C}} \sigma) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}_X(\mathrm{GSp}(n,k)^+)$ *and* $\mathcal{R}_n(\mathrm{GO}(X))$.

If π is in $\mathcal{R}_X(\mathrm{GSp}(n,k)^+)$ or σ is in $\mathcal{R}_n(\mathrm{GO}(X))$, then we denote the corresponding elements of $\mathcal{R}_n(\mathrm{GO}(X))$ and $\mathcal{R}_X(\mathrm{GSp}(n,k)^+)$ by $\theta(\pi)$ and $\theta(\sigma)$, respectively.

The problem of whether the extended Weil representation defines a well behaved correspondence between $\mathrm{Irr}(\mathrm{GSp}(n,k))$ and $\mathrm{GO}(X)$ when $\mathrm{GSp}(n,k)^+$ is a proper subgroup of $\mathrm{GSp}(n,k)$ is also dealt with in [R]. To describe the results, suppose that $\mathrm{GSp}(n,k)^+$ is a proper subgroup of $\mathrm{GSp}(n,k)$, i.e., that $\det(X) \neq 1$. Then the multiplicity free assumption is unnecessary since $[\mathrm{GSp}(n,k)^+ : k^\times \cdot \mathrm{Sp}(n,k)] = [\mathrm{GO}(X) : k^\times \cdot \mathrm{O}(X)] = 2$. See, for example, [GK]. One would like to know if the condition

$$\mathrm{Hom}_R(\omega, \pi \otimes_{\mathbf{C}} \sigma) \neq 0$$

defines a bijection between $\mathcal{R}_X(\mathrm{GSp}(n,k))$, the set of all π in $\mathrm{Irr}(\mathrm{GSp}(n,k))$ such that some constituent of $\pi|_{\mathrm{Sp}(n,k)}$ lies in $\mathcal{R}_X(\mathrm{Sp}(n,k))$, and $\mathcal{R}((\mathrm{GO}(X)))$. In [R] it is shown that this condition defines such a bijection if and only if a certain criterion is satisfied.

To state this criterion, we need to introduce the other nondegenerate symmetric bilinear space X' of dimension m and determinant $\det(X)$. From the Witt decomposition theorem we see that X' can be taken to have the same vector space as X, but with symmetric bilinear form multiplied by an element of $k^\times$. Assume that X' has this form. Then $\mathrm{GO}(X) = \mathrm{GO}(X')$, and the restrictions of the Weil representations ω and ω' associated to X and X', respectively, to $\mathrm{O}(X) = \mathrm{O}(X')$ are identical. It follows that $\mathcal{R}_n(\mathrm{O}(X)) = \mathcal{R}_n(\mathrm{O}(X'))$ and $\mathcal{R}_n(\mathrm{GO}(X)) = \mathcal{R}_n(\mathrm{GO}(X'))$. However, the correspondences defined by ω and ω' may differ. In [R] it is proven that the above condition defines a bijection if and only if the correspondences defined by ω and ω' are disjoint, i.e., $\mathcal{R}_X(\mathrm{Sp}(n,k)) \cap \mathcal{R}_{X'}(\mathrm{Sp}(n,k)) = \emptyset$.

Suppose $\mathcal{R}_X(\mathrm{Sp}(n,k)) \cap \mathcal{R}_{X'}(\mathrm{Sp}(n,k)) = \emptyset$. From [R] we have the following. Let g be a representative for the nontrivial coset of $\mathrm{GSp}(n,k)/\mathrm{GSp}(n,k)^+$. Let σ be in $\mathcal{R}_n(\mathrm{GO}(X)) = \mathcal{R}_n(\mathrm{GO}(X'))$, and let π and π' in $\mathrm{Irr}(\mathrm{GSp}(n,k)^+)$ correspond to σ with respect to ω and ω', respectively. Then $g \cdot \pi = \pi'$, and

$$\mathrm{Ind}_{\mathrm{GSp}(n,k)^+}^{\mathrm{GSp}(n,k)} \pi$$

in $\mathcal{R}_X(\mathrm{GSp}(n,k))$ corresponds to σ.

When the criterion is expected to hold depends on m and n. If the underlying bilinear spaces lie in the stable range, i.e., if $m \geq 4n+2$, then the criterion does not hold. From [HKS], we have have the following conjecture.

Conjecture 1.3 (Theta dichotomy). *If $m \le 2n$, then*

$$\mathcal{R}_X(\mathrm{Sp}(n,k)) \cap \mathcal{R}_{X'}(\mathrm{Sp}(n,k)) = \emptyset.$$

For progress on the conjecture, see [KR] and [HKS]. The theta dichotomy conjecture follows from another strong and precise conjecture of S.S. Kudla. See section 7.

2. Four Dimensional Symmetric Bilinear Spaces. In this section we recall the characterization of the group of similitudes of a four dimensional symmetric bilinear space in terms of the units of a quaternion algebra. For the remainder of this paper, d will will be an element of $k^\times/k^{\times 2}$. If $d = 1$ then let $K = k \times k$; if $d \ne 1$ then let $K = k(\sqrt{d})$. Let $\mathrm{Gal}(K/k) = \{1, -\}$.

Four dimensional symmetric bilinear spaces can be constructed from quaternion algebras over K. Let B be a quaternion algebra defined over K, with canonical involution $*$. We say that a k linear ring automorphism s of B is a **Galois action** on B if $s^2 = 1$ and $s(ax) = \overline{a}s(x)$ for a in K and x in B. Let s be a Galois action on B. Define $X(s)$ to be the set of all x in B such that $s(x) = x^*$. Then $X(s)$ is a four dimensional vector space over k, and equipped with the restriction of the symmetric bilinear form corresponding to the reduced norm of B, $X(s)$ is a nondegenerate symmetric bilinear space. The determinant and Hasse invariant of $X(s)$ are d and $\epsilon(d)\epsilon(s)$, respectively. Here, to define $\epsilon(s)$, let $B(s)$ be the fixed points of s. Then $B(s)$ is a quaternion algebra over k, and $\epsilon(s) = 1$ if $B(s)$ is split and $\epsilon(s) = -1$ if $B(s)$ is ramified.

The elements of $k^\times \times B^\times$ give elements $\mathrm{GSO}(X)$. Define a left action ρ of $k^\times \times B^\times$ on $X(s)$ by

$$\rho(t,g)x = t^{-1}gxs(g)^*.$$

Then $\rho(t,g)$ is in $\mathrm{GSO}(X(s))$ for (t,g) in $k^\times \times B^\times$. There is an inclusion of $K^\times$ in $k^\times \times B^\times$ that sends a to $(\mathrm{N}_k^K(a), a)$.

Theorem 2.1. *For every four dimensional nondegenerate symmetric bilinear space X of determinant d over k there exists a quaterion algebra B over K and a Galois action s on B such that $X \cong X(s)$ as symmetric bilinear spaces. For every quaternion algebra B over K and Galois action s on B the sequence*

$$1 \to K^\times \to k^\times \times B^\times \xrightarrow{\rho} \mathrm{GSO}(X(s)) \to 1$$

is exact.

Proof. Let B be the even Clifford algebra of X. To see that that B is a quaternion algebra over K, and construct s, let x_1, x_2, x_3, x_4 be an orthogonal basis for X. Let Z be the center of B. Then $Z = k + kx_1x_2x_3x_4$, and $(x_1x_2x_3x_4)^2 = d$. Hence, Z is isomorphic to K as a k algebra, and B is an algebra over K. Let $\mathbf{i} = x_2x_3$ and $\mathbf{j} = x_3x_1$. Clearly, $\mathbf{i}^2$ and $\mathbf{j}^2$ are in $k^\times$, and $\mathbf{ij} = -\mathbf{ji}$. Let $\mathbf{k} = \mathbf{ij}$. Then $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are linearly independent over k, and $B_0 = k + k\mathbf{i} + k\mathbf{j} + k\mathbf{k}$ is a quaternion algebra over k. The map from $K \otimes_k B_0$ to B that sends $a \otimes x$ to ax is an isomorphism of K

algebras, and so B is a quaternion algebra over K. Define s by $s(a \otimes x) = \overline{a} \otimes x$. Then s is a Galois action on B.

To prove the first statement, we note that X represents 1 [O], and so we may assume that $(x_4, x_4) = 1$. Define a map T from X to $X(s)$ by $T(x) = xx_4$. Then T is a well defined isometry. The second statement follows from 4.6.1 of V of [Kn]. $\square$

Using these results, we will now define concrete realizations of the two four dimensional nondegenerate symmetric bilinear spaces $X(d, \epsilon)$ of determinant d and Hasse invariant ϵ in $\{\pm 1\}$. Suppose first $d = 1$. Let B be $\mathrm{M}_2(k) \times \mathrm{M}_2(k)$ or $D \times D$. Define a Galois action on B by $s(x, y) = (y, x)$. Then $X(s)$ is obviously isomorphic to $\mathrm{M}_2(k)$ or D. We find that $\mathrm{M}_2(k)$ and D, regarded as symmetric bilinear spaces with forms corresponding to the reduced norm, have determinant 1 and Hasse invariant $\epsilon(d)$ and $-\epsilon(d)$, respectively. We let $X(1, \epsilon(d)) = \mathrm{M}(k)$ and $X(1, -\epsilon(d)) = D$. Since N_k^K is surjective, the above exact sequence simplifies to

$$1 \to k^\times \to \mathrm{Gl}(2,k) \times \mathrm{Gl}(2,k) \xrightarrow{\rho} \mathrm{GSO}(X(1, \epsilon(1)) \to 1$$

and

$$1 \to k^\times \to D^\times \times D^\times \xrightarrow{\rho} \mathrm{GSO}(X(1, -\epsilon(1)) \to 1,$$

where ρ is now defined by $\rho(g, g')x = gxg'^*$, and the inclusion of $k^\times$ sends x to (x, x^{-1}).

Suppose next that $d \neq 1$. Let $B = \mathrm{M}_2(K)$. Then $B = K \otimes_k \mathrm{M}_2(k)$ and $B = K \otimes_k D$. Here we regard D as a subalgebra of B by letting

$$D = \{ \begin{pmatrix} a & b\delta \\ \overline{b} & \overline{a} \end{pmatrix} : a, b \in K\}$$

where δ is a representative for the nontrivial coset of $k^\times / \mathrm{N}_k^K(K^\times)$. Let s and s' be the Galois actions on B corresponding to $\mathrm{M}_2(k)$ and D, respectively, as in the proof of Theorem 2.1. Explicitly,

$$s\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}, \qquad s'\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overline{d} & \delta\overline{c} \\ \overline{b}/\delta & \overline{a} \end{pmatrix}.$$

Then $X(s)$ and $X(s')$ have determinant d and Hasse invariants $\epsilon(d)$ and $-\epsilon(d)$, respectively. We let $X(d, \epsilon(d)) = X(s)$ and $X(d, -\epsilon(d)) = X(s')$. There are exact sequences

$$1 \to K^\times \to k^\times \times \mathrm{Gl}(2, K) \xrightarrow{\rho} \mathrm{GSO}(X(d, \pm\epsilon(d)) \to 1.$$

Explicitly,

$$X(d, \epsilon(d)) = \{ \begin{pmatrix} a & b\sqrt{d} \\ c\sqrt{d} & \overline{a} \end{pmatrix} : a \in K, b, c \in k\},$$

and

$$X(d, -\epsilon(d)) = \{ \begin{pmatrix} b & -\delta a \\ \overline{a} & c \end{pmatrix} : a \in K, b, c \in k\}.$$

For the remainder of this paper, ϵ will be in $\{\pm 1\}$, and $X = X(d, \epsilon)$. Because of the remarks in section 1 concerning the theta correspondence for similitudes when $d \neq 1$, we will disregard the case $d \neq 1$ and $\epsilon = -\epsilon(d)$. Thus, if $d \neq 1$, then $X = X(d, \epsilon(d))$. We will let ω denote the extended Weil representation associated to X and the nonnegative integer n. If necessary, the dependence of ω on n will be indicated by a subscript.

3. Representations. In this section we make some definitions and elementary observations concerning the relationship between representations of $\mathrm{GO}(X)$ and $\mathrm{GSO}(X)$ and the quaternion algebras from the last section. We remind the reader that the case $d \neq 1$ and $\epsilon = -\epsilon(d)$ for our purposes can be and will be ignored. We also point out that by [HPS], Lemma 7.2, the restriction of representations of $\mathrm{GO}(X)$ to $\mathrm{O}(X)$ is multiplicity free.

Suppose first that $d = 1$. Let $\mathrm{Irr}_f(\mathrm{Gl}(2,k) \times \mathrm{Gl}(2,k))$ be the set of pairs of representations in $\mathrm{Irr}(\mathrm{Gl}(2,k))$ with the same central character. Define $\mathrm{Irr}_f(D^\times \times D^\times)$ similarly. There are bijections

$$\mathrm{Irr}(\mathrm{GSO}(X(1,\epsilon(1))) \xrightarrow{\cong} \mathrm{Irr}_f(\mathrm{Gl}(2,k) \times \mathrm{Gl}(2,k))$$

and

$$\mathrm{Irr}(\mathrm{GSO}(X(1,-\epsilon(1))) \xrightarrow{\cong} \mathrm{Irr}_f(D^\times \times D^\times)$$

that take π to the representation that sends (g,g') to $\pi(\rho(g,g'))$. If (τ,τ') is in $\mathrm{Irr}_f(\mathrm{Gl}(2,k)\times \mathrm{Gl}(2,k))$ or $\mathrm{Irr}_f(D^\times \times D^\times)$, then the corresponding element $\pi(\tau,\tau')$ of $\mathrm{Irr}(\mathrm{GSO}(X(1,\pm\epsilon(d)))$ has as space the space of $\tau \otimes_{\mathbb{C}} \tau'$ and is defined by

$$\pi(\tau,\tau')(\rho(g,g')) = \tau(g) \otimes \tau'(g').$$

The central character of $\pi(\tau,\tau')$ is $\omega_\tau = \omega_{\tau'}$, and the contragredient of $\pi(\tau,\tau')$ is $\pi(\tau,\tau')^\vee = \pi(\tau^\vee,\tau'^\vee)$.

Suppose that $d \neq 1$. Let $\mathrm{Irr}_f(\mathrm{Gl}(2,K))$ be the set of elements of $\mathrm{Irr}(\mathrm{Gl}(2,K))$ with Galois invariant central character. Recall that if a quasi-character of $K^\times$ is Galois invariant, then it factors through N_k^K via exactly two quasi-characters of $k^\times$. There is a two to one surjective map

$$\mathrm{Irr}(\mathrm{GSO}(X(d,\epsilon(d))) \to \mathrm{Irr}_f(\mathrm{Gl}(2,K))$$

that take π to the representation that has space the space of π and is defined by $g \mapsto \pi(\rho(1,g))$. If τ is in $\mathrm{Irr}(\mathrm{Gl}(2,K))$, and χ and χ' are the two quasi-characters of $k^\times$ such that $\omega_\tau = \chi \circ \mathrm{N}_k^K$ and $\omega_\tau = \chi' \circ \mathrm{N}_k^K$, then the two elements $\pi(\tau,\chi)$ and $\pi(\tau,\chi')$ of $\mathrm{Irr}(\mathrm{GSO}(X(d,\epsilon(d)))$ lying over τ are defined by

$$\pi(\tau,\chi)(\rho(t,g)) = \chi(t)^{-1}\tau(g) \qquad \pi(\tau,\chi')(\rho(t,g)) = \chi'(t)^{-1}\tau(g).$$

The central character of $\pi(\tau,\chi)$ is χ, and the contragredient of $\pi(\tau,\chi)$ is $\pi(\tau,\chi)^\vee = \pi(\tau^\vee,\chi^{-1})$.

Having described the representations of $\mathrm{GSO}(X)$, we consider their relationship to representations of $\mathrm{GO}(X)$. Let π be in $\mathrm{Irr}(\mathrm{GSO}(X))$. If the induced representation of π to $\mathrm{GO}(X)$ is irreducible, we say that π is **regular**, and if the induced representation of π to $\mathrm{GO}(X)$ is reducible we say that π is **invariant**. Let h_0 in $\mathrm{GO}(X)$ be such that $h_0^2 = 1$ and h_0 is not in $\mathrm{GSO}(X)$. If V is the space of π, we can regard the induced representation of π to $\mathrm{GO}(X)$ as the representation with space $V \oplus V$ and action

$$\begin{aligned} h \cdot (v \oplus v') &= \pi(h)v \oplus \pi(h_0 h h_0)v', \\ h_0 \cdot (v \oplus v') &= v' \oplus v, \end{aligned}$$

for $h \in \mathrm{GSO}(X)$. It follows that π is regular if and only if $h_0 \cdot \pi \not\cong \pi$, and π is invariant if and only if $h_0 \cdot \pi \cong \pi$. If π is regular, we denote the induced representation of π to $\mathrm{GO}(X)$ by π^+. If π is invariant, then the induced representation of π to $\mathrm{GO}(X)$ is the direct sum of two irreducible representations that extend π; these representations are twists of each other by the unitary character sign. If π is invariant and T is a map on the space of π intertwining π and $h_0 \cdot \pi$ such that T^2 is the identity, then the actions of the two extensions π_1 and π_2 of π to $\mathrm{GO}(X)$ on h_0 are given by $\pi_1(h_0) = T$ and $\pi_2(h_0) = -T$, respectively. Every element σ of $\mathrm{Irr}(\mathrm{GO}(X))$ is either induced from a regular representation of $\mathrm{GSO}(X)$, or is an extension of an invariant representation of $\mathrm{GSO}(X)$; moreover, the first possibility occurs if and only if $\sigma \otimes_{\mathbf{C}} \mathrm{sign} \cong \sigma$.

We can describe regular and invariant representations in terms of the above characterizations. For the remainder of the paper we will let h_0 be the map that sends x to x^*.

Proposition 3.1. *Let π be in $\mathrm{Irr}(\mathrm{GSO}(X))$. If $d = 1$, then π is invariant if and only if $\pi = \pi(\tau, \tau)$ for some τ in $\mathrm{Irr}(\mathrm{Gl}(2,k))$ or $\mathrm{Irr}(B^\times)$. If $d \neq 1$, then π is invariant if and only if $\pi = \pi(\tau, \chi)$ for some Galois invariant τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$.*

Proof. Suppose $d = 1$ and $\pi = \pi(\tau, \tau')$. Since $h_0\rho(g, g')h_0 = \rho(g', g)$ for g and g' in $\mathrm{Gl}(2,k)$ or $D^\times$, we have $h_0 \cdot \pi = \pi(\tau, \tau')$. Suppose that $d \neq 1$ and $\pi = \pi(\tau, \chi)$. Since $h_0\rho(t, g)h_0 = \rho(t, \overline{g})$ for (t, g) in $k^\times \times \mathrm{Gl}(2, K)$, we have $h \cdot \pi = \pi(\tau \circ \sigma, \chi)$. $\square$

4. Distinguished representations and the correspondence. In this section we will define what it means for an invariant representation of $\mathrm{GSO}(X)$ to be distinguished, and we will consider what effect being distinguished has on what extensions of the representation to $\mathrm{GO}(X)$ occur in the theta correspondence. The idea that certain extensions of a distinguished representation cannot occur in the theta correspondence is due to [HK]. This appears in Theorem 4.3 below. We go a step further, and show how an extension of a distinguished representation can be proven to occur in the theta correspondence. See Theorem 4.4.

These results may generalize. The definition of being distinguished generalizes to representations of $\mathrm{GSO}(X)$ if X is an arbitrary nondegenerate even dimensional symmetric bilinear space, and the proofs of Theorem 4.3 and Theorem 4.4 are general. The key question, which I do not the answer to, is whether Lemma 4.2 generalizes. For more remarks about generalizations, see the end of this section.

Let π be in $\mathrm{Irr}(\mathrm{GSO}(X))$. To define what it means for π to be distinguished, suppose y in X is anisotropic. Then the stabilizer in $\mathrm{SO}(X)$ of y can be identified with $\mathrm{SO}(Y)$, where Y is the orthogonal complement to y, and we will write $\mathrm{SO}(Y)$ for this stabilizer. We say that π is **distinguished** if π is invariant, and there is an anisotropic y in X such that

$$\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) \neq 0,$$

and, if $\det(X) \neq 1$, then Y is isotropic.

In fact, a representation is distinguished if and only if it is distinguished with respect to a certain anisotropic y_0 in X. Define y_0 in the following way. If $d = 1$, let $y_0 = 1$. If $d \neq 1$, also let $y_0 = 1$. Using the Witt cancellation theorem and the Witt extension theorem, one can show that if y is as in the last paragraph then there exists h in $\mathrm{GSO}(X)$ such that $h(y) = y_0$. It follows that a representation is distinguished if and only if it is distinguished with respect to y_0.

If a representation is distinguished, then its extensions to $\mathrm{GO}(X)$ can be identified. Suppose that π in $\mathrm{Irr}(\mathrm{GSO}(X))$ is invariant and distinguished. Each extension of π to $\mathrm{GO}(X)$ provides an action the nontrivial element of $\mathrm{O}(Y)/\mathrm{SO}(Y) \cong \{\pm 1\}$ on $\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1})$. The actions will be multiplication by ± 1, respectively. We denote by π^+ the extension inducing multiplication by 1, and by π^- the extension inducing multiplication by -1. From the last paragraph, the definitions of π^+ and π^- do not depend on the choice of y. For the remainder of this paper, we let Y be the orthogonal complement to y_0.

The group $\mathrm{SO}(Y)$ can be concretely described. If $d = 1$, then $\mathrm{SO}(Y)$ is the image under ρ of the subgroup $\{(g, g^{*-1}) : g \in \mathrm{Gl}(2,k)\}$ or $\{(g, g^{*-1}) : g \in D^\times\}$. If $d \neq 1$, then by Hilbert's Theorem 90, $\mathrm{SO}(Y)$ is the image under ρ of the subgroup $\{(\det(g), g) : g \in \mathrm{Gl}(2,k)\}$. We also note that h_0 fixes y_0, and thus is contained in $\mathrm{O}(Y)$. Together, $\mathrm{SO}(Y)$ and h_0 generate $\mathrm{O}(Y)$.

In the case $d = 1$, the next proposition completely identifies all the distinguished representations. We will consider the case $d \neq 1$ in greater detail in the next section.

Proposition 4.1. *Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X))$. Assume that π is invariant. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) \leq 1.$$

If $d = 1$, then π is distinguished. If $d \neq 1$ and $\pi = \pi(\tau, \chi)$, then π is distinguished if and only if

$$\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi \circ \det) \neq 0.$$

Proof. Suppose that $d = 1$. Since π is invariant, it follows that $\pi = \pi(\tau, \tau)$ for some τ in $\mathrm{Irr}(\mathrm{Gl}(2,k))$ or τ in $\mathrm{Irr}(B^\times)$. Now $\tau^\vee \cong \omega_\tau^{-1} \otimes_{\mathbb{C}} \tau$. It follows that there is an isomorphism

$$\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) \cong \mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau \otimes_{\mathbb{C}} \tau^\vee, \mathbf{1}),$$

or an isomorphism

$$\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) \cong \mathrm{Hom}_{D^\times}(\tau \otimes_{\mathbb{C}} \tau^\vee, \mathbf{1}).$$

Here $\mathrm{Gl}(2,k)$ or $D^\times$ is embedded on the diagonal. It is well known that the second homomorphism space has dimension one.

Suppose that $d \neq 1$. Then there is an isomorphism

$$\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1}) \cong \mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi \circ \det).$$

By an argument as in [H], this space has dimension less than or equal to 1. $\square$

The following lemma will be essential in determining which extensions of a distinguished representation occur in the theta correspondence. In a different form, the following lemma is due to D. Prasad [P].

Lemma 4.2 (Prasad). *Let $n = 1$ or 2. Then any distribution on X^n invariant under* $\mathrm{SO}(Y)$ *is invariant under* $\mathrm{O}(Y)$.

Proof. We first claim that it suffices to show that any distribution on $\mathrm{M}_2(k)^n$ or D^n invariant under conjugation by $\mathrm{Gl}(2,k)$ or $D^\times$, respectively, is invariant under $*$. To this end, we define a map L from X^n to $\mathrm{M}_2(k)^n$ or D^n in the following way. If $d = 1$, we let L be the identity. Suppose $d \neq 1$ and $\epsilon = \epsilon(d)$. For x in X, define $l(x)$ in $\mathrm{M}_2(k)$ by

$$l(x) = \frac{x + s(x)}{2} + \frac{x - s(x)}{2\sqrt{d}}.$$

Define L by $L(x) = l(x)$ if $n = 1$ and $L(x \oplus y) = l(x) \oplus l(y)$ if $n = 2$. Clearly, L is an isomorphism of k vector spaces, and $L(h_0 x) = L(x)^*$ for x in X^n. Moreover,

$$L(hx) = gL(x)g^{-1}$$

if h in $\mathrm{SO}(Z)$ and $h = \rho(g, g^{*-1})$ in the case $d = 1$, and $h = \rho(\det(g), g)$ in the case $d \neq 1$. Our claim follows.

Now we show that any distribution on $\mathrm{M}_2(k)^n$ or D^n invariant under $\mathrm{Gl}(2,k)$ or $D^\times$ is invariant under $*$. First consider D. Let f be a distribution on D^n invariant under conjugation by $D^\times$. Since $D^\times/k^\times$ is compact, there is a Haar measure on $D^\times/k^\times$ such that $f(\varphi) = f(\varphi')$ for φ in $\mathcal{S}(D^n)$, where φ' is defined by

$$\varphi'(x) = \int_{D^\times/k^\times} \varphi(gxg^{-1})\, dg.$$

Let φ be in $\mathcal{S}(D^n)$ and x be in D^n. By the proof of Proposition 3.3 of [P] there exists g_0 in $G^\times$ such that $g_0 x g_0^{-1} = x^*$. So

$$\begin{aligned}
(\varphi^*)'(x) &= \int_{D^\times/k^\times} \varphi(gx^*g^{-1})\, dg \\
&= \int_{D^\times/k^\times} \varphi(gg_0x(gg_0)^{-1})\, dg \\
&= \varphi'(x).
\end{aligned}$$

It follows that $f(\varphi^*) = f(\varphi)$.

Now consider the case of $\mathrm{M}_2(k)$. We argue as in the proof of Proposition 4.5 of [P]. If $n = 1$, we use the map from $\mathrm{M}_2(k)$ to k^2 that sends x to $(\mathrm{tr}(x), \det(x))$. If $n = 2$, we map from $\mathrm{M}_2(k)^2$ to k^5 that sends (x, y) to

$$(\mathrm{tr}(x), \mathrm{tr}(y), \det(x), \det(y), \mathrm{tr}(xy)).$$

It can be verified that the proof of Proposition 4.5 of [P] goes through in this last case. $\square$

The next theorem shows that one of the extensions of a distinguished representation cannot occur in the theta correspondence when n is 1 or 2. As we pointed out above the idea is due to [HK]. In [HK] the case $n = 1$ was considered.

Theorem 4.3. *Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X))$. If π is distinguished then π^- is not in $\mathcal{R}_n(\mathrm{GO}(X))$ for $n = 1$ and 2.*

Proof. We begin with two comments concerning π. First, $\pi|_{\mathrm{SO}(X)}$ is multiplicity free. For let

$$\pi|_{\mathrm{SO}(X)} = m \cdot \pi_1 \oplus \cdots \oplus m \cdot \pi_M,$$

where the $\pi_i \in \mathrm{Irr}(\mathrm{SO}(X))$ are mutually inequivalent, and m and M are positive integers. Then

$$\sum_{i=1}^{M} m \cdot \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SO}(Y)}(\pi_i, \mathbf{1}) = 1,$$

which implies that $m = 1$, and that $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SO}(Y)}(\pi_i, \mathbf{1}) = 1$ for exactly one i, say $i = 1$, and $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SO}(Y)}(\pi_i, \mathbf{1}) = 0$ for $i > 1$. Second, suppose that V_i is the space of π; we assert that $\pi^+(h_0)V_i = V_i$ for all i. Let us prove this first when $i = 1$. Let $\pi^+(h_0)V_1 = V_i$. Let f in $\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi_1, \mathbf{1})$ be nonzero. Define a linear functional f' on V_i by $f'(v) = f(\pi^+(h_0)v)$. Then f' is in $\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi_i, \mathbf{1})$. Since $f' \neq 0$, $i = 1$. Let i be arbitrary. There exists h in $\mathrm{GSO}(X)$ such that $\pi(h)V_1 = V_i$. We have $\pi^+(h_0)V_i = \pi^+(h_0 h)V_1 = \pi^+(h_0 h h_0)V_1 = \pi(h)\pi(h^{-1}h_0 h h_0)V_1 = \pi(h)V_1 = V_i$, since $h^{-1}h_0 h h_0$ is in $\mathrm{SO}(X)$.

Suppose that π^- is in $\mathcal{R}_n(\mathrm{GO}(X))$ for $n = 1$ or 2. Then there exists a nonzero $\mathrm{O}(Y)$ map T from ω_n to π^-. Let V be the space of π. We may assume that the composition T_1 of T with the projection of V onto V_1 is nonzero. Let $f \in \mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, \mathbf{1})$ be nonzero. Consider the composition $f \circ T_1$. This is a nonzero $\mathrm{SO}(Y)$ invariant distribution on X^n. By Lemma 4.2, $f \circ T_1$ is invariant under h_0. But since T_1 is an $\mathrm{O}(X)$ map and by the definition of τ^-, the composition of h_0 with $f \circ T_1$ is $-f \circ T_1$. Since $f \circ T_1 \neq 0$, this is a contradiction. □

Then next theorem gives a sufficient condition for one of the extensions of a distinguished representation to occur in the theta correspondence.

Theorem 4.4. *Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X))$. Suppose π is regular or distinguished, and $n = 1$ or 2. Then*

$$\mathrm{Hom}_{\mathrm{SO}(X)}(\omega_n, \pi) \neq 0 \implies \mathrm{Hom}_{\mathrm{O}(X)}(\omega_n, \pi^+) \neq 0.$$

Proof. Suppose first that π is regular. Let V be the space of π. We use the model for π^+ from the last section. Let L in $\mathrm{Hom}_{\mathrm{SO}(X)}(\omega_n, \pi)$ be nonzero. Define $L' : \omega_n \to \pi^+$ by $L'(\varphi) = L(\varphi) \oplus L(\omega_n(h_0)\varphi)$. Then L' is in $\mathrm{Hom}_{\mathrm{O}(X)}(\omega_n, \pi^+)$ and L' is nonzero.

Suppose that π is invariant is distinguished. We will use the notation of the proof of Theorem 4.3. Let L in $\mathrm{Hom}_{\mathrm{SO}(Y)}(\omega_n, \pi)$ be nonzero. We may assume that the composition of L with the projection of V onto V_1 is nonzero. To complete the proof it suffices to show that $L_1 \circ \omega_n(h_0) = \pi^+(h_0) \circ L_1$. We first show that $\omega_n(h_0)\ker(L_1) = \ker(L_1)$. Suppose not, i.e., suppose that $L_1(\omega_n(h_0)\ker(L_1)) \neq 0$. Then by the irreducibility of π_1, $L_1(\omega_n(h_0)\ker(L_1)) = V_1$. Let f in $\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi_1, \mathbf{1})$ be nonzero. Consider $f \circ L_1$. This distribution is nonzero and $\mathrm{SO}(Y)$ invariant. By

Lemma 4.2, $f \circ L_1$ is invariant under h_0, so that $f(V_1) = f(L_1(\omega_n(h_0)\ker(L_1)) = L_1(\ker(L_1)) = 0$, contradicting $f \neq 0$. Now since $\ker(L_1)$ is invariant under $\omega_n(h_0)$, it follows that $\mathcal{S}(X^n)/\ker(L_1)$ is an $\mathrm{O}(X)$ space. Via the $\mathrm{SO}(X)$ isomorphism given by L_1 between $\mathcal{S}(X^n)/\ker(L_1)$ and V_1 we can define an action of h_0 on V_1 so that L_1 is an $\mathrm{O}(X)$ map. By Theorem 4.3, this extension must be π^+. □

A similar argument proves the following statement. Let π be in $\mathrm{Irr}(\mathrm{GSO}(X))$ and Π be in $\mathrm{Irr}(\mathrm{GSp}(n,k)^+)$, for $n = 1$ or 2. Assume that π is regular or distinguished. Then

$$\mathrm{Hom}_{R'}(\omega_n, \Pi \otimes_{\mathbb{C}} \pi) \neq 0 \implies \mathrm{Hom}_R(\omega_n, \Pi \otimes_{\mathbb{C}} \pi^+) \neq 0.$$

Here R' is the subset of elements of R whose first entries are in $\mathrm{GSO}(X)$.

This result has some interesting consequences. It implies that if a regular or distinguished element of $\mathrm{Irr}(\mathrm{GSO}(X))$ corresponds to an element of $\mathrm{Irr}(\mathrm{GSp}(n,k)^+)$, in the obvious sense, then the corresponding element of $\mathrm{Irr}(\mathrm{GSp}(n,k)^+)$ is unique. In particular, since all elements of $\mathrm{Irr}(\mathrm{GSO}(X))$ are either regular or distinguished when $\det(X) = 1$, it follows that in this case if Π is as above, then Π is always uniquely determined. When $\det(X) = 1$ and $n = 1$ this helps one to understand the Jacquet-Langlands correspondence from the point of view of the theta correspondence. See section 7 and [S]. When $\det(X) = 1$ and $n = 2$, using the relation to the alternate approach to similitudes using the induced Weil representation [R], this gives a different argument for part of the proof of the strong multiplicity one theorem for regular representations of $\mathrm{GSp}(2)$ as in [So2]. It would be interesting to see if a complete proof could be obtained along these lines. This would require that the results of this section be extended to the case when X is the split six dimensional space. To do so, it will probably be necessary to use a subgroup of $\mathrm{O}(X)$ defined differently from $\mathrm{SO}(Y)$. This is the case for $\dim_k X = 2$, where $\mathrm{SO}(X)$ plays the role of $\mathrm{SO}(Y)$. For more remarks about this case, see section 7.

5. Distinguished $\mathrm{Gl}(2,K)$ representations.

In the last section we reduced the problem of determining the distinguished representations of $\mathrm{GSO}(X)$ in the case $d \neq 1$ to a problem concerning the corresponding representations of $\mathrm{Gl}(2,K)$. The problem of determining distinguished $\mathrm{Gl}(2,K)$ representations has essentially been solved by several authors. See [H] and [F]. Ultimately, the consideration of distinguished $\mathrm{Gl}(2,K)$ representations goes back to a global result of [HLR]. However, since a complete account does not appear in the literature we need to give an exposition.

We begin by defining some notation and recalling some facts. Essentially, we will follow [G]. In this section we assume that $d \neq 1$ so that K is a quadratic extension of k. Let π_K be a uniformizer for K, and let ψ_K be a nontrivial Galois invariant additive character of K. If τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$ is infinite dimensional, let $K(\tau,\psi_K)$ be the Kirillov model of τ with respect to ψ_K. Let τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$ be infinite dimensional. For g in $\mathrm{Gl}(2,K)$, ζ a quasi-character of $K^\times$, f in $K(\tau,\psi_K)$, and s in $\mathbb{C}$, let

$$Z(g,\zeta,f,s) = \int_{K^\times} \tau(g)f(x)\zeta(x)|x|^{s-1/2}\,dx.$$

This integral converges absolutely if $\Re(s)$ is sufficiently large. Moreover, the function defined by the integral for sufficiently large $\Re(s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ with at most two poles. If ζ is a quasi-character of $K^\times$ then there exists a meromorphic function $\gamma(\tau \otimes_{\mathbf{C}} \zeta, s, \psi_K)$ on the complex plane such that

$$\gamma(\tau \otimes_{\mathbf{C}} \zeta, s, \psi_K) Z(g, \zeta, f, s) = Z(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, \zeta^{-1}\omega_\tau^{-1}, f, 1-s)$$

for g in $\mathrm{Gl}(2, K)$ and f in $K(\tau, \psi_K)$. For ζ a quasi-character of $K^\times$ let

$$\epsilon(\tau \otimes_{\mathbf{C}} \zeta, s, \psi_K) = \gamma(\tau \otimes_{\mathbf{C}} \zeta, s, \psi_K) \frac{L(\tau \otimes_{\mathbf{C}} \zeta, s)}{L(\tau \otimes_{\mathbf{C}} \omega_\tau^{-1}\zeta, 1-s)}.$$

Here the L factors are as in [G]. The function $\epsilon(\tau \otimes_{\mathbf{C}} \zeta, s, \psi_K)$ is entire, and has no zeros. The notation for irreducible principal series and special representations of $\mathrm{Gl}(2, K)$ will be as in [GL]. Let $\pi(\mu_1, \mu_2)$ be a principal series representation of $\mathrm{Gl}(2, K)$. Then $\pi(\mu_1, \mu_2)$ is Galois invariant if and only if $\mu_1 \circ - = \mu_1$ and $\mu_2 \circ - = \mu_2$, or $\mu_1 \circ - = \mu_2$. Let $\sigma(\mu_1, \mu_2)$ be a special representation. Then $\sigma(\mu_1, \mu_2)$ is Galois invariant if and only if $\mu_1 \circ - = \mu_1$ and $\mu_2 \circ - = \mu_2$.

Lemma 5.1. *Let τ in $\mathrm{Irr}(\mathrm{Gl}(2, K))$ be infinite dimensional and Galois invariant. Let $\omega_\tau = \chi \circ \mathrm{N}_k^K$ and let ζ be a quasi-character of $K^\times$ whose restriction to $k^\times$ is χ. If τ is not a principal series representation $\pi(\mu_1, \mu_2)$ with μ_1 and μ_2 Galois invariant, then the integral $Z(g, \zeta^{-1}, f, 1/2)$ is absolutely convergent for all g in $\mathrm{Gl}(2, K)$ and f in $K(\tau, \psi_K)$.*

Proof. The claim follows if τ is supercuspidal. Assume that τ is a principal series representation. Then $\tau = \pi(\mu_1, \mu_2)$ with $\mu_1 \circ - = \mu_2$. It suffices to show that for $f \in \mathcal{S}(K)$ the integral

$$\int_{K^\times} |x|^{1/2} \mu_1(x) f(x) \zeta(x)^{-1} \, d^\times x$$

is absolutely convergent. An estimate shows that this integral converges absolutely if

$$|\mu_1(\pi_K)\zeta(\pi_K)^{-1}| < |\pi_K|^{-1/2}.$$

Since $|\mu_1(\pi_K)\zeta(\pi_K)^{-1}|^2 = 1$, our claim follows. Suppose that τ is a special representation. Then $\tau = \sigma(\mu_1, \mu_2)$ with $\mu_1 \circ - = \mu_1$, $\mu_2 \circ - = \mu_2$ and $\mu_1 = \mu_2 | \ |$. Again, it suffices to show that the above integral is absolutely convergent. We have $|\mu_1(\pi_K)\zeta(\pi_K)^{-1}| = |\pi_K|^{1/2} < |\pi_K|^{-1/2}$. $\square$

Lemma 5.2. *Let τ, χ and ζ be as in the last lemma. Then $\gamma(\tau \otimes_{\mathbf{C}} \zeta^{-1}, s, \psi_K)$ is defined at $1/2$ and*

$$\gamma(\tau \otimes_{\mathbf{C}} \zeta^{-1}, 1/2, \psi_K) = \epsilon(\tau \otimes_{\mathbf{C}} \zeta^{-1}, 1/2, \psi_K).$$

Proof. By definition,

$$\epsilon(\tau \otimes_{\mathbf{C}} \zeta^{-1}, s, \psi_K) = \gamma(\tau \otimes_{\mathbf{C}} \zeta^{-1}, s, \psi_K) \frac{L(\tau \otimes_{\mathbf{C}} \zeta^{-1}, s)}{L(\tau \otimes_{\mathbf{C}} \omega_\tau^{-1}\zeta, 1-s)}.$$

Since $\epsilon(\tau \otimes_{\mathbf{C}} \zeta^{-1}, s, \psi_K)$ is an entire function, and since the L functions are defined at $1/2$ by Lemma 5.1, it suffices to show that

$$\frac{L(\tau \otimes_{\mathbf{C}} \zeta^{-1}, 1/2)}{L(\tau \otimes_{\mathbf{C}} \omega_\tau^{-1}\zeta, 1/2)} = 1.$$

If τ is supercuspidal this is clear. Suppose that τ is a principal series representation $\pi(\mu_1, \mu_2)$ with $\mu_1 \circ - = \mu_2$. Then

$$\frac{L(\tau \otimes_{\mathbf{C}} \zeta^{-1}, 1/2)}{L(\tau \otimes_{\mathbf{C}} \omega_\tau^{-1}\zeta, 1/2)} = \frac{L(\mu_1\zeta^{-1}, 1/2)L(\mu_2\zeta^{-1}, 1/2)}{L(\mu_1^{-1}\zeta, 1/2)L(\mu_2^{-1}\zeta, 1/2)}.$$

It will suffice to show that $\mu_1(\pi_K)^2 = \zeta(\pi_K)^2$ if $\mu_1\zeta^{-1}$ is unramified and $\mu_2(\pi_K)^2 = \zeta(\pi_K)^2$ if $\mu_2\zeta^{-1}$ is unramified. By symmetry, it is enough to prove one of these statements. Suppose $\mu_1\zeta^{-1}$ is unramified. If K/k is unramified, then this follows since we can take π_K in $k^\times$, and $\mu_1\mu_2 = \zeta\zeta \circ -$ and $\mu_1 \circ - = \mu_2$. Suppose that K/k is ramified. Since the residual characteristic of k is odd, we can assume that $\overline{\pi_K} = -\pi_K$ and π_K^2 is a uniformizer of k. Then $\mu_1(\pi_K)^2 = \mu_1(-1)\mu_1(\pi_K)\mu_2(\pi_K) = \mu_1(-1)\zeta(\pi_K)\zeta(\overline{\pi_K}) = \mu_1(-1)\zeta(-1)\zeta(\pi_K)^2 = \zeta(\pi_K)^2$, since $\zeta(-1) = \mu_1(-1)$ because $\mu_1\zeta^{-1}$ is unramified. The case when τ is a special representation is analogous; for details, see the similar case treated in the remark below. □

The last lemma does not hold for all irreducible principal series representations $\pi(\mu_1, \mu_2)$ with μ_1 and μ_2 Galois invariant. Indeed, we claim that if $\tau = \pi(\mu_1, \mu_2)$ is an irreducible principal series representation with μ_1 and μ_2 Galois invariant, then

$$\gamma(\tau \otimes_{\mathbf{C}} \zeta^{-1}, 1/2, \psi_K) = \epsilon(\tau \otimes_{\mathbf{C}} \zeta^{-1}, 1/2, \psi_K)$$

except if $\mu_1\zeta^{-1}$ is unramified and $\mu_1(\pi_K)\zeta(\pi_K)^{-1} = |\pi_K|^{-1/2}$, or $\mu_2\zeta^{-1}$ is unramified and $\mu_2(\pi_K)\zeta(\pi_K)^{-1} = |\pi_K|^{-1/2}$; in these last cases,

$$\gamma(\tau \otimes_{\mathbf{C}} \zeta^{-1}, 1/2, \psi_K) = -\epsilon(\tau \otimes_{\mathbf{C}} \zeta^{-1}, 1/2, \psi_K).$$

To prove these claims we proceed as in the proof of Lemma 5.2. We need to compute

$$\lim_{s \to 1/2} \frac{L(\tau \otimes_{\mathbf{C}} \zeta^{-1}, s)}{L(\tau \otimes_{\mathbf{C}} \omega_\tau^{-1}\zeta, 1-s)} = \lim_{s \to 1/2} \frac{L(\mu_1\zeta^{-1}, s)L(\mu_2\zeta^{-1}, s)}{L(\mu_1^{-1}\zeta, 1-s)L(\mu_2^{-1}\zeta, 1-s)}.$$

We first show that $\mu_1\zeta^{-1}$ is unramified if and only if $\mu_2\zeta^{-1}$ is. Suppose that $\mu_1\zeta^{-1}$ is unramified. Then $\zeta(u) = \mu_1(u)$ for all $u \in \mathfrak{O}_K^\times$. Since $\mu_1(\ker(\mathrm{N}_k^K)) = 1$ and $\ker(\mathrm{N}_k^K)$ is contained in $\mathfrak{O}_K^\times$, $\zeta(\ker(\mathrm{N}_k^K)) = 1$. So, $\zeta \circ - = \zeta$. Now $\mu_1\mu_2 = \zeta\zeta \circ - = \zeta^2$. Hence, $\mu_1\zeta^{-1} = (\mu_2\zeta^{-1})^{-1}$, and $\mu_2\zeta^{-1}$ is unramified. The converse follows by symmetry. Note that we also have shown that if $\mu_1\zeta^{-1}$ and $\mu_2\zeta^{-1}$ are unramified then $\mu_1(\pi_K)\mu_2(\pi_K) = \zeta(\pi_K)^2$, i.e., $\mu_1(\pi_K)\zeta(\pi_K)^{-1} = \mu_2(\pi_K)^{-1}\zeta(\pi_K)$. If now $\mu_1\zeta^{-1}$ and $\mu_2\zeta^{-1}$ are ramified or $\mu_1\zeta^{-1}$ and $\mu_2\zeta^{-1}$ are unramified and $\mu_1(\pi_K)\zeta(\pi_K)^{-1} \neq |\pi_K|^{-1/2}$ and $\mu_2(\pi_K)\zeta(\pi_K)^{-1} \neq |\pi_K|^{-1/2}$, then the limit is 1. Suppose $\mu_1\zeta^{-1}$ and $\mu_2\zeta^{-1}$ are unramified and $\mu_1(\pi_K)\zeta(\pi_K)^{-1} = |\pi_K|^{-1/2}$ or $\mu_2(\pi_K)\zeta(\pi_K)^{-1} =$

$|\pi_K|^{-1/2}$. Then exactly one of $\mu_1(\pi_K)\zeta(\pi_K)^{-1}$ and $\mu_2(\pi_K)\zeta(\pi_K)^{-1}$ is $|\pi_K|^{-1/2}$. Without loss of generality, we may assume that $\mu_1(\pi_K)\zeta(\pi_K)^{-1} = |\pi_K|^{-1/2}$. Then

$$\begin{aligned}\lim_{s\to 1/2} \frac{L(\mu_1\zeta^{-1},s)L(\mu_2\zeta^{-1},s)}{L(\mu_1^{-1}\zeta,1-s)L(\mu_2^{-1}\zeta,1-s)} &= \lim_{s\to 1/2} \frac{L(\mu_1\zeta^{-1},s)}{L(\mu_1\zeta^{-1},1-s)} \lim_{s\to 1/2} \frac{L(\mu_2\zeta^{-1},s)}{L(\mu_2\zeta^{-1},1-s)} \\ &=(-1)\cdot 1 = -1.\end{aligned}$$

The next theorem follows essentially from [H] and from [T], as interpreted in [HST]. The previous discussion shows that in the following theorem it is essential to use ϵ instead of γ factors. Note also that ψ_K differs from the additive character in [H]. There it is assumed that ψ_K is trivial on k.

Theorem 5.3 (Hakim-Flicker). *Let τ in $\mathrm{Irr}(\mathrm{Gl}(2,K))$ be infinite dimensional and Galois invariant. Let $\omega_\tau = \chi \circ \mathrm{N}_k^K$. Then the following are equivalent:*

(1) $\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi\circ\det) \neq 0$;

(2) *For every quasi-character ζ of $K^\times$ whose restriction to $k^\times$ is χ,*

$$\epsilon(\tau\otimes_{\mathbb{C}}\zeta^{-1}, 1/2, \psi_K) = \chi(-1);$$

(3) *τ is the base change of an element of $\mathrm{Irr}(\mathrm{Gl}(2,k))$ with central character $\chi\omega_{K/k}$.*

Proof. Assume first that $\tau \neq \pi(\mu_1,\mu_2)$ with μ_1 and μ_2 Galois invariant.

(1) $\iff$ (2): The equivalence follows from Lemmas 5.1 and 5.2, and an argument essentially as in [H].

(2) $\iff$ (3): Since τ is Galois invariant τ and $\tau \neq \pi(\mu_1,\mu_2)$ with μ_1 and μ_2 Galois invariant, τ is the base change of a discrete series representation of $\mathrm{Gl}(2,k)$ that has central character χ or $\chi\omega_{K/k}$. The equivalence of (2) and (3) is 4 of Lemma 14 of [HST].

Now suppose that $\tau = \pi(\mu_1,\mu_2)$ with μ_1 and μ_2 Galois invariant. We will show that (1), (2) and (3) all hold. The statement (2) follows from Lemma 14 of [HST]. To see (3), note that μ_1 and μ_2 factor through N_k^K via, say, μ_1' and μ_2', respectively. By replacing μ_1' by $\omega_{K/k}\mu_1'$, if necessary, we may assume that $\mu_1'\mu_2' = \chi$. Since $\mu_1\mu_2^{-1} \neq |\ |_K^{\pm 1}$ it follows that $\mu_1'\mu_2'^{-1} \neq |\ |_k^{\pm 1}$. It follows $\pi(\mu_1',\mu_2')$ is defined, and the base change of $\pi(\mu_1',\mu_2')$ is τ. To show (1), we proceed as in Proposition 9 of [F]. Let

$$g_0 = \begin{pmatrix} -\sqrt{d} & \sqrt{d} \\ 1 & 1 \end{pmatrix},$$

and

$$T = \{\begin{pmatrix} a & bd \\ b & a \end{pmatrix} : a, b \in k, a^2 - db^2 \neq 0\}, \quad T' = \{\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} : z \in K^\times\}.$$

Then $g_0 T' g_0^{-1} = T$ and

$$g_0^{-1}\,\mathrm{Gl}(2,k) g_0 = \{\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : a\overline{a} - b\overline{b} \neq 0\}.$$

Define $L : \pi(\mu_1, \mu_2) \to \mathbb{C}$ by

$$L(f) = \int_{T \backslash \mathrm{Gl}(2,k)} f(g_0^{-1} g) \chi(\det(g))^{-1} \, dg.$$

A computation shows that the integrand is well defined. Moreover, one can show that $T \backslash \mathrm{Gl}(2,k)$ has finite measure and that the integrand is bounded, so that the integral converges. Finally, L is nonzero and contained in $\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi \circ \det)$. $\square$

6. The main theorem. We will now prove the main theorem. The method of the proof is entirely analogous to the global technique of computing a Fourier coefficient of a global theta lift. See, for example, [HST].

In defining distinguished representations we used anisotropic vectors in X and their stabilizers in $\mathrm{O}(X)$; we now will consider vectors in X^2 and their stabilizers in $\mathrm{O}(X)$. Let l be a positive integer. We will say that $x \in X^l$ is **nondegenerate** if the components of x generate a nondegenerate subspace of X, or, equivalently, if $\det(x_i, x_j) \neq 0$. If z in X^2 is nondegenerate, then the stabilizer of z in $\mathrm{O}(X)$ can be identified with $\mathrm{O}(Z)$, where Z is the orthogonal complement to the space generated by the components of z. Also, it is easy to show that if z in X^2 is nondegenerate, then $\mathrm{SO}(X) \cdot z = \mathrm{O}(X) \cdot z$, and $\mathrm{SO}(X) \cdot z = \mathrm{O}(X) \cdot z$ is closed.

Lemma 6.1. *Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X))$. In the case $d \neq 1$ assume that π is infinite dimensional. Then there exists a nondegenerate z in X^2 such that*

$$\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi, \mathbf{1}) \neq 0.$$

Proof. Suppose first $d = 1$ and $\epsilon = \epsilon(1)$. Let $\pi = \pi(\tau, \tau')$. Suppose that τ and τ' are infinite dimensional. Let

$$z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then z is nondegenerate, and

$$\mathrm{SO}(Z) = \{\rho(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}) : a \in k^\times\}.$$

We will use the Kirillov models $K(\tau, \psi)$ and $K(\tau', \psi)$ of τ and τ' with respect to our additive character ψ, respectively. Let n be so large that

$$\int_{k^\times} f(x)|x|^n \, dx$$

converges absolutely for f in $K(\tau, \psi)$ and f in $K(\tau', \psi)$. Define $L : \pi \to \mathbf{1}$ by

$$L(f \otimes f') = \int_{k^\times} f(x)|x|^n \, dx \cdot \int_{k^\times} f'(x)|x|^n \, dx.$$

Then L is a well defined nonzero $\mathbb{C}$ linear map, and L is $\mathrm{SO}(Z)$ invariant.

Suppose next that exactly one of τ and τ', say τ, is infinite dimensional. Since τ' is finite dimensional, τ' is one dimensional, and there exists a quasi-character β' of $k^\times$ such that $\tau' = \beta' \circ \det$. By hypothesis, $\beta'^2 = \omega_{\tau'} = \omega_\tau$. Suppose that τ is a supercuspidal or special representation. We claim that

$$\int_{k^\times} \beta'(x)^{-1} f(x)\, dx$$

converges absolutely for f in $K(\tau, \psi)$. This is clear if τ is supercuspidal. If τ is the special representation $\sigma(\mu_1, \mu_2)$ with $\mu_1 = \mu_2|\ |$ then this follows from the estimate $|\pi_k|^{1/2}|\beta'(\pi_k)|^{-1}|\mu_1(\pi_k)| = |\pi_k| < 1$. Now define $L : \pi \to \mathbf{1}$ by

$$L(f \otimes z) = z \int_{k^\times} \beta'(x)^{-1} f(x)\, dx.$$

Then L is a nonzero element of $\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi, \mathbf{1})$. Suppose that τ is a principal series representation. In this case, we require another nondegenerate element of X^2. Every quadratic extension E of k is contained in $\mathrm{M}_2(k)$ as a k algebra, and for every quadratic extension E of k contained in $X = \mathrm{M}_2(k)$, $\mathrm{Gal}(E/k) = \{1, *\}$, and there exists a nondegenerate z in X^2 such that

$$\mathrm{SO}(Z) = \{\rho(x, x^{*-1}) : x \in E^\times\}.$$

Fix a quadratic extension E of k in X and such a z in X^2. Let α be the quasi-character of $E^\times$ defined by $\alpha(x) = \beta'(\det(x))$. Then α extends ω_τ. By [T], we have $\mathrm{Hom}_{E^\times}(\tau, \alpha) \neq 0$ if and only if $\epsilon(\mathrm{BC}_{E/k}(\tau) \otimes_{\mathbb{C}} \alpha^{-1}, 1/2, \psi_E) = \omega_\tau(-1)$. By Lemma 14 of [HST], $\epsilon(\mathrm{BC}_{E/k}(\tau) \otimes_{\mathbb{C}} \alpha^{-1}, 1/2, \psi_E) = \omega_\tau(-1)$, so that $\mathrm{Hom}_{E^\times}(\tau, \beta' \circ \det) \neq 0$. Let $f \in \mathrm{Hom}_{E^\times}(\tau, \beta' \circ \det)$ be nonzero. Define $L : \pi \to \mathbf{1}$ by $L(v \otimes z) = z f(v)$. Then L is a nonzero element of $\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi, \mathbf{1})$.

Suppose that τ and τ' are both finite dimensional, i.e., one dimensional. Let β and β' be quasi-characters of $k^\times$ such that $\tau = \beta \circ \det$ and $\tau' = \beta' \circ \det$. Since $\omega_\tau = \omega_{\tau'}$, we have $\beta^2 = \beta'^2$. This implies that $\beta = \beta'$ or $\beta = \omega_{E/k}\beta'$ for some quadratic extension E of k, since the residual characteristic of k is odd. Let E be contained in X and let z in X^2 be as above. Since $\det(x) = \mathrm{N}_k^E(x)$ for x in $E^\times$, it follows that $\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi, \mathbf{1}) \neq 0$.

Now suppose $d = 1$ and $\epsilon = -\epsilon(1)$. Since $\mathrm{SO}(X)$ is compact, it will suffice to show that there exists nonzero v in π and nondegenerate z in X^2 such that $\pi(h)v = v$ for $h \in \mathrm{SO}(Z)$. Since for every quadratic extension E of k a statement as above holds, to prove the existence of the required v and z it will suffice to show that there exists a quadratic extension E of k contained in D, a quasi-character ϕ of $E^\times$, and nonzero vectors w in τ and w' in τ' such that $\tau(x)w = \phi(x)w$ and $\tau'(x)w' = \phi(x^*)w'$ for x in $E^\times$.

If τ and τ' are one dimensional then an argument as in the case $\epsilon = \epsilon(1)$ works.

Suppose $\dim \tau > 1$ and $\dim \tau' > 1$. We will use terminology and results from [T]. We first assert that we can assume that τ and τ' are minimal. To see this, let $\alpha = \omega_\tau = \omega_{\tau'}$. Consider $\alpha|_{1+\pi_k\mathfrak{O}_k}$. For some large n, we can regard α as a character of $1 + \pi_k\mathfrak{O}_k / 1 + \pi_k^n\mathfrak{O}_k$. This is a finite group of odd order. It follows that squaring is an automorphism of the group of characters of this group. Hence, there exists a quasi-character η of $k^\times$ such that $\eta^2 = \alpha$ on $1 + \pi_k\mathfrak{O}_k$. Consider $\tau \otimes_{\mathbf{C}} \eta^{-1}$ and $\tau' \otimes_{\mathbf{C}} \eta^{-1}$. The common central character of these representations has conductor less than or equal to 1. Since any element of $\mathrm{Irr}(D^\times)$ of dimension larger than 1 with central character of conductor less than or equal to 1 is minimal, $\tau \otimes_{\mathbf{C}} \eta^{-1}$ and $\tau' \otimes_{\mathbf{C}} \eta^{-1}$ are minimal. Since our claim holds for $\tau \otimes_{\mathbf{C}} \eta^{-1}$ and $\tau' \otimes_{\mathbf{C}} \eta^{-1}$ if and only if it holds for τ and τ', we may assume that τ and τ' are minimal.

Let $\mathrm{JL}(\tau)$ and $\mathrm{JL}(\tau')$ be the representations corresponding to τ and τ' under the Jacquet-Langlands correspondence, respectively. Since $\dim\tau > 1$ and $\dim\tau' > 1$, these representations are supercuspidal. Let $a(\mathrm{JL}(\tau))$ and $a(\mathrm{JL}(\tau'))$ be the conductors of $\mathrm{JL}(\tau)$ and $\mathrm{JL}(\tau')$, respectively. Without loss of generality , we may assume that $\dim(\tau) \geq \dim(\tau')$. Using the formulas for $\dim\tau$ and $\dim\tau'$ in terms of $a(\mathrm{JL}(\tau))$ and $a(\mathrm{JL}(\tau'))$, respectively, one can show that $a(\mathrm{JL}(\tau)) \geq a(\mathrm{JL}(\tau'))$. Note that the formula in [T] for $\dim\tau$ when $a(\mathrm{JL}(\tau))$ is odd appears incorrectly: it should be $(q+1)q^{(c-3)/2}$ instead of $(q+1)^{(c-3)/2}$. Let E be a quadratic extension of k whose ramification index e has the same parity as $a(\mathrm{JL}(\tau))$. Let S be the set of all quasi-characters of $E^\times$ whose conductors are less than or equal to $e(a(\mathrm{JL}(\tau))-1)/2$ and which extend α, and let S' be the set of all quasi-characters of $E^\times$ whose conductors are less than or equal to $[e(a(\mathrm{JL}(\tau'))-1)/2+1/2]$ and which extend α. Since $a(\mathrm{JL}(\tau)) \geq a(\mathrm{JL}(\tau'))$ we have $S' \subset S$. By the proof of Lemma 3.2 of [T], $\tau|_{E^\times}$ is the direct sum of the elements of S. By the proof of Lemma 3.1 of [T] every quasi-character of $E^\times$ that occurs in $\tau'|_{E^\times}$ is contained in S'. It follows that there exists a quasi-character ϕ of $E^\times$ that occurs in $\tau|_{E^\times}$ and $\tau'|_{E^\times}$. Since the conductor of $\phi \circ *$ is the same as the conductor of ϕ, it follows that $\phi \circ *$ also occurs in $\tau|_{E^\times}$, which proves our claim.

The case when, say, $\dim(\tau) > 1$ and $\dim(\tau') = 1$ remains. Let $\tau' = \beta' \circ \mathrm{N}$. Then $\beta'^2 = \alpha$. It follows that the common central character of $\tau \otimes_{\mathbf{C}} \beta'^{-1}$ and $\tau' \otimes_{\mathbf{C}} \beta'^{-1} = \mathbf{1}$ is trivial. Thus, we may assume that τ is minimal and $\tau' = \mathbf{1}$. Let S be as in the last paragraph. Since α is trivial, it follows that the trivial character of $E^\times$ lies in S, and so we can take ϕ to be the trivial character of $E^\times$.

Suppose now $d \neq 1$. Let $\pi = \pi(\tau, \chi)$. By assumption, τ is infinite dimensional. Let

$$z = \begin{pmatrix} 0 & \sqrt{d} \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ \sqrt{d} & 0 \end{pmatrix}.$$

A computation shows that

$$\mathrm{SO}(Z) = \{\rho(1, \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}) : u \in \ker(\mathrm{N}_k^K)\}$$

Since $\mathrm{SO}(Z)$ is compact it will suffice to show that there exists a nonzero vector v in τ such that

$$\tau\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} v = v$$

for u in $\ker(\mathrm{N}_k^K)$. We will use the Kirillov model $K(\tau, \psi_K)$ of τ. Let f be the characteristic function of $\mathfrak{O}_K^\times$. Then f is in $K(\tau, \psi_K)$, and since $\ker(\mathrm{N}_k^K)$ is contained in $\mathfrak{O}_K^\times$, we have $f(ux) = f(x)$ for x in $K^\times$ and u in $\ker(\mathrm{N}_k^K)$. Thus, f is the desired vector. $\square$

Lemma 6.2. *Suppose that $d \neq 1$. Let π be in* $\mathrm{Irr}(\mathrm{GSO}(X))$. *Assume that π is infinite dimensional, invariant, but not distinguished. Let π_1 and π_2 be the two extensions of π to* $\mathrm{GO}(X)$. *Then there exists a nondegenerate z in X^2 such that*

$$\mathrm{Hom}_{\mathrm{O}(Z)}(\pi_1, \mathbf{1}) \neq 0, \qquad \mathrm{Hom}_{\mathrm{O}(Z)}(\pi_2, \mathbf{1}) \neq 0.$$

Proof. Let $\pi = \pi(\tau, \chi)$. Then τ is infinite dimensional. Let the notation be as in section 5. Let

$$z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix};$$

Then

$$\mathrm{SO}(Z) = \{\rho(a, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) : a \in k^\times\},$$

and $\mathrm{O}(Z)$ is generated by $\mathrm{SO}(Z)$ and

$$\rho(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \circ h_0.$$

Since π is invariant, by Proposition 3.1, τ is Galois invariant. From the explicit form of $K(\tau, \psi_K)$ it follows that $K(\tau, \psi_K)$ is invariant under composition by $-$, and a computation shows that $(\tau(g)f) \circ - = \tau(\overline{g})(f \circ -)$. By the remarks in section 3 and the proof of Proposition 3.1, we may assume that $\pi_1(h_0)$ is given by $\pi_1(h_0)f = f \circ -$ and $\pi_2(h_0)$ is given by $\pi_2(h_0)f = -f \circ -$. Since π is not distinguished, by Proposition 4.1 we have that $\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\tau, \chi \circ \det) = 0$. By Theorem 5.3, it follows that τ is not the base change of an element of $\mathrm{Irr}(\mathrm{Gl}(2, k))$ with central character $\chi\omega_{K/k}$. In particular, τ is not $\pi(\mu_1, \mu_2)$ for some Galois invariant quasi-characters μ_1 and μ_2 of $K^\times$. Let ζ be a quasi-character of $K^\times$ that extends χ. By Lemma 5.1,

$$Z(g, \zeta^{-1}, f, 1/2) = \int_{K^\times} \tau(g)f(x)\zeta(x)^{-1}\, dx$$

converges absolutely for all g in $\mathrm{Gl}(2, K)$ and f in $K(\tau, \psi_K)$. Define $L_\zeta : \pi \to \mathbf{1}$ by

$$L_\zeta(f) = Z(1, \zeta^{-1}, f, 1/2).$$

Then L_ζ is nonzero, and a computation shows that L_ζ is in $\mathrm{Hom}_{\mathrm{SO}(X)_z}(\pi, \mathbf{1})$. Moreover,

$$L_\zeta(\pi_1(\rho(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \circ h_0)f) = Z(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \zeta\omega_\tau^{-1}, f, 1/2),$$

and

$$L_\zeta(\pi_2(\rho(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \circ h_0)f) = -Z(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \zeta\omega_\tau^{-1}, f, 1/2),$$

for f in π. By the local functional equation for τ and Lemma 5.2, we thus have

$$L_\zeta(\pi_1(\rho(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \circ h_0)f) = \epsilon(\tau \otimes_{\mathbf{C}} \zeta^{-1}, 1/2, \psi) L_\zeta(f)$$

and

$$L_\zeta(\pi_2(\rho(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \circ h_0)f) = -\epsilon(\tau \otimes_{\mathbf{C}} \zeta^{-1}, 1/2, \psi) L_\zeta(f)$$

for f in $K(\tau, \chi)$. Since τ is not the base change of an element of $\mathrm{Irr}(\mathrm{Gl}(2,k))$ with central character $\chi\omega_{K/k}$, by Lemma 14 of [HST], there exists quasi-characters ζ and ζ' of $K^\times$ extending χ such that

$$\epsilon(\tau \otimes_{\mathbf{C}} \zeta^{-1}, 1/2, \psi) = \chi(-1), \qquad \epsilon(\tau \otimes_{\mathbf{C}} \zeta'^{-1}, 1/2, \psi) = -\chi(-1).$$

This completes the proof. □

Theorem 6.3. *Let σ be in $\mathrm{Irr}(\mathrm{GO}(X))$. In the case $\det(X) \neq 1$, assume σ is infinite dimensional. Then σ is in $\mathcal{R}_2(\mathrm{GO}(X))$ if and only if σ is not of the form π^- for some distinguished π in $\mathrm{Irr}(\mathrm{GSO}(X))$.*

Proof. By Theorem 4.3, if σ is in $\mathcal{R}_2(\mathrm{GO}(X))$, then σ is not of the form π^- for some distinguished π. Let π in $\mathrm{Irr}(\mathrm{GSO}(X))$, and if $d \neq 1$, then assume that π is infinite dimensional. We need to show that if π is regular then π^+ is in $\mathcal{R}_2(\mathrm{GO}(X))$, if π is invariant and distinguished then π^+ is in $\mathcal{R}_2(\mathrm{GO}(X))$, and if π is invariant but not distinguished, then both extensions of π to $\mathrm{GO}(X)$ lie in $\mathcal{R}_2(\mathrm{GO}(X))$.

Suppose $d = 1$. By Theorem 4.4, it will suffice to show that $\mathrm{Hom}_{\mathrm{SO}(X)}(\omega, \pi) \neq 0$. By Lemma 6.1, there exists a nondegenerate z in X^2 such that $\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi^\vee, \mathbf{1}) \neq 0$. There is an $\mathrm{SO}(X)$ isomorphism of $\mathcal{S}(\mathrm{SO}(X) \cdot z)$ with

$$\text{c-Ind}_{\mathrm{SO}(Z)}^{\mathrm{SO}(X)} \mathbf{1}.$$

By 1.8 of [BZ], it follows that there is a surjective $\mathrm{SO}(Z)$ map from ω to this induced representation. By Frobenius reciprocity as in Proposition 2.29 of [BZ],

$$\begin{aligned} \mathrm{Hom}_{\mathrm{SO}(X)}(\text{c-Ind}_{\mathrm{SO}(Z)}^{\mathrm{SO}(X)} \mathbf{1}, \pi) &\cong \mathrm{Hom}_{\mathrm{SO}(Z)}(\mathbf{1}, (\pi^\vee)|_{\mathrm{SO}(Z)}^\vee) \\ &\cong \mathrm{Hom}_{\mathrm{SO}(Z)}(\pi^\vee, \mathbf{1}). \end{aligned}$$

Since the last space is nonzero, it follows that $\mathrm{Hom}_{\mathrm{SO}(X)}(\omega, \pi)$ is nonzero.

Suppose now $d \neq 1$. If π is regular or invariant and distinguished, then an argument as in the last paragraph suffices. If π is invariant but not distinguished, then using Lemma 6.2 and the technique of the last paragraph with $\mathrm{SO}(X)$ replaced by $\mathrm{O}(X)$, one can construct nonzero elements of $\mathrm{Hom}_{\mathrm{O}(X)}(\omega, \pi_1)$ and $\mathrm{Hom}_{\mathrm{O}(X)}(\omega, \pi_2)$, where π_1 and π_2 are the extensions of π to $\mathrm{GO}(X)$. □

7. A case of a conjecture of Kudla. S.S. Kudla has made some important conjectures about the first appearance of a representation in the theta correspondence. In this section we essentially prove a case of one these conjectures. In the case $d \neq 1$ our result is not as complete because our understanding of the theta correspondence between $\mathrm{Gl}(2,k)^+$ and $\mathrm{Irr}(\mathrm{GO}(X))$ is not strong as in the case $d = 1$.

To describe the conjectures, suppose for the moment that X is an arbitrary nondegenerate even dimensional symmetric bilinear space over k. By the existence of the stable range, for every σ in $\mathrm{Irr}(\mathrm{O}(X))$ there exists a nonnegative integer n such that σ lies in $\mathcal{R}_n(\mathrm{O}(X))$. For σ in $\mathrm{Irr}(\mathrm{O}(X))$ let $n(\sigma)$ be the smallest integer such integer.

Conjecture 7.1 (S.S. Kudla). *If σ is in $\mathrm{Irr}(\mathrm{O}(X))$ then*

$$n(\sigma) + n(\sigma \otimes_{\mathbf{C}} \mathrm{sign}) = \dim_k X.$$

There is also a conjecture for elements of $\mathrm{Irr}(\mathrm{Sp}(n,k))$. To state this conjecture we need some more notation. Fix d in $k^\times/k^{\times 2}$. Then there are, up to equivalence, exactly two anisotropic even dimensional symmetric bilinear spaces X_+ and X_- of determinant d. From X_+ and X_- we can create two series of even dimensional symmetric bilinear spaces by adding hyperbolic planes to X_+ and X_-. For π in $\mathrm{Irr}(\mathrm{Sp}(n,k))$, let $m_+(\pi)$ be the smallest nonnegative even integer m such that π occurs in the theta correspondence with the m dimensional space with anisotropic component X_+; define $m_-(\pi)$ similarly.

Conjecture 7.2 (S.S. Kudla). *If π is in $\mathrm{Irr}(\mathrm{Sp}(n,k))$ then*

$$m_+(\pi) + m_-(\pi) = 4n + 4.$$

One can make completely analogous definitions and conjectures for the theta correspondence for similitudes. It is easy to see that Conjectures 7.1 and 7.2 hold for the correspondence for isometries if and only if they hold for the correspondence for similitudes.

Suppose X is again as in defined in Section 2. To prove Conjecture 7.1 in this case, we need to understand $\mathcal{R}_1(\mathrm{GO}(X))$ and $\mathcal{R}_3(\mathrm{GO}(X))$. To characterize $\mathcal{R}_1(\mathrm{GO}(X))$ we need to recall some facts about the theta correspondence when the dimension of the underlying bilinear spaces is two, and about the theta correspondence between $\mathrm{Irr}(\mathrm{GO}(X))$ and $\mathrm{Irr}(\mathrm{Gl}(2,k)^+)$ in the case $d \neq 1$.

Let V be a nondegenerate two dimensional symmetric bilinear space of determinant d. Then $\mathrm{GSO}(V)$ is abelian, and all the elements of $\mathrm{Irr}(\mathrm{GSO}(V))$ are one dimensional. We define regular and invariant representations exactly as in section 3. If α in $\mathrm{Irr}(\mathrm{GSO}(V))$ is regular, α^+ will again denote the induced representation of α to $\mathrm{GO}(X)$. Moreover, we say that α in $\mathrm{Irr}(\mathrm{GSO}(V))$ is distinguished if and only if

$$\mathrm{Hom}_{\mathrm{SO}(V)}(\alpha, 1) \neq 0.$$

Thus, $\mathrm{SO}(V)$ plays the role that $\mathrm{SO}(Y)$ did in section 4, and if α is in $\mathrm{Irr}(\mathrm{GSO}(V))$ is distinguished then we define α^+ and α^- just as in section 4. A result entirely analogous to the main theorem holds: If β is in $\mathrm{Irr}(\mathrm{GO}(X))$, then β is in $\mathcal{R}_1(\mathrm{GO}(V))$ if and only if β is not of the form α^- for some distinguished α in $\mathrm{Irr}(\mathrm{GSO}(V))$. Moreover, by Theorem 1.9 of [Ca], Conjecture 1.3 (theta dichotomy) holds for $X = V$ and $2n = 2$, and the remarks preceeding Conjecture 1.3 apply. If one makes the identification of V with K then elements of $\mathrm{GSO}(X)$ can be identified with quasi-characters of $K^\times$. The map that takes a quasi-character α of $K^\times$ to $\theta(\alpha^+)^\vee$ is just the usual map of that associates to a quasi-character an element of $\mathrm{Irr}(\mathrm{Gl}(2,k))$.

The case when V is anisotropic contains information about the restriction of representations of $\mathrm{Gl}(2,k)$ that we use in the proof of the next theorem. Let π be in $\mathrm{Irr}(\mathrm{Gl}(2,k))$. It is well known that the restriction of π to $\mathrm{Sl}(2,k)$ is multiplicty free, and that $\pi|_{\mathrm{Sl}(2,k)}$ is reducible if and only if π is a theta lift an element of $\mathrm{Irr}(\mathrm{GO}(V))$ for some anisotropic V. Let π be a theta lift of σ in $\mathrm{Irr}(\mathrm{GO}(V))$. Then from Lemma 4.2 of [R] and the remarks in section 1 it follows that the restriction of π to $\mathrm{Sl}(2,k)$ has two irreducible components if and only if $\sigma \not\cong \alpha^+$ with α such that $\alpha|_{\mathrm{SO}(V)} \neq 1$, and $\alpha|^2_{\mathrm{SO}(V)} = 1$. Let α in $\mathrm{Irr}(\mathrm{GSO}(X))$ be such that $\alpha|_{\mathrm{SO}(V)} \neq 1$ and $\alpha|^2_{\mathrm{SO}(V)} = 1$, and assume $\pi = \theta(\alpha^+)$. Then again from Lemma 4.2 of [R] the restriction of π to $\mathrm{Sl}(2,k)$ has four components. Finally, from Theorem 1.9 (d) of [Ca] it follows that every such π, that is, every π in $\mathrm{Irr}(\mathrm{Gl}(2,k))$ whose restriction to $\mathrm{Sl}(2,k)$ has four components, is a theta lift from every anisotropic V.

We also need to make some remarks about the theta correspondence between $\mathrm{Irr}(\mathrm{GO}(X))$ and $\mathrm{Irr}(\mathrm{Gl}(2,k)^+)$ in the case $d \neq 1$. This was considered in [Co] using the extended Weil representation Ω of $\mathrm{Gl}(2,k) \times \mathrm{GO}(X)$; see [R] for the definition. By an argument as in the proof of Proposition 3.5 of [R], as representations of $\mathrm{Gl}(2,k) \times \mathrm{GSO}(X)$,

$$\Omega \cong \text{c-Ind}_{R'}^{\mathrm{Gl}(2,k)\times\mathrm{GSO}(X)} \omega,$$

where R' is as in the remark after Theorem 4.4. Using Frobenius reciprocity, the main result of [Co] now states that for every infinite dimensional Π in $\mathrm{Irr}(\mathrm{Gl}(2,k))$, if $\mathrm{BC}(\Pi^\vee)$ is the base change of $\Pi^\vee$ to $\mathrm{Gl}(2,K)$, and $\pi = \pi(\mathrm{BC}(\Pi^\vee), \omega_{K/k}\omega_{\Pi^\vee})$ then

$$\mathrm{Hom}_{R'}(\omega, \Pi \otimes_{\mathbf{C}} \pi) \neq 0.$$

If $\mathrm{BC}(\Pi^\vee)$ is infinite dimensional then by Proposition 4.1 and Theorem 5.3, π is distinguished. By the remark following Theorem 4.4, it follows that if $\mathrm{BC}(\Pi^\vee)$ is infinite dimensional then

$$\mathrm{Hom}_{R}(\omega, \Pi \otimes_{\mathbf{C}} \pi^+) \neq 0.$$

This restricted understanding compels us to make the following definition. We let $\mathrm{Irr}_{\mathrm{BC\ inf}}(\mathrm{GO}(X))$ be the set of σ in $\mathrm{Irr}(\mathrm{GO}(X))$ such that if σ is contained in $\mathcal{R}_1(\mathrm{GO}(X))$ then for any element Π of $\mathrm{Irr}(\mathrm{Gl}(2,k))$ that has $\theta(\sigma)$ as an irreducible constituent, $\mathrm{BC}(\Pi^\vee)$ is infinite dimensional. The elements of $\mathrm{Irr}(\mathrm{GO}(X))$ not contained in $\mathrm{Irr}_{\mathrm{BC\ inf}}(\mathrm{GO}(X))$ are limited. For example, $\mathrm{Irr}_{\mathrm{BC\ inf}}(\mathrm{GO}(X))$ contains all the supercuspidal representations, as we show in the next section. Using [K] and the knowledge of the elements of $\mathrm{Irr}(\mathrm{Gl}(2,k))$ whose base changes to $\mathrm{Gl}(2,K)$ are finite dimensional, one could compute all the possible elements of $\mathrm{Irr}(\mathrm{GO}(X))$ which are not in $\mathrm{Irr}_{\mathrm{BC\ inf}}(\mathrm{GO}(X))$.

Lemma 7.3. *Let σ be in $\mathrm{Irr}(\mathrm{GO}(X))$. In the case $d \neq 1$ assume that σ is infinite dimensional, σ is in $\mathrm{Irr}_{\mathrm{BC\ inf}}(\mathrm{GO}(X))$, and Conjecture 7.2 for $n = 1$. Then σ is in $\mathcal{R}_1(\mathrm{GO}(X))$ if and only if σ is of the form π^+ for some distinguished π.*

Proof. If σ in $\mathrm{Irr}(\mathrm{GO}(X))$ is of the form π^+ for some distinguished π then σ is in $\mathcal{R}_1(\mathrm{GO}(X))$ by an argument as in the proof of Theorem 6.3, with $\mathrm{SO}(Y)$ playing the role of $\mathrm{SO}(Z)$.

Suppose that σ is in $\mathcal{R}_1(\mathrm{GO}(X))$. Suppose $d = 1$ and $\epsilon = \epsilon(1)$. By [S],

$$\theta(\{\pi^+ : \pi \in \mathrm{Irr}(\mathrm{GSO}(X)) \text{ is distinguished}\}) = \mathrm{Irr}(\mathrm{Gl}(2,k)).$$

From Theorem 1.2, it follows that σ is of the form π^+ for some distinguished π.

Suppose $d = 1$ and $\epsilon = -\epsilon(1)$. Suppose that σ is not of the form π^+ for some distinguished π. By Theorem 4.3, σ is of the form π^+ for some regular π. It follows that $\mathbf{1}$ does not occur in $\pi|_{\mathrm{O}(X)}$, and by Lemma 8.1, $\theta(\sigma)$ is supercuspidal. Now by [S] and the discussion following Theorem 4.4,

$$\theta(\{\pi^+ : \pi \in \mathrm{Irr}(\mathrm{GSO}(X)) \text{ is distinguished}\})$$

contains the set of supercuspidal representations of $\mathrm{Gl}(2,k)$. By Theorem 1.2 it follows σ is of the form π^+ for some distinguished π, a contradiction.

Suppose now that $d \neq 1$ and $\epsilon = \epsilon(d)$. Suppose first $\theta(\sigma)$ extends to a representation Π of $\mathrm{Gl}(2,k)$. Then if the notation is as in the discussion preceeding the lemma, we find that $\sigma = \theta(\Pi|_{\mathrm{Gl}(2,k)^+}) = \pi^+$.

Suppose that $\theta(\sigma$ $)$ induces irreducibly to $\mathrm{Gl}(2,k)$, and assume that σ is n form π^+ for some distinguished π. Let Π be the induction of $\theta(\sigma)$ to $\mathrm{Gl}(2,k)$. Again, there is a nonzero R' map from ω to $\Pi \otimes_{\mathbf{C}} \pi$. Let g in $\mathrm{Gl}(2,k)$ be a representative for the nontrivial coset of $\mathrm{Gl}(2,k)/\mathrm{Gl}(2,k)^+$. It follows that at least one of

$$\mathrm{Hom}_{R'}(\omega, \theta(\sigma) \otimes_{\mathbf{C}} \pi), \qquad \mathrm{Hom}_{R'}(\omega, g\theta(\sigma) \otimes_{\mathbf{C}} \pi).$$

is nonzero. If the first space is nonzero then we find as in the last paragraph that $\sigma = \pi^+$, a contradiction. It follows that the first space is zero and the second is nonzero. This implies that

$$\mathrm{Hom}_{\mathrm{Sl}(2,k)}(\omega', \theta(\sigma)) \neq 0,$$

where ω' is the extended Weil representation corresponding to the other four dimensional symmetric bilinear space of determinant d. Hence, $m_+(\theta(\sigma)), m_-(\theta(\sigma)) \leq 4$. By Conjecture 7.2 for $n = 1$, this implies that $m_+(\theta(\sigma)) = m_-(\theta(\sigma)) = 4$. It follows that Π is not a lift from a two dimensional symmetric bilinear space with determinant d. However, the restriction of Π to $\mathrm{Gl}(2,k)^+$ is reducible, and so Π is a lift from an anisotropic two dimensional symmetric bilinear space of determinant different from d. This, along with the fact that Π has a reducible restriction to $\mathrm{Gl}(2,k)^+$, implies that the restriction to $\mathrm{Sl}(2,k)$ of Π has four distinct irreducible components. By our above remarks, Π is a lift from a two dimensional symmetric bilinear space of the same determinant as X. This is a contradiction. $\square$

Lemma 7.4. *Let σ be in $\mathrm{Irr}(\mathrm{GO}(X))$. Assume that $\sigma|_{\mathrm{O}(X)} \neq \mathrm{sign}$, and in the case $d \neq 1$, assume that σ is infinite dimensional. Then σ is in $\mathcal{R}_3(\mathrm{GO}(X))$.*

Proof. By Theorem 6.3 and the principle of persistence [V], p. 67, it suffices to show that if π is in $\mathrm{Irr}(\mathrm{GSO}(X))$ is distinguished and $\pi|_{\mathrm{SO}(X)} \neq \mathbf{1}$, then π^- is in $\mathcal{R}_3(\mathrm{GO}(X))$. Let x in X^3 be such that the components of x form a basis for the orthogonal complement to Y from section 4. Then the stabilizer of x in $\mathrm{O}(X)$ is $\{1, h_1\}$, where $h_1 = -h_0$. By an argument as in the proof of Theorem 6.3, it suffices to show that there exists a nonzero vector v in the space of π such that $\pi^-(h_1)v = v$; to prove this, it suffices to show that $\pi^-(h_1) \neq -1$. To this end, suppose that $\pi^-(h_1) = -1$. Then for h in $\mathrm{GSO}(X)$ we have $\pi(h) = \pi(h_1 h h_1^{-1})$.

Suppose now $d = 1$. Let $\pi = \pi(\tau, \tau)$. Then $\tau(g) \otimes \tau(g') = \tau(g') \otimes \tau(g)$ for $g \in \mathrm{Gl}(2, k)$ or $D^\times$. But by the assumption that $\pi|_{\mathrm{SO}(X)} \neq \mathbf{1}$ it follows that the dimension of σ is larger than one. This is a contradiction.

Similarly, if $d \neq 1$, there is a contradiction. □

Theorem 7.5. *Let σ be in $\mathrm{Irr}(\mathrm{GO}(X))$. In the case $d \neq 1$ assume that σ is infinite dimensional, σ is in $\mathrm{Irr}_{\mathrm{BC\ inf}}(\mathrm{GO}(X))$, and Conjecture 7.2 for $n = 1$. Then*

$$n(\sigma) + n(\sigma \otimes_{\mathbf{C}} \mathrm{sign}) = 4.$$

Proof. Let π in $\mathrm{Irr}(\mathrm{GSO}(X))$ be a constituent of the restriction of σ to $\mathrm{GSO}(X)$. Suppose first π is regular so that $\sigma = \pi^+$. By Lemma 7.3, we have $n(\sigma) = n(\sigma \otimes_{\mathbf{C}} \mathrm{sign}) \geq 2$. By Theorem 6.3, it follows that $n(\pi^+) = n(\pi^+ \otimes_{\mathbf{C}} \mathrm{sign}) = 2$.

Suppose next that π is distinguished. Without loss of generality, we may assume that $\sigma = \pi^+$. Suppose $\pi|_{\mathrm{SO}(X)} \neq \mathbf{1}$. Then by Lemma 7.3, $n(\sigma) = 1$ and by Theorem 4.3 and Lemma 7.4, $n(\sigma \otimes_{\mathbf{C}} \mathrm{sign}) = n(\pi^-) = 3$. Suppose that $\pi|_{\mathrm{SO}(X)} = \mathbf{1}$. Then $\sigma|_{\mathrm{O}(X)} = \mathbf{1}$, and by the appendix of [Ra], $n(\sigma) = 0$ and $n(\sigma \otimes_{\mathbf{C}} \mathrm{sign}) = n(\pi^-) = 4$.

Finally, suppose that $d \neq 1$ and π is invariant but not distinguished. Then if the notation is as in Lemma 6.2, by Lemma 7.3 and Theorem 6.3, $n(\sigma) = n(\sigma \otimes_{\mathbf{C}} \mathrm{sign}) = 2$. □

The following table summarizes the results when $d = 1$.

$d = 1$, $\sigma \in \mathrm{Irr}(\mathrm{GO}(X))$		
σ	$n(\sigma)$	$n(\sigma \otimes_{\mathbf{C}} \mathrm{sign})$
$\sigma\|_{\mathrm{O}(X)} = \mathbf{1}$	0	4
$\sigma\|_{\mathrm{O}(X)} \neq \mathbf{1}$, $\sigma = \pi^+$, π invariant	1	3
$\sigma = \pi^+$, π regular	2	2
$\sigma\|_{\mathrm{O}(X)} \neq \mathrm{sign}$, $\sigma = \pi^-$, π invariant	3	1
$\sigma\|_{\mathrm{O}(X)} = \mathrm{sign}$	4	0

The next table summarizes the information when $d \neq 1$. We remind the reader that in this case we need to assume Conjecture 7.2 with $n = 1$.

$d \neq 1$, $\sigma \in \mathrm{Irr}_{\mathrm{BC\ inf}}(\mathrm{GO}(X))$ infinite dimensional		
σ	$n(\sigma)$	$n(\sigma \otimes_{\mathbb{C}} \mathrm{sign})$
$\sigma = \pi^+$, π distinguished	1	3
$\sigma\|_{\mathrm{GSO}(X)}$ invariant, not distinguished	2	2
$\sigma = \pi^+$, π regular	2	2
$\sigma = \pi^-$, π distinguished	3	1

8. Supercuspidal representations of $\mathrm{GSp}(2,k)$**.** Using the theta correspondence we have been considering, we will now construct series of supercuspidal representations of $\mathrm{GSp}(2,k)$. Since the statements of the theorems are lengthy we will not restate here.

The proof of Theorems 8.2 and 8.3 depend on the following result from [K]. In the statement we use some notation from the previous section. The reader should note that this statement also holds for other dual pairs. See [V].

Lemma 8.1 (Kudla). *Suppose that X is an even dimensional nondegenerate symmetric bilinear space. Let σ be in $\mathrm{Irr}(\mathrm{O}(X))$. If σ is supercusidal, then $\theta(\sigma)$ in $\mathrm{Irr}(\mathrm{Sp}(n(\sigma),k))$ is supercuspidal.*

Using Lemma 8.1, and the tables from the last section, it is easy now to prove Theorems 8.2 and 8.3. We make two remarks. First, in the case $d \neq 1$, every supercuspidal representation in $\mathrm{Irr}(\mathrm{GO}(X))$ is in $\mathrm{Irr}_{\mathrm{BC\ inf}}(\mathrm{GO}(X))$. To see this, suppose that σ in $\mathrm{Irr}(\mathrm{GO}(X))$ is supercuspidal and σ is in $\mathcal{R}_1(\mathrm{GO}(X))$. By Lemma 8.1 $\theta(\sigma)$ in $\mathrm{Irr}(\mathrm{Gl}(2,k)^+)$ is supercuspidal. Since the base change of any supercuspidal element of $\mathrm{Irr}(\mathrm{Gl}(2,k))$ is infinite dimensional, σ is in $\mathrm{Irr}_{\mathrm{BC\ inf}}(\mathrm{GO}(X))$. Second, we note that in the proof of Theorem 8.3 part (2) to verify that there exactly one π lying over τ is not distinguished one uses Theorem 5.3 and part (c) of Theorem 1 from [GL].

Appendix. In the following tables we illustrate the main theorem by showing the passage of a representations of $D^\times \times D^\times$, $\mathrm{Gl}(2,k) \times \mathrm{Gl}(2,k)$ and $\mathrm{Gl}(2,K)$ from these groups to $\mathrm{GSO}(X)$, $\mathrm{GO}(X)$ and finally $\mathrm{GSp}(2,k)^+$. In the case $d \neq 1$, we assume that τ is infinite dimensional. Also, passage to $\mathrm{GSp}(2,k)$ in the case $d \neq 1$ requires Conjecture 1.3 (Theta dichotomy); see section 1.

d = 1

(τ,τ')	$\longrightarrow$	$\pi(\tau,\tau')$	$\longrightarrow$	$\pi(\tau,\tau')^+$	$\longrightarrow$	$\theta(\pi(\tau,\tau')^+)$
$\mathrm{Gl}(2,k) \times \mathrm{Gl}(2,k)$ or $D^\times \times D^\times$		$\mathrm{GSO}(X)$		$\mathrm{GO}(X)$		$\mathrm{GSp}(2,k)$

1. $(\tau,\tau') \in \mathrm{Irr}_f(\mathrm{Gl}(2,k) \times \mathrm{Gl}(2,K))$ or $\mathrm{Irr}_f(D^\times \times D^\times)$, $\tau \not\cong \tau'$.

$$\begin{tikzcd} & & \pi(\tau,\tau)^+ \arrow[r] & \theta(\pi(\tau,\tau)^+) \\ (\tau,\tau) \arrow[r] & \pi(\tau,\tau) \arrow[ur] \arrow[dr] & & \\ & & \pi(\tau,\tau)^- \quad \text{does not lift} & \end{tikzcd}$$

$\mathrm{Gl}(2,k) \times \mathrm{Gl}(2,k)$ or $D^\times \times D^\times$	$\mathrm{GSO}(X)$	$\mathrm{GO}(X)$	$\mathrm{GSp}(2,k)$

2. $\tau \in \mathrm{Irr}(\mathrm{Gl}(2,k))$ or $\mathrm{Irr}(D^\times)$.

$\mathbf{d \neq 1}$

$$
\begin{tikzcd}
 & \pi(\tau,\chi) \arrow[r] & \pi(\tau,\chi)^{+} \arrow[r] & \theta(\pi(\tau,\chi)^{+}) \\
\tau \arrow[ur] \arrow[dr] & & & \\
 & \pi(\tau,\chi') \arrow[r] & \pi(\tau,\chi')^{+} \arrow[r] & \theta(\pi(\tau,\chi')^{+}) \\
\mathrm{Gl}(2,K) & \mathrm{GSO}(X) & \mathrm{GO}(X) & \mathrm{GSp}(2,k)^{+}
\end{tikzcd}
$$

1. $\tau \in \mathrm{Irr}_f(\mathrm{Gl}(2,K))$ not Galois invariant.

$$
\begin{tikzcd}
 & & \pi(\tau,\chi)^{+} \arrow[rr] & & \theta(\pi(\tau,\chi)^{+}) \\
 & \pi(\tau,\chi) \arrow[ur] \arrow[dr] & & & \\
 & & \pi(\tau,\chi)^{-} & \text{does not lift} & \\
\tau \arrow[uur] \arrow[ddr] & & & & \\
 & & \pi(\tau,\chi')^{+} \arrow[rr] & & \theta(\pi(\tau,\chi')^{+}) \\
 & \pi(\tau,\chi') \arrow[ur] \arrow[dr] & & & \\
 & & \pi(\tau,\chi')^{-} & \text{does not lift} & \\
\mathrm{Gl}(2,K) & \mathrm{GSO}(X) & \mathrm{GO}(X) & & \mathrm{GSp}(2,k)^{+}
\end{tikzcd}
$$

2. $\tau \in \mathrm{Irr}_f(\mathrm{Gl}(2,K))$ Galois invariant, χ and χ' distinguished.

$$
\begin{tikzcd}
 & & \pi(\tau,\chi)^{+} \arrow[rr] & & \theta(\pi(\tau,\chi)^{+}) \\
 & \pi(\tau,\chi) \arrow[ur] \arrow[dr] & & & \\
 & & \pi(\tau,\chi)^{-} & \text{does not lift} & \\
\tau \arrow[uur] \arrow[ddr] & & & & \\
 & & \pi(\tau,\chi')_{1} \arrow[rr] & & \theta(\pi(\tau,\chi')_{1}) \\
 & \pi(\tau,\chi') \arrow[ur] \arrow[dr] & & & \\
 & & \pi(\tau,\chi')_{2} \arrow[rr] & & \theta(\pi(\tau,\chi')_{2}) \\
\mathrm{Gl}(2,K) & \mathrm{GSO}(X) & \mathrm{GO}(X) & & \mathrm{GSp}(2,k)^{+}
\end{tikzcd}
$$

3. $\tau \in \mathrm{Irr}_f(\mathrm{Gl}(2,K))$ Galois invariant, χ distinguished, not χ' distinguished.

References

[BZ] I.N. Bernshtein and A.V. Zelevinskii, *Representations of the group* $\mathrm{Gl}(n, F)$ *where* F *is a nonarchimedean local field*, Russian Math. Surveys **31** (1976), 1–68.

[C] P. Cartier, *Representations of $\mathfrak{p}$-adic groups: a survey*, in Automorphic Forms, Representations, L-functions, Proc. Symposia Pure Math. vol. XXXIII - Part 1, American Mathematical Society, Providence, 1979.

[Ca] W. Casselman, *On the representations of* $Sl_2(k)$ *related to binary quadratic forms*, Amer. J. Math. **94** (1972), 810–834.

[Co] M. Cognet, *Representation de Weil et changement de base quadratique*, Bull. Soc. Math. France **113** (1985), 403–457.

[F] Y. Flicker, *On distinguished representations*, J. Reine Angew. Math. **418** (1991), 139–172.

[G] S.S. Gelbart, *Automorphic Forms on Adele Groups*, Annals of Mathematics Studies 83, Princeton University Press, Princeton, 1975.

[GK] S.S. Gelbart and A.W. Knapp, *L-indistinguishability and R groups for the special linear group*, Adv. in Math. **43** (1982), 101–121.

[GL] P. Géradin and J.P. Labesse, *The solution of a base change problem for* Gl(2) *(following Langlands, Saito, Shintani)*, in Automorphic Forms, Representations, L-functions, Proc. Symposia Pure Math. vol. XXXIII - Part 2, American Mathematical Society, Providence, 1979.

[H] J. Hakim, *Distinguished p-adic representations*, Duke Math. J. **66** (1991), 1–22.

[HK] M. Harris and S.S. Kudla, *The central critical value of a triple product L-function*, Annals of Math. **133** (1991), 605–672.

[HKS] M. Harris, S.S. Kudla and W.J. Sweet, *Theta dichotomy for unitary groups*, Preprint (1994).

[HLR] G. Harder, R. Langlands and M. Rapoport, *Algebraische Zyklen auf Hilbert-Blumenthal-Flachen*, J. Reine Angew. Math. **366** (1986), 53–120.

[HPS] R. Howe and I.I. Piatetski-Shapiro, *Some examples of automorphic forms on* Sp_4, Duke Math. J. **50** (1983), 55–106.

[HST] M. Harris, D. Soudry and R. Taylor, *l-adic representations associated to modular forms over imaginary quadratic fields I: lifting to* $\mathrm{GSp}_4(\mathbb{Q})$, Invent. Math. **112** (1993), 377–411.

[K] S.S. Kudla, *On the local theta correspondence*, Invent. Math. **83** (1986), 229–255.

[Kn] M.-A. Knus, *Quadratic and Hermitian Forms over Rings*, Grundlehren der mathematischen Wissenschaften 294, Springer-Verlag, Berlin-Heidelberg-New York, 1991.

[KR] S.S. Kudla and S. Rallis, *A regularized Siegel-Weil formula: the first term identity*, Annals of Math. **140** (1994), 1–80.

[MVW] C. Moeglin, M.-F. Vigneras and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Mathematics 1291, Springer-Verlag, Berlin-Heidelberg-New York, 1987.

[O] O.T. O'Meara, *Introduction to Quadratic Forms*, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[P] D. Prasad, *Trilinear forms for representations of* Gl(2) *and local* ϵ*-factors*, Compositio Math. **75**, 1–46.

[PSS] I.I. Piatetski-Shapiro and D. Soudry, *L and* ϵ *factors for* GSp(4), J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1982), 505–530.

[R] B. Roberts, *The theta correspondence for similitudes*, preprint (submitted for publication) (1994).

[Ra] S. Rallis, *On the Howe duality conjecture*, Compositio Math. **51** (1984), 333–399.

[S] H. Shimizu, *Theta series and automorphic forms on* Gl(2), J. Math. Soc. Japan **24** (1972), 638–683.

[So1] D. Soudry, *The L and* γ *factors for generic representations of* $\mathrm{GSp}(4, k) \times \mathrm{Gl}(2, k)$ *over a local nonarchimedean field* k, Duke Math. J. **51** (1984), 355–394.

[So2] D. Soudry, *A uniqueness theorem for representations of* GSO(6) *and the strong multiplicity one theorem for generic representations of* GSp(4), Israel J. of Math. **58** (1987), 257–287.

[T] J. Tunnell, *Local ε-factors and characters of* Gl(2), Amer. J. Math. **105** (1983), 1277–1307.

[V] M.F. Vignéras, *Correspondances entre representations automorphes de* Gl(2) *sur une estension quadratic de* GSp(4) *sur* $\mathbb{Q}$, *conjecture locale de Langlands pour* GSp(4), in The Selberg Trace Formula and Related Topics, American Mathematical Society, Providence, R.I., 1986.

[W] J.-L. Waldspurger, *Demonstration d'une conjecture de duality de Howe dans le case p-adiques,* $p \neq 2$, in Israel Math. Conf. Proc. vol. 2, 1990, pp. 267–324.

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