Multiple Point Seshadri Constants and the Dimension of Adjoint Linear Series

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ABSTRACT. In this note multiple point Seshadri constants measuring the positivity of ample line bundles on complex projective varieties at a finite number of points are defined. A lower bound which is asymptotically optimal for a large number of points is proven for the constant at very general points. As an application estimates on the number of sections in adjoint linear systems are deduced.

1 Introduction.

Starting with and motivated in part by the famous Fujita conjectures, there has been a lot of activity recently concerning effectivity statements for ample or adjoint line bundles on smooth complex projective varieties, most prominently Siu's effective version of the big Matsusaka Theorem and various effective numerical criteria for freeness or very ampleness of adjoint linear systems due to Demailly, Ein-Lazarsfeld, Kollár, Siu and others (cf. [De] and the references therein).

Here, less ambitiously, the concept of multiple point Seshadri constants is used to obtain "effective" Riemann-Roch type estimates on the number of sections in adjoint linear series. Namely, for a nef and big divisor L on an n-dimensional smooth projective variety Y with canonical divisor K_Y, by Riemann-Roch and vanishing the number of sections $h^0(Y, \mathcal{O}_Y(K_Y + rL))$ for $r > 0$ is a polynomial of degree n with leading coefficient $\frac{L^n}{n!}$ in r. Our "effective" version here (cf. Corollary 3.4) is, that for any $r \geq n^2 \geq 9$, the estimate

$$h^0(Y, \mathcal{O}_Y(K_Y + rL)) \geq \frac{7}{8}\frac{r^n L^n}{n^n} - 1$$

holds. This, together with some variants concerning spanned line bundles, surfaces and minimal n-folds of general type, follows from universal lower bounds for multiple point Seshadri constants at very general points which are defined as follows.

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Let L be a big and nef line bundle on an n-dimensional (irreducible) complex projective variety X, and m an integer. For pairwise distinct $x_1, \ldots, x_m \in X$ define the *multiple point Seshadri constant* at $x_1, \ldots, x_m$ by

$$\epsilon(L, x_1, \ldots, x_m) := \inf_C \frac{L.C}{\sum_i \operatorname{mult}_{x_i} C},$$

where the infimum is taken over all integral curves C with $C \cap \{x_1, \ldots, x_m\} \neq \emptyset$.

Another way of saying this is that $\epsilon(L, x_1, \ldots, x_m)$ is the maximum of all real numbers ϵ such that

$$H = f^*L - \epsilon \sum_{i=1}^{m} E_i$$

considered as an $\mathbb{R}$-divisor is nef on the the blow up $f : Bl_{\{x_1,\ldots,x_m\}}(X) \longrightarrow X$ of X along $x_1, \ldots, x_m$, where E_i denote the exceptional divisors. Since nef divisors have non-negative self-intersection, this immediately gives the upper bound

$$\epsilon(L, x_1, \ldots, x_m) \leq \frac{\sqrt[n]{L^n}}{\sqrt[n]{m}}.$$

It turns out that for $m >> 0$ and very general points this bound is asymptotically sharp.

Here by *very general* points we mean that $(x_1, \ldots, x_m)$ is outside the union of countably many proper subvarieties of $X \times \cdots \times X$, and by *general* that $(x_1, \ldots, x_m)$ is outside a Zariski closed subset.

Write for short $\epsilon(L, n, m)$ for the multiple point Seshadri constant of L at m very general points. Note that, by the open nature of ampleness, the multiple point Seshadri constants at general and very general points are related in the following way: for any $\delta > 0$ one has

$$\epsilon(L, x_1, \ldots, x_m) \geq \epsilon(L, n, m) - \delta$$

for general points $x_1, \ldots, x_m$ (cf. [EKL, Lemma 1.4] for the precise argument).

It is convenient to state our result in terms of the 1–point constant $\epsilon(L, n, 1)$ at a very general point of X. The main result of [EKL] was to establish the lower bound $\epsilon(L, n, 1) \geq 1/n$ for arbitrary X. It is conjectured that even $\epsilon(L, n, 1) \geq 1$ might be true. Here we prove with an elementary argument:

Theorem 1.1. *Let L be a nef and big line bundle on an n-dimensional complex-projective variety X and $m \geq 2$ an integer. Then*

$$\epsilon(L, n, m) \geq \min\left\{ \epsilon(L, n, 1), \frac{\sqrt[n]{L^n}}{2}, \frac{\sqrt[n]{L^n (m-1)^{n-1}}}{m} \right\}.$$

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2 Proof of Theorem 1.1.

(2.1). The first part of the proof is along the lines of [EKL]. To begin with we remark that there is no loss of generality in supposing that X is in fact smooth. To see this choose a resolution

$$f : Y \longrightarrow X$$

of singularities and consider the pullback $L' = f^*L$ instead of L. Finally note that $\epsilon(L, f(y_1), \ldots, f(y_m)) = \epsilon(L', y_1, \ldots, y_m)$ for any $y_1, \ldots, y_m$ such that f is an isomorphism near the y_i.

(2.2). Suppose the Theorem is not true. Let X^m denote the m-fold cartesian product of X minus the diagonals. Then, as in [EKL, (3.3), (3.4)], the fact that, for any real number $\beta > 0$, the set of pairs

$$\left\{(C, x) \mid C \subset X \text{ an integral curve, } x = (x_1, \ldots, x_m) \in X^m,\ \beta \cdot \sum_{i=1}^{m} \operatorname{mult}_{x_i} C > (C.L)\right\}$$

is parametrized by countably many irreducible quasi-projective varieties implies that there exist a $\delta > 0$ and a Zariski open $U \subset X^m$ such that for all $x = (x_1, \ldots, x_m) \in U$ there is an *m-exceptional* curve C_x *based at* $x_1, \ldots, x_m$, i.e. an integral curve satisfying

$$(1 - \delta) \sum_{i=1}^{m} \operatorname{mult}_{x_i} C_x > \max\left\{ \frac{1}{\epsilon(L, n, 1)}, \frac{2}{\sqrt[n]{L^n}}, \frac{m}{\sqrt[n]{L^n (m-1)^{n-1}}} \right\} (L.C_x).$$

Fix such a $\delta > 0$. It then follows that there is an irreducible variety S and an irreducible family $\mathcal{C} \subset X \times S$ of integral curves together with a dominant quasi-finite morphism $g = (g_1, \ldots, g_m) : S \longrightarrow X^m$ such that the fibre $C_s \subset X$ of $\mathcal{C}$ over $s \in S$ is an m-exceptional curve based at $g_1(s), \ldots, g_m(s)$.

Such a family $\mathcal{C}$ will be called *m-exceptional.*

(2.3). Next we claim that there exists an integer m', $2 \le m' \le m$, and an m'-exceptional family $\mathcal{C}' \subset X \times S'$ of curves whose members C'_s pass through each of the $g_i(s)$, $1 \le i \le m'$, for every $s \in S'$.

To prove this start with the m-exceptional family $\mathcal{C}$ from (2.2). First we can assume that, for sufficiently general $s \in S$, the curve C_s passes at least through two of the $g_i(s)$, since otherwise we would obtain a contradiction to the main result of [EKL] which bounds the 1–point constant.

Observe that $\mathcal{C}$ and the graphs Γ_i of the g_i are closed in $X \times S$ and therefore also $pr_S(\Gamma_i \cap \mathcal{C})$, where pr_S denotes the projection $X \times S \longrightarrow S$. Renumbering we then can assume that $pr_S(\Gamma_j \cap \mathcal{C}) = S$ for $j = 1, \ldots, m'$, in other words $g_j(s) \in C_s$ for all $s \in S$ and such j, where $2 \leq m' \leq m$ by the above. Choosing an appropriate dense open subset $S'' \subset S$ we can arrange $g_k(s) \notin C_s$ for $k = m' + 1, \ldots, m$ and all $s \in S''$.

Now pick a subvariety $S' \subset S''$ such that $g' = (g_1, \ldots, g_{m'}) : S' \longrightarrow X^{m'}$ is quasi-finite. It remains to show that all curves of the resulting family $\mathcal{C}' = \mathcal{C}|_{X \times S'}$ are indeed m'-exceptional. But this follows from the elementary observation that

$$\frac{(m'-1)^{n-1}}{(m')^n} \geq \min\left\{\frac{1}{2^n}, \frac{(m-1)^{n-1}}{m^n}\right\}$$

for all postitive integers n and all $2 \leq m' \leq m$. The latter can be shown by minimizing the real function $f(a) = \frac{(a-1)^{n-1}}{a^n}$ in the real intervall $[2, m]$.

(2.4). We proceed by proving that m-exceptional families $\mathcal{C}$ with the property of (2.3) do not exist. The idea is to find an m-exceptional curve of our family which intersects a given divisor in X having high multiplicity at $m-1$ points of the curve properly, and use the fact that in this case the products of the multiplicities at the points of intersection give a lower bound for the local intersection numbers.

To this end consider for $i = 1, \ldots, m$ the functions

$$s \mapsto \text{mult}_{g_i(s)} C_s.$$

We claim that these have constant values r_i on open dense subsets of S. This can be proven e.g. by using the relative Samuel stratification (cf. [LeTe, Theorem (4.15)]) of the morphism $\mathcal{C} \longrightarrow S$, which gives a finite partition of $\mathcal{C}$ into locally closed subsets on which the relative Hilbert–Samuel functions are constant. Restricting this stratification to the graphs $\Gamma_i \subset \mathcal{C}$ then gives a partition of Γ_i into locally closed subsets on which the Hilbert–Samuel functions of the local rings $\mathcal{O}_{C_s, g_i(s)}$ are constant, and this implies that the multiplicities are also constant (cf. [Fu, §4.3]). The arguments in [LeTe, §4] in fact show that $s \mapsto \text{mult}_{g_i(s)} C_s$ are Zariski upper-semicontinuous in S.

Therefore we can pick a general $s' \in S$ such that $r_i = \text{mult}_{g_i(s')} C_{s'} \leq \text{mult}_{g_i(s)} C_s$ for all i and $s \in S$. Assume that $r_1 \leq r_2 \leq \cdots \leq r_m$.

(2.5). Recall that it is $\frac{p^n}{n!} + o(p^n)$ conditions to impose multiplicity at leat p at a given point, and that different points impose independent conditions; hence the Theorem of Riemann-Roch shows that, for $k >> 0$, there exists a divisor $D' \in |kL|$ having multiplicity at least

$$(1-\delta)\frac{k\sqrt[n]{L^n}}{\sqrt[n]{m-1}}$$

at $g_2(s'), \ldots, g_m(s')$.

(2.6). Finally put $T := g^{-1}(X, g_2(s'), \ldots, g_m(s'))$. Since g is quasi-finite and the g_i are dominant, T is of dimension $\dim(g_1(T)) = \dim(X)$. Therefore the subfamily $(C_t)_{t \in T}$ constists of m-exceptional curves having at least multiplicity r_i at $g_i(t) = g_i(s')$ for $i = 2, \ldots, m$, and their first base points $g_1(t), t \in T$ are dense in X. In particular, we can find a $t \in T$ such that $C_t \not\subseteq D'$. But this gives a contradiction because of

$$k(L.C_t) = D'.C_t \geq (1-\delta)\frac{k\sqrt[n]{L^n}}{\sqrt[n]{m-1}}\sum_{i=2}^{m} r_i \geq (1-\delta)\frac{(m-1)k\sqrt[n]{L^n}}{m\sqrt[n]{m-1}}\sum_{i=1}^{m}\mathrm{mult}_{g_i(t)}C_t$$

which proves the Theorem.

3 Applications.

We start with a supplementary result concerning surfaces. In case X is a smooth projective surface Ein and Lazarsfeld proved that $\epsilon(A, x) \geq 1$ for an ample line bundle A off a countable subset of X (cf. [EL]). From this one obtains easily:

Lemma 3.1. *$\epsilon(L, 2, m) \geq 1$ if and only if $L^2 \geq m$.*

Proof. (Ein-Lazarsfeld) Since we are only interested in very general points the arguments given in [EL] in fact show that $\epsilon(L, 2, 1) \geq 1$ holds for any nef and big line bundle L on a projective (possibly singular) surface. Therefore, after blowing up X along a very general point via the map $f_x : X' \longrightarrow X$ with exceptional divisor E_x the line bundle $L_x = f_x^* L - E_x$ is again nef with $L_x^2 = L^2 - 1$. Iterating this procedure gives the desired result. □

One application of the concept of multiple point Seshadri constants is to provide a fairly good estimate on the number of sections of adjoint linear series.

Proposition 3.2. *Let L be a nef and big line bundle on a smooth projective variety X of dimension $n \geq 2$ and $m \geq 2$ an integer. Then*

$$h^0(X, \mathcal{O}_X(K_X + rL)) \geq m$$

whenever

$$r \geq n \cdot \max\left\{\frac{1}{\epsilon(L,n,1)}, \frac{2}{\sqrt[n]{L^n}}, \frac{m}{\sqrt[n]{L^n(m-1)^{n-1}}}\right\}.$$

Proof. Let $f : Y \longrightarrow X$ be the blowing up of X along m very general points $x_1, \ldots, x_m$ with exceptional divisors $E_1, \ldots, E_m$. Since

$$H = f^* L - \epsilon(L, n, m)\sum_{i=1}^{m} E_i$$

is nef and big if $\epsilon(L,n,m)^n < \frac{L^n}{m}$, the same holds for

$$\left(r - \frac{n}{\epsilon(L,n,m)}\right) f^*L + \frac{n}{\epsilon(L,n,m)}\left(f^*L - \epsilon(L,n,m)\sum_{i=1}^{m} E_i\right)$$

if $r \geq \frac{n}{\epsilon(L,n,m)}$. Then Kawamata–Viehweg vanishing gives

$$H^1(X, \mathcal{O}_X(K_X + rL) \otimes \mathcal{I}_{x_1} \otimes \cdots \otimes \mathcal{I}_{x_m}) = H^1(Y, \mathcal{O}_Y(K_Y + rf^*L - n\sum_{i=1}^{m} E_i)) = 0$$

if $r > \frac{n}{\epsilon(L,n,m)}$, or $r = \frac{n}{\epsilon(L,n,m)}$ and $\epsilon(L,n,m)^n < \frac{L^n}{m}$. In other words the linear series $|K_X + rL|$ separates the points $x_1, \ldots, x_m$ under these conditions which are implied by Theorem 1.1 and our assumption. In particular one obtains $h^0(X, \mathcal{O}_X(K_X + rL)) \geq m$. $\square$

Remark 3.3. Suppose L is a big line bundle on X which is spanned. Since L is spanned, the complete linear series $|L|$ induces a morphism $\varphi : X \longrightarrow Z \subset \mathbb{P}^N$ which is generically finite because L is big. Therefore any curve through a sufficiently general point is mapped by φ onto a curve in Z. Fixing such a pair $x \in C$, we can choose a hyperplane section $H \subset Z$ through $\varphi(x)$ meeting $\varphi(C)$ properly, and this gives rise to $x \in D = \varphi^*H \in |L|$ meeting C properly. Therefore $L.C \geq \text{mult}_x D \cdot \text{mult}_x C \geq \text{mult}_x C$, and this shows $\epsilon(L,n,1) \geq 1$ for spanned and big L.

Corollary 3.4. *Let X be a smooth complex projective variety of dimension $n \geq 2$.*

(1) *Suppose L is nef, $n \geq 3$ and $r \geq n^2$. Then*

$$h^0(X, \mathcal{O}_X(K_X + rL)) \geq \frac{7}{8}\frac{r^n L^n}{n^n} - 1.$$

(2) *Suppose L is spanned and $L^n \geq \left(\frac{2n}{n+1}\right)^n$. Then*

$$h^0(X, \mathcal{O}_X(K_X + (n+1)L)) \geq L^n + 1.$$

(3) *Suppose L is nef and X is a surface. Then*

$$h^0(X, \mathcal{O}_X(K_X + 2L)) \geq L^2 - 1.$$

Proof. (1) This follows directly from Proposition 3.2 and the bound

$$\epsilon(L,n,1) \leq 1/n \leq \frac{\sqrt[n]{L^n}}{2}$$

plus some elementary estimates. One has $n^n \geq (8n)^n(8n-1)^{1-n}$ for $n \geq 3$, and therefore a first estimate $h^0(X, \mathcal{O}_X(K_X + n^2L)) \geq 8n$. Then one proves

$$\frac{m^n}{(m-1)^{n-1}} \leq \Big(1 + \frac{1}{c-1}\Big)m$$

if $m \geq cn$ with a $c \geq 8$ by expanding $(\frac{m}{m-1})^{n-1}$. Now given $r \geq n^2$ we can determine the wanted m satisfying $\frac{r^nL^n}{n^n} \geq \frac{m^n}{(m-1)^{n-1}}$ as follows. We already know that $m \geq 8n$, so we can choose m to be the largest integer satisfying

$$\frac{7}{8}\frac{r^nL^n}{n^n} \geq m.$$

(2) This follows from Proposition 3.2 and Remark 3.3.

(3) This follows in the same spirit from Lemma 3.1. □

Remark 3.5. Let L be big and nef and X be smooth. According to Corollary 3.4 the multiple point approach improves the estimate

$$h^0(X, \mathcal{O}_X(K_X + n(n+s)L)) \geq \binom{n+s}{n}$$

for the number of sections in adjoint linear series which was obtained in [EKL] using the generation of s-jets at one very general point. However, for L^n and r in a certain range it is possible to obtain better lower bounds for $h^0(X, \mathcal{O}_X(K_X + rL))$ by interpolating between the two methods. For example, if

$$r \geq (n+k) \cdot \max\left\{ \frac{1}{\epsilon(L,n,1)}, \frac{2}{\sqrt[n]{L^n}}, \frac{m}{\sqrt[n]{L^n(m-1)^{n-1}}} \right\},$$

then as above one proves that $|K_X + rL|$ generates k-jets at m very general points, in particular

$$h^0(X, \mathcal{O}_X(K_X + rL)) \geq m \cdot \binom{n+k}{n}$$

for such r.

Finally we state a variant of Corollary 3.4 which gives estimates of the dimension of some linear series on varieties with mild singularities.

Proposition 3.6. *Let X be a n–dimensional normal projective variety with at most log-terminal singularities, and r a positive integer such that rK_X is a nef and big Cartier–divisor (which is the case e.g. if X is a minimal n-fold of general type and index r). Then, for all $q > n^2$,*

$$h^0(X, \mathcal{O}_X(qrK_X)) \geq \frac{7}{8}\frac{(q-1)^n}{n^n}(rK_X)^n - 1.$$

Scetch of proof. (cf. also [EKL, (4.6)]) Let

$$f : Y \longrightarrow X$$

be a resolution of singularities of X such that

$$K_Y + \Delta \equiv f^* K_X + P$$

with a fractional divisor Δ supported on a divisor with normal crossings and P an integral effective f–exceptional divisor.

Now $K_Y + \Delta + (qr-1)f^*K_X$ is numerically equivalent to an integral divisor, and since

$$\epsilon((qr-1)f^*K_X, n, m) = \frac{qr-1}{r}\epsilon(f^*(rK_X), n, m),$$

a variant of Proposition 3.2 using vanishing for $\mathbb{Q}$–divisors implies that

$$h^0(Y, \mathcal{O}_Y(K_Y+\Delta+(qr-1)f^*K_X)) = h^0(Y, \mathcal{O}_Y(qrf^*K_X+P)) = h^0(X, \mathcal{O}_X(qrK_X)) \geq m$$

whenever

$$qr - 1 \geq rn \cdot \max\left\{ \frac{1}{\epsilon(rK_X, n, 1)}, \frac{2}{\sqrt[n]{(rK_X)^n}}, \frac{m}{\sqrt[n]{(rK_X)^n (m-1)^{n-1}}} \right\}.$$

Then the claim follows as in the proof of Corollary 3.4. $\square$

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