ASYMPTOTIC METRICS FOR $SU(N)$-MONOPOLES WITH MAXIMAL SYMMETRY BREAKING

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ABSTRACT. We compute the asymptotic metrics for moduli spaces of $SU(N)$ monopoles with maximal symmetry breaking. These metrics are exponentially close to the exact monopole metric [1] as soon as, for each simple root, the individual monopoles corresponding to that root are well separated.

1. INTRODUCTION

It has been known since the work of Taubes [30] that an $SU(2)$-monopole of charge m with well-separated zeros of the Higgs field approximates a collection of m monopoles of charge 1. This fact is reflected in the asymptotic behaviour of the natural hyperkähler metric on the moduli space M_m of charge m monopoles. Namely, in the asymptotic region of M_m, the monopole metric is exponentially close to another hyperkähler metric, whose geodesics determine scattering of m particles with electric, magnetic and scalar charges. This metric was found by Gibbons and Manton in [13] and a proof that it differs from the exact monopole metric by an exponentially small amount was given in [2]. For $m = 2$, the Gibbons-Manton metric is just the product of a flat metric and the Taub-NUT metric with a negative mass parameter.

Monopoles exist for any compact Lie group and Taubes' estimates work equally well for $SU(N)$-monopoles with maximal symmetry breaking, that is monopoles whose Higgs field has distinct eigenvalues at infinity. This time a moduli space $M_{m_1,\dots,m_{N-1}}(\mu_1,\dots,\mu_N)$, where m_i are positive integers and $\mu_1 < \cdots < \mu_N$, is obtained by identifying gauge-equivalent monopoles whose Higgs field at infinity defines a map from the 2-sphere to the adjoint orbit O of $\mathrm{diag}(i\mu_1,\dots,i\mu_N)$ and whose degree is $(m_1,\dots,m_{N-1}) \in H_2(O,\mathbb{Z})$. We should think of particles making up the monopole as coming in $N-1$ distinguishable types, with m_i being the number of particles of type i.

In this paper we shall compute the asymptotic metric on $M_{m_1,\dots,m_{N-1}}(\mu_1,\dots,\mu_N)$ [1]. This metric turns out to be a hybrid between the Gibbons-Manton metric and the metric on the moduli space of monopoles of charge $(1,1,\dots,1)$ which was computed in [25, 26, 9]. Particles of the same type interact as in the Gibbons-Manton metric while the particles of different types as in the $(1,1,\dots,1)$-metric (i.e. neighbouring types interact as in the Taub-NUT metric with a positive mass parameter and non-neighbouring types do not interact). The precise formula for the asymptotic metric is given in the next section. Somewhat surprisingly, the monopole metric on $M_{m_1,\dots,m_{N-1}}(\mu_1,\dots,\mu_N)$ is exponentially close to the asymptotic metric as soon

[1] Strictly speaking, we only show that our asymptotic metric is close to the metric on the moduli space of solutions to Nahm's equations. Thus we can conclude that the geodesics of the monopole metric and of the asymptotic metric are close to each other (see Remark 3.5). The metrics themselves are close if the Nahm transform for $SU(N)$-monopoles, $N > 2$, is an isometry.

as, for each i, particles of type i are far apart. Particles of different types can be as close to each other as we wish (in fact sometimes they can have the same position). Moreover, as we observe in section 9, if particles of a single type, say i, are far apart, then the monopole metric is exponentially close to yet another hyperkähler metric. This metric is simpler than the monopole metric, but still given by transcendental functions. It is only when the particles of each type are far apart, that the asymptotic metric becomes algebraic and we are able to compute it explicitly.

The proof uses the idea of our previous work [2], i.e. replacing solutions to Nahm's equations corresponding to monopoles with solutions defined on half-line. This gives a new moduli space whose metric is then computed by twistor methods.

The paper is organized as follows. In the next section we collect some facts about T^m-invariant hyperkähler metrics in dimension $4m$ of which our asymptotic metric is an example. In section 3 we recall the description, due to Nahm [27] and to Hurtubise and Murray [20], of $SU(N)$-monopoles in terms of Nahm's equations. We also define there the moduli space of solutions to Nahm's equations whose metric will be the asymptotic metric. This moduli space is a hyperkähler quotient of the product of several simpler moduli spaces, and in the following four sections we compute the metrics on these. In section 8 we discuss the topology of these moduli spaces. Finally, in section 9, we put the results together to obtain an explicit formula for the asymptotic metric (Theorem 9.2). We also prove there that the rate of approximation is exponential (Theorem 9.1) and discuss the topology of the asymptotic moduli space. The appendix deals with the question of identifying certain hyperkähler quotients with corresponding complex-symplectic quotients which is needed in section 8.

2. Hyperkähler quotients and T^n-invariant hyperkähler metrics

The Gibbons-Manton metric [13, 14] is an example of a $4m$-dimensional (pseudo)-hyperkähler metric admitting a tri-Hamiltonian (hence isometric) action of the n-dimensional torus T^m. Such metrics have particularly nice properties and were studied by several authors [24, 18, 29]. On the set where the action of T^m is free such a metric can be locally written in the form:

$$g = \Phi d\overline{\mathbf{x}} \cdot d\overline{\mathbf{x}} + \Phi^{-1}(d\overline{t} + A)^2, \tag{2.1}$$

where $\overline{\mathbf{x}}$ is the hyperkähler moment map, $d\overline{t}$ is dual to the $(m \times 1)$-matrix of Killing vector fields and the matrix Φ and the 1-form A depend only on the $\mathbf{x}_i$ and satisfy certain linear PDE's. In particular, Φ determines the metric up to a gauge equivalence. The set where the T^m-action is free can be viewed as a T^m-bundle over an open subset of $\mathbb{R}^{3m}$. For the Gibbons-Manton metric this open subset is the configuration space of $\tilde{C}_m(\mathbb{R}^3)$ of m distinct points in $\mathbb{R}^3$ (i.e. $\mathbb{R}^{3m}$ without the generalized diagonal) and

$$\Phi_{ij} = \begin{cases} -\frac{1}{\nu} - \sum_{k\neq i} \frac{1}{\|\mathbf{x}_i - \mathbf{x}_k\|} & \text{if } i = j \\ \frac{1}{\|\mathbf{x}_i - \mathbf{x}_j\|} & \text{if } i \neq j. \end{cases} \tag{2.2}$$

Here ν is the mass parameter. We can, in particular, take $\nu = \infty$ and $m = 2$. Then the linearity of the equations for Φ and A implies that, for any mapping $(i,j) \mapsto s_{ij}$ of $\{1,\dots,m\} \times \{1,\dots,m\}$ such that $s_{ij} = s_{ji}$ and $s_{ii} = 0$ for $i,j = 1,\dots,m$, and for any constants c_i, $i = 1,\dots,m$, the following matrix Φ defines a T^m-invariant (pseudo)-hyperkähler metric:

$$\Phi_{ij} = \begin{cases} c_i + \sum_{k\neq i} \frac{s_{ik}}{\|\mathbf{x}_i - \mathbf{x}_k\|} & \text{if } i = j \\ -\frac{s_{ij}}{\|\mathbf{x}_i - \mathbf{x}_j\|} & \text{if } i \neq j. \end{cases} \tag{2.3}$$

The asymptotic metric on $M_{m_1,\dots,m_{N-1}}(\mu_1,\dots,\mu_N)$ will turn out to be of this form. Namely $m = m_1 + \cdots + m_{N-1}$ and, if we define the type $t(i)$ of an $i \leq m$ by $t(i) = \min\{k; i \leq \sum_{s\leq k} m_s\}$, then

$$c_i = \mu_{k+1} - \mu_k \ \text{ if } t(i) = k \tag{2.4}$$

and

$$s_{ij} = \begin{cases} -2 & \text{if } t(i) = t(j) \\ 1 & \text{if } |t(i) - t(j)| = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

3. Moduli spaces of solutions to Nahm's equations

We shall define in this section several moduli spaces of solutions to Nahm's equations. All of these spaces carry hyperkähler metrics. In particular, we shall recall, after Nahm [27] and Hurtubise and Murray [20], the description of the moduli spaces of $SU(N)$ monopoles with maximal symmetry breaking in terms of solutions to Nahm's equations. We shall also describe the asymptotic moduli spaces. We remark that from the point of view of hyperkähler geometry many interesting metrics are obtained by replacing below the unitary group with an arbitrary compact Lie group. Such a generalization is straightforward, but, as our focus is on monopoles, we shall restrict ourselves to the unitary case.

Nahm's equations are the following ODE's:

$$\dot{T}_i + [T_0, T_i] + \frac{1}{2} \sum_{j,k=1,2,3} \epsilon_{ijk}[T_j, T_k] = 0\,, \quad i = 1,2,3. \tag{3.1}$$

The functions T_0, T_1, T_2, T_3 are defined on some interval and are skew-hermitian and analytic. If the rank of the T_i is n, then the space of solutions is acted upon by the gauge group $\mathcal{G}$ of $U(n)$-valued functions $g(t)$:

$$\begin{aligned} T_0 &\mapsto \operatorname{Ad}(g)T_0 - \dot{g}g^{-1} \\ T_i &\mapsto \operatorname{Ad}(g)T_i\,, \qquad i = 1,2,3. \end{aligned} \tag{3.2}$$

We define first fundamental moduli spaces of $\mathfrak{u}(n)$-valued solutions $F_n(m;c)$ and $\tilde{F}_n(m;c)$. Here m is a nonnegative integer less than or equal to n and c is a positive real number (c can be negative or zero for $\tilde{F}_n(m;c)$). The moduli spaces $F_n(m;c)$ correspond to monopoles with minimal symmetry breaking and are the basic building blocks from which all moduli spaces of framed $SU(N)$-monopoles with maximal symmetry breaking can be obtained by means of the hyperkähler quotient construction. The spaces $\tilde{F}_n(m;c)$ play a similar role for the asymptotic metrics. They are defined as follows:

- Solutions in $F_n(m;c)$ are defined on $(0,c]$, while solutions in $\tilde{F}_n(m;c)$ are defined on $(0,\infty]$.

- For a solution (T_0,T_1,T_2,T_3) in either $F_n(m;c)$ or $\tilde{F}_n(m;c)$, T_0 and the $m\times m$ upper-diagonal blocks of T_1,T_2,T_3 are analytic at $t=0$, while the $(n-m)\times(n-m)$ lower-diagonal blocks have simple poles with residues defining the standard $(n-m)$-dimensional irreducible representation of $\mathfrak{su}(2)$. The off-diagonal blocks are of the form $t^{(n-m-1)/2}\times(\textit{analytic in } t)$.
- A solution in $F_n(m;c)$ is analytic at $t=c$, while a solution in $\tilde{F}_n(m;c)$ approaches a diagonal limit at $+\infty$ exponentially fast. Furthermore $(T_1(+\infty),T_2(+\infty),T_3(+\infty))$ is a regular triple, i.e. its centralizer consists of diagonal matrices.
- The gauge group for $F_n(m;c)$ consists of gauge transformations g with $g(0)=g(c)=1$, while the gauge group for $\tilde{F}_n(m;c)$ has the Lie algebra consisting of functions $\rho:[0,+\infty)\to\mathfrak{u}(n)$ such that
  (i) $\rho(0)=0$ and $\dot\rho$ has a diagonal limit at $+\infty$;
  (ii) $(\dot\rho-\dot\rho(+\infty))$ and $[\tau,\rho]$ decay exponentially fast for any regular diagonal matrix $\tau\in\mathfrak{u}(n)$;
  (iii) $c\dot\rho(+\infty)+\lim_{t\to+\infty}(\rho(t)-t\dot\rho(+\infty))=0$.

Remark 3.1. Alternately, the space $\tilde{F}_n(m;c)$ can be viewed as the moduli space of solutions defined on $[-c,+\infty]$ with the gauge group given by the transformation which are exponentially close to $\exp(ht)$ for some diagonal h.

The tangent space at a solution (T_0,T_1,T_2,T_3) can be identified, for both $F_n(m;c)$ and $\tilde{F}_n(m;c)$, with the space of solutions to the following system of linear equations:

$$\begin{aligned}
&\dot t_0+[T_0,t_0]+[T_1,t_1]+[T_2,t_2]+[T_3,t_3]=0,\\
&\dot t_1+[T_0,t_1]-[T_1,t_0]+[T_2,t_3]-[T_3,t_2]=0,\\
&\dot t_2+[T_0,t_2]-[T_1,t_3]-[T_2,t_0]+[T_3,t_1]=0,\\
&\dot t_3+[T_0,t_3]+[T_1,t_2]-[T_2,t_1]-[T_3,t_0]=0.
\end{aligned}\tag{3.3}$$

$F_n(m;c)$ carries a hyperkähler metric is defined by

$$\|(t_0,t_1,t_2,t_3)\|^2=\int_0^c\sum_0^3\|t_i\|^2,\tag{3.4}$$

while $\tilde{F}_n(m;c)$ possesses an indefinite (and possibly degenerate) hyperkähler metric given by:

$$\|(t_0,t_1,t_2,t_3)\|^2=c\sum_0^3\|t_i(+\infty)\|^2+\int_0^{+\infty}\sum_0^3\left(\|t_i(s)\|^2-\|t_i(+\infty)\|^2\right)ds.\tag{3.5}$$

The moduli space $F_n(m;c)$ has a tri-Hamiltonian action of $U(n)\times U(m)$ given by gauge transformations g with arbitrary values at $t=c$ and with $g(0)$ being block-diagonal with the off-diagonal blocks equal to 0 and the $(n-m)\times(n-m)$ lower-diagonal block being identity. Both $U(n)$ and $U(m)$ act freely. The hyperkähler moment map for the action of $U(n)$ is $(-T_1(c),-T_2(c),-T_3(c))$, while the one for the action of $U(m)$ is $\pi(T_1(0),T_2(0),T_3(0))$, where π denotes the projection onto the $m\times m$ upper-diagonal block.

The moduli space $\tilde{F}_n(m;c)$ has a similarily defined free tri-Hamiltonian action of $U(m)$. In addition, it has a free tri-Hamiltonian action of the diagonal torus $T^n\leq U(n)$ given by gauge transformations which are asymptotic to $\exp(-th+\lambda h)$ for a diagonal h and real λ. The moment map for this action is $(T_1(+\infty),T_2(+\infty),T_3(+\infty))$.

We shall now consider hyperkähler quotients of various products of $F_n(m;c)$ and $\tilde{F}_n(m;c)$. We observe that the hyperkähler quotient construction of say, $F_n(m;c)\times F_n(l;c')$ matches solutions $(T_0(t),T_1(t),T_2(t),T_3(t))$ in $F_n(m;c)$ with $(-T_0(c+c'-t),-T_1(c+c'-t),-T_2(c+c'-t),-T_3(c+c'-t))$ for a $(T_0(t),T_1(t),T_2(t),T_3(t))$ in $\tilde{F}_n(m;c)$. The resulting space can be identified with the moduli space of solutions to Nahm's equations on $[0,c+c']$ having appropriate poles at $t=0$ and at $t=c+c'$. We recall that the triple backslash denotes hyperkähler quotient (in all our constructions the moment map is canonical and we quotient its 0-set). We have basic hyperkähler isomorphisms:

$$\begin{aligned} \big(F_n(m;c)\times F_n(n;c')\big)///U(n) &\simeq F_n(m;c+c') \\ \big(F_n(m;c)\times \tilde{F}_n(n;c')\big)///U(n) &\simeq \tilde{F}_n(m;c+c'). \end{aligned} \tag{3.6}$$

The group acts diagonally on the product. In particular all $\tilde{F}_n(m;c)$ can be obtained from the $\tilde{F}_n(n;c)$ and the $F_n(m;c)$.

We now define auxilliary moduli spaces $F_{n,m}(c,c')$, $F_{\tilde{n},m}(c,c')$, $F_{n,\tilde{m}}(c,c')$, $F_{\tilde{n},\tilde{m}}(c,c')$. Here n,m are arbitrary positive integers and c,c' are arbitrary positive real numbers. The spaces are defined as follows:

- if $n<m$, then $F_{n,m}(c,c')=\big(F_n(n;c)\times F_m(n;c')\big)///U(n)$. The spaces with a tilde over n or m are obtained by replacing the corresponding F with $\tilde{F}$;
- if $n>m$, then $F_{n,m}(c,c')=\big(F_n(m;c)\times F_m(m;c')\big)///U(m)$ and similarly for the other spaces;
- if $n=m$, then $F_{n,n}(c,c')$ is the hyperkähler quotient of $F_n(n;c)\times F_n(n;c')\times \mathbb{H}^n$ by the diagonal action of $U(n)$.

Remark 3.2. Thus these moduli spaces consist of $\mathfrak{u}(\mathfrak{m})$-valued solutions T_i^- on $[-x,0)$ and of $\mathfrak{u}(\mathfrak{n})$-valued solutions T_i^+ on $(0,y]$, where $x=c'$ or $-\infty$ and $y=c$ or $y=+\infty$, with matching conditions at $t=0$: if $n>m$ (resp. $n<m$), then the limit of the $m\times m$ upper-diagonal block of T_i^+ (resp. T_i^-) at $t=0$ is equal to the limit of T_i^- (resp. T_i^+); if $n=m$, then there exists a vector $(V,W)\in\mathbb{C}^{2n}$ such that $(T_2^++iT_3^+)(0_+)-(T_2^-+iT_3^-)(0_-)=VW^T$ and $T_1^+(0_+)-T_1^-(0_-)=(|V|^2-|W|^2)/2$. The gauge transformations $g(t)$ satisfy similar matching conditions: if $n\neq m$, then the upper-diagonal $m\times m$ block is continuous, the lower-diagonal block is identity at $t=0$ and the off-diagonal blocks vanish to order $(n-m-1)/2$; if $n=m$, then $g(t)$ is continuous at $t=0$.

Notice that $F_{n,m}(c,c')$ is isomorphic, as a hyperkähler manifold, to $F_{m,n}(c',c)$ and similarly for $F_{\tilde{n},\tilde{m}}(c,c')$. We can now define the moduli spaces we are really interested in. Let us fix an integer N and consider functions $\sigma:\{1,\dots,N-1\}\to\mathbb{N}\sqcup\mathbb{N}$ is arbitrary and $\mu:\{1,\dots,N\}\to\mathbb{R}$ is increasing. We shall denote the second copy of $\mathbb{N}$ by $\tilde{\mathbb{N}}$ and write its elements as $\tilde{1},\tilde{2},\tilde{3},\dots$. We define the moduli space $F_\sigma(\mu)$ as a hyperkähler quotient of $F_{\sigma_1}(c_1)\times F_{\sigma_2}(c_2,c_2')\times\ldots\times F_{\sigma_{N-1}}(c_{N-1},c_{N-1}')\times F_{\sigma_N}(c_N')$. Here $c_i+c_{i+1}'=\mu(i+1)-\mu(i)$, $\sigma_1,\sigma_N\in\mathbb{N}\sqcup\mathbb{N}$ and $\sigma_i:\{1,2\}\to\mathbb{N}\sqcup\mathbb{N}$ for $2\le i\le N-1$. Furthermore $\sigma_1=\sigma(1)$, $\sigma_i(1)=\sigma(i)$, $\sigma_i(2)=\sigma(i-1)$ for $2\le i\le N-1$, and $\sigma_N=\sigma(N-1)$. Finally $F_n(c),F_{\tilde{n}}(c)$ denote $F_n(0;c)$ and $\tilde{F}_n(0;c)$. The group by which we quotient is a product of unitary groups and of tori acting on this product space: we take the diagonal action of $U(n)$ on $F_{\sigma_i}(c_i,c_i')\times F_{\sigma_{i+1}}(c_{i+1},c_{i+1}')$ (or $F_{\sigma_1}(c_1)\times F_{\sigma_2}(c_2,c_2')$ for $i=1$ and similarily for $i=N-1$) if $\sigma(i)=n$ and the diagonal action of T^n if $\sigma(i)=\tilde{n}$.

Remark 3.3. The moduli space $F_\sigma(\mu)$ should be viewed as consisting of solutions to Nahm's equations on $N-1$ "intervals" I_i, $i=1,\dots,N_1$ with matching conditions at the boundary points. If $\sigma(i)\in\mathbb{N}$, then $I_i=[\mu(i),\mu(i+1)]$, while if $\sigma(i)\in\tilde{\mathbb{N}}$, then $I_i=[\mu(i),+\infty)\cup(-\infty,\mu(i+1)]$ (see Remark 3.1). The solutions satisfy matching conditions of Remark 3.2 at each μ_i and are continuous at each infinity. The gauge transformations matching conditions of Remark 3.2 at each μ_i and are exponentially close to $\exp(h_it+p_i)$ near $\pm\infty$ in I_i, $\sigma(i)\in\tilde{\mathbb{N}}$, for some diagonal matrices h_i,p_i.

A theorem of Hurtubise and Murray [20], giving a full proof of the correspondence found by Nahm [27], and generalizing the $SU(2)$ case due to Hitchin [17] can be phrased as follows (this formulation uses the connectivity of the moduli space of $SU(N)$-monopoles due to Jarvis [21]):

Theorem 3.4. *The moduli space $M_{m_1,\dots,m_{N-1}}(\mu_1,\dots,\mu_N)$ of framed $SU(N)$ monopoles of charge $(m_1,\dots,m_{N-1})$ and the symmetry breaking at infinity equal to $(\mu_1,\dots,\mu_N)$, μ_i distinct, is diffeomorphic to the moduli space $F_\sigma(\mu)$ with $\sigma(i)=m_i$ and $\mu(i)=\mu_i$.* □

Remark 3.5. It is expected, but at present not known (except for $N=2$ [28]), that this diffeomorphism is an isometry. Nevertheless, the twistorial character of Hurtubise and Murray's construction shows that this diffeomorphism preserves the three complex structures and, hence, the Levi-Civita connection. Thus, the geodesics are the same.

Our aim is to show that the metric on $F_\sigma(\mu)$ with $\sigma(i)=m_i$ and $\mu(i)=\mu_i$ is asymptotic to the metric on $F_{\tilde\sigma}(\mu)$ with $\tilde\sigma(i)=\tilde m_i$ and $\mu(i)=\mu_i$. We shall first compute the metric on $F_{\tilde\sigma}(\mu)$.

4. Complex structures on $F_n(m;c)$ and $\tilde F_n(n;c)$

All moduli spaces described in the previous section have an isometric action of $SU(2)$ or $SO(3)$ rotating the complex structures and therefore all complex structures are equivalent. We shall consider the complex structure I and describe the complex coordinates (and the complex symplectic form $\omega_2+i\omega_3$) on $F_n(m;c)$ and $\tilde F_n(n;c)$. All other moduli spaces can be described as open subsets of complex-symplectic quotients of products of these.
We set $\alpha=T_0+iT_1$ and $\beta=T_2+iT_3$. The Nahm equations can be then written as one complex and one real equation:

$$\frac{d\beta}{dt}=[\beta,\alpha] \tag{4.1}$$

$$\frac{d}{dt}(\alpha+\alpha^*)=[\alpha^*,\alpha]+[\beta^*,\beta]. \tag{4.2}$$

First, we consider $F_n(m;c)$ (cf. [19, 2]).
Let $E_1,\dots,E_n$ denote the standard basis of $\mathbb{C}^n$. There is a unique solution w_1 of the equation

$$\frac{dw}{dt}=-\alpha w \tag{4.3}$$

with

$$\lim_{t\to0}\left(t^{-(n-m-1)/2}w_1(t)-E_{m+1}\right)=0. \tag{4.4}$$

Setting $w_i(t) = \beta^{i-1}(t)w_1(t)$, we obtain a solution to (4.3) with

$$\lim_{t\to 0}\left(t^{i-(n-m+1)/2}w_i(t) - E_{m+i}\right) = 0.$$

In addition there are solutions $u_1, \ldots, u_m$ to (4.3) whose last $n-m$ components vanish to order $(n-m+1)/2$, and which are linearly independent at $t=0$. The complex gauge transformation $g(t)$ with $g^{-1} = (u_1, \ldots, u_m, w_1, \ldots, w_{n-m})$ makes α identically zero and sends $\beta(t)$ to the constant matrix (cf. [19])

$$B = \left(\begin{array}{ccc|cccc} & & & 0 & \ldots & 0 & g_1 \\ & h & & \vdots & & \vdots & \vdots \\ & & & 0 & \ldots & 0 & g_m \\ \hline f_1 & \ldots & f_m & 0 & \ldots & 0 & e_1 \\ 0 & \ldots & 0 & 1 & \ddots & & e_2 \\ \vdots & & \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 1 & e_{n-m} \end{array}\right). \tag{4.5}$$

The mapping $(\alpha, \beta) \to (g(c), B)$ gives a biholomorphism between $(F_n(m;c), I)$ and $Gl(n,\mathbb{C}) \times \mathfrak{gl}(m,\mathbb{C}) \times \mathbb{C}^{n+m}$ [4]. The action of $Gl(n,\mathbb{C})$ is given by the right translations, and the action of $Gl(m,\mathbb{C})$ is given by $p\cdot\big(h,f,g,e,g(c)\big) = (php^{-1}, fp^{-1}, pg, e, pg(c))$, where for the last term we embedded $Gl(m,\mathbb{C})$ in $Gl(n,\mathbb{C})$ as the $m\times m$ upper-diagonal block. We can compute the complex symplectic form $\omega = \omega_2 + i\omega_3$. We denote by $b, \hat{b}$ vectors tangent to the space of B's in (4.5) and by $\rho, \hat{\rho}$ right-invariant vector fields on $Gl(n,\mathbb{C})$. We have [4]:

$$\omega\left((\rho, b), (\hat{\rho}, \hat{b})\right) = \operatorname{tr}\big(\rho\hat{b} - \hat{\rho}b - B[\rho,\hat{\rho}]\big). \tag{4.6}$$

Now we consider the complex structure of $\tilde{F}_n(n;c)$. Let $\mathfrak{n}$ be a unipotent algebra corresponding to the Cartan algebra of diagonal matrices. We consider the open dense subset $\tilde{F}(\mathfrak{n})$ of $\tilde{F}_n(n;c)$ defined as the set of all solutions $(\alpha,\beta) = (T_0 + iT_1, T_2 + iT_3)$ such that the intersection of the sum of positive eigenvalues of $\operatorname{ad}(iT_1(+\infty))$ with with the centralizer $C(\beta(+\infty))$ is contained in $\mathfrak{n}$. We observe that, since $(T_1(+\infty), T_2(+\infty), T_3(+\infty))$ is a regular triple, the projection of $T_1(+\infty)$ onto $C(\beta(+\infty))$ is a regular element. Now, as in [2], we use results of Biquard [7] to deduce that $\tilde{F}(\mathfrak{n})$ is biholomorphic to an open subset of $Gl(n,\mathbb{C})\times_N \mathfrak{b}$ where $N = \exp\mathfrak{n}$ and $\mathfrak{b} = \mathfrak{d} + \mathfrak{n}$, $\mathfrak{d}$ denoting the diagonal matrices. Briefly, the element g of $Gl(n,\mathbb{C})$ is given by the value at $t=0$ of the complex gauge transformation $g(t)$ which makes $(0, \beta(+\infty) + n)$ into (α,β).
The charts $\tilde{F}(\mathfrak{n})$ are glued as follows: $[g, d+n] \sim [g', d'+n']$ if and only if $n\in\mathfrak{n}, n'\in\mathfrak{n}'$, and either $\mathfrak{n}'\subset\mathfrak{n}$ and there exists an $m\in N$ such that $gm^{-1} = g'$, $\operatorname{Ad}(m)(d+\mathfrak{n}) = d' + \mathfrak{n}'$ or vice versa (i.e. $\mathfrak{n}\subset\mathfrak{n}'$ etc.).

We remark that $F_\emptyset$ is an open dense subset biholomorphic to an open subset of $Gl(n,\mathbb{C})\times\mathfrak{d}$. We shall denote this subset by $\tilde{F}_n^{\mathrm{reg}}(n;c)$. If $b_d, \hat{b}_d$ denote vectors tangent to the space of diagonal matrices and $\rho,\hat{\rho}$ denote this time left-invariant vector fields on $Gl(n,\mathbb{C})$, then the form ω is given by [2]:

$$\omega = -\operatorname{tr}\big(b_d\hat{\rho} - \rho\hat{b}_d - [\rho,\hat{\rho}]\beta_d\big). \tag{4.7}$$

All other moduli spaces $\tilde{F}_n(m;c)$ and $F_\sigma(\mu)$ can be viewed as hyperkähler quotients of products of $F_n(m;c)$ and $\tilde{F}_n(n;c)$. Thus, as complex-symplectic manifolds, they are isomorphic to open subsets of complex-symplectic quotients of the corresponding complex-symplectic manifolds computed above. The description of the latter quotients is straightforward. Let us remark that Hurtubise showed in [19] that if $\sigma(i) = m_i$, $i = 1, \dots, N-1$, then $F_\sigma(\mu)$ (i.e. the moduli space of $SU(N)$-monopoles) is biholomorphic to the space of based rational maps from $\mathbb{C}P^1$ to $SU(N)/T$ (maximal torus) of degree $(m_1, \dots, m_{N_1})$.

5. Complex-symplectic structure of $F_{\tilde{n},\tilde{m}}(c,c')$, $n > m$

Our aim is to calculate the metric on $F_\sigma(\mu)$ where $\sigma(i) = \tilde{m}_i$. This space has dimension $4p = 4(m_1 + \ldots + m_{N-1})$ and admits a tri-Hamiltonian action of T^p. By the definition of $F_\sigma(\mu)$, it is a hyperkähler quotient, by a torus, of the product spaces $\tilde{F}_n(0;c)$ and $F_{\tilde{n},\tilde{m}}(c,c')$. The metric on $\tilde{F}_n(0;c)$ was calculated in [2] - it is the Gibbons-Manton metric [13] with the mass parameter $-1/c$. It remains to calculate the metric on $F_{\tilde{n},\tilde{m}}(c,c')$. For convenience we shall write

$$\tilde{F}_{n,m}(c,c') := F_{\tilde{n},\tilde{m}}(c,c').$$

The dimension of this space is $4(n+m)$ and it has a tri-Hamiltonian action of an $(n+m)$-dimensional torus.

The space $\tilde{F}_{n,m}(c,c')$ should be thought as consisting of solutions to Nahm's equations on $(-\infty,0)\cup(0,\infty)$, which are $\mathfrak{u}(m)$-valued on $(-\infty,0)$, $\mathfrak{u}(n)$-valued on $(0,\infty)$, and satisfy appropriate matching conditions at zero. In what follows we shall usually say "$\tilde{F}_{n,m}(c,c')$ is biholomorphic to ... " rather than "$\tilde{F}_{n,m}(c,c')$ is biholomorphic to an open subset of ... ". This never leads to any problems.

We consider the space $\tilde{F}_{n,m}(c,c')$ for $n > m$. From its description as a complex-symplectic quotient, $\tilde{F}_{n,m}(c,c')$ is given by charts of the form $\{(b_-,(g,b_+)\} \in \mathfrak{b} \times \big(Gl(n,\mathbb{C}) \times_{N'} \mathfrak{b}'\big)$ such that gb_+g^{-1} is of the form (4.5) with $h = b_-$. Let us consider the chart on which $\mathfrak{b} = \mathfrak{d}_m$ and $\mathfrak{b}' = \mathfrak{d}_n$ ($\mathfrak{d}_m$ and $\mathfrak{d}_n$ denote $m\times m$ and $n\times n$ diagonal matrices). Let us write the elements of $\mathfrak{d}_m$ as $\mathrm{diag}(\kappa_1,\dots,\kappa_m)$ and the elements of $\mathfrak{d}_n$ as $\mathrm{diag}(\beta_1,\dots,\beta_n)$. Let $q_+(z) = \prod(z-\beta_i)$ and $q_-(z) = \prod(z-\kappa_i)$. We assume that the roots of both these polynomials are distinct and we consider multiplication by z on $\mathbb{C}[z]/(q_+)$. It is a linear operator which, in the basis

$$\frac{\prod_{j\neq i}(z-\beta_j)}{\prod_{j\neq i}(\beta_i-\beta_j)}, \qquad i = 1,\dots,n, \tag{5.1}$$

is the diagonal matrix $\mathrm{diag}(\beta_1,\dots,\beta_n)$. On the other hand, in the basis

$$\frac{\prod_{j\neq i}(z-\kappa_j)}{\prod_{j\neq i}(\kappa_i-\kappa_j)}, q_-(z),\dots,z^{n-m-1}q_-(z), \qquad i = 1,\dots,m, \tag{5.2}$$

the multiplication by z is given by a matrix of the form (4.5) with $f_i = 1/\prod_{j\neq i}(\kappa_i - \kappa_j)$. Let Z be the matrix transforming the basis (5.1) into (5.2). Then any g which sends $\mathrm{diag}(\beta_1,\dots,\beta_n)$ to a matrix of the form (4.5) can be written as

$$\mathrm{diag}(v_1^{-1},\dots,v_m^{-1},1,\dots,1)Z\,\mathrm{diag}(u_1,\dots,u_n). \tag{5.3}$$

We shall now compute Z. We introduce one more basis of $\mathbb{C}[z]/(q_+)$:

$$1,\dots,z^{n-1}. \tag{5.4}$$

The passage from (5.1) to (5.4) is given by $V(\beta_1,\dots,\beta_n)^{-1}$, where $V(\beta_1,\dots,\beta_n)$ is the Vandermonde matrix, i.e. its (i,j)-th entry is $(\beta_i)^{j-1}$. We then compute the passage from (5.4) to (5.2) as given by the matrix

$$L=\left(\begin{array}{c|c} V_\kappa & W\\ \hline 0 & H\end{array}\right), \tag{5.5}$$

where $V_\kappa = V(\kappa_1,\dots,\kappa_m)$, $W_{ij} = (\kappa_i)^{m+j-1}$ and H is upper-triangular with $H_{ij} = (-1)^{i-j}H_{i-j}$, where H_k denotes the k-th complete symmetric polynomial in $\kappa_1,\dots,\kappa_m$, i.e. the sum of all monomials of degree k.

Remark 5.1. The factorization $Z = LV^{-1}$ is unique only if $\beta_i \neq \kappa_j$ for all i,j. If, for instance, $\beta_1 = \kappa_1$, then the above g sends $\operatorname{diag}(\beta_1,\dots,\beta_n)$ to a matrix of the form (4.5) with $g_1 = 0$. However, there is then another g which makes $f_1 = 0$.

We calculate the complex symplectic form on $\tilde{F}_{n,m}(c,c')$. The chart where $\beta_i \neq \beta_j$, $\kappa_r \neq \kappa_s$, $\beta_i \neq \kappa_s$ for all $i,j = 1,\dots,n$, $i\neq j$, $r,s = 1,\dots,m$, $r\neq s$, can be described as consisting of pairs $\big((g_-,\kappa_d),(g_+,\beta_d)\big)$, where $\kappa_d = \operatorname{diag}(\kappa_1,\dots,\kappa_m)$, $\beta_d = \operatorname{diag}(\beta_1,\dots,\beta_n)$, $g_- = V_\kappa^{-1}\operatorname{diag}(v_1,\dots,v_m)$, $g_+ = V_\kappa^{-1}LV_\beta^{-1}\operatorname{diag}(u_1,\dots,u_n)$. According to the formula (4.7), the complex symplectic form in this chart is equal to:

$$\omega = -\operatorname{tr}\big(k_d\tilde{\rho}_- - \rho_-\tilde{k}_d - \kappa_d[\rho_-,\tilde{\rho}_-] + b_d\tilde{\rho}_+ - \rho_+\tilde{b}_d - \beta_d[\rho_+,\tilde{\rho}_+]\big).$$

Here k_d,ρ_-,b_d,ρ_+ are dual to, respectively, $\kappa_d,(g_-)^{-1}dg_-, d\beta_d,(g_+)^{-1}dg_+$. The first three terms can be computed as in [2] and give

$$\sum_{i=1}^m \frac{dv_i}{v_i}\wedge d\kappa_i - \sum_{i<j}\frac{d\kappa_i\wedge d\kappa_j}{\kappa_i-\kappa_j}. \tag{5.6}$$

To compute the remaining two terms let us write $X = V_\kappa^{-1}L$ and $Y = V_\beta^{-1}\operatorname{diag}(u_1,\dots,u_n)$. Let us also write $\beta_c = Y\beta_dY^{-1}$ and b_c,x,y for vector fields dual to $\beta_c, X^{-1}dX, Y^{-1}dY$. Then $\rho_+ = Y^{-1}xY + y$ and $b_c = Yb_dY^{-1} + Y[y,\beta_d]Y^{-1}$. Thus the last three terms in the above formula can be rewritten as

$$\begin{aligned}&-\operatorname{tr}\big(b_d\tilde{\rho}_+ - \rho_+\tilde{b}_d - \beta_d[\rho_+,\tilde{\rho}_+]\big) = -\operatorname{tr}\Big(b_d\tilde{y} - y\tilde{b}_d + b_dY^{-1}\tilde{x}Y - Y^{-1}xY\tilde{b}_d\\ &-\beta_d[y,\tilde{y}] - \beta_d[y,Y^{-1}\tilde{x}Y] - \beta_d[Y^{-1}xY,\tilde{y}] - Y\beta_dY^{-1}[x,\tilde{x}]\Big) = \omega_+ - \operatorname{tr}\Big(Yb_dY^{-1}\tilde{x} - xY\tilde{b}_dY^{-1}\\ &-Y\beta_dY^{-1}[YyY^{-1},\tilde{x}] - Y\beta_dY^{-1}[x,Y\tilde{y}Y^{-1}] - Y\beta_dY^{-1}[x,\tilde{x}]\Big) = \omega_+ - \operatorname{tr}\big(b_c\tilde{x} - x\tilde{b}_c - \beta_c[x,\tilde{x}]\big).\end{aligned}$$

Here $\omega_+ = -\operatorname{tr}\big(b_d\tilde{y} - y\tilde{b}_d - \beta_d[y,\tilde{y}]\big)$, which, again as in [2], is equal to

$$\sum_{i=1}^n \frac{du_i}{u_i}\wedge d\beta_i - \sum_{i<j}\frac{d\beta_i\wedge d\beta_j}{\beta_i-\beta_j}. \tag{5.7}$$

For the remaining terms we observe that $d\beta_c$ is upper triangular and $X^{-1}dX$ is strictly upper-triangular. Hence the remaining terms vanish and the complex-symplectic form on $\tilde{F}_{n,m}(c,c')$ is given by:

$$\omega = \sum_{i=1}^n \frac{du_i}{u_i}\wedge d\beta_i - \sum_{i<j}\frac{d\beta_i\wedge d\beta_j}{\beta_i-\beta_j} + \sum_{i=1}^m \frac{dv_i}{v_i}\wedge d\kappa_i - \sum_{i<j}\frac{d\kappa_i\wedge d\kappa_j}{\kappa_i-\kappa_j}. \tag{5.8}$$

6. Twistor space and the metric of $\tilde{F}_{n,m}(c,c')$, $n>m$

First, we compute the twistor space of $F_n(m;c)$. Let ζ be the affine coordinate on $\mathbb{C}P^1$. The twistor space of any hyperkähler manifold admitting an action of $SU(2)$ rotating the complex structures can be trivialized using just two charts $\zeta\neq\infty$ and $\zeta\neq 0$. For a moduli space of solutions to Nahm's equations this is achieved by putting $\eta=\beta+(\alpha+\alpha^*)\zeta-\beta^*\zeta^2$, $u=\alpha-\beta^*\zeta$ over $\zeta\neq\infty$ and $\tilde\eta=\beta/\zeta^2+(\alpha+\alpha^*)/\zeta-\beta^*$, $\tilde u=-\alpha^*-\beta/\zeta$ over $\zeta\neq 0$. Then, over $\zeta\neq 0,\infty$, we have $\tilde\eta=\eta/\zeta^2$, $\tilde u=u-\eta/\zeta$. Moreover, the real structure is $\zeta\mapsto -1/\bar\zeta$, $\eta\mapsto -\eta^*/\bar\zeta^2$, $u\mapsto -u^*+\eta^*/\bar\zeta$ (cf. [11, 8]).

We consider the matrix (4.5). We have

Lemma 6.1. *The (i,j)-th entry of $\exp Bt/\zeta$ is of the form $\frac{1}{(i-j)!}(t/\zeta)^{i-j}+O(t^{i-j+1})$ for $i>j>m$.*

Proof. We write $(Bt/\zeta)^k$ in the block form as

$$\begin{pmatrix} P(k) & Q(k)\\ R(k) & S(k)\end{pmatrix}.$$

One then checks by induction that:

$$S(k)_{(r,s)}=\begin{cases}0 & \text{if } k<r-s\\ (t/\zeta)^{r-s} & \text{if } k=r-s\\ O(t^{r-s+1}) & \text{if } k>r-s\end{cases}$$

and similarily for the other blocks. □

Therefore the $(m+1)$-th column, denoted by $\tilde p_{m+1}$ of $g^{-1}(t)\exp\{Bt/\zeta\}$ is of the form $t^{n-m-1}p+O(t^{n-m})$, for some vector p. This means that p belongs to the $-(n-m-1)/2$-eigenspace of $\operatorname{Res}\tilde u$, and so is of the form aE_n (E_n is the n-th vector of the standard basis), for some constant a. Computing the t^{n-m-1}-term of the last entry of $\tilde p_{m+1}$ gives $(\operatorname{Res}\eta)^{n-m-1}E_{m+1}\big(\zeta^{n-m-1}(n-m-1)!\big)^{-1}=\zeta^{-(n-m-1)}$, and so $a=\zeta^{-(n-m-1)}$. Thus, as in (4.4), $\tilde w_1=\zeta^{n-m-1}p_{m+1}$ is a solution $\tilde w_1(t)$ to $\frac{d}{dt}\tilde w_1=-\tilde u\tilde w_1$ with

$$\lim_{t\to 0}\big(t^{-(n-m-1)/2}w_1(t)-E_n\big)=0.$$

In the same vein we see that $\tilde w_i(t)=\zeta^{n-m+1-2i}\tilde p_{m+i}$, where $\tilde p_{m+i}$ is the $(m+i)$-th column of $g^{-1}(t)\exp\{Bt/\zeta\}$. In other words $\tilde g(t)=d(\zeta)\exp\{Bt/\zeta\}g(t)$ where

$$d(\zeta)=\operatorname{diag}\{1,\dots,1,\zeta^{-(n-m-1)},\dots,\zeta^{(n-m-1)}\}.$$

Similar computations show that the real structure sends B to $-r(\zeta)\big(B^*/\bar\zeta^2\big)r(\zeta)^{-1}$ and g to $r(\zeta)\exp\{B^*/\bar\zeta\}(g^*)^{-1}$ where

$$r_{ij}(\zeta)=\begin{cases}0 & \text{if } i+j\neq n+m+1\\ (-1)^{j-1}\bar\zeta^{n+m+1-2j} & \text{if } i+j=n+m+1.\end{cases}\tag{6.1}$$

Now we consider the subset of $\tilde F_n(n;c)$, where the eigenvalues of $\beta(+\infty)$ are distinct. We have assigned to each element of this set the pair $(\beta(+\infty);g)$. We know that $\tilde\beta(+\infty)=\beta(+\infty)/\zeta^2$. The argument in section 2 shows then that $\tilde g=g\exp\{-c\beta(+\infty)/\zeta\}$. The real structure sends g to $(g^*)^{-1}\exp\{c\beta(+\infty)^*/\bar\zeta\}$.

Now we wish to calculate the twistor space of $\tilde F_{n,m}(c,c')$ for $n>m$. This space is a hyperkähler quotient of $F_n(m;1)\times\tilde F_n(m;c-1)\times\tilde F_m(m;c')$. On the subset corresponding to the same chart as in section 5 (i.e. $\mathfrak{b} = \mathfrak{d}_m$ and $\mathfrak{b}' = \mathfrak{d}_n$), the coordinates are given by $\beta(-\infty), \beta(+\infty)$ and g such that $g\beta(+\infty)g^{-1}$ is of the form (4.5) with $h = \beta(-\infty)$. This g can be written as $(g_3)^{-1}g_1g_2$, where g_1 (resp. g_2, resp. g_3) is the g considered above for $F_n(m;1)$ (resp. $\tilde{F}_n(m;c-1)$, resp. $\tilde{F}_m(m;c')$). It follows that

$$\tilde{g} = \exp\{c'\beta(-\infty)/\zeta\}(g_3)^{-1}d(\zeta)\exp\{B/\zeta\}g_1g_2\exp\{-(c-1)\beta(+\infty)/\zeta\}$$

which can be rewritten as

$$\exp\{c'\beta(-\infty)/\zeta\}d(\zeta)(g_3)^{-1}g_1g_2\exp\{-c\beta(+\infty)/\zeta\}.$$

Therefore

$$\tilde{g} = \exp\{c'\beta(-\infty)/\zeta\}d(\zeta)g\exp\{-c\beta(+\infty)/\zeta\}.$$

We now compute the twistor space in coordinates $\kappa_1, \dots, \kappa_m, v_1, \dots, v_m, \beta_1, \dots, \beta_n, u_1, \dots, u_n$ of section 5. We know that $\tilde{\kappa}_i = \kappa_i/\zeta^2$ and $\tilde{\beta}_j = \beta_j/\zeta^2$. The matrix g is given by (5.3), where $Z = LV^{-1}$ where L is described by (5.5) and V is the Vandermonde matrix for $\beta_1, \dots, \beta_n$. We obtain equations (here $v_i = \tilde{v}_i = 1$ and $\kappa_i = \tilde{\kappa}_i = 0$ for $i > m$):

$$\tilde{v}_i^{-1}\tilde{u}_j\tilde{Z}_{ij} = \exp\{(c'\kappa_i - c\beta_j)/\zeta\}d_i(\zeta)Z_{ij}v_i^{-1}u_j.$$

In addition $\tilde{Z}_{ij} = Z_{ij}$ if $i \le m$ and $\tilde{Z}_{ij} = \zeta^{2i-2}Z_{ij}$ if $i > m$. Hence

$$\tilde{v}_i^{-1}\tilde{u}_j = \begin{cases} \exp\{(c'\kappa_i - c\beta_j)/\zeta\}v_i^{-1}u_j & \text{if } i \le m \\ \zeta^{-n-m+1}\exp\{-c\beta_j/\zeta\}v_i^{-1}u_j & \text{if } i > m. \end{cases}$$

As $v_i = \tilde{v}_i = 1$ for $i > m$, we finally obtain

$$\tilde{v}_i = \zeta^{-n-m+1}\exp\{-c'\kappa_i/\zeta\}v_i \tag{6.2}$$
$$\tilde{u}_j = \zeta^{-n-m+1}\exp\{-c\beta_j/\zeta\}u_j. \tag{6.3}$$

Finally, the real structure is computed as in [2]:

$$\beta_i \mapsto -\bar{\beta}_i/\bar{\zeta}^2, \qquad u_i \mapsto \bar{u}_i^{-1}(1/\bar{\zeta})^{n+m-1}e^{c\bar{\beta}_i/\bar{\zeta}}\prod_{j\neq i}(\bar{\beta}_i - \bar{\beta}_j)\prod_{j=1}^{m}(\bar{\beta}_i - \bar{\kappa}_j) \tag{6.4}$$

$$\kappa_i \mapsto -\bar{\kappa}_i/\bar{\zeta}^2, \qquad v_i \mapsto \bar{v}_i^{-1}(1/\bar{\zeta})^{n+m-1}e^{c'\bar{\kappa}_i/\bar{\zeta}}\prod_{j\neq i}(\bar{\kappa}_i - \bar{\kappa}_j)\prod_{j=1}^{n}(\bar{\kappa}_i - \bar{\beta}_j) \tag{6.5}$$

We now have to calculate the real sections. First of all we have

$$\beta_i(\zeta) = z_i + 2x_i\zeta - \bar{z}_i\zeta^2 \qquad \text{for } i = 1, \dots, n, \tag{6.6}$$
$$\kappa_i(\zeta) = z_{n+i} + 2x_{n+i}\zeta - \bar{z}_{n+i}\zeta^2 \qquad \text{for } i = 1, \dots, m, \tag{6.7}$$

where $p_i = (z_i, x_i) \in \mathbb{C} \times \mathbb{R}$ are such that $p_i \neq p_j$ if $i \neq j$. These curves of genus 0 should be thought of as spectral curves of individual monopoles. Let S_i denote either β_i or κ_i. Two curves S_i and S_j intersect in a pair of distinct points a_{ij} and a_{ji}, where

$$a_{ij} = \frac{(x_i - x_j) + r_{ij}}{\bar{z}_i - \bar{z}_j}, \quad r_{ij} = \sqrt{(x_i - x_j)^2 + |z_i - z_j|^2}. \tag{6.8}$$

As in [2], if $i, j \le n$, then u_i has a zero at a_{ji} and is nonzero at a_{ij}. Similarily, if $i, j > n$, then v_{i-n} has a zero at a_{ji} and is nonzero at a_{ij}. Let us consider what happens when $i \leq n$ and $j > n$ (and no other curves intersect S_i at a_{ji}). First of all, computing the characteristic polynomial of (4.5) gives [19]:

$$\det(\eta - B) = \det(\eta - h)(\eta^{n-m} - e_{n-m}\eta^{n-m-1} - \ldots - e_1) - f(\eta - h)_{\mathrm{adj}} g, \tag{6.9}$$

from which we conclude that $f_{j-n}g_{j-n}$ is zero at both a_{ij} and a_{ji}. This implies, since $f_i = v_i / \prod_{s\neq i}(\kappa_i - \kappa_s)$, that v_{j-n} is zero precisely when f_{j-n} is zero. Now, if the passage from $\mathrm{diag}(\beta_1, \ldots, \beta_n)$ to (4.5) is given by the matrix G of the form (5.3) with $Z = LV^{-1}$, then $G_{j-n,s} = 0$ if $s \neq i$ and $G_{j-n,i} = v_{j-n}^{-1}u_i$. This has two implications: 1) u_i is zero if and only if v_{j-n} is, and 2) $g_j = 0$ in (4.5). Hence, in this situation, $v_{j-n} \neq 0$. Thus u_i and v_{j-n} are zero at exactly one of the two points of intersection of S_i and S_j. Furthermore, since $\kappa_k(\zeta)$ does not intersect $\beta_l(\zeta)$ at a_{ij} or a_{ji} if $k \neq j - n$ or $l \neq i$, we conclude that $f_k g_k \neq 0$ at a_{ij} or a_{ji} if $k \neq j - n$. Thus $v_k(a_{ij}) \neq 0$ and $v_k(a_{ji}) \neq 0$ for $k \neq j - n$. Since $G(a_{ij})$ and $G(a_{ji})$ are invertible we also have $u_l(a_{ij}) \neq 0$ and $u_l(a_{ji}) \neq 0$ for $\neq i$.

Summing up, $u_i(\zeta)$, $i \leq n$, and $v_i(\zeta)$, $i \leq m$, are of the form:

$$u_i(\zeta) = A_i \prod_{\substack{j\leq n\\ j\neq i}} (\zeta - a_{ji}) \prod_{j>n} (\zeta - c_{ij}) e^{c(x_i - \bar{z}_i\zeta)}, \tag{6.10}$$

$$v_i(\zeta) = B_i \prod_{\substack{j\leq m\\ j\neq i}} (\zeta - a_{j+n,i+n}) \prod_{j\leq n} (\zeta - c_{i+n,j}) e^{c'(x_i - \bar{z}_i\zeta)}. \tag{6.11}$$

Here c_{ij} can be either a_{ij} or a_{ji} and is at present undetermined. The reality condition implies that

$$A_i\bar{A}_i = \prod_{\substack{j\leq n\\ j\neq i}} (x_i - x_j + r_{ij}) \prod_{j>n} \bigl(\pm(x_i - x_j) + r_{ij}\bigr),$$

$$B_i\bar{B}_i = \prod_{\substack{j\leq m\\ j\neq i}} (x_{i+n} - x_{j+n} + r_{i+n,j+n}) \prod_{j\leq n} \bigl(\pm(x_{i+n} - x_j) + r_{i+n,j}\bigr),$$

where the undetermined signs are positive if $c_{ij} = a_{ji}$ and negative if $c_{ij} = a_{ij}$. By continuity, these formulas extend to the case when more than two S_i intersect at a point.

We can now compute the metric on $\tilde{F}_{n,m}(c, c')$ up to the above sign indeterminacy. This metric is of the form (2.1) and it is enough to compute the matrix Φ. The complex symplectic form on the twistor space is given by the formula (5.8) and it follows from it that each factor in (6.10) (resp. (6.11)) together with the corresponding factor of $|A_i|$ (resp. $|B_i|$) gives a separate contribution to Φ (the coefficient of ζ in the expansion of ω is the Kähler form ω_1). The factors of u_i, indexed by $j \leq n$, together with the exponential term, describe exactly the twistor space of the Gibbons-Manton metric (2.2) with mass parameter $-1/c$, as computed in Proposition 6.2 of [2], providing that the complex symplectic form is taken to be the first two terms of (5.8). Thus these terms contribute

$$c - \sum_{\substack{j\leq n\\ j\neq i}} \frac{1}{r_{ij}}$$

to Φ_{ii}, $i \le n$, and $1/r_{ij}$ to Φ_{ij} for $i, j \le n$, $i \ne j$. An exactly parallel statement holds for the factors of v_i indexed by $j \le m$ plus the exponential term. The remaining factors contribute terms in $\frac{du_i}{u_i} \wedge d\beta_i$ and $\frac{dv_i}{v_i} \wedge d\kappa_i$. The calculation in the proof of Theorem 6.4 in [2] shows that the contribution of these factors to the matrix Φ are Gibbons-Manton-like terms with positive or negative sign. Thus we conclude that the matrix Φ for $\tilde{F}_{n,m}(c, c')$ is given by (2.3) with

$$c_i = \begin{cases} c & \text{if } i \le n \\ c' & \text{if } i > n, \end{cases} \tag{6.12}$$

$$s_{ij} = \begin{cases} -1 & \text{if } i, j \le n \text{ or } i, j > n \\ (-1)^{\epsilon_{ij}} & \text{if } i \le n, j > n \text{ or } i > n, j \le n. \end{cases} \tag{6.13}$$

Here $\epsilon_{ij} = 0$ if $c_{ij} = a_{ij}$ and $\epsilon_{ij} = 1$ if $c_{ij} = a_{ji}$. We shall eventually see that all ϵ_{ij} are equal to 0.

7. The metric on $\tilde{F}_{n,n}(c, c')$

This space is the hyperkähler quotient of $\tilde{F}_n(n; c) \times \tilde{F}_n(n; c') \times \mathbb{H}^n$ by the diagonal action of $U(n)$. According to section 4 its complex charts can be described as $\big(b_-, (g, b_+), (V, W)\big) \in \mathfrak{b} \times \big(Gl(n, \mathbb{C}) \times_{N'} \mathfrak{b}'\big) \times \mathbb{C}^{2n}$ with $g b_+ g^{-1} = b_- + VW^T$.

Our first step is to calculate the complex-symplectic form on the set where $\mathfrak{b} = \mathfrak{b}' = \mathfrak{d}$. Let us write $\beta_d^+ = \operatorname{diag}(\beta_1^+, \dots, \beta_n^+)$ for b_+ and $\beta_d^- = \operatorname{diag}(\beta_1^-, \dots, \beta_n^-)$ for b_- on this set. The choice of our chart implies that $\beta_i^+ \ne \beta_j^+$ and $\beta_i^- \ne \beta_j^-$ for $i \ne j$. In addition we suppose that $\beta_i^- \ne \beta_j^+$ for all $i, j \le n$. Then one of β_d^-, β_d^+, say β_d^-, is invertible. Since all components of W must be nonzero (otherwise the spectra of β_d^- and β_d^+ are not disjoint), W is cyclic for β_d^-. Consider the basis given by columns of $\big((\beta_d^-)^{n-1} W, (\beta_d^-)^{n-2} W, \dots, W\big)^T$, in which β^- is of form (4.5) (with $m = 0$) and $W^T = (0, 0, \dots, 1)$. Thus, β^+ is also of form (4.5). We can therefore describe this chart as consisting of pairs $\big((g_-, \beta_d^-), (g_+, \beta_d^+)\big)$ with $g_- \beta_d^- (g_-)^{-1}$ and $g_+ \beta_d^+ (g_+)^{-1}$ both of form (4.5) (with $m = 0$). We have $g_\pm = V(\beta_1^\pm, \dots, \beta_n^\pm)^{-1} \operatorname{diag}(u_1^\pm, \dots, u_n^\pm)$. The form ω, via the complex-symplectic quotient, can be written as ($b_d^\pm, \rho_\pm$ are dual to $\beta_d^\pm, (g_\pm)^{-1} dg_\pm$):

$$\omega = -\operatorname{tr}\big(b_d^+ \tilde{\rho}_+ - \rho_+ \tilde{b}_d^+ - \beta_d^+[\rho_+, \tilde{\rho}_+] + b_d^- \tilde{\rho}_- - \rho_- \tilde{b}_d^- - \beta_d^-[\rho_-, \tilde{\rho}_-]\big),$$

which can be computed as in [2] giving:

$$\omega = \sum_{i=1}^n \frac{du_i^-}{u_i^-} \wedge d\beta_i^- - \sum_{i<j} \frac{d\beta_i^- \wedge d\beta_j^-}{\beta_i^- - \beta_j^-} + \sum_{i=1}^n \frac{du_i^+}{u_i^+} \wedge d\beta_i^+ - \sum_{i<j} \frac{d\beta_i^+ \wedge d\beta_j^+}{\beta_i^+ - \beta_j^+}. \tag{7.1}$$

We now wish to compute the twistor space of $\tilde{F}_{n,n}(c, c')$. We proceed as in the previous section. A calculation done there shows that the coordinates g, β_d^-, β_d^+ (here $g \beta_d^+ g^{-1} = \beta_d^- + VW^T$) change from $\zeta \ne \infty$ to $\zeta \ne 0$ as:

$$\tilde{\beta}_d^\pm = \beta_d^\pm / \zeta^2, \qquad \tilde{g} = \exp\{c' \beta_d^-\} g \exp\{-c \beta_d^+\}.$$

Moreover, since the twistor space of $\mathbb{H}^n$ is simply $O(1) \otimes \mathbb{C}^{2n}$, we have

$$\tilde{V} = V/\zeta, \qquad \tilde{W} = W/\zeta.$$

We wish to pass to coordinates $\beta_i^\pm, u_i^\pm$, $i=1,\dots,n$. The passage from the basis in which $\beta^\pm$ are diagonal to the one in which they are of the form (4.5) is achieved by the matrix

$$H=\left((\beta_d^-)^{n-1}W,(\beta_d^-)^{n-2}W,\dots,W\right)^T.$$

Thus $g_-=Hd$ and $g_+=Hdg$ for some diagonal d. We have $\tilde d=d\exp\{-c'\beta_d^-\}$ (d is g_- in the chart in which b_- is diagonal). On the other hand we have written $g_\pm=V(\beta_1^\pm,\dots,\beta_n^\pm)^{-1}\operatorname{diag}(u_1^\pm,\dots,u_n^\pm)$. Comparing the two formulas in the two charts, we conclude that $u_i^\pm(\zeta)$ changes from $\zeta\neq\infty$ to $\zeta\neq 0$ as:

$$\tilde u_i^+=\zeta^{1-2n}\exp\{-c\beta_i^+/\zeta\}u_i^+,\quad i=1,\dots,n,$$

$$\tilde u_i^-=\zeta^{1-2n}\exp\{-c'\beta_i^-/\zeta\}u_i^-,\quad i=1,\dots,n.$$

The real structure is given by the formulas (6.4) and (6.5) with $m=n$, $\beta_i=\beta_i^+,\kappa_i=\beta_i^-,u_i=u_i^+,v_i=u_i^-$.

We have to know what happens to the $u_i^\pm(\zeta)$ when two curves $\beta_i^\pm(\zeta)$ intersect. As in the previous section we write $S_i(\zeta)=\beta_i^+(\zeta)=z_i+2x_i\zeta-\bar z_i\zeta^2$, $S_{n+i}(\zeta)=\beta_i^-(\zeta)=z_{n+i}+2x_{n+i}\zeta-\bar z_{n+i}\zeta^2$, and we denote the intersection points of S_i and S_j by a_{ij},a_{ji}. These are given by the formula (6.8).

Consider first the intersection of S_i and S_j where both i and j are either less than or equal to n, or both are greater than n (and no other S_k intersect at a_{ij},a_{ji}). We can still assume generically that the spectra of β_d^- and β_d^+ are disjoint and, hence, W is cyclic for β_d^-. Then H is invertible at a_{ij},a_{ji} and so are $g_\pm$. We compute, as in [2], that each $u_i^\pm$ has a zero at the intersection point a_{ji} and is nonzero at a_{ij}, and all other $u_s^\pm$, $s\neq i,j$ are nonzero at both a_{ij} and a_{ji}.

Now consider the intersection of S_i and S_j where $i\le n$ and $j>n$. In the chart in which β^- is diagonal, we had $g\beta_d^+g^{-1}=\beta_d^-+VW^T$. Thus $\det(\beta_d^--\beta_i^+1+VW^T)=0$. Since VW^T has rank one, and so all its $k\times k$, $k>1$, minors vanish, we have for any diagonal matrix $d=\operatorname{diag}(d_1,\dots,d_n)$ the formula

$$\det(d+VW^T)=\prod_k d_k+\sum_k\left(V_kW_k\prod_{l\neq k}d_l\right). \tag{7.2}$$

In our case $d_j=0$ and $d_k\neq 0$ for $k\neq j$. We conclude that V_jW_j vanish at both a_{ij} and a_{ji}. However both V_j and W_j are sections of $O(1)$ and so have exactly one zero. Thus W_j vanishes at either a_{ij} or a_{ji} (and only one of them). Furthermore, if we consider the diagonal matrix $d=\beta_d^--\beta_s^-1$, $s\neq j$, then, by the above argument, the nonvanishing of $\det(d+VW^T)$ implies that V_sW_s does not vanish at either a_{ij} or a_{ji} if $s\neq j$. In summary, it is precisely the j-th column of H that vanishes at either a_{ij} or a_{ji}. Thus the same statement holds for both g_- and g_+, and, as $g_\pm=V(\beta_1^\pm,\dots,\beta_n^\pm)^{-1}\operatorname{diag}(u_1^\pm,\dots,u_n^\pm)$, we conclude that both u_j^- and u_j^+ vanish at either a_{ij} or a_{ji} and no other $u_s^\pm$ vanishes at either a_{ij} or a_{ji}. This means that u_i^+ is given by the formula (6.10) and u_i^- is given by the formula (6.11) (with $n=m$). Once more, the formulas extend to the non-generic case. The remainder of the previous section can be now repeated word by word, and we conclude that the metric on $\tilde F_{n,n}(c,c')$ is of the form (2.1) with Φ given by (2.3) where the c_i and s_{ij} are given by (6.12) and (6.13).

8. Topology of $\tilde{F}_{n,m}(c,c')$

We shall discuss the topology of $\tilde{F}_{n,m}(c,c')$. This space can be viewed as a moduli space of solutions to Nahm's equations defined on $(-\infty,0]\cup(0,+\infty)$ with the appropriate matching at 0. The tri-Hamiltonian action of $T^{n+m}=T^n\times T^m$ gives us the moment map to $\mathbb{R}^3\otimes\mathbb{R}^{n+m}$ which is simply

$$\Big(\big(T_1(+\infty),-T_1(-\infty)\big),\big(T_2(+\infty),-T_2(-\infty)\big),\big(T_2(+\infty),-T_2(-\infty)\big)\Big).$$

Before stating the result let us recall that a basis of the second homology $H_2\big(\tilde{C}_p(\mathbb{R}^3),\mathbb{Z}\big)$ of a configuration space $\tilde{C}_p(\mathbb{R}^3)$ is given by the $p(p-1)/2$ 2-spheres

$$S^2_{ij}=\{(\mathbf{x}_1,\dots,\mathbf{x}_p)\in\mathbb{R}^3\otimes\mathbb{R}^p;|\mathbf{x}_i-\mathbf{x}_j|=const,\ \ \mathbf{x}_k=const\ \text{ if }\ k\neq i,j\}\tag{8.1}$$

where $i<j$. We have:

Proposition 8.1. *The above moment map induces a homeomorphism between the orbit space of $\tilde{F}_{n,m}(c,c')$ and $\tilde{C}_n(\mathbb{R}^3)\times\tilde{C}_m(\mathbb{R}^3)$. The set of principal T^{n+m}-orbits of $\tilde{F}_{n,m}(c,c')$ maps to $\tilde{C}_{n+m}(\mathbb{R}^3)$ and as a T^{n+m}-bundle is determined by the element $(h_1,\dots,h_{n+m})$ of $H^2\big(\tilde{C}_{n+m}(\mathbb{R}^3),\mathbb{Z}^{n+m}\big)$ given by*

$$h_k(S^2_{ij})=\begin{cases}s_{ij} & \text{if } k=i\\ -s_{ij} & \text{if } k=j\\ 0 & \text{otherwise,}\end{cases}$$

where the s_{ij} are given by (6.13).

Proof. Let us fix an element (τ^+,τ^-) of $\tilde{C}_n(\mathbb{R}^3)\times\tilde{C}_m(\mathbb{R}^3)$. Identify τ^+ with a regular triple $(\tau_1^+,\tau_2^+,\tau_3^+)$ of diagonal $n\times n$ matrices and similarly τ^- with a regular triple $(\tau_1^-,\tau_2^-,\tau_3^-)$ of diagonal $m\times m$ matrices. As in Proposition 5.2 of [2] the space of T^{n+m}-orbits mapping to (τ^+,τ^-) can be identified with the set of solutions to Nahm's equations with $T_0\equiv 0$ and having values conjugate to $(\tau^+,-\tau^-)$ at $+\infty$ and at $-\infty$. If $n>m$, this space is diffeomorphic to the hyperkähler quotient X of the product $M(\tau_1^+,\tau_2^+,\tau_3^+)\times F_n(m;1)\times M(\tau_1^-,\tau_2^-,\tau_3^-)$ by $U(n)\times U(m)$, where the M's are Kronheimer's hyperkähler structures on $Gl(n,\mathbb{C})/(T^n)^{\mathbb{C}}$ and $Gl(m,\mathbb{C})/(T^m)^{\mathbb{C}}$ [23]. If $n=m$, this space is diffeomorphic to the hyperkähler quotient X of the product $M(\tau_1^+,\tau_2^+,\tau_3^+)\times\mathbb{H}^n\times M(\tau_1^-,\tau_2^-,\tau_3^-)$ by $U(n)$.

The first statement will be proved if we can show that these hyperkähler quotients are single points. First we show that the corresponding complex-symplectic quotient, with respect to a generic complex structure I (i.e. one in which $M(\tau_1^\pm,\tau_2^\pm,\tau_3^\pm)$ are biholomorphic to regular adjoint orbits), are single points.

(1) $n>m$**.** Let $M(\tau_1^+,\tau_2^+,\tau_3^+)$ be complex-symplectic isomorphic to the adjoint orbit O^+ of $\mathrm{diag}(\beta_1^+,\dots,\beta_n^+)$ (β_i^+ distinct) and $M(\tau_1^-,\tau_2^-,\tau_3^-)$ to the adjoint orbit O^- of $\mathrm{diag}(\beta_1^-,\dots,\beta_m^-)$ (β_i^- distinct). First of all, the complex symplectic quotient of $M(\tau_1^+,\tau_2^+,\tau_3^+)\times F_n(m;1)$ by $Gl(n,\mathbb{C})$ can be identified with the set U of elements of O^+ which are of the form (4.5). Then the zero-set of the complex moment map for the action of $Gl(m,\mathbb{C})$ on $U\times O^-$ can be identified with the set Y of matrices of the form (4.5) which belong to O^+ and such that h belongs to O^-. We have to show that Y is a single orbit of $Gl(m,\mathbb{C})$. Since the β_i^- are distinct we can diagonalize h. Then the equation (6.9) shows that the e_i's and the products f_ig_i are determined. Thus we obtain a single $(\mathbb{C}^*)^m$-orbit.

(2) $n = m$**.** We make the same assumption about the complex-symplectic structure of the two M's. The zero set of the complex moment map for the action of $Gl(n, \mathbb{C})$ on $O^+ \times O^- \times \mathbb{C}^{2n}$ is the set $\{(a, b, V, W) \in O^+ \times O^- \times \mathbb{C}^n \times \mathbb{C}^n;\, a = b + VW^T\}$. Again we have to show that this set is a single orbit of $Gl(n, \mathbb{C})$. Let us diagonalize b and use the formula (7.2) with $d = b - \eta 1$. Substituting β_i^- for η shows that $V_i W_i$ is determined, $i = 1, \ldots, n$. We obtain a single $(\mathbb{C}^*)^n$-orbit.

We remark that the above proof shows that the action of $G^{\mathbb{C}}$, where $G^{\mathbb{C}}$ is $Gl(n, \mathbb{C}) \times Gl(m, \mathbb{C})$ in case (1) or $Gl(n, \mathbb{C})$ in case (2), on the zero-set of the complex moment map has closed orbits of the form $G^{\mathbb{C}}/T^{\mathbb{C}}$ for some subtorus T of G.

Thus, to prove the first statement, we have to show that the complex-symplectic and the hyperkähler quotient coincide. The proof of this requires a substantial detour from the main line of argument and will be given in the appendix. Let us remark that Hurtubise's argument [19] for matching solutions to Nahm's equations on two (or more) intervals cannot be adapted to the case of two half-lines (in this case his Lemma 2.19 will not provide any information).

It is clear from the description of the sections of the twistor space - formulas (6.10) and (6.11) - that the action is free precisely over $\tilde{C}_{n+m}(\mathbb{R}^3)$. To determine the principal bundle, one merely has to repeat the calculation in the proof of Proposition 6.3 in [2]. □

Corollary 8.2. *The action of T^{n+m} on $\tilde{F}_{n,m}(c, c')$ extends to the global action of $(\mathbb{C}^*)^{n+m}$ with respect to any complex structure.*

Proof. This is equivalent to showing that, if we fix $\zeta \in \mathbb{C}P^1$, then the $u_i(\zeta)$ and $v_j(\zeta)$ of (6.10) and (6.11) can take arbitrary complex values (with appropriate degenerations at the intersection points of the $\beta_i(\zeta)$). If, for example $\zeta = 0$, then the z_i are fixed and one solves for the x_i. One shows that a solution always exists and by the previous result the corresponding point lies in $\tilde{F}_{n,m}(c, c')$. □

9. Asymptotic comparison of metrics

We consider the moduli space $M_{m_1, \ldots, m_{N-1}}(\mu_1, \ldots, \mu_N)$ of $SU(N)$ monopoles with maximal symmetry breaking. We wish to compare the metric on $F_{m_1, \ldots, m_{N-1}}(\mu_1, \ldots, \mu_N)$ (whose Levi-Civita connection coincides with that on $M_{m_1, \ldots, m_{N-1}}(\mu_1, \ldots, \mu_N)$) with the metric on $F_{\tilde{m}_1, \ldots, \tilde{m}_{N-1}}(\mu_1, \ldots, \mu_N)$. As discussed in Remark 3.3, this space consists of solutions to Nahm's equations on the union of I_k where $I_k = [\mu_k, +\infty) \cup (-\infty, \mu_{k+1}]$ with matching conditions at the endpoints of each I_k. It will be convenient to write

$$I_k = [[\mu_k, \mu_{k+1}]]$$

and denote the "middle point" $\pm\infty$ by ∞_k. We shall also use double brackets for any connected subset of $[[\mu_k, \mu_{k+1}]]$.

The space $F_{\tilde{m}_1, \ldots, \tilde{m}_{N-1}}(\mu_1, \ldots, \mu_N)$ should be thought of as consisting of $m = m_1 + \cdots + m_{N-1}$ particles with phases. The positions of particles are $\mathbf{x}_i^k = (x_i^k, \operatorname{Re} z_i^k, \operatorname{Im} z_i^k)$, $i \leq m_k$, $k = 1, \ldots N-1$, where $\operatorname{diag}(x_1^k, \ldots, x_{m_k}^k) = \sqrt{-1}T_1(\infty_k)$ and $\operatorname{diag}(z_1^k, \ldots, z_{m_k}^k) = (T_2 + \sqrt{-1}T_3(\infty_k)$. We put

$$R_k = \min\{|\mathbf{x}_i^k - \mathbf{x}_j^k|;\, i \neq j\}. \tag{9.1}$$

We shall write $F^{\mathrm{reg}}_{\tilde{m}_1, \ldots, \tilde{m}_{N-1}}(\mu_1, \ldots, \mu_N)$ for the subset of $F_{\tilde{m}_1, \ldots, \tilde{m}_{N-1}}(\mu_1, \ldots, \mu_N)$ where $z_i^k \neq z_j^k$ for $i \neq j$ and $k = 1, \ldots, N-1$. This depends on the chosen complex structure (which is I in the case at hand). If we write for this complex structure, as in section 4, α for T_0+iT_1, β for T_2+iT_3, then we can define the subset $F^{\mathrm{reg}}_{m_1,\dots,m_{N-1}}(\mu_1,\dots,\mu_N)$ of $F_{m_1,\dots,m_{N-1}}(\mu_1,\dots,\mu_N)$ as the set of (α,β) such that the eigenvalues of β restricted to the k-th interval, $k=1,\dots,N-1$, are distinct. Finally, let Σ be the product of symmetric groups $\prod_{k=1}^{N-1}\Sigma_{m_k}$. We have:

Theorem 9.1. *There exists a biholomorphism ϕ from $F^{\mathrm{reg}}_{\tilde m_1,\dots,\tilde m_{N-1}}(\mu_1,\dots,\mu_N)/\Sigma$ to $F^{\mathrm{reg}}_{m_1,\dots,m_{N-1}}(\mu_1,\dots,\mu_N)$ such that on the set where the action of T^m, $m=\sum_{k=1}^{N-1}m_k$, is free we have*

$$|\phi^*g-\tilde g| = O\big(e^{-\lambda R}\big) \tag{9.2}$$

where $\tilde g, g$ denote the metrics on $F^{\mathrm{reg}}_{\tilde m_1,\dots,\tilde m_{N-1}}(\mu_1,\dots,\mu_N)$ and $F^{\mathrm{reg}}_{m_1,\dots,m_{N-1}}(\mu_1,\dots,\mu_N)$ respectively, $R=\min\{R_k; k=1,\dots,N-1\}$ and λ is a constant.
Furthermore, the estimate (9.2) *can be differentiated term by term in the coordinates given by positions and phases of particles.*

Remarks . 1. For a possible generalization see the discussion at the end of the section.
2. The points of $F^{\mathrm{reg}}_{\tilde m_1,\dots,\tilde m_{N-1}}(\mu_1,\dots,\mu_N)$ where the action of T^m is not free may not be smooth; however the λ in (9.2) will be uniform and thus the estimate will extend to the smooth tangent directions at these points.

Proof. We shall write M for $F^{\mathrm{reg}}_{m_1,\dots,m_{N-1}}(\mu_1,\dots,\mu_N)$ and $\tilde M$ for $F^{\mathrm{reg}}_{\tilde m_1,\dots,\tilde m_{N-1}}(\mu_1,\dots,\mu_N)$.
Part 1: Construction of ϕ. This is completely analogous to the $SU(2)$ case. One goes via an intermediate moduli space M_I consisting of solutions (α,β) to the complex Nahm equation which are constant and diagonal on each $[[\mu_k+C,\mu_{k+1}-C]]$ for some $C<\min\{(\mu_{k+1}-\mu_k)/2; k=1,\dots,N-1\}$ and satisfy appropriate matching conditions at each μ_k, modulo gauge transformations $g(t)$ which satisfy the matching conditions of Remark 3.3. In particular $g(t)=\exp\{h_kt-p_k\}$ near ∞_k for some complex diagonal matrices for diagonal h_k,p_k. The passage from M_I to M is given by restricting these solutions to the union of $[[\mu_k,(\mu_k+\mu_{k+1})/2]]\cup[[(\mu_k+\mu_{k+1})/2,\mu_{k+1}]]$, viewing them as solutions to the complex Nahm equation on $[\mu_1,\mu_{N-1}]$ and solving the real equation as in [19].

The map from M to M_I/Σ is defined by using a complex gauge transformation to make an element (α,β) of M constant and diagonal on each $[\mu_k+C,\mu_{k+1}-C]$, cut off at the center of each interval and extend trivially onto $[[\mu_k,\mu_{k+1}]]$.

The passage from $\tilde M$ to M_I is given, as in section 4, by making a solution constant and diagonal on each $[[\mu_k+C,\mu_{k+1}-C]]$ by a complex gauge transformation with $g(\infty_k)=1$ and $g(t)=1$ on each $[\mu_k,\mu_k+C/2]\cup[\mu_{k+1}-C/2,\mu_{k+1}]$. The inverse mapping is given by first solving the real Nahm equation by a complex transformation which is exponentially close to $\exp\{h_kt-p_k\}$ near ∞_k for some diagonal h_k,p_k. To see that this can be done we argue as follows (cf. [2]). By the argument in the proof of Proposition 8.1, we can first solve the real equation on each $(\infty_{k-1},\mu_k)\cup(\mu_k,\infty_k)$ by bounded gauge transformation $g_k(t)$ satisfying the matching condition at μ_k. Now, from Corollary 8.2 and the definition of $\tilde M$ as the hyperkähler (and so complex-symplectic) quotient of the $\tilde F_(n,m)(c,c')$, we conclude that there is a global action of $(\mathbb{C}^*)^m$ on $\tilde M$. Using this action allows us to replace the g_k by a gauge transformation $g(t)$ which is exponentially close to $\exp\{h_kt-p_k\}$ near each ∞_k and which also solves the real equation.
Part 2: Estimates on solutions. Thus we constructed the biholomorphism and

it remains to prove the estimates. This is of course a local problem and we can assume that we are in a small enough open set U in which the (local) action of the Lie algebra of T^m is free. Then the phases and positions of particles in U identify U with an open subset of $(0,\mathbf{x})$ in $\mathbb{R}^m \times \tilde{C}_m(\mathbb{R}^3)$. We shall write the coordinates on $\mathbb{R}^m \times \mathbb{R}^{3m}$ as $\rho_i, \mathbf{x}_i$, $i=1,\dots,m$, with $\mathbf{x}_i=(x_i,z_i)$ in the decomposition $\mathbb{R}^3 = \mathbb{R}\times\mathbb{C}$ corresponding to the chosen complex structure I.
For each element of $\tilde{M}$ in U we choose a representative (T_0,T_1,T_2,T_3) such that for each k and each generator $\rho_i \in \mathbb{R}^{m_k}$ $(\mathbb{R}^m = \prod_k \mathbb{R}^{m_k})$ we have

$$\frac{\partial}{\partial\rho_i} T_0(+\infty_k) = d\mu_1(IX_{\rho_i}) \tag{9.3}$$

where μ_1 is the moment map for the action of T^m (and so of $\mathbb{R}^m$) with respect to ω_1 and X_{ρ_i} is the Killing vector field corresponding to ρ_i. Obviously it does not matter which complex structure we use. In this trivialization $\{0\}\times\tilde{C}_m(\mathbb{R}^3)$ corresponds to solutions with $T_0(\infty_k)=0$, $k=1,\dots,N-1$.

We first obtain estimates on solutions to Nahm's equations. Recall that the biholomorphism ϕ was defined as the composition $\phi=\phi_2\phi_1$ with $\phi_1:\tilde{M}\to M_I$ and $\phi_2 : M_I \to M$. A solution to Nahm equations on a half-line $[x,\infty)$, with a regular triple as the limit at infinity, approaches its limit exponentially fast [23]. Furthermore, by section 1 of [5] the estimate $T_i(t) = O\big(\exp\{-\lambda R t\}$ is uniform in the T_i on any $[x+\epsilon,+\infty)$, $\epsilon>0$. Here R is the separation distance of particles as in (9.1). Now, ϕ_1 was defined by a complex gauge transformation $g(t)$ making α and β constant and diagonal with $g(\infty_k)=1$ and $g(t)=1$ on each $[\mu_k,\mu_k+C/2]\cup[\mu_{k+1}-C/2,\mu_{k+1}]$. Thus we conclude that $(\tilde{T}_0,\tilde{T}_1,\tilde{T}_2,\tilde{T}_3)=\phi_1(T_0,T_1,T_2,T_3)$ satisfies

$$|\tilde{T}_i(t)-T_i(t)| = \begin{cases} 0 & \text{if } t\in[\mu_k,\mu_k+C/2]\cup[\mu_{k+1}-C/2,\mu_{k+1}] \\ O\big(\exp\{-\lambda R_k t\}\big) & \text{if } t\in[[\mu_k+C,\mu_{k+1}-C]]. \end{cases} \tag{9.4}$$

We now consider ϕ_2. After cutting off the solutions, we obtain a solution $(\hat{\alpha},\hat{\beta})$ on $[\mu_1,\mu_N]$ to the complex Nahm equation which satisfies

$$F(\hat{\alpha},\hat{\beta}) := \frac{d}{dt}(\hat{\alpha}+\hat{\alpha}^*) + [\hat{\alpha},\hat{\alpha}^*] + [\hat{\beta},\hat{\beta}^*] = O(e^{-\lambda R}).$$

Using Lemma 2.10 in [12] and a simple comparison theorem ([3], Lemma 2.8), one shows that the real equation can be solved on each $[\mu_k,\mu_{k+1}]$ by a complex gauge transformation $g_k(t)$ with $g_k(\mu_k)=g_k(\mu_{k+1})=1$ and $g_k^*g_k$ uniformly bounded by $O(e^{-\lambda R})$. Furthermore, the proof Corollary 2.19 and of Lemma 2.20 of [12] shows that the first derivative of $g_k^*g_k$ is also bounded like $O(e^{-\lambda R})$ near μ_k,μ_{k+1}. Thus the resulting solutions $(\check{\alpha},\check{\beta})$ differ near each μ_k from $(\hat{\alpha},\hat{\beta})$ by $O(e^{-\lambda R})$ (again using the gauge freedom for the skew-hermitian part of $\check{\alpha}$). Working in the lower-triangular gauge, as in [5], shows that this estimate holds on whole $[\mu_1,\mu_N]$. The estimate on the first derivative of $g_k^*g_k$ at μ_k,μ_{k+1} also shows that, while $\check{\alpha}$ does not satisfy the matching conditions at the μ_k, the jumps are of order $O(e^{-\lambda R})$. One now goes through the proofs of Lemma 2.19 and Propositions 2.20 and 2.21 in [19] to show that one can solve the real equation *and* the matching conditions by a complex gauge transformation $g(t)$ on $[\mu_1,\mu_N]$ such that $g^*(t)g(t)$ differs uniformly from 1 by $O(e^{-\lambda R})$. One repeats the argument with the derivatives to conclude that $(\tilde{\tilde{T}}_0,\tilde{\tilde{T}}_1,\tilde{\tilde{T}}_2,\tilde{\tilde{T}}_3=\phi_2(\tilde{T}_0,\tilde{T}_1,\tilde{T}_2,\tilde{T}_3)$ satisfies

$$|\tilde{\tilde{T}}_i(t)-\tilde{T}_i(t)| = O(e^{-\lambda R}) \tag{9.5}$$

on all of $[\mu_1,\mu_N]$ for $i=0,1,2,3$ (for $i=0$ one uses gauge freedom to guarantee the estimate).

Part 3: Estimates for the metric. We now wish to estimate the metric and the derivatives of its coefficients. First, if we view the chosen representatives (T_0,T_1,T_2,T_3), $(\tilde{T}_0,\tilde{T}_1,\tilde{T}_2,\tilde{T}_3)$ and $(\tilde{\tilde{T}}_0,\tilde{\tilde{T}}_1,\tilde{\tilde{T}}_2,\tilde{\tilde{T}}_3)$ as functions of the $\rho_i,\mathbf{x}_i$, then going through the successive linearizations of the procedure just described we conclude that the estimates (9.4) and (9.5) can be differentiated with respect to ρ_i,x_i,z_i (here (9.3) is essential). Let us write $(s_0^{(p)},s_1^{(p)},s_2^{(p)},s_3^{(p)})$ for any partial derivative of degree p of T_i or the $\tilde{\tilde{T}}_i$ (with respect to ρ_i,x_i,z_i). The integral (3.4) or (3.5) with the $s_i^{(p)}$ replacing the t_i does not define a coefficient, or its derivative, of the metrics $\tilde{g}$ or g, since $(s_0^{(1)},s_1^{(1)},s_2^{(1)},s_3^{(1)})$ does not satisfy the first of the equations (3.3). We have to first modify $(s_0^{(1)},s_1^{(1)},s_2^{(1)},s_3^{(1)})$ by an infinitesimal gauge transformation $\rho(t)$. One obtains the estimates on $\rho,\dot{\rho}$ and their partial derivatives as in section 3 of [3], using the equations (3.3) and their successive linearizations (together with linearizations of the equations in [19] on the jumps of the $\check{\alpha}^{(p)}$). There is one point worth emphasizing. At each stage we obtain a differential inequality for $y(t)=\|\rho^{(p-1)}(t)\|$: $\ddot{y}(t)\geq -Ke^{\lambda R}$ if we are dealing with a finite interval and $\ddot{y}(t)\geq -Ke^{-(t+1)\lambda R}$ if we are dealing with a half line. We want to conclude that our solution will be exponentially small on the whole interval or half-line. This is no problem on a finite interval, since then $\rho^{(p-1)}$ is exponentially small at the endpoints. On a half-line, however, $\rho(t)$ is apriori an infinitesimal gauge transformation, and so exponentially close to $\exp\{ht+p\}$ for some diagonal h,p. Here we use condition (9.3). Suppose that $s=(s_0^{(1)},s_1^{(1)},s_2^{(1)},s_3^{(1)})$ corresponds to the tangent vector $v=(v_0,v_1,v_2,v_3)$ in $\mathbb{R}^{4m}$. In order to solve the first of equations (3.3) we modify s by $\rho(t)$. We shall then obtain a tangent vector (t_0,t_1,t_2,t_3) corresponding to the vector v. However, using the hypercomplex structure, we see that $t_0(\infty)=d\mu_1(X_{v_0})$. Thus, from our choice (9.3), $s_0(\infty_k)=t_0(\infty_k)$ and so $\dot{\rho}(\infty_k)=0$ for each $k=1,\dots,N-1$. In order to deal with p we observe that ρ is unique (cf. proof of Proposition 4.1 in [2]; the uniqueness is essentially equivalent to smoothness of our open set U). Now we go back to the description of $\tilde{M}$ as a hyperkähler quotient of a product Q of $\tilde{F}_{n,m}(c,c')$. Then s_0 and t_0 are descended from vectors tangent to Q and we can repeat the argument to show that we can find a bounded infinitesimal gauge transformation $\rho_k(t)$ on each $[[\mu_k,\mu_{k+1}]]$ taking the restriction of s_0 to the restriction of t_0. However, each ρ_k is exponentially close, by Remark 3.1, to $\exp(h_kt)$, for some diagonal h_k. Thus each ρ_k is exponentially close to zero near ∞_k (that the rate of decay is $O(\big(e^{-(t+1)\lambda R_k}\big)$ follows from the fact that (t_0,t_1,t_2,t_3) satisfies (3.3)); see also proof of Lemma 3.8 in [23]). Combining the ρ_k must give us ρ by uniqueness. Thus our infinitesimal gauge transformatio

n is exponentially close to zero near each ∞_k and the inequality $\ddot{y}(t)\geq -Ke^{-(t+1)\lambda R}$ on $[0,\infty)$ for the norm y of ρ does guarantee that $\rho(t)$ is bounded as $O\big(e^{-(t+1)\lambda R}\big)$ on the whole half-line (if $\rho(0)=0$). The same argument works for $\rho^{(p)}(t)$, $p\geq 1$. $\square$

As remarked after the statement of the above theorem, there is a likely generalization of this result. Suppose that it is only particles of a given type, say k_0, that separate (recall that the type of particle i is the smallest k for which $i\leq m_k$). Then the metric on $F_\sigma(\mu)=F_{m_1,\dots,m_{N-1}}(\mu_1,\dots,\mu_N))$ should get close to the metric on $F_{\tilde{\sigma}}(\mu)$, where $\tilde{\sigma}(k)=m_k$ if $k\neq i_0$ and $\tilde{\sigma}(k_0)=\tilde{m}_{k_0}$. Similarily, if the particles of types $k_1,\dots,k_s$ separate, the metric should be close to the metric on $F_{\tilde\sigma}(\mu)$, where $\tilde\sigma(k)=m_k$ if $k\neq k_1,\dots,k_s$ and $\tilde\sigma(k_j)=\tilde m_{k_j}$, for $j=1,\dots,s$. All of these moduli spaces have dimension $4(m_1+\ldots+m_{N-1})$. In general the metric on $F_{\tilde\sigma}(\mu)$ will be simpler than the one on $F_\sigma(\mu)$ (it has a tri-Hamiltonian action of a $(\sum_{i=1}^s m_{k_i})$-dimensional torus), but it is only in the case when $\{k_1,\dots,k_s\}=\{1,\dots,N-1\}$ that the metric is algebraic. The only difficulty in proving that the metrics are exponentially close lies in the construction of the biholomorphism ϕ. We used there, in an essential way, the fact that the action of T^m on $F_{\tilde\sigma}(\mu)$ extends to a global action of $(\mathbb{C}^*)^n$ and this followed from our knowledge of the twistor space of $F_{\tilde\sigma}(\mu)$. We can nevertheless conclude that in the general case, there is a biholomorphism ϕ in a neighbourhood of each point of $F_{\tilde\sigma}(\mu)$ such that the estimate (9.2) holds with $R=\min\{R_{k_i};i=1,\dots,s\}$. We cannot conclude that this estimate can be differentiated term by term, since we do not know precisely the positions of particles of types different than $k_1,\dots,k_s$, but we can conclude that the Riemannian curvature tensor also satisfies the estimate (9.2). This follows, as above, by arguments modelled on section 4 of [3].

We can finally compute the asymptotic monopole metric, i.e. the metric on $F_{\tilde m_1,\dots,\tilde m_{N-1}}(\mu_1,\dots,\mu_N)$:

Theorem 9.2. *The hyperkähler metric on $F_{\tilde m_1,\dots,\tilde m_{N-1}}(\mu_1,\dots,\mu_N)$ is determined by the matrix Φ of the form* (2.3) *with the c_i and s_{ij} given by* (2.4) *and* (2.5)*.*

Proof. From its definition in section 3 $F_{\tilde m_1,\dots,\tilde m_{N-1}}(\mu_1,\dots,\mu_N)$ is the hyperkähler quotient, by a product of tori, of the product

$$\tilde F_{m_1}(c_1)\times\tilde F_{m_2,m_1}(c_2,c_2')\times\cdots\times\tilde F_{m_{N-1},m_{N-2}}(c_{N-1},c_{N-1}')\times\tilde F_{m_N}(c_N')$$

where $c_i+c_{i+1}'=\mu_{i+1}-\mu_i$, $i=1,\dots,N-1$. The matrices Φ for each factor are of the form (2.3) with the s_{ij} given by (6.13) (for the first and last factor the metric is the Gibbons-Manton metric given by (2.2)). On the hyperkähler quotient these matrices are simply added together (after viewing each of them as a submatrix of an $m\times m$ matrix, $m=m_1+\cdots+m_{N-1}$). Thus the result is proven as soon as we show that all ϵ_{ij} of (6.13) are zero. Suppose that this is not true. Then the metric $F_{\tilde m_1,\dots,\tilde m_{N-1}}(\mu_1,\dots,\mu_N)$ is not positive definite when the i-th and the j-th particle are close together and all the remaining ones are far apart from each other and from these two. However, since these two particles are of different types, the previous theorem shows that the metric in this region is exponentially close to the monopole metric and so it must be positive-definite. The resulting contradiction proves the result. □

Finally we shall discuss the topology of the asymptotic moduli space. First of all, from Proposition 8.1, the orbit space of $F_{\tilde m_1,\dots,\tilde m_{N-1}}(\mu_1,\dots,\mu_N)$ is $\prod\tilde C_{m_k}(\mathbb{R}^3)$, and so particles of different types can take the same position. Now recall from section 2 that the type $t(i)$ of the particle i is defined as $\min\{k;i\leq\sum_{s\leq k}m_s\}$. It follows easily from Proposition 8.1 that the set of principal orbits of $T^{\overline m}$, $m=m_1+\cdots+m_{N-1}$, on $F_{\tilde m_1,\dots,\tilde m_{N-1}}(\mu_1,\dots,\mu_N)$ is a bundle P over

$$C=\{(\mathbf{x}_1,\dots,\mathbf{x}_m)\in\mathbb{R}^3\otimes\mathbb{R}^m;|t(i)-t(j)|\leq1\implies\mathbf{x_i}\neq\mathbf{x_j}\}.$$

The basis of the second integer homology of C is given by the spheres S_{ij} defined by (8.1), where now i,j run over the set $\{(i,j);i<j \text{ and } |t(i)-t(j)|\leq1\}$. As in Lemma 7.1 in [2], we obtain that the bundle P is determined by the element

$(h_1, \dots, h_m)$ of $H^2(C, \mathbb{Z}^m)$ such that

$$h_k(S_{ij}) = \begin{cases} s_{ij} & \text{if } k = i \\ -s_{ij} & \text{if } k = j \\ 0 & \text{otherwise,} \end{cases}$$

where the s_{ij} are given by (2.5).

Appendix A. Complex-symplectic vs. hyperkähler quotients

In order to finish the proof of Proposition 8.1 we have to show that certain hyperkähler and complex-symplectic quotients coincide. This question reduces, on the zero-set of the complex moment map, to identifying the Kähler quotient with the ordinary geometric quotient, i.e. to showing that all orbits of the complexified group are stable. The following useful criteria are given in [16], Lemma 3.3, and [15], Sections A.1.3 and A.2.3.

Proposition A.1. *Let H be a connected closed subgroup of a compact semisimple Lie group G and suppose that $M = G^{\mathbb{C}}/H^{\mathbb{C}}$ is equipped with a G-invariant Kähler form ω defined by a global G-invariant Kähler potential K for ω. Then the single $G^{\mathbb{C}}$-orbit M is stable if and only if K is proper.*

Proposition A.2. *Suppose that the complex torus $M = (\mathbb{C}^*)^n$ is equipped with a T^n-invariant Kähler form ω. Then:*

(i) *The T^n-action is Hamiltonian if and only if there exists a global T^n-invariant Kähler potential K for ω;*
(ii) *If K has a quadratic growth at infinity, as a function on $\mathbb{R}^n$ where $(\mathbb{C}^*)^n = T^n e^{\mathbb{R}^n}$, then M is stable.*

In order to use these criteria we shall view the spaces involved in the proof of Proposition 8.1 as hyperkähler quotients of simpler manifolds.

Recall that, for a regular triple (τ_1, τ_2, τ_3) of $n \times n$ diagonal matrices, $M(\tau_1, \tau_2, \tau_3)$ denotes Kronheimer's hyperkähler structure on $Gl(n, \mathbb{C})/T^{\mathbb{C}}$ (T is the diagonal torus in $U(n)$) with the cohomology class of ω_i equal to τ_i, $i = 1, 2, 3$ (after identifying $H^2(Gl(n, \mathbb{C})/T^{\mathbb{C}}, \mathbb{R})$ with $\mathrm{Lie}(T)$). We have

Proposition A.3. *$M(\tau_1, \tau_2, \tau_3)$ is isomorphic to a hyperkähler quotient of a flat quaternionic vector space by a product G of unitary groups. With respect to generic complex structure the action of $G^{\mathbb{C}}$ on the zero-set of the complex moment map is free and its orbits are closed.*

Proof. Kobak and Swann [22] show how to construct nilpotent orbits as hyperkähler quotients by a product G of unitary groups. Changing the level set of the moment map, from zero to appropriate values, gives a hyperkähler manifold N which, with respect to a generic complex structure, is isomorphic, as a complex-symplectic manifold, to $M(\tau_1, \tau_2, \tau_3)$ (if the levels of the complex moment map for the abelian factors of G are $t_1, \dots, t_{n-1}$, then the resulting orbit has eigenvalues $0, t_1, t_1 + t_2, \dots, (t_1 + \dots + t_{n-1})$). As a Riemannian manifold, N is complete and the uniqueness result of [6] shows that $N \simeq M(\tau_1, \tau_2, \tau_3)$ as hyperkähler manifolds.

The flat space which we start with can be viewed as a space of matrices and the second statement follows by putting the products of matrices defining the zero set of the complex moment map into tne Jordan normal form. $\square$

We can finally finish the proof of Proposition 8.1. We considered there a hyperkähler quotient X of the product of two $M(\tau_1,\tau_2,\tau_3)$'s and of either $\mathbb{H}^n$ or of $F_n(m;1)$. By the above proposition, X can be viewed as a hyperkähler quotient of either an $\mathbb{H}^p$ (when $n=m$), for some p, or of the product of $\mathbb{H}^p$ and of $F_n(m;1)$ (when $n>m$). Furthermore, the complexification $H^{\mathbb{C}}$ of the group H by which we quotient acts freely and with closed orbits on the zero-set of the (generic) complex moment map. In addition the usual Kähler potential K_1, given by the square of the distance from the origin, on $\mathbb{C}^{2p}\simeq\mathbb{H}^p$ is proper and $Sp(p)$-invariant. Moreover it has quadratic (in fact exponential) growth on any closed orbit of a subtorus of $Sp(p,\mathbb{C})$.

Since the $H^{\mathbb{C}}$-orbits are closed on the zero set of the complex moment map, Propositions A.1 and A.2 show that all these $H^{\mathbb{C}}$-orbits are stable for $n=m$.

The space $F_n(m;1)$ also has an $\big(U(n)\times U(m)\big)$-invariant Kähler potential for any complex structure. This is a general phenomenon for hyperkähler manifolds with an $SU(2)$-action rotating the complex structures, see [18]. For the complex structure I this Kähler potential is given by:

$$K=\int_0^1\big(\|T_1(t)\|^2+\|T_2(t)\|^2\big)dt.$$

Since the factor $\big(U(n)\times U(m)\big)$ of H is acts diagonally on $\mathbb{H}^p\times F_n(m;1)$, the Kähler potential K_2 on each orbit of $Gl(n,\mathbb{C})\times Gl(m,\mathbb{C})$ is the sum of the restrictions of K_1 and K. Since K_1 is proper and has quadratic growth on each closed toral orbit and since K is positive, it follows that K_2 is proper (on each orbit) and has quadratic growth on each closed toral orbit. Thus the complex-symplectic quotient coincides with the hyperkähler quotient in this case as well.

References

[1] M.F. Atiyah and N.J. Hitchin, *The geometry and dynamics of magnetic monopoles*, Princeton University Press, Princeton (1988).
[2] R. Bielawski, 'Monopoles and the Gibbons-Manton metric', *Comm. Math. Phys.*, to appear.
[3] R. Bielawski, 'Asymptotic behaviour of $SU(2)$ monopole metrics', *J. reine angew. Math.*, 468 (1995), 139–165.
[4] R. Bielawski, 'Hyperkähler structures and group actions', *J. London Math. Soc.*, 55 (1997), 400–414.
[5] R. Bielawski, 'Monopoles, particles and rational functions', *Ann. Glob. Anal. Geom.*, 14 (1996), 123–145.
[6] R. Bielawski, 'Invariant hyperkähler metrics with a homogeneous complex structure', *Math. Proc. Cambridge Phil. Soc.* 122 (1997), 473–482.
[7] O. Biquard, 'Sur les équations de Nahm et les orbites coadjointes des groupes de Lie semi-simples complexes', *Math. Ann.* 304 (1996), 253–276.
[8] O. Biquard, 'Twisteurs des orbites coadjointes', Ecole Polytechnique preprint (1997).
[9] G. Chalmers, 'Multi-monopole moduli spaces for $SU(N)$ gauge group', hep-th/9605182.
[10] A.S. Dancer, 'Nahm's equations and hyperkähler geometry', *Comm. Math. Phys.* 158 (1993), 545–568.
[11] A.S. Dancer, 'A family of hyperkähler manifolds', *Quart. J. Math. Oxford* 45 (1994), 463–478.
[12] S.K. Donaldson, 'Nahm's equations and the classification of monopoles', *Commun. Math. Phys.* 96 (1984), 387–407.
[13] G.W. Gibbons and N.S. Manton, 'The moduli space metric for well-separated BPS monopoles', *Phys. Lett. B* 356 (1995), 32–38.
[14] G.W. Gibbons and P. Rychenkova, 'HyperKähler quotient construction of BPS monopole moduli spaces', *Comm. Math. Phys.* 186 (1997), 581–599.
[15] V. Guillemin, *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*, Birkhäuser, Boston (1994).

[16] P. Heinzner, A.T. Huckleberry, and F. Loose, 'Kählerian extensions of the symplectic reduction', *J. reine angew. Math.* 455 (1994), 123–140.
[17] N.J. Hitchin, 'On the construction of monopoles', *Comm. Math. Phys.* 89 (1983), 145–190.
[18] N.J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, 'Hyperkähler metrics and supersymmetry', *Comm. Math. Phys.* 108 (1985), 535–586.
[19] J.C. Hurtubise, 'The classification of monopoles for the classical groups', *Comm. Math. Phys.* 120 (1989), 613–641.
[20] J.C. Hurtubise and M.K. Murray, 'On the construction of monopoles for the classical groups', *Comm. Math. Phys.* 122 (1989), 35–89.
[21] S. Jarvis, 'A rational map for Euclidean monopoles via radial scattering', Oxford preprint (1996).
[22] P.Z. Kobak and A. Swann, 'Classical nilpotent orbits as hyper-Kähler quotients', *Internat. J. Math.*, 7 (1996), 193–210.
[23] P.B. Kronheimer, 'A hyper-kählerian structure on coadjoint orbits of a semisimple complex group', *J. London Math. Soc.* 42 (1990), 193–208.
[24] U. Lindström and M. Roček, 'Scalar tensor duality and $N = 1, 2$ nonlinear σ-models', *Nucl. Phys.* 222B (1983), 285-308.
[25] K. Lee, E.J. Weinberg and P. Yi, 'Moduli space of many BPS monopoles for arbitrary gauge groups,' *Phys. Rev. D* 54 (1996), 1633–1643.
[26] M.K. Murray, 'A note on the $(1, 1, \ldots, 1)$ monopole metric', *J. Geom. Phys.* 23 (1997), 31–41.
[27] W. Nahm, 'The construction of all self-dual monopoles by the ADHM method', in *Monopoles in quantum field theory*, World Scientific, Singapore (1982).
[28] H. Nakajima, 'Monopoles and Nahm's equations' in *Einstein metrics and Yang-Mills connections*, Marcel Dekker, New York (1993).
[29] H. Pedersen and Y.S. Poon, 'Hyper-Kähler metrics and a generalization of the Bogomolny equations,' *Comm. Math. Phys.* 117 (1988), 569–580.
[30] C.H. Taubes, 'Min-max theory for the Yang-Mills-Higgs equations,' . *Comm. Math. Phys.* 97 (1985), 473-540.

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