MAPS BETWEEN p–COMPLETED

CLASSIFYING SPACES. II.

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1. Introduction.

The aim of this paper is to apply the programme from [1] to investigate maps between p–completed classifying spaces of compact connected Lie groups. However instead of cohomology with rational coefficients we shall use complex K–theory with p–adic coefficients.

We assume throughout that G and G' are compact, connected Lie groups with maximal tori T, T' and Weyl groups W, W'.

Definition 1.1 (see [2] page 2) Let R be a commutative ring. We say that a homomorphism of R–modules

$$\phi : \pi_1(T) \otimes R \longrightarrow \pi_1(T') \otimes R$$

is admissible if for each $w \in W$ there exists $w' \in W'$ such that $\phi \circ w = w' \circ \phi$.

Definition 1.2 We say that two admissible maps ϕ and ψ from $\pi_1(T) \otimes R$ to $\pi_1(T') \otimes R$ are equivalent if there exists $w' \in W'$ such that $\phi = w' \circ \psi$.

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[1] The results of this paper grew up in the correspondence with Professor J. Frank Adams during the period 1987–1988. These results form chapter 6 of our second thesis presented at Universitat Autònoma de Bracelona. We would like to thank very much Professor J. Frank Adams for correspondence, suggestions and constant encouragement.

It is clear that the relation defined above is an equivalence relation on the set of admissible maps from $\pi_1(T) \otimes R$ to $\pi_1(T') \otimes R$.

We shall denote by $\mathrm{Hom}_R(T,T')$ the set of equivalence classes of admissible maps from $\pi_1(T) \otimes R$ to $\pi_1(T') \otimes R$.

We recall from [2] Theorem 1.1 that for any map $f : (BG)_p \longrightarrow (BG')_p$ there is an admissible map $f_* : \pi_1(T) \otimes Z_p \longrightarrow \pi_1(T') \otimes Z_p$, unique up to the action of W', so f determines uniquely an equivalence class f_* in $\mathrm{Hom}_{Z_p}(T,T')$ which we shall denote by $\chi(f)$.

Theorem 1.3 If p does not divide the order of W then the map

$$\chi : [(BG)_p,(BG')_p] \longrightarrow \mathrm{Hom}_{Z_p}(T,T')$$

is bijective.

For any space we set

$$(*) \qquad H^*(X,Q_p) := H^*(X;Z_p) \otimes Q \ ,$$

where Q_p is a field of p–adic numbers.

We point out that $H^*((BG)_p,Q_p)$ defined by the formula $(*)$ is equal to the singular cohomology of BG with Q_p–coefficients.

Theorem 1.4 If p does not divide the order of W then the natural map

$$\Phi : [(BG)_p,(BG')_p] \longrightarrow \mathrm{Hom}(H^*((BG')_p,Q_p),H^*((BG)_p,Q_p))$$

is injective.

In [1], to study maps between classifying spaces, cohomology groups were used. However if one deals only with one fixed prime it is necessary to use complex K–theory if one wants to get analogous results.

We denote by $K^0(\ ,R)$ the 0^{th}–term of complex K–theory with R–coefficients. Let O_R be the set of operations in $K^0(\ ,R)$. The functor $K^0(\ ,R)$ is equipped with the natural augmentation $K^0(\ ,R) \longrightarrow R$.

Let $\mathrm{Hom}_{O_R}(K^0(X,R); K^0(Y,R))$ be the set of R–algebra homomorphism which commute with the action of O_R and augmentations.

Theorem 1.5 If p does not divide the order of W then the natural map

$$\Psi : [(BG)_p,(BG')_p] \longrightarrow \mathrm{Hom}_{O_{Z_p}}(K^0((BG')_p,Z_p), K^0((BG)_p, Z_p))$$

is bijective.

(We point out that $K^0((BG)_p,Z_p) = K^0(BG,Z_p)$.)

Corollary 1.6 Let n be the order of W. If p does not divide n then the maps

$$\Psi : [BG,(BG')_{1/n}] \longrightarrow \mathrm{Hom}_{O_{Z[1/n]}}(K^0((BG')_{1/n},Z_{[1/n]}),K^0(BG,Z_{[1/n]}))$$

and

$$\chi : [BG,(BG')_{1/n}] \longrightarrow \mathrm{Hom}_{Z_{[1/n]}}(T,T')$$

are bijective.

The next result shows that there is a direct relation between Theorem 1.3 and Theorems 1.4 and 1.5.

Theorem 1.7 Let f and g be two maps from $(BG)_p$ to $(BG')_p$. Then the following conditions are equivalent:

i) $\chi(f) = \chi(g)$ in $\mathrm{Hom}_{Z_p}(T,T')$,

ii) $K^0(f,Z_p) = K^0(g,Z_p)$,

iii) $H^*(f,Q_p) = H^*(g,Q_p)$.

The rest of the paper contains proofs of the announced results. We shall prove the results in almost reversed order to that in which we announced them. However we feel this was a natural way to present our results, while the proofs, require a change of order.

We would like to thank the referee for pointing several misprints. The referee also informed us that the related results were obtaind by J. Aguade, D. Notbohm and L. Smith.

2. Proofs.

Proof of Theorem 1.7.

i) $\Rightarrow$ iii). The condition $\chi(f) = \chi(g)$ implies $H^*(f,Q_p) = H^*(g,Q_p)$ because $H^*((BG)_p,Q_p) \longrightarrow H^*((BT)_p,Q_p)$ and the similar map for G' are injective.

iii) $\Rightarrow$ i). It follows from [2] Theorem 1.1 and iii) that the compositions

$$H^*((BT)_p,Q_p) \underset{\bar{g}^*}{\overset{\bar{f}^*}{\leftleftarrows}} H^*((BT')_p,Q_p) \longleftarrow H^*((BG')_p,Q_p)$$

are equal. The same arguments as in the proof of Theorem 1.7 in [1] shows that $\chi(f) = \chi(g)$.

ii) $\Leftrightarrow$ iii). For a connected, compact Lie group G the Chern character

$$ch : K^0((BG)_p,Z_p) \longrightarrow H^{ev}((BG)_p,Q_p)$$

is injective and after tensoring with Q becomes an isomorphism. Therefore the functoriality of the Chern character implies that ii) is equivalent to iii).

Proof of Theorem 1.4

First we collect some facts which we shall use in the proof.

Let $\pi = \varinjlim_n \pi_n$ be a direct limit of finite groups. We set

$$R(\pi) = \varprojlim_n R(\pi_n) \ ,$$

where $R(\pi_n)$ is the complex representation ring of a finite group π_n.

Let us notice that

2.1. $$K^0(B\pi,Z_p) = \varprojlim_n K^0(B\pi_n,Z_p) \ .$$

The natural map

$$R(\pi_n) \longrightarrow K^0(B\pi_n, Z_p)$$

induces

$$R(\pi) \longrightarrow K^0(B\pi, Z_p) \ .$$

2.2. If $\pi = \varinjlim_n \pi_n$ is a p–group i.e. each π_n is a p–group then

$$\varprojlim_n (R(\pi_n) \otimes Z_p) \longrightarrow K^0(B\pi, Z_p)$$

is an isomorphism, and the map

$$R(\pi) \longrightarrow K^0(B\pi, Z_p)$$

is injective.

For a torus T we denote by T(n) a subgroup of T consisting of elements which are in the kernel of the multiplication by p^n. We set $T(\infty) = \bigcup_{n=1}^{\infty} T(n)$.

The proof of the theorem will follow closely the line of the proof of Theorem 1 in [6] with few modifications. Let $f,g : (BG)_p \longrightarrow (BG')_p$ be two maps such that $H^*(f,Q_p) = H^*(g,Q_p)$. Let $i : BT(\infty) \longrightarrow (BG\ )_p$ be the map induced by the inclusion of T into G. Then it follows from [3] Theorem 1.1 that the maps $f \circ i$ and $g \circ i$ are induced by homomorphisms ϕ, $\psi : T(\infty) \longrightarrow G'$. The Chern character $ch : K^0(BT(\infty),Z_p) \longrightarrow H^{ev}(BT(\infty),Q_p)$ is injective. The equality $H^*(B\phi,Q_p) = H^*(B\psi,Q_p)$ implies that $K^0(B\phi,Z_p) = K^0(B\psi,Z_p)$. Let us notice that

$T(\infty)$ is a p–group. Hence 2.2 implies that ϕ and ψ induce the same homomorphism of representations rings

$$R(\phi) = R(\psi) : R(G') \longrightarrow R(T(\infty)) \ .$$

We can assume that the images of ϕ and ψ are in T'. Let $i_\alpha : Z/p^\infty \longrightarrow T(\infty)$ be an inclusion. Then the restrictions of ϕ and ψ to $i_\alpha(Z/p^\infty)$ are conjugate by some element of W. There is an uncountable number of families $i_1(Z/p^\infty),\dots,i_m(Z/p^\infty)$ which generate $T(\infty)$, but the group W is finite. Hence ϕ and ψ are conjugate by some element of W. This implies that the maps $B\phi$ and $B\psi$ are homotopic. The Weyl group W acts on $T(\infty)$ and therefore it acts also on $BT(\infty)$, and we can assume that this action is free. The space $BT(\infty)/W$ is homologically equivalent at a prime p to $(BG)_p$. Hence Theorem 1 from [5] implies that f and g are homotopic.

3. K–theory.

To realize the programme from [1] in a case of a single prime it is necessary to replace cohomology by K–theory. The reason is very simple. In a global situation considered in [1] one has the Steenrod operations for almost all primes. In a single prime case, $H^*((BG)_p,Q_p)$ can have only Steenrod operations for one prime, but $K^0((BG)_p,Q_p)$ has Adams operations ψ^k for all $k \in Z$. It is well known that ψ^k operations are closely related to Steenrod operations for all primes. This principle allows us to get the similar results in a single prime case, as the results got in [1] in almost all primes case.

Lemma 3.1 Let us set $X(n) = B(Z/p^\infty)^n$. There is a bijection

$$[X(m),X(n)] \cong \mathrm{Hom}_{O_{Z_p}}(K^0(X(n),Z_p),K^0(X(m),Z_p)) \ .$$

Proof. It is well known that $K^0(X(n),Z_p) = Z_p[[H_1,\dots,H_n]]$ and $K^0(X(m),Z_p) = Z_p[[K_1,\dots,K_m]]$ where the generator H_i (resp. K_i) corresponds to the i^{th}–projection $(Z/p^\infty)^n$ (resp. $(Z/p^\infty)^m) \longrightarrow Z/p^\infty$.

Let $f \in \mathrm{Hom}_{O_{Z_p}}(K^0(X(n),Z_p),K^0(X(m),Z_p))$. Then $f(H_k) = \sum a_{i_1,\dots,i_m} K_1^{i_1} \cdot \ldots \cdot K_m^{i_m}$, where the summation is over all indices $(i_1,\dots,i_m)$ such that $i_1 + \dots + i_m > 0$ and $i_\ell \geq 0$ for $\ell = 1,\dots,m$. The fact that f commutes with ψ^2 implies the equality

(*)

$$(1 + \sum a_{i_1,\dots,i_m} K_1^{i_1} \cdot \ldots \cdot K_m^{i_m})^2 - 1 = \sum a_{i_1,\dots,i_m}((K_1^2 + 2K_1)^{i_1} \cdot \ldots \cdot (K_m^2 + 2K_m)^{i_m} \ .$$

Comparing coefficients at $K_1^{i_1} \cdot \ldots \cdot K_m^{i_m}$ we get

$$2a_{i_1,\dots,i_m} + P = 2^{i_1+\dots+i_m} a_{i_1,\dots,i_m} + P' \ ,$$

where P and P' are polynomials in $a_{j_1,\dots,j_m}$ with $j_1 + \dots + j_m < i_1 + \dots i_m$ and $j_\ell \leq i_\ell$ for $\ell = 1,\dots,m$. Therefore $a_{j_1,\dots j_m}$ are uniquely determined by $c_{1,k} = a_{1,0,\dots,0}$, $c_{2,k} = a_{0,1,0\dots0}, \dots, c_{m,k} = a_{0,\dots,0,1}$.

Let us notice that

$$(**) \qquad f(H_k) = (1+K_1)^{c_{1,k}} \cdot (1+K_2)^{c_{2,k}} \cdot \ldots \cdot (1+K_m)^{c_{m,k}} - 1$$

satisfies $(*)$, hence f is given by the formula $(**)$. An element f defines a matrice $C = (c_{i,j})$ with coefficients in Z_p. Such a matrice C defines a homomorphism from $(Z/p^{\infty})^m$ to $(Z/p^{\infty})^n$. This homomorphism induces f. This finishes the proof of the lemma.

Proof of Theorem 1.5.

The fact that ϕ is injective follows immediately from Theorem 1.7 and Theorem 1.4.

Let $f : K^0((BG')_p, Z_p) \longrightarrow K^0((BG)_p, Z_p)$ be a Z_p–algebra homomorphism commuting with augmentations and the action of O_{Z_p}. Let $i : K^0((BG)_p, Z_p) \longrightarrow K^0(BT(\infty), Z_p)$ and $i' : K^0((BG')_p, Z_p) \longrightarrow K^0(BT'(\infty), Z_p)$ be induced by inclusions of maximal tori into Lie groups. It follows from [4] Theorem 4.1 that there is a Z_p–algebra homomorphism

$$F : K^0(BT'(\infty), Z_p) \longrightarrow K^0(BT(\infty), Z_p)$$

commuting with O_{Z_p}–action such that $i \circ f = F \circ i'$. From the proof of Theorem 4.1 it follows that F is compatible with augmentations. Lemma 3.1 implies that there is $g : BT(\infty) \longrightarrow BT'(\infty)$ which on $K^0(\ , Z_p)$ induces F. The arguments like those used in the proof of Theorem 1.7 in [1] shows that the map

$$\phi : \pi_1(T) \otimes Z_p \longrightarrow \pi_1(T') \otimes Z_p$$

induced by g is admissible.

This implies that the composition

$$k : BT(\infty) \xrightarrow{g} BT'(\infty) \longrightarrow (BG')_p$$

has the property, that for any $w \in W$, the map $k \circ w$ is homotopic to k. We can assume that the action of W on $BT(\infty)$ is free. Hence it follows from [5] Theorem 1 that there is $K : (BG)_p \longrightarrow (BG')_p$ such that K is restricted to $BT(\infty)$ is homotopic to k. It is an obvious verification that $K^0(K, Z_p) = f$.

Proof of Theorem 1.3.

The injectivity of the map χ follows from Theorem 1.7 and Theorem 1.4.

To show that χ is surjective one uses [5] Theorem 1 in the same way as in the proof of Theorem 1.5.

Proof of Corollary 1.6.

The corollary follows from the Sullivan arithmetic square and the lack of phantoms in the situations considered.

REFERENCES

1. J.F. Adams, Z. Mahmud, Maps between classifying spaces, Inventiones Math. **35** (1976), 1–41.
2. J.F. Adams, Z. Wojkowiak, Maps Between p–Completed Classifying Spaces, Proc. of the Royal Society of Edinburgh, 112A, 1989, 231–235.

3. W. Dwyer, A. Zabrodsky, Maps between classifying spaces, in "Algebraic Topology, Barcelona 1986", Lecture Notes in Mathematics 1298, Springer–Verlag, 1987, pp. 106–119.
4. C. Wilkerson, Lambda–rings, binomial domains, and vector bundles over $CP(\infty)$, Communications in Algebra 10 (3) (1982), 311–328.
5. Z. Wojtkowiak, Maps from $B\pi$ into X, Quart. J. Math. Oxford (2) 39 (1988), 117–127.
6. Z. Wojtkowiak, A remark on maps between classifying spaces of compact Lie groups, Canadian Mathematical Bulletin, 31 (4) (1988), 452–458

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