Holomorphic Automorphisms Of Quadrics Of Codimension 2 In $\mathbb{C}^5$

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1. Introduction

Let $(z_1, \ldots, z_n, w_1, \ldots, w_k)$ be coordinates in $\mathbb{C}^{n+k}$. A quadric of codimension k in $\mathbb{C}^{n+k}$ will be given by the equations

$$v^j = \sum_{\mu,\nu=1}^{n} H^j_{\mu\nu} z^\mu \bar{z}^\nu = \langle z, z\rangle^j, j = 1, \ldots, k, \tag{1}$$

where $\langle z, z\rangle^j$ is a hermitian form in $z = (z_1, \ldots, z_n)$ and $w^j = u^j + iv^j, j = 1, \ldots, k$.

According to the definition of Baouendi-Trèves-Beloshapka, Q is called Levi-non-degenerate iff the forms $v^j = \langle z, z\rangle^j$ are linearly independent and

$$\langle z, a\rangle^j = 0 \text{ for } j = 1, \ldots, k \text{ and for all } z \in \mathbb{C}^n$$

implies $a = 0$.

It was proved by Beloshapka [2] that the nondegeneracy condition is equivalent to the finiteness of the group of holomorphic automorphisms. He also described the Lie algebras of these groups [3].

Since the quadrics are homogenious, we may restrict our interest to the so-called isotropy groups, the groups of automorphisms preserving a fixed point (say the origin).

In [4] the authors found the automorphism groups in the case $n = k = 2$, using a matrix substitution into the scheme of Chern-Moser's normalizations of the equation of the Heisenberg sphere in $\mathbb{C}^2$.

The same method allowed in [5] to find the automorphism groups of some quadrics with $n = k = 3$, among them Beloshapka's nullquadric.

In the present paper we give a classifcation of all types of quadrics with $n = 3, k = 2$ and their automorphism groups. The substituion scheme also works in this case with $n \neq k$.

It follows from a result by Abrosimov [1], that quadrics in general position with $n > 2, k = 2$ have only linear automorphisms. More precisely, if H^2 is non-degenerate in usual sense (this can be assumed if there is a linear combination of H^1 and H^2 being

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non-degenerate) and the matrix $H^1(H^2)^{-1}$ has more than two different eigenvalues then all holomorphic automorphisms of the corresponding quadric are linear.

In our case 4 of 10 different types have nonlinear automorphisms. Two of them are direct products. The other two quadrics (one of them is a nullquadric) give a counterexample to Beloshapka's conjecture, that the nullquadrics might have the largest automorphism groups.

2. Classification of the quadrics and the linear automorphisms

We will classify the possible types of Levi-nondegenerate quadrics with $n = 3, k = 2$ under the action

$$\begin{aligned} z^* &= Cz \\ w^* &= \rho w, \end{aligned}$$

where $C \in \mathrm{GL}(3, \mathbb{C}), \rho \in \mathrm{GL}(2, \mathbb{R})$.

Theorem 1. *Any nondegenerate quadric of codimension 2 in $\mathbb{C}^5$ is equivalent to one of the following, pairwise nonequivalent quadrics:*

$$
\begin{array}{rrcl}
(i) & v^1 & = & |z^1|^2 + |z^2|^2 \\
 & v^2 & = & |z^2|^2 + |z^3|^2 \\
(ii) & v^1 & = & |z^1|^2 - |z^2|^2 \\
 & v^2 & = & |z^2|^2 - |z^3|^2 \\
(iii) & v^1 & = & |z^1|^2 - |z^2|^2 \\
 & v^2 & = & |z^2|^2 + z^1\bar{z}^3 + z^3\bar{z}^1 \\
(iv) & v^1 & = & z^1\bar{z}^2 + z^2\bar{z}^1 \\
 & v^2 & = & |z^1|^2 - |z^3|^2 \\
(v) & v^1 & = & z^1\bar{z}^2 + z^2\bar{z}^1 \\
 & v^2 & = & |z^1|^2 + |z^3|^2 \\
(vi) & v^1 & = & z^1\bar{z}^2 + z^2\bar{z}^1 \\
 & v^2 & = & |z^2|^2 + z^1\bar{z}^3 + z^3\bar{z}^1 \\
(vii) & v^1 & = & |z^1|^2 + |z^2|^2 \\
 & v^2 & = & |z^3|^2 \\
(viii) & v^1 & = & |z^1|^2 - |z^2|^2 \\
 & v^2 & = & |z^3|^2 \\
(ix) & v^1 & = & |z^1|^2 \\
 & v^2 & = & |z^2|^2 + z^1\bar{z}^3 + z^3\bar{z}^1 \\
(x) & v^1 & = & z^1\bar{z}^2 + z^2\bar{z}^1 \\
 & v^2 & = & z^1\bar{z}^3 + z^3\bar{z}^1
\end{array}
$$

At first we consider the case when there exists a linear combination of the two forms which is positive definite. Then we can without loss of generality assume that

$$v^1 = |z^1|^2 + |z^2|^2 + |z^3|^2$$

After some coordinate transformation preserving this first form the second form can be written

$$v^2 = a_{11}|z^1|^2 + a_{22}|z^2|^2 + a_{33}|z^3|^2.$$

Substituting the second from by some linear combination we obtain

$$v^2 = (a_{22} - a_{11})|z^2|^2 + (a_{33} - a_{11})|z^3|^2.$$

Since the two forms are linear independent it follows the at least one of the coefficients in the second form is different from 0, say a_{33}. Therefore we can assume that the two forms after some transformation ρ in $\mathbb{C}^2_w$ are

$$\begin{aligned} v^1 &= |z^1|^2 + (1-\kappa)|z^2|^2 \\ v^2 &= |z^3|^2 + \kappa|z^2|^2 \end{aligned}$$

where $\kappa = a_{22}/a_{33}$

If $\kappa = 0$ or $\kappa = 1$ we obtain case *vii*, otherwise, after some ρ transformation, case *i*.

We assume now, that there does not exist any linear combination of the two forms being positive definite. We prove that also in this case thre exists a linear combination of rank not exceeding 2:

Lemma 1. *Let a nondegenerate quadric of codimension 2 in $\mathbb{C}^5$ be given by* 1. *Then there exist coordinates in $\mathbb{C}^3$ and a linear combination of the two forms of rank not exceeding 2.*

Proof. We choose coordinates such that the first form is diagonal. If it is positive definite or of rank < 3, the lemma is proved. Suppose, it has the signature $v^1 = |z^1|^2 + |z^2|^2 - |z^3|^2$ and the second form is arbitrary: $v^2 = \sum a_{ij} z^i \bar{z}^j$. After some linear transformation in $\mathbb{C}^3$ preserving the first form, the second form satisfies the conditions: $a_{33} = a_{12} = 0$ and a_{13}, a_{23} are real.

Now we consider the linear combination $v^1 + tv^2$. We have to show that the determinant of the corresponding matrix vanishes for a suitable t. This determinant is a polynomial $p(t)$. Therefore it is sufficient to prove that it is not constant: If $p(t)$ is constant then $a_{11} = -a_{22}, |a_{23}| = |a_{13}|$ and $a_{13}^2 + a_{23}^2 = a_{11}^2$. But this means that v^2 has rank < 3. This completes the proof of the lemma.

We suppose now that the first form is $v^1 = |z^1|^2 + |z^2|^2$. Without loss of generality we may then assume that $v^2 = a_{11}|z^1|^2 + a_{22}|z^2|^2 + a_{33}|z^3|^2 + a_{13}(z^1\bar{z}^3 + z^3\bar{z}^1) + a_{23}(z^2\bar{z}^3 + z^3\bar{z}^2)$, where a_{13} and a_{23} are real. Since we consider the case that there is no positive definite linear combination of the two forms, we conclude that $a_{33} = 0$.

It follows from the condition that the quadric is nondegenerate that a_{13} and a_{23} cannot both equal to 0, hence there exists a transformation

$$\begin{aligned} z^1 &\mapsto z^1 \\ z^2 &\mapsto z^2 \\ z^3 &\mapsto z^3 + \alpha z^1 + \beta z^2 \end{aligned}$$

leading to $a_{11} = a_{22}$.

After trivial transformations $v^2 = z^1\bar{z}^3 + z^3\bar{z}^1$. This is case *v*.

We consider the case that $v^1 = |z^1|^2 - |z^2|^2$ and $v^2 = \sum a_{ij} z^i \bar{z}^j$.

Then two cases are possible: 1. $a_{33} \neq 0$ (then without loss of generality $a_{33} = 1$), or 2. $a_{33} = 0$.

In the first case we apply

$$\begin{aligned} z^1 &\mapsto z^1 \\ z^2 &\mapsto z^2 \\ z^3 &\mapsto z^3 - a_{13} z^1 - a_{23} z^2 \end{aligned}$$

then $v^2 = a_{11}|z^1|^2 + a_{22}|z^2|^2 + a_{12} z^1 \bar{z}^2 + \bar{a}_{12} z^2 \bar{z}^1 + |z^3|^2$.

By means of some transformation with respect to z^1, z^2 we obtain

$$\begin{aligned} v^1 &= z^1 \bar{z}^2 + z^2 \bar{z}^1 \\ v^2 &= a_{11}|z^1|^2 + a_{22}|z^2|^2 + |z^3|^2. \end{aligned}$$

The following cases are possible

$$\begin{array}{rr} & a_{11} = a_{22} = 0 \quad (viii) \\ a_{11} = 0, a_{22} > 0 & a_{22} = 0, a_{11} > 0 \quad (iv) \\ & a_{11} a_{22} > 0 \quad (i) or (ii) \\ & a_{11} a_{22} < 0 \quad (iii). \end{array}$$

Let $a_{33} = 0$. Then $a_{13} \neq 0$, or $a_{23} \neq 0$, and, without loss of generality $\operatorname{Im} a_{13} = \operatorname{Im} a_{23} = 0$. If $|a_{13}|^2 - |a_{23}|^2$, we apply

$$\begin{aligned} a_{13} z^1 + a_{23} z^2 &\mapsto z^1 \\ a_{23} z^1 + a_{13} z^2 &\mapsto z^2 \\ z^3 &\mapsto z^3 \end{aligned}$$

and obtain

$$\begin{aligned} v^1 &= |z^1|^2 - |z^2|^2 \\ v^2 &= \sum_{i,j=1,2} a_{ij} z^i \bar{z}^j + z^1 \bar{z}^3 + z^3 \bar{z}^1. \end{aligned} \tag{2}$$

One easily eliminates a_{22}. Then

$$\begin{aligned} z^1 &\mapsto z^1 \\ z^2 &\mapsto z^2 \\ z^3 &\mapsto z^3 - \frac{a_{11}}{2} z^1 - a_{12} z^2 \end{aligned}$$

leads to $v^2 = z^1\bar{z}^3 + z^3\bar{z}^1$. This is case (iv).
It remains to consider the case $|a_{13}| = |a_{23}|$.
We apply

$$\begin{aligned} z^1 &\mapsto z^1 + z^2 \\ z^2 &\mapsto z^1 - z^2 \\ z^3 &\mapsto z^3 \end{aligned}$$

and then

$$\begin{aligned} z^1 &\mapsto z^1 \\ z^2 &\mapsto z^2 \\ z^3 &\mapsto z^3 - \frac{a_{11}}{2} z^1 - a_{12} z^2. \end{aligned}$$

This leads to

$$\begin{aligned} v^1 &= z^1\bar{z}^2 + z^2\bar{z}^1 \\ v^2 &= a_{22}|z^2|^2 + z^1\bar{z}^3 + z^3\bar{z}^1. \end{aligned}$$

This is either case v or case x.
It remains to consider the case, when the first form is $v^1 = |z^1|^2$. Suppose $a_{22} \neq 0$ and $a_{33} \neq 0$. By means of some transformation of the form

$$\begin{aligned} z^1 &\mapsto z^1 \\ z^2 &\mapsto z^2 + \alpha z^1 + \beta z^3 \\ z^3 &\mapsto z^3 + \gamma z^1 + \delta z^2 \end{aligned}$$

one can eliminate a_{12}, a_{13} and a_{23}. Then there exists a linear combination of the forms with $a_{11} = 0$. We obtain the following cases: $v^2 = |z^2|^2 + |z^3|^2$ (case vii), $v^2 = |z^2|^2 - |z^3|^2$ (case $viii$).

We consider the case $a_{22} = 0$ $a_{33} \neq 0$ (this is equivalent to $a_{33} = 0$ $a_{22} \neq 0$). Then a_{13} can be eliminated and $v^2 = |z^3|^2 + z^1\bar{z}^2 + z^2\bar{z}^1$ (case ix).

Now let $a_{22} = 0$ $a_{33} = 0$. After some obvious transformation we obtain $v^2 = z^1\bar{z}^2 + z^2\bar{z}^1$. This quadric is degenerate.

We have to show that the 10 cases are not equivalent by pairs.

Below we will give the linear groups of (C,ρ) transformations. The dimension of these Lie groups is invariant for a quadric. In cases *i*, *ii*, *iii* this dimension is 4, in cases *iv*, *v*,*vi* it is 5, in cases *vii*, *viii* it is 7 and in cases *ix*, *x* it is 8.

Case *i* has no vector with $\langle z,z\rangle = 0$; case *ii* has a 4-dimensional variety and case *iii* a 3-dimensional variety of such vectors.

The cases *iv*, *v* and *vi* are different, because the variety of vectors with $\langle z,z\rangle = 0$ is 4-dimensional in case *iv*, 2-dimensional in case *v*, and 3-dimensional in case *vi*.

The cases *vii* and *viii* are direct products of hyperquadrics in $\mathbb{C}^2$ and $\mathbb{C}^3$. The signatures of the quadrics in $\mathbb{C}^3$ are different, hence cases *vii* and *viii* are different.

The same argument as in cases *iv*, *v* and *vi* shows, that cases *ix* and *x* are different.

We give now the groups of linear (C,ρ) transformations in the 10 cases. We denote real parameters by greek letters and complex parameters by latin letters.

In case *i* and *ii* C has the form:

$$\lambda \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix}$$

and $\rho = \lambda^2 \,\mathrm{id}$.

In case *iii* C equals

$$a \begin{pmatrix} \cosh\alpha & 0 & \sinh\alpha \\ 0 & e^{i\phi} & 0 \\ \sinh\alpha 0 & 0 & \cosh\alpha \end{pmatrix},$$

and $\rho = |a| \,\mathrm{id}$.

In case *iv* and *v* C has the form

$$\begin{pmatrix} \lambda e^{i\phi_1} & i\alpha e^{i\phi_1} & 0 \\ 0 & \mu e^{i\phi_1} & 0 \\ 0 & 0 & \mu e^{i\phi_2} \end{pmatrix}$$

and ρ has the form

$$\begin{pmatrix} \lambda\mu & 0 \\ 0 & \mu^2 \end{pmatrix}$$

In case *vi* the matrices C have the form

$$C = a \begin{pmatrix} 1 & 0 & 0 \\ i\theta & \lambda & 0 \\ i\delta - \frac{\theta^2}{2} & \gamma - i\theta\lambda & \lambda^2 \end{pmatrix}$$

and

$$\rho = |a|^2 \begin{pmatrix} \lambda & 0 \\ 2\gamma\lambda & 1 \end{pmatrix}.$$

In cases *vii,viii* the transformation groups are direct products of the corresponding groups for hyperquadrics.

In case *ix* we obtain

$$C = a \begin{pmatrix} 1 & 0 & 0 \\ b & e^{i\phi} & 0 \\ c & -\bar{b}e^{i\phi} & \lambda \end{pmatrix}$$

and

$$\rho = |a|^2 \begin{pmatrix} 1 & 0 \\ |b|^2 + 2\,\mathrm{Re}\,c & \lambda \end{pmatrix}.$$

In case *x*

$$C = a \begin{pmatrix} 1 & 0 & 0 \\ i\alpha & \beta & \gamma \\ i\theta & \delta & \xi \end{pmatrix}$$

and

$$\rho = |a|^2 \begin{pmatrix} \beta & \gamma \\ \delta & \xi \end{pmatrix}.$$

3. Matrix substitutuions

It follows from Beloshapka's uniqueness theorem [3] that in the cases *i-vi* any automorphism is linear.

In the cases *vii-ix* we present matrix substitutions which realize 8 dimensional subgroups. In fact, only case *ix* is interesting because *vii* and *viii* are direct products. Case *ix* is of special interest because it has a group of dimension 16, the maximally possible.

It was a conjecture of Beloshapka that the groups of nullquadrics are the maximal. The quadric *ix* is a counterexample.

The isotropy group of *x* will be obtained in the next section. It has only dimension 10.

The matrix substitutions are

in case *vii*

$$Z = \begin{pmatrix} z^1 & z^3 \\ z^2 & z^2 \\ z^3 & z^1 \end{pmatrix}$$

$$\bar{Z} = \begin{pmatrix} \bar{z}^1 & \bar{z}^2 & \bar{z}^3 \\ \bar{z}^3 & \bar{z}^2 & \bar{z}^1 \end{pmatrix}$$

$$W = \begin{pmatrix} w^1 & w^2 \\ w^2 & w^1 \end{pmatrix}$$

$$\bar{W} = \begin{pmatrix} \bar{w}^1 & \bar{w}^2 \\ \bar{w}^2 & \bar{w}^1 \end{pmatrix}$$

in case *viii*

$$Z = \begin{pmatrix} z^1 & z^3 \\ z^2 & -z^2 \\ z^3 & z^1 \end{pmatrix}$$

$$\bar{Z} = \begin{pmatrix} \bar{z}^1 & -\bar{z}^2 & \bar{z}^3 \\ \bar{z}^3 & \bar{z}^2 & \bar{z}^1 \end{pmatrix}$$

$$W = \begin{pmatrix} w^1 & w^2 \\ w^2 & w^1 \end{pmatrix}$$

$$\bar{W} = \begin{pmatrix} \bar{w}^1 & \bar{w}^2 \\ \bar{w}^2 & \bar{w}^1 \end{pmatrix}$$

in case *ix*

$$Z = \begin{pmatrix} z^1 & 0 \\ z^2 & 0 \\ z^3 & z^1 \end{pmatrix}$$

$$\bar{Z} = \begin{pmatrix} \bar{z}^1 & 0 & 0 \\ \bar{z}^3 & \bar{z}^2 & \bar{z}^1 \end{pmatrix}$$

$$W = \begin{pmatrix} w^1 & 0 \\ w^2 & w^1 \end{pmatrix}$$

$$\bar{W} = \begin{pmatrix} \bar{w}^1 & 0 \\ \bar{w}^2 & \bar{w}^1 \end{pmatrix}$$

The complex 3-vector a will be represented as 2×3 matrix like the corresponding z, and the real 2-vector r as 2×2 matrix like the corresponding w.

Then the Poincaré formula

$$
\begin{aligned}
Z &\mapsto (Z+AW)(\mathrm{id} - 2i\bar{A}Z - (R+i\bar{A}A)W)^{-1} \\
W &\mapsto W(\mathrm{id} - 2i\bar{A}Z - (R+i\bar{A}A)W)^{-1}
\end{aligned}
$$

gives 8-dimensional subgroups of the automorphism groups. Together with the linear automorphisms they cover the whole groups.

4. The nullquadric

It follows from Beloshapka's uniqueness theorem that the isotropy group of the nullquadric Q_0 has dimension 10. We obtained a subgroup of dimension 8 consisting of linear automorphisms.

We present now a 2-dimensional subgroup of automorphisms with identical CR-projection of the differential in 0.

Therefore, let S be the hyperquadric in $\mathbb{C}^3$ determined by

$$
\operatorname{Im} w = 2 \operatorname{Re} z^1 \bar{z}^2
$$

Then Q_0 is the fibred product of two copies of S over $\mathbb{C}$ with respect to the projection $S \to \mathbb{C}$ defined by $(z^1, z^2, w) \mapsto z^1$.

The automorphisms

$$
\begin{aligned}
z^1 &\mapsto \frac{z^1}{1 - 2i\bar{a}z^1} \\
z^2 &\mapsto \frac{z^2 + aw}{1 - 2i\bar{a}z^1} \\
w &\mapsto \frac{w}{1 - 2i\bar{a}z^1},
\end{aligned}
$$

with $a \in \mathbb{C}$, can be lifted to automorphisms of Q_0, because the z^1 component depends only on z^1. It is easy to see that they have identical CR projection in 0.

We write down the final formula

$$
\begin{aligned}
z^1 &\mapsto \frac{z^1}{1-2i\bar{a}z^1} \\
z^2 &\mapsto \frac{z^2+aw^1}{1-2i\bar{a}z^1} \\
z^3 &\mapsto \frac{z^3+aw^2}{1-2i\bar{a}z^1} \\
w^1 &\mapsto \frac{w^1}{1-2i\bar{a}z^1} \\
w^2 &\mapsto \frac{w^2}{1-2i\bar{a}z^1}.
\end{aligned}
$$

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