REFLEXIVE MODULES ON QUOTIENT SURFACE SINGULARITIES

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Introduction

D. Mumford in characteristic 0 [9] and J. Lipman [7] in characteristic $p > 0$ proved that a surface singularity X is rational if and only if X has finitely many rank one reflexive modules up to isomorphism. This implies a characterization of quotient surface singularities as those ones which have finitely many indecomposable(with respect to direct sum) reflexive modules (1.2).

For rational double points on $\mathbb{C}$, J. Mc Kay [8] gave a one - to - one correspondence between vertices of the extended diagram associated to the finite subgroup $G \subset SL(2,\mathbb{C})$ defining X as a quotient $\mathbb{C}^2/G$, irreducible representations of G and indecomposable reflexive modules on X. Trying to understand geometrically this correspondence, G. Gonzalez - Sprinberg and J.L. Verdier [4] associated geometrically to each indecomposable reflexive module the first Chern class of the pull-back on the minimal resolution $f:\tilde{X} \to X$. They show that the Chern class intersects exactly one rational curve of the exceptional locus, and that it determines the module. After this, H. Knörrer [6] reinterpeted this geometrical construction group-theoritically. This, is a sense, completes a cycle, as the work of J. Mc Kay was from the point of view of representation theory. Finally M. Artin and J.L. Verdier [1]gave an elegant and theoritical proof of the fact that the first Chern class determines the module.

This led H. Knörrer and J.L. Verdier to ask whether it remains true that for a general quotient surface singularity X the first Chern class of an indecomposeable reflexive module determines it.

We give here a negative answer to this question (3.3). To this aim we give a numerical criterion for an invertible sheaf on a resolution $f:\tilde{X} \to X$ to be the pull-back of a reflexive rank one module. Then we "dualize" the construction of Artin-Verdier in order to introduce a sort of obstruction for the question to have a positive answer (2.4).

The method relies on the techniques of cyclic covers we developed in [3]. However, this works only for rank one modules. Therefore one would need similar method for higher rank modules to get more precise informations about them. Nevertheless we get in § 4 a list of properties the pull-back by f of an higher rank reflexive module on a rational singularity has to fulfill.

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§ 1 Quotient surface singularities

Let X be a rational surface singularity on $\mathbb{C}$ and $U = \operatorname{Reg} X$ be the regular locus.

(1.1) In [3,(1.7)] we state a known fact: X is a quotient singularity if and only if the canonical cover is a rational singularity. For the same reason given there we can say:

Lemma. X is a quotient singularity if and only if each cover of X, which is étale on U, has rational singularities.

Corollary (1.2). X is a quotient singularity if and only if it has finitely many indecomposable reflexive modules (up to direct sum).

Proof. This result is due to J. Herzog [5]. Since we do not know any reference for a geometrical (and very elementary) proof, we give it here.

If X has finitely many indecomposable reflexive modules, then ω_X has a finite order and we can consider the canonical cover $\pi: X' \to X$ [3,(1.4)]. Since for every reflexive module M on X', $\pi_* M$ is reflexive, especially for the rank one ones, $\operatorname{Pic} \operatorname{Reg} X'$ has to be finite and therefore, X' has to be rational.

If X is a quotient singularity of group G, call $\mu: \mathbb{C}^2 \to X$ the Galois map which is étale on U. Let $i: U \to X$ be the embedding, $V = \mu^{-1}(U)$ and $j: V \to \mathbb{C}^2$. One has a one-to-one correspondence between:

(i) (indecomposable) locally free sheaves on U

(ii) (indecomposable) reflexive modules on X

(iii) (G-indecomposable) locally free G-sheaves on V

(iv) (irreducible) representations of G.

The equivalences are given by:

(i) → (ii) $F \to i_* F$

(ii) → (i) $F \to i^* F$

(i) → (iii) $F \to \mu^* F$

(iii) → (i) $G \to (\mu_* G)^G$

(iii) → (iv) $G \to j_* G/m \cdot j_* G = H$

where m is the maximal ideal of 0 in $\mathbb{C}^2$. (One has only to observe that the reflexive hull $j_* G$ of G is locally free - actually even free - and that G preserves the stalk at 0, because 0 is a fix point).

(iv) → (iii) $H \to j^* (\mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} H)$

§ 2 Reflexive modules on quotient singularities

Let X be a rational surface singularity, $i:U \to X$ the embedding of the regular locus, $f:\tilde{X} \to X$ any desingularization.

Lemma and definition (2.1) A sheaf M on $\tilde{X}$ can be written as $M = \frac{f^*M}{\text{torsion}}$, where M is a reflexive module if and only if

i) M is locally free

ii) M is generated by its global sections

iii) $R^1 f_* M^\vee \otimes \omega_{\tilde{X}} = 0$, where $M^\vee$ is the dual of M.

If M fulfills i),ii),iii), M is said to be full.

Proof. It should be well known and is simply a reformulation of [1, (1.7) and (1.9)]. One has the exact sequence

$$0 \to f_* M \to i_* M|_U \to H^1_E(M) \to R^1 f_* M .$$

If M is full, then $R^1 f_* M = 0$ by (ii) and (iii) is Serre-dual to $H^1_E(M)$. Conversely if (iii) is true, then $f_* M = M$ is reflexive. One has the natural map

$$f^*M \to M$$

which factors over

$$\frac{f^*M}{\text{torsion}} \longrightarrow M$$

if (i) is true and becomes injective. If (ii) is true, one has a surjection

$$\oplus \mathcal{O}_{\tilde{X}} \longrightarrow\!\!\!\!\to M$$

which gives a surjection by applying f_*

$$\oplus \mathcal{O}_X = \oplus f_* \mathcal{O}_{\tilde{X}} \longrightarrow\!\!\!\!> M$$

This gives surjections

$$\oplus \mathcal{O}_{\tilde{X}} = \oplus f^* \mathcal{O}_X \longrightarrow\!\!\!\!> f^*M \longrightarrow\!\!\!\!> \mathcal{M}$$

In other words, $\frac{f^*M}{\text{torsion}}$ is generated by the global sections of $\mathcal{M}$. Therefore, they are equal.

(2.2) Now we want to rewrite the construction of Artin-Verdier ([1, (1.2), (1.9), (1.11)]) in the dual language.

For a full sheaf $\mathcal{M}$ of rank r, take r generic sections to define its first Chern class D as the zero locus of the discriminant of the r sections.

$$(*) \quad 0 \to \mathcal{M}^{v} \to \bigoplus_{1}^{r} \mathcal{O}_{\tilde{X}} \to \mathcal{O}_D \to 0$$

Taking global sections, it is not right exact. Let C be the image of the r global sections in $f_*\mathcal{O}_D$, ring of the normalization of some curve living on X. The sequence

$$(**) \quad 0 \to M^{v} \to \bigoplus_{1}^{r} \mathcal{O}_X \to C \to 0$$

is exact because $f_*\mathcal{M}^{v} = M^{v}$, coming from (2.1(iii)) and $R^1 f_*\mathcal{M} \otimes \omega_{\tilde{X}} = 0$.

__Lemma__ (2.2). __The__ __$\mathcal{O}_X$-submodule__ C __of__ $f_*\mathcal{O}_D$ __determines__ M^{v} __uniquely__ __up__ __to__ __isomorphism__ __as__ __the__ __module__ __of__ __relations__ __of__ __a__ __minimal__ __number__ __of__ __sections__ __generating__ C.

__Proof__. Let $\mathcal{O}_{\tilde{X}}(-Z) = \frac{f^*\mathfrak{m}}{\text{torsion}}$ be the fundamental cycle, where $\mathfrak{m}$ is the maximal ideal of the singularity. Then

length $C/\mathfrak{m}\cdot C = D \cdot Z$

and one has $r \geqslant D\cdot Z$. Artin and Verdier simply say:

- if $r > D\cdot Z$, then $(r - D\cdot Z)$ factors $\mathcal{O}_{\tilde{X}}$ split in $M^{\vee}$.
- if $r = D\cdot Z$, then $M^{\vee}$ is simply the module of relations.

In case X is Gorenstein (2.1 (iii)) says that $\mathcal{C} = f_*\mathcal{O}_D$ and therefore the first Chern class determines the module.

(2.3) Suppose now that the first Chern class determines the indecomposable reflexive module, that means by (2.2) determines $\mathcal{C}$. Then it determines a fortiori $R^1 f_* \mathcal{M}^{\vee}$.

Given now $\mathcal{M}$ indecomposable and full such that the first Chern class D is irreducible, i.e.: $D\cdot E_{i_0} = 1$ and $D\cdot E_i = 0$ for $i \neq i_0$. Assume $R^1 f_* \mathcal{M}^{\vee} \neq 0$. Take $\mathcal{C} \subsetneq \mathcal{C}' \subset f_*\mathcal{O}_D$ any submodule $\mathcal{C}'$, and complete the r sections of $\mathcal{C}$ to r' generating sections of $\mathcal{C}'$.

<u>Lemma.</u> <u>There exists a full sheaf $\mathcal{M}'$ such that the following diagram consists only of exact sequences.</u>

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \overset{r'-r}{\underset{1}{\oplus}} \mathcal{O}_{\tilde{X}} & \xrightarrow{\ \sim\ } & \overset{r'-r}{\underset{1}{\oplus}} \mathcal{O}_{\tilde{X}} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{M}'^{\vee} & \longrightarrow & \overset{r'}{\underset{1}{\oplus}} \mathcal{O}_{\tilde{X}} & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathcal{M}^{\vee} & \longrightarrow & \overset{r}{\underset{1}{\oplus}} \mathcal{O}_{\tilde{X}} & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Moreover one can write $M' = N \oplus \bigoplus_1^s \mathcal{O}_{\tilde{X}}$, where N is full, indecomposable, and not isomorphic to M.

Proof. The construction (2.2) gives the existence of a locally free sheaf M' which is generated by global sections. Since the map $\bigoplus_1^r f_*\omega_{\tilde{X}} \to f_*\mathcal{O}_D \otimes \omega_{\tilde{X}}$ is surjective ((2.1)(iii)), the map $\bigoplus_1^{r'} f_*\omega_{\tilde{X}} \to f_*(\mathcal{O}_D \otimes \omega_{\tilde{X}})$ is a fortiori surjective, and therefore f_*M' is reflexive. May be, r' is larger than the minimal number r'' of sections of C'. But anyway, the splitting $M'^{\vee} = (f_*N)^{\vee} \oplus \bigoplus_1^{r'-r''} \mathcal{O}_X$ gives an indecomposable N, otherwise each summand would give a non trivial constribution to the irreducible D. On the other side, N has the same first Chern class as M, and is not isomorphic because C is not isomorphic to C'.

(2.4) In order to find a counter-example to the question of Knörrer-Verdier, (2.3) just says that one needs an example for which D is irreducible and $R^1 f_* M^{\vee} \neq 0$.

§ 3. Rank one modules

(3.1) Let X be a rational singularity, $f: \tilde{X} \to X$ be any desingularization, L be any invertible sheaf such that $\deg_{E_i} L \geq 0$ for all exceptional curves E_i. L will be said a.p (arithmetically positive) in the sequel. Since Pic U is finite and the intersection matrix is negative definite, there is some power N and some effective divisor A whose support is the exceptional locus such that : $L^N = \mathcal{O}_{\tilde{X}}(-A)$

Proposition (3.2). Let L be an invertible a.p sheaf on $\tilde{X}$, *where* X *is a quotient singularity. Let* $\mathcal{L}^N = \mathcal{O}_{\tilde{X}}(-A)$ *as in* (3.1). *Then*

i) $\mathcal{L}$ *is full if and only if* $\chi(\mathcal{O}_{[\frac{A}{N}]} \otimes \mathcal{L}([\frac{A}{N}])) = 0$.

ii) $R^1 f_* \mathcal{L}^{-1} = -\chi(\mathcal{O}_{\{\frac{A}{N}\}} \times \mathcal{L}^{-1})$

Here one denotes as in [3] by $[\frac{A}{N}]$ the integral part of the $\mathbb{Q}$-divisor $\frac{A}{N}$ and $\{\frac{A}{N}\} = -[-\frac{A}{N}]$

Proof. Let $L^{[i]}$ be the reflexive hull of $f_* \mathcal{L}^i$. The $\mathcal{O}_X$-module

$E = \overset{N-1}{\underset{0}{\oplus}} L^{[i]}$

has an $\mathcal{O}_X$-algebra structure (by the identification $\mathcal{O}_X \simeq L^{[N]}$) which is normal because the $L^{[i]}$ are reflexive. The corresponding cover $\mathrm{Spec}_X E$ has rational singularities. The $\mathcal{O}_{\tilde{X}}$ - module

$\mathcal{E} = \overset{N-1}{\underset{0}{\oplus}} \mathcal{L}^i([\frac{i \cdot A}{N}])$

has an $\mathcal{O}_{\tilde{X}}$ - algebra structure (by the section $\mathcal{L}^N \to \mathcal{O}_{\tilde{X}}$ whose divisor is A) which is normal and has only rational singularities ([3,(1.5)] and corresponding references). Therefore the map composed of $\mathrm{Spec}_{\tilde{X}} \mathcal{E} \to \tilde{X}$ and f factorizes on $\mathrm{Spec}_X E$. One gets

$$R^q f_* \mathcal{L}^i([\frac{i \cdot A}{N}]) = \begin{cases} L^{[i]} & q = 0 \\ 0 & q = 1 \end{cases}$$

for $0 \le i \le (N-1)$.

i) From the exact sequence

$$0 \to \mathcal{L} \to \mathcal{L}([\frac{A}{N}]) \to \mathcal{O}_{[\frac{A}{N}]} \otimes \mathcal{L}([\frac{A}{N}]) \to 0$$

one gets $h^1(\mathcal{O}_{[\frac{A}{N}]} \otimes \mathcal{L}([\frac{A}{N}])) = 0$, and therefore the condition $h^0(\mathcal{O}_{[\frac{A}{N}]} \otimes \mathcal{L}([\frac{A}{N}])) = 0$ implies that $f_* \mathcal{L} = L$ and $R^1 f_* \mathcal{L} = 0$.

Assume that L is not generated by global sections. Then one can write $L = N(F)$ for a full invertible sheaf N and an effective divisor F. Therefore one has $\chi(\mathcal{O}_F \otimes L) = 0 = \chi(\mathcal{O}_F) + L\cdot F$. This implies $F = 0$ because $F\cdot L \geq 0$ by assumption.

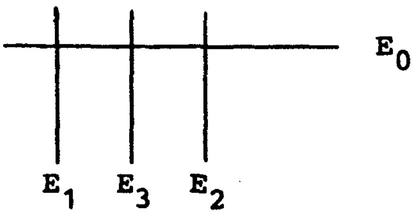
Now if L is full the exact sequence gives trivially the condition wanted.

ii) One has $L^{N-1}(A) = L^{-1}$ which gives the exact sequence

$$0 \to L^{N-1}\left(\left[\frac{(N-1)\cdot A}{N}\right]\right) \to L^{-1} \to \mathcal{O}_{\{\frac{A}{N}\}} \otimes L^{-1} \to 0$$

and one argues along the same line as in i).

(3.3) Counter example.



$E_0^2 = E_1^2 = E_2^2 = -2$

$E_3^2 = -3$

Take $L = M$ defined by $L \cdot E_0 = 1$ and $L\cdot E_i = 0$ for $i > 0$. Then $[\frac{A}{N}] = E_0$ and $\mathcal{O}_{E_0} \otimes L(E_0) = \mathcal{O}_{E_0}(-1)$. Therefore L is full ((3.2) i)). Moreover one sees immediately that $L \otimes \omega_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-Z)$, where Z is the fundamental cycle. Therefore $L = \frac{f^*\omega^V}{\text{torsion}}$ and $R^1f_*L^{-1} = h^1(\omega_Z) = 1$. According to (2.3) and (2.4), one constructs M of rank 2 by the following diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathfrak{m} & \to & \mathcal{O}_X & \to & \mathbb{C} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & f_*M^V & \to & \overset{2}{\underset{1}{\oplus}}\mathcal{O}_X & \to & f_*\mathcal{O}_D & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & f_*L^{-1} & \to & \mathcal{O}_X & \to & \mathcal{O}_{f_*D} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where D is the first Chern class of L and $\mathfrak{m}$ the maximal ideal of Sing X. From the vertical left sequence and from $f_*L^{-1} = \omega_X$, one sees that M^V is the reflexive hull of the 1-holomorphic forms on Reg X (see [2]). Therefore M is the dual of the 1-holomorphic forms.

§ 4 Some properties of higher rank modules.

X is in this § a rational surface singularity of multiplicity μ. On a desingularization $f:\tilde{X} \to X$ one considers a full sheaf M of rank r, $L = \overset{r}{\wedge} M$, D the first Chern class, the full sheaf "dual" $M' = \dfrac{f^*(f_*M)^V}{\text{torsion}}$, $L' = \overset{r}{\wedge} M'$, the fundamental cycle $-Z$ and the maximal ideal as before. We assume M to be without $\mathcal{O}_{\tilde{X}}$ summand.

One has seen in § 2 and § 3 that $R^1f_*M^V$ is an important invariant of M which does not depend only on D. That is what we compute now.

Proposition (4.1)

i) $$R^1f_*M^V = D\cdot Z - r$$
$$= R^1f_*L^{-1} - R^1f_*L'^{-1} \otimes \omega_{\tilde{X}}$$
$$R^1f_*M^V(-Z) = 0$$
$$R^1f_*M^V = h^1(\mathcal{O}_Z \otimes M^V)$$

ii) $f_*M^V = f_*M^V(-Z) = (f_*M)^V$

$h^0(\mathcal{O}_Z \otimes M^V) = 0$

iii) $f_*\mathcal{O}_D(-Z) \subset C \subset f_*\mathcal{O}_D$

$m \cdot C = f_*\mathcal{O}_D(-Z)$

iv) $\frac{D\cdot Z}{(\mu-1)} \leq r \leq D\cdot Z$

<u>Proof.</u> The first equality is already proven in (2.2). From the exact sequence

$$0 \to L^{-1} \to M^V \to \overset{r-1}{\underset{1}{\oplus}} \mathcal{O}_{\tilde{X}} \to 0$$

one gets the exact sequences

$$0 \to L^V \to M^V \to J \to 0$$

$$0 \to J \to \overset{r-1}{\underset{1}{\oplus}} \mathcal{O}_X \to R^1 f_* L^{-1} \to R^1 f_* M^V \to 0$$

From

$$0 \to \overset{r-1}{\underset{1}{\oplus}} \mathcal{O}_{\tilde{X}} \to M' \to L' \to 0$$

one gets

$$0 \to \overset{r-1}{\underset{1}{\oplus}} \mathcal{O}_X \to M^V \to f_* L' \to 0$$

and moreover one has ((2.1) iii))

$$0 \to f_* L' \to L^V \to R^1 f_* L'^{-1} \otimes \omega_{\tilde{X}} \to 0$$

Assuming everthing to be projective and computing Euler-Poincaré characteristics, one gets the second equality.

From the exact sequence

$$0 \to M^{v}|_{Z} \to \bigoplus_{1}^{r} \mathcal{O}_{\tilde{X}}|_{Z} \to \mathcal{O}_{D}|_{Z} \simeq \mathbb{C}^{D\cdot Z} \to 0$$

one gets $h^1(M^{v}|_{Z}) \geq D\cdot Z - r$.

But one has $h^1(M^{v}|_{Z}) \leq R^1 f_* M^{v} = D\cdot Z - r$.

Therefore the map

$$R^1 f_* M^{v}(-Z) \to R^1 f_* M^{v}$$

has to be zero and moreover one has $h^0(M^{v}|_{Z}) = 0$. So one has $R^1 f_* M^{v}(-Z) = 0$.

From

$$\begin{array}{ccc} \bigoplus_{1}^{r} f_* \mathcal{O}_{\tilde{X}}(-Z) & \longrightarrow\!\!\!\!\to & f_* \mathcal{O}_D(-Z) \\ \downarrow & & \downarrow \\ \bigoplus_{1}^{r} f_* \mathcal{O}_{\tilde{X}} & \longrightarrow\!\!\!\!\to & C \to f_* \mathcal{O}_D \end{array}$$

one gets the factorization

$$f_* \mathcal{O}_D(-Z) \to C$$

One has length $C/f_*\mathcal{O}_D(-Z) = Z.D - R^1 f_* M^{v}$

$= r$

$\leq$ length $C/m\cdot C = r$.

Therefore $f_* \mathcal{O}_D(-Z) = m\cdot C$

Finally (iv) is simply the right exactness of the sequence ((2.2)(*)) tensorized with $\omega_{\tilde{X}}$, after applying f_*.

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