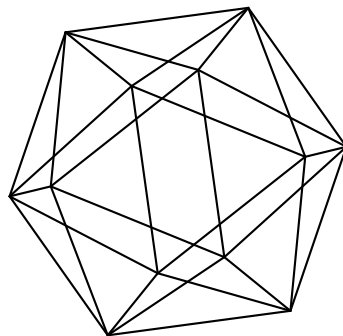


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by

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Abstract: We study the homotopy category $\mathrm{hmf}(R, W)$ of matrix factorizations of non-zero elements $W \in R^\times$, where R is an elementary divisor domain. When R has prime elements and W factors into a square-free element W_0 and a finite product of primes of multiplicity greater than one and which do not divide W_0 , we show that $\mathrm{hmf}(R, W)$ is triangle-equivalent with an orthogonal sum of the triangulated categories of singularities $\mathrm{D}_{\mathrm{sing}}(A_n(p))$ of the local Artinian rings $A_n(p) = R/\langle p^n \rangle$, where p runs over the prime divisors of W of order $n \geq 2$. This result holds even when R is not Noetherian. The triangulated categories $\mathrm{D}_{\mathrm{sing}}(A_n(p))$ are Krull-Schmidt and we describe them explicitly. We also study the cocycle category $\mathrm{zmf}(R, W)$, showing that it is additively generated by elementary matrix factorizations. Finally, we discuss a few classes of examples.

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Introduction

The study of open-closed topological Landau-Ginzburg models [1,2,3,4] defined on a Stein manifold X [5] leads naturally to the problem of understanding categories of finitely-generated projective factorizations over the non-Noetherian ring $\mathcal{O}(X)$ of holomorphic complex-valued functions defined on X . The simplest interesting models of this type arise when X is an arbitrary borderless, smooth and connected non-compact Riemann surface Σ (which may have infinite genus), with superpotential given by a non-vanishing holomorphic function $W : \Sigma \rightarrow \mathbb{C}$. In this situation, the ring $R = \mathcal{O}(X)$ is a so-called elementary divisor domain (see Appendix B), i.e. it has the property that any matrix with entries from R admits a Smith normal form. Since any

elementary divisor domain is a Bézout domain, this implies that any finitely-generated projective R -module is free (see [6]), hence the relevant category reduces to the usual homotopy category $\mathrm{hmf}(R, W)$ of finite rank matrix factorizations over R .

In the present paper, we consider a similar problem for *any* elementary divisor domain R which has prime elements, showing that the triangulated structure of the category $\mathrm{hmf}(R, W)$ can be determined explicitly for any element $W \in R^\times$ which is *critically-finite*, i.e. which can be written as a product $W = W_0 W_c$, where the *non-critical part* W_0 is a square-free element of R^\times and the *critical part* W_c is a (non-empty) finite product of prime elements of R , each of which has multiplicity strictly greater than one (we also require that W_0 and W_c are coprime). More precisely, we will prove the following result, which can be viewed as a non-Noetherian extension of the Buchweitz correspondence [7] to elementary divisor domains:

Theorem 0.1 *Let R be an elementary divisor domain which has prime elements and W be a critically-finite element of R with critical part $W_c = p_1^{n_1} \dots p_N^{n_N}$, where p_1, \dots, p_N (with $N \geq 1$) are prime elements of R which are not mutually associated in divisibility and $n_i \geq 2$. Then there exist equivalences of triangulated categories:*

$$\mathrm{hmf}(R, W) \simeq \bigvee_{i=1}^N \underline{\mathrm{mod}}_{R/\langle p^{n_i} \rangle} \simeq \bigvee_{i=1}^N \mathrm{D}_{\mathrm{sing}}(R/\langle p^{n_i} \rangle) \quad , \quad (0.1)$$

where $\underline{\mathrm{mod}}_{R/\langle p^{n_i} \rangle} \simeq \mathrm{D}_{\mathrm{sing}}(R/\langle p^{n_i} \rangle)$ denotes the projectively-stabilized category of finitely-generated modules (a.k.a. the category of singularities) of the ring $R/\langle p^{n_i} \rangle$, a ring which is Artinian.

Our proof relies on the fact that matrices over an elementary divisor domain admit a Smith normal form, which allows us to reduce the problem to understanding certain properties of elementary matrix factorizations (i.e. those matrix factorizations whose reduced rank equals one). The latter were studied in [8] for any Bézout domain. The triangulated categories $\mathrm{D}_{\mathrm{sing}}(R/\langle p^{n_i} \rangle)$ are Krull-Schmidt and they admit Auslander-Reiten triangles; their Auslander-Reiten quivers are determined in Section 2. Together with Theorem 0.1, this gives a complete description of the category $\mathrm{hmf}(R, W)$ when the hypothesis of the theorem is satisfied.

The paper is organized as follows. In Section 1, we recall a few definitions and constructions for matrix factorizations over Bézout domains. In Section 2, we discuss finitely-generated modules over the quotient of a Bézout domain by a principal primary ideal. Section 3 considers the homotopy category of matrix factorizations over an elementary divisor domain for a critically-finite W , giving the proof of Theorem 0.1. Section 4 discusses some examples, while the appendices collect information about matrices over greatest common divisor (GCD) domains and about elementary divisor domains (EDD).

Notations and conventions. We use the same notations and conventions as in [8]. In particular, given an element x of a unital commutative ring R , the symbol $(x) \in R/U(R)$ (where $U(R)$ is the group of units of R) denotes the class of x under association in divisibility. When R is a GCD domain (see Appendix A) and $x_1, \dots, x_n \in R$, the symbol $(x_1, \dots, x_n) \in R/U(R)$ denotes the association in divisibility class formed by the greatest common divisors of x_1, \dots, x_n . The symbol $\langle x_1, \dots, x_n \rangle$ denotes the ideal generated by x_1, \dots, x_n . The symbol \mathbb{Z}_2 stands for the field $\mathbb{Z}/2\mathbb{Z}$, whose elements we denote by $\hat{0}$ and $\hat{1}$. The symbol \mathbb{N} denotes the set of natural numbers including zero, while $\mathbb{N}^* \stackrel{\mathrm{def.}}{=} \mathbb{N} \setminus \{0\}$.

1. Matrix factorizations over a Bézout domain

Categories of matrix factorizations over a Bézout domain were studied in [8], to which we refer the reader for more detail. In this section, we recall some definitions and constructions which will be used later on. Let R be a Bézout domain and $W \in R^\times$ be a non-zero element of R .

1.1. Categories of matrix factorizations over R . As in [8], we consider the following categories:

- The R -linear and \mathbb{Z}_2 -graded differential category $\text{MF}(R, W)$ of finite rank matrix factorizations of W over R . Its objects are pairs $a = (M, D)$ with M a free \mathbb{Z}_2 -graded R -module of finite rank and D an odd endomorphism of M such that $D^2 = \text{Wid}_M$. Since W is non-vanishing, the even and odd components of M have equal rank, which we denote by $\rho(a)$ and call the *reduced rank* of a ; we have $\text{rk} M = 2\rho(a)$. Choosing a \mathbb{Z}_2 -homogeneous basis of M allows us to identify M with the R -supermodule $R^{\rho(a)|\rho(a)}$ whose \mathbb{Z}_2 -homogeneous components are both equal to the free module $R^{\oplus\rho(a)}$. Then D is identified with a square matrix of size $2\rho(a)$ in block off-diagonal form:

$$D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix} ,$$

where u and v are square matrices of size $\rho(a)$ with entries in R . The condition $D^2 = \text{Wid}_M$ amounts to the relations:

$$uv = vu = WI_{\rho(a)} , \quad (1.1)$$

where $I_{\rho(a)}$ is the identity matrix of size $\rho(a)$. Since $W \neq 0$, these conditions imply that the matrices u and v have maximal rank. Given two objects $a_1 = (M_1, D_1)$ and $a_2 = (M_2, D_2)$ of $\text{MF}(R, W)$, the \mathbb{Z}_2 -graded R -module of morphisms from a_1 to a_2 is given by the inner Hom:

$$\text{Hom}_{\text{MF}(R, W)}(a_1, a_2) = \underline{\text{Hom}}_R(M_1, M_2) = \text{Hom}_{\hat{R}}^{\hat{0}}(M_1, M_2) \oplus \text{Hom}_{\hat{R}}^{\hat{1}}(M_1, M_2) ,$$

endowed with the differential determined by the condition:

$$\partial_{a_1, a_2}(f) = D_2 \circ f - (-1)^\kappa f \circ D_1 , \quad \forall f \in \text{Hom}_R^\kappa(M_1, M_2) ,$$

where $\kappa \in \mathbb{Z}_2$.

- The R -linear and \mathbb{Z}_2 -graded cocycle, coboundary and total cohomology categories $\text{ZMF}(R, W)$, $\text{BMF}(R, W)$ and $\text{HMF}(R, W)$ of $\text{MF}(R, W)$.
- The subcategories $\text{mf}(R, W)$, $\text{zmf}(R, W)$, $\text{bmf}(R, W)$ and $\text{hmf}(R, W)$ obtained from $\text{MF}(R, W)$, $\text{ZMF}(R, W)$, $\text{BMF}(R, W)$ and $\text{HMF}(R, W)$ by restricting to morphisms of even degree. Notice that $\text{hmf}(R, W)$ is the usual homotopy category of finite rank matrix factorizations.

It is clear that $\text{MF}(R, W)$, $\text{BMF}(R, W)$ and $\text{ZMF}(R, W)$ admit double direct sums (and hence all finite direct sums of at least two elements). On the other hand, $\text{HMF}(R, W)$ is an additive category. Two matrix factorizations a_1 and a_2 of W over R are called *strongly isomorphic* if they are isomorphic in the category $\text{zmf}(R, W)$. It is clear that two strongly isomorphic factorizations are also isomorphic in $\text{hmf}(R, W)$, but the converse need not hold. Matrix factorizations for which $M = R^{\rho|\rho}$ form a dg subcategory of $\text{MF}(R, W)$ which is dg-equivalent with $\text{MF}(R, W)$. We will often tacitly identify $\text{MF}(R, W)$ with this subcategory. Given two matrix factorizations $a_1 = (R^{\rho_1|\rho_1}, D_1)$ and $a_2 = (R^{\rho_2|\rho_2}, D_2)$ of W with $D_i = \begin{bmatrix} 0 & v_i \\ u_i & 0 \end{bmatrix}$ and $u_i, v_i \in \text{Mat}(\rho_i, \rho_i, R)$, a morphism $f \in \text{Hom}_{\text{mf}(R, W)}(a_1, a_2)$ has matrix form:

$$f = \begin{bmatrix} f_{\hat{0}\hat{0}} & 0 \\ 0 & f_{\hat{1}\hat{1}} \end{bmatrix}$$

with $f_{\hat{0}\hat{0}}, f_{\hat{1}\hat{1}} \in \text{Mat}(\rho_1, \rho_2, R)$ and we have:

$$\mathfrak{d}_{a_1, a_2}(f) = D_2 \circ f - f \circ D_1 = \begin{bmatrix} 0 & v_2 \circ f_{\hat{1}\hat{1}} - f_{\hat{0}\hat{0}} \circ v_1 \\ u_2 \circ f_{\hat{0}\hat{0}} - f_{\hat{1}\hat{1}} \circ u_1 & 0 \end{bmatrix} .$$

1.2. The triangulated structure of $\text{hmf}(R, W)$. The category $\text{hmf}(R, W)$ is naturally triangulated with an involutive suspension functor. This triangulated structure is defined as follows (see [9] for a detailed treatment).

Definition 1.1 Let $a = (M, D)$ be a matrix factorization of W , where

$$D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix} .$$

The suspension of a is the matrix factorization $\Sigma M \stackrel{\text{def.}}{=} (M', D')$, where:

$$(M')^{\hat{0}} \stackrel{\text{def.}}{=} M^{\hat{1}} \quad , \quad (M')^{\hat{1}} \stackrel{\text{def.}}{=} M^{\hat{0}} ,$$

and:

$$D' \stackrel{\text{def.}}{=} \begin{bmatrix} 0 & -u \\ -v & 0 \end{bmatrix} .$$

Given two matrix factorizations $a_1 = (M_1, D_1)$ and $a_2 = (M_2, D_2)$ of W and a morphism $f \in \text{Hom}_{\text{hmf}(R, W)}(a_1, a_2)$, its suspension Σf coincides with f when the latter is viewed as an element of $\text{Hom}_R^{\hat{0}}(M_1^{\hat{1}}, M_2^{\hat{0}})$.

It is easy to check that Σ is an endofunctor of $\text{hmf}(R, W)$ which satisfies $\Sigma^2 = \text{id}_{\text{hmf}(R, W)}$.

Definition 1.2 Let $a_i = (M_i, D_i)$ for $i \in \{1, 2\}$ be two matrix factorizations of W with $D_i = \begin{bmatrix} 0 & v_i \\ u_i & 0 \end{bmatrix}$ and $f : a_1 \rightarrow a_2$ be a morphism in $\text{hmf}(R, W)$ with $f = \begin{bmatrix} f_{\hat{0}\hat{0}} & 0 \\ 0 & f_{\hat{1}\hat{1}} \end{bmatrix}$. Then:

- The mapping cone $C(f)$ of f is the matrix factorization $C(f) = (M, D)$ of W , where:

$$M \stackrel{\text{def.}}{=} M^{\hat{0}} \oplus M^{\hat{1}} \quad \text{with} \quad M^{\hat{0}} \stackrel{\text{def.}}{=} M_1^{\hat{1}} \oplus M_2^{\hat{0}} \quad , \quad M^{\hat{1}} \stackrel{\text{def.}}{=} M_1^{\hat{0}} \oplus M_2^{\hat{1}}$$

and:

$$D \stackrel{\text{def.}}{=} \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix} , \quad \text{with} \quad u \stackrel{\text{def.}}{=} \begin{bmatrix} -v_1 & 0 \\ f_{\hat{1}\hat{1}} & u_2 \end{bmatrix} , \quad v \stackrel{\text{def.}}{=} \begin{bmatrix} -u_1 & 0 \\ f_{\hat{0}\hat{0}} & v_2 \end{bmatrix} .$$

- The morphism $\varphi_f : a_2 \rightarrow C(f)$ is defined via the following diagram:

$$\begin{array}{ccccc} M_2^{\hat{0}} & \xrightarrow{u_2} & M_2^{\hat{1}} & \xrightarrow{v_2} & M_2^{\hat{0}} \\ \downarrow \iota_1 & & \downarrow \iota_2 & & \downarrow \iota_1 \\ M_1^{\hat{1}} \oplus M_2^{\hat{0}} & \xrightarrow{u} & M_1^{\hat{0}} \oplus M_2^{\hat{1}} & \xrightarrow{v} & M_1^{\hat{1}} \oplus M_2^{\hat{0}} \end{array} ,$$

where $\iota_1 : M_2^{\hat{0}} \rightarrow M_1^{\hat{1}} \oplus M_2^{\hat{0}}$ and $\iota_2 : M_2^{\hat{1}} \rightarrow M_1^{\hat{0}} \oplus M_2^{\hat{1}}$ are the inclusions.

- The morphism $\psi_f : C(f) \rightarrow \Sigma a_1$ is defined via the following diagram:

$$\begin{array}{ccccc} M_1^{\hat{1}} \oplus M_2^{\hat{0}} & \xrightarrow{u} & M_1^{\hat{0}} \oplus M_2^{\hat{1}} & \xrightarrow{v} & M_1^{\hat{1}} \oplus M_2^{\hat{0}} \\ \downarrow \pi_1 & & \downarrow \pi_2 & & \downarrow \pi_1 \\ M_1^{\hat{1}} & \xrightarrow{-v} & M_1^{\hat{0}} & \xrightarrow{-u} & M_1^{\hat{1}} \end{array} ,$$

where π_1 and π_2 are the natural projections.

The following result is well-known (see [9] for details):

Theorem 1.3 *The category $\text{hmf}(R, W)$ is triangulated when equipped with the suspension functor Σ and with the collection of distinguished triangles given by sequences isomorphic with those of the form:*

$$a_1 \xrightarrow{f} a_2 \xrightarrow{\varphi_f} C(f) \xrightarrow{\psi_f} \Sigma a_1 ,$$

where $f : a_1 \rightarrow a_2$ is any morphism in $\text{hmf}(R, W)$.

Proposition 1.4 *Let s be a unit of R . Then there exists a triangulated equivalence:*

$$\text{hmf}(R, sW) \simeq \text{hmf}(R, W) .$$

Proof. Let $\Phi_s : \text{zmf}(R, W) \rightarrow \text{zmf}(R, sW)$ be the functor which takes a factorization $a = (R^{\rho|\rho}, D)$ of W with $D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$ into the factorization $\Phi_s(a) = (R^{\rho|\rho}, D^s)$ of sW , where $D^s = \begin{bmatrix} 0 & sv \\ u & 0 \end{bmatrix}$ is a factorization of sW and leaves unchanged the morphism $f = \begin{bmatrix} f_{\hat{0}\hat{0}} & 0 \\ 0 & f_{\hat{1}\hat{1}} \end{bmatrix}$ from a_1 to a_2 into itself. Using the explicit expression:

$$\mathfrak{d}_{a_1, a_2}^s(f) = D_2^s \circ f - f \circ D_1^s = \begin{bmatrix} 0 & sv_2 \circ f_{\hat{1}\hat{1}} - sf_{\hat{0}\hat{0}} \circ v_1 \\ u_2 \circ f_{\hat{0}\hat{0}} - f_{\hat{1}\hat{1}} \circ u_1 & 0 \end{bmatrix} , \quad (1.2)$$

we conclude that:

$$D_2 \circ f - f \circ D_1 = 0 \iff D_2^s \circ f - f \circ D_1^s = 0 .$$

This implies that the functor Φ_s is well-defined and¹:

$$\text{Hom}_{\text{zmf}(R, W)}(a_1, a_2) = \text{Hom}_{\text{zmf}(R, sW)}(\Phi_s(a_1), \Phi_s(a_2)) .$$

The coboundary categories $\text{bmf}(R, W)$ and $\text{bmf}(R, sW)$ are also related to each other in a similar way. More precisely, equation (1.2) gives:

$$\mathfrak{d}_{a_1, a_2}^s(f) = 0 \iff \mathfrak{d}_{a_1, a_2}(f) = 0 ,$$

which implies the equality $\text{bmf}(R, W)(a_1, a_2) = \text{bmf}(R, sW)(\Phi_s(a_1), \Phi_s(a_2))$. As a result, the functor Φ_s gives an equivalence of categories from $\text{hmf}(R, W)$ to $\text{hmf}(R, sW)$. Since the modules of morphisms naturally coincide, we also conclude that Φ_s maps distinguished triangles into distinguished triangles. This follows immediately from what we proved here and from Theorem 1.3. \square

¹ Notice that the right hand side is always a subset of the left hand side for any element $s \in R$. The equality holds since s is a unit.

1.3. Localizations. Let $S \subset R$ be a multiplicative subset of R containing the identity $1 \in R$. Let $\lambda_S : R \rightarrow R_S$ denote the natural ring homomorphism from R to the localization $R_S \stackrel{\text{def.}}{=} S^{-1}R$ of R at S . For any $r \in R$, let $r_S \stackrel{\text{def.}}{=} \lambda_S(r) = \frac{r}{1} \in R_S$ denote its extension. For any R -module N , let $N_S = S^{-1}N = N \otimes_R R_S$ denote the localization of N at S . For any morphism of R -modules $f : N \rightarrow N'$, let $f_S \stackrel{\text{def.}}{=} f \otimes_R \text{id}_{R_S} : N_S \rightarrow N'_S$ denote the localization of f at S . For any \mathbb{Z}_2 -graded R -module $M = M^{\hat{0}} \oplus M^{\hat{1}}$, we have $M_S = M_S^{\hat{0}} \oplus M_S^{\hat{1}}$, since the localization functor is exact. In particular, localization at S induces a functor from the category of \mathbb{Z}_2 -graded R -modules to the category of \mathbb{Z}_2 -graded R_S -modules.

Let $a = (M, D)$ be a matrix factorization of W . The *localization of a at S* (see [8]) is the following matrix factorization of W_S over the ring R_S :

$$a_S \stackrel{\text{def.}}{=} (M_S, D_S) \in \text{ObMF}(R_S, W_S) \quad .$$

This extends to an even dg functor $\text{loc}_S : \text{MF}(R, W) \rightarrow \text{MF}(R_S, W_S)$, which is R -linear and preserves direct sums. In turn, the latter induces functors $\text{ZMF}(R, W) \rightarrow \text{ZMF}(R_S, W_S)$, $\text{BMF}(R, W) \rightarrow \text{BMF}(R_S, W_S)$, $\text{HMF}(R, W) \rightarrow \text{HMF}(R_S, W_S)$ and $\text{hmf}(R, W) \rightarrow \text{hmf}(R_S, W_S)$, which we again denote by loc_S .

1.4. Critically-finite elements. Since R is a Bézout (and hence a GCD) domain, the irreducible elements of R are prime, which implies that any factorizable element (i.e. an element with finite factorization into irreducibles) of R has a unique prime factorization up to association. A divisor d of the element $W \in R^\times$ which is not a unit is called *critical* if $d^2 | W$. The *critical ideal* \mathfrak{J}_W of W is the ideal consisting of all elements of R which are divisible by every critical divisor of W :

$$\mathfrak{J}_W \stackrel{\text{def.}}{=} \{r \in R \mid d|r \ \forall d \in R^\times \text{ such that } d^2 | W\} \quad (1.3)$$

The following notion was introduced in [8]:

Definition 1.5 *A non-zero non-unit W of R is called:*

- non-critical, if W has no critical divisors;
- critically-finite if it has a factorization of the form:

$$W = W_0 W_c \quad \text{with} \quad W_c = p_1^{n_1} \dots p_N^{n_N} \quad , \quad (1.4)$$

where $N \geq 1$, $n_j \geq 2$, p_1, \dots, p_N are critical prime divisors of W with $(p_i) \neq (p_j)$ for $i \neq j$ and W_0 is non-critical and coprime with W_c .

The elements W_0 , W_c and p_i in the factorization (1.4) are determined by W up to association, while n_i are uniquely determined by W .

Remark 1.1. Let W be a critically-finite element of R with decomposition (1.4). Then the Chinese remainder theorem gives an isomorphism of rings:

$$R/\langle W \rangle \simeq R/\langle W_0 \rangle \oplus R/\langle W_c \rangle \quad .$$

When R is a Bézout domain, the ring:

$$R/\langle W_c \rangle \simeq R/\langle p_1^{n_1} \rangle \oplus \dots \oplus R/\langle p_N^{n_N} \rangle \simeq R/\langle p_1^{n_1} \dots p_N^{n_N} \rangle$$

is Artinian and Gorenstein since $R/\langle p_i^{n_i} \rangle$ are Gorenstein Artinian rings (see Section 2). However, the rings $R/\langle W_0 \rangle$ and $R/\langle W \rangle$ need not be Noetherian.

1.5. Elementary matrix factorizations. A matrix factorization $a = (M, D)$ of W over R is called *elementary* if it has unit reduced rank, i.e. if $\rho(a) = 1$. Any elementary factorization is strongly isomorphic to one of the form $e_v \stackrel{\text{def.}}{=} (R^{1|1}, D_v)$, where v is a divisor of W and $D_v \stackrel{\text{def.}}{=} \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$, with $u \stackrel{\text{def.}}{=} W/v \in R$. Let $\text{EF}(R, W)$ denote the full subcategory of $\text{MF}(R, W)$ whose objects are the elementary factorizations of W over R . Let $\text{ZEF}(R, W)$ and $\text{HEF}(R, W)$ denote respectively the cocycle and total cohomology categories of $\text{EF}(R, W)$. We also use the notations $\text{zef}(R, W) \stackrel{\text{def.}}{=} \text{ZEF}^{\hat{0}}(R, W)$ and $\text{hef}(R, W) \stackrel{\text{def.}}{=} \text{HEF}^{\hat{0}}(R, W)$ for the subcategories obtained by keeping only the even morphisms. An elementary factorization is indecomposable in $\text{zmf}(R, W)$, but it need not be indecomposable in $\text{hmf}(R, W)$.

2. Finitely-generated modules over the quotient of a Bézout domain by a principal primary ideal

Let R be a Bézout domain and $p \in R$ be a prime element. In this section, we study the category of finitely-generated modules over the quotient ring $R/\langle p^n \rangle$ (with $n \geq 2$) and its stable category.

2.1. The rings $A_n(p)$. Fix an integer $n \geq 2$ and consider the quotient ring²:

$$A_n(p) \stackrel{\text{def.}}{=} R/\langle p^n \rangle .$$

Let $\mathfrak{m}_n(p) = pA_n(p) = \langle p \rangle / \langle p^n \rangle$ and $\mathfrak{k}_p = R/\langle p \rangle$. The following result was proved in [8].

Lemma 2.1 *The following statements hold:*

1. *The principal ideal $\langle p \rangle$ generated by p is maximal.*
2. *The primary ideal $\langle p^n \rangle$ is contained in a unique maximal ideal of R .*
3. *The quotient $A_n(p)$ is a quasi-local ring with maximal ideal $\mathfrak{m}_n(p)$ and residue field \mathfrak{k}_p .*
4. *$A_n(p)$ is a generalized valuation ring.*

Remark 2.1. Let $Z(A_n(p))$ be the set of zero divisors, $N(A_n(p))$ be the nilradical and $J(A_n(p))$ be the Jacobson radical of $A_n(p)$. Then we have (see [6, Exercise 1.1]):

$$Z(A_n(p)) = N(A_n(p)) = J(A_n(p)) = \mathfrak{m}_n(p) .$$

Proposition 2.2 *$A_n(p)$ is an Artinian local principal ideal ring, whose ideals are $\langle p^i \rangle / \langle p^n \rangle$ for $i = 0, \dots, n$.*

Proof. Let I be an ideal of R such that $\langle p^n \rangle \subsetneq I \subsetneq \langle p \rangle$. Since $A_n(p)$ is a generalized valuation ring by Lemma 2.1, its ideals are totally ordered by inclusion. Hence there exists an $i \in \{2, \dots, n-1\}$ such that $\langle p^i \rangle \subset I \subsetneq \langle p^{i-1} \rangle$. Suppose that $I \setminus \langle p^i \rangle$ is non-empty and take any element $x \in I \setminus \langle p^i \rangle$. Then $x = rp^{i-1}$ for some $r \in R$ such that p doesn't divide r , i.e. $(r, p) = (1)$. Since R is a Bézout domain, there exist $a, b \in R$ such that $ar + bp = 1$. Multiplying with p^{i-1} , this gives $p^{i-1} = ax + bp^i$, which belongs to I since both x and p^i belong to I . Thus $p^{i-1} \in I$, which implies

² This ring will later on also be denoted by A for ease of notation.

$\langle p^{i-1} \rangle \subset I$ and hence $I = \langle p^{i-1} \rangle$, contradicting the fact that the inclusion $I \subset \langle p^{i-1} \rangle$ is strict. It follows that every ideal of $R/\langle p^n \rangle$ has the form $\langle p^i \rangle/\langle p^n \rangle$ for some $i \in \{0, \dots, n\}$. In particular, $R/\langle p^n \rangle$ is an Artinian (and hence Noetherian) local ring. Since R is a Noetherian Bézout ring, it is also a principal ideal ring. \square

Remark 2.2. Since $A_n(p)$ has non-trivial divisors of zero, it cannot be a regular local ring. It was shown in [10] that the global dimension of a generalized valuation ring which is not an integral domain is necessarily infinite. Thus $\text{gl dim}(A_n(p)) = \infty$. Also notice that $A_n(p)$ has length n as a module over itself.

For simplicity, in the remainder of this section we denote $A_n(p)$ by Λ , the residue field $\mathbf{k}_n(p)$ by \mathbf{k} and the maximal ideal $\mathbf{m}_n(p)$ by \mathbf{m} .

2.2. The category mod_Λ . Let mod_Λ be the category of finitely-generated modules over $\Lambda = A_n(p)$. Since Λ is Artinian, the following statements are equivalent for a Λ -module M by the Akizuki-Hopkins-Lewitzki theorem:

- M is Noetherian.
- M is Artinian.
- M is finitely-generated.
- M has finite composition length.

Let $\Lambda_i = \langle p^{n-i} \rangle/\langle p^n \rangle = p^{n-i}\Lambda$ with $i \in \{0, \dots, n\}$ be the ideals of Λ , thus $\Lambda_0 = 0$, $\Lambda_{n-1} = \mathbf{m}$ and $\Lambda_n = \Lambda$. These form the finite ascending sequence:

$$0 = \Lambda_0 \subset \Lambda_1 \subset \dots \subset \Lambda_{n-1} \subset \Lambda_n = \Lambda \quad . \quad (2.1)$$

Let $V_i \stackrel{\text{def.}}{=} \Lambda/\Lambda_{n-i} \simeq_R R/\langle p^i \rangle$ (with $i = 0, \dots, n$) be the cyclically-presented cyclic Λ -modules with annihilators $\text{Ann}(V_i) = \Lambda_{n-i}$. We have natural isomorphisms of R -modules $\varphi_i : V_i \xrightarrow{\sim} \Lambda_i$ given by taking the element $x + \langle p^i \rangle$ ($x \in R$) of $V_i \simeq_R R/\langle p^i \rangle$ to the element $p^{n-i}x + \langle p^n \rangle$ of Λ_i . Unlike the ideals Λ_i (which can be viewed as *non-unital* Λ -algebras), the modules V_i have a *unital* Λ -algebra structure with unit $1_\Lambda + \Lambda_{n-i}$. This unit is not preserved by the R -module isomorphisms φ_i . It is clear that the non-zero cyclic modules V_1, \dots, V_n are indecomposable, with endomorphism rings given by the local rings:

$$\text{End}_\Lambda(V_i) \simeq R/\langle p^i \rangle \quad , \quad \forall i \in \{1, \dots, n\} \quad .$$

Recall that a commutative ring R is called an *FGC* (*finitely-generated commutative*) *ring* if every finitely-generated R -module is isomorphic with a finite direct sum of cyclic modules. For any FGC ring R , the finite direct sum decomposition of a finitely-generated R -module into non-zero indecomposable cyclic modules is unique up to permutation and isomorphism of the indecomposable cyclic summands [11].

Proposition 2.3 *Λ is an FGC ring whose indecomposable non-zero finitely-generated Λ -modules are the cyclic modules V_1, \dots, V_n . Moreover, the decomposition of a finitely-generated Λ -module into non-zero cyclic modules is unique up to permutation and isomorphism of factors, hence mod_Λ is a Krull-Schmidt category.*

Proof. It is well-known that *any* module over a principal ideal ring decomposes as a direct sum of cyclic modules [12]. In particular, Λ is an FGC ring. Uniqueness of the decomposition into non-zero cyclic modules up to permutation and isomorphism of factors follows from [11, Proposition 3.4] since Λ is a generalized valuation ring. The indecomposable finitely-generated Λ -modules coincide with the cyclic modules V_1, \dots, V_n . See [13, Theorem 3.2]. \square

Proposition 2.4 *The only non-zero indecomposable Λ -module which is projective is $V_n \simeq \Lambda_n = \Lambda$.*

Proof. Since any projective module over a local ring is free, it follows that a finitely-generated Λ -module is projective iff it is free of finite rank. Such a module is indecomposable iff it has rank one. Another way to see this is as follows. Since the non-zero indecomposable Λ -modules are V_i with $i \in \{1, \dots, n\}$, it suffices to show that V_i is projective iff $i = n$. The module $\Lambda_n = \Lambda$ is projective since it is free. Thus it suffices to show that V_1, \dots, V_{n-1} are not projective. Recall that $\Lambda_{n-i} = p^i \Lambda$ is a principal Λ -module. It is well-known that such a module is projective iff there exists an idempotent $e \in \Lambda$ such that $p^i \Lambda = e \Lambda$. Suppose that this is the case for some $i \in \{1, \dots, n-1\}$. Then we must have:

$$p^{2i} \Lambda = e^2 \Lambda = e \Lambda = p^i \Lambda \quad . \quad (2.2)$$

If $2i \leq n$, this amounts to $\Lambda_{2i} = \Lambda_i$, which is impossible since the inclusions in (2.1) are strict. If $2i \geq n$, then we have $p^{2i} \Lambda = 0$ and relation (2.2) amounts to $p^i \Lambda = 0$, which is impossible since i belongs to the set $\{1, \dots, n-1\}$. \square

2.3. Uniseriality. Notice that Λ is a uniserial ring and that the indecomposable cyclic modules $V_i \simeq_{\Lambda} \Lambda_i$ are uniserial modules of length i . The unique composition series of Λ_i is given by:

$$0 = \Lambda_0 \subset \dots \subset \Lambda_i \quad .$$

In particular, the only simple Λ -module is $\Lambda_1 \simeq_R V_1 \simeq_R \mathbf{k}$. We have:

$$V_{i+1}/V_i \simeq \Lambda_{i+1}/\Lambda_i \simeq \mathbf{k}$$

and the only composition factor of $\Lambda_i \simeq_{\Lambda} V_i$ is \mathbf{k} , with multiplicity i .

2.4. The Frobenius property. The following result shows that Λ is a Frobenius ring.

Proposition 2.5 *The ring Λ is a commutative Frobenius ring. In particular, Λ is self-injective and hence it is a Gorenstein ring of dimension zero. Thus:*

$$\mathrm{Ext}_{\Lambda}^i(\mathbf{k}, \Lambda) \simeq_{\Lambda} \begin{cases} \mathbf{k} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases} \quad .$$

Proof. It is clear that Λ has a unique minimal ideal, namely Λ_1 . Since Λ is a local Artinian ring, it follows that Λ is Frobenius. This implies that R is self-injective and hence Gorenstein of dimension zero. \square

Since Λ is Noetherian and self-injective (i.e. quasi-Frobenius, which for a commutative ring is the same as being Frobenius), it follows that a Λ -module is injective iff it is projective. In particular, mod_{Λ} is a Frobenius category. Notice that $K_{\Lambda} = \Lambda$ is a canonical Λ -module. In particular, all finitely-generated Λ -modules are reflexive.

2.5. *The Auslander-Reiten quiver of mod_A .* The following result allows us to describe the morphisms between the modules V_i .

Proposition 2.6 *Let R be a Bézout domain and $a, b \in R^\times$. Then there exists an isomorphism of R -modules:*

$$q_{ab} : \text{Hom}_R(R/\langle a \rangle, R/\langle b \rangle) \xrightarrow{\sim} R/\langle a, b \rangle$$

which is determined up to multiplication by a unit of R . If $a, b, c \in R^\times$ are three elements and $f \in \text{Hom}_R(R/\langle a \rangle, R/\langle b \rangle)$, $g \in \text{Hom}_R(R/\langle b \rangle, R/\langle c \rangle)$, then we have:

$$q_{ac}(g \circ f) \stackrel{\text{def.}}{=} s_{abc} q_{bc}(g) q_{ab}(f) \quad ,$$

where $s_{abc} \in \frac{(a,c)(b)}{(b,c)(a,b)}$.

Proof. The cyclic module $R/\langle a \rangle$ is generated by the element $\epsilon_a = 1 \bmod \langle a \rangle$, while $R/\langle b \rangle$ is generated by $\epsilon_b = 1 \bmod \langle b \rangle$. Consider the injective R -module morphism $\varphi_{ab} : \text{Hom}_R(R/\langle a \rangle, R/\langle b \rangle) \rightarrow R/\langle b \rangle$ which associates to $f \in \text{Hom}_R(R/\langle a \rangle, R/\langle b \rangle)$ the unique element $\varphi_{ab}(f) \in R/\langle b \rangle$ such that $f(\epsilon_a) = \varphi_{ab}(f)\epsilon_b$. Let $r \in R$ be an element such that $\varphi_{ab}(f) = r \bmod \langle b \rangle$. Since $a\varphi_{ab}(f)\epsilon_b = af(\epsilon_a) = f(a\epsilon_a) = f(0) = 0$, we have $a\varphi_{ab}(f) = 0$ in the ring $R/\langle b \rangle$, which is equivalent with the condition $b|ar$. Writing $a = a_1d_{ab}$ and $b = b_1d_{ab}$ with $d_{ab} \in (a, b)$ and $(a_1, b_1) = (1)$, this is equivalent with the condition $b_1|r$, i.e. $r \in \langle b_1 \rangle$. Hence the image of φ_{ab} equals $\langle b_1 \rangle / \langle b \rangle$. The map $\langle b_1 \rangle \ni r \rightarrow r/b_1 \in R$ induces an isomorphism of R -modules $\psi_{ab} : \langle b_1 \rangle / \langle b \rangle \xrightarrow{\sim} R/\langle a, b \rangle$. Then $q_{ab} \stackrel{\text{def.}}{=} \psi_{ab} \circ \varphi_{ab} : \text{Hom}_R(R/\langle a \rangle, R/\langle b \rangle) \rightarrow R/\langle a, b \rangle$ is the desired isomorphism of R -modules, which acts as $q_{ab}(f) = \frac{\varphi_{ab}(f)}{b_1} = \frac{d_{ab}\varphi_{ab}(f)}{b}$. Since d_{ab} is determined up to multiplication by a unit of R , the same holds for $q_{ab}(f)$.

Given three non-vanishing elements a, b, c of R and morphisms f, g as in the proposition, we have:

$$(g \circ f)(\epsilon_a) = g(\varphi_{ab}(f)\epsilon_b) = \varphi_{bc}(g)\varphi_{ab}(f)\epsilon_c \quad ,$$

which gives $\varphi_{ac}(g \circ f) = \varphi_{bc}(g)\varphi_{ab}(f)$. Thus:

$$q_{ac}(g \circ f) = \frac{d_{ac}\varphi_{bc}(g)\varphi_{ab}(f)}{c} = \frac{d_{ac}cb}{cd_{bc}d_{ab}} q_{bc}(g)q_{ab}(f) = s_{abc} q_{bc}(g)q_{ab}(f) \quad ,$$

where:

$$s_{abc} = \frac{d_{ac}b}{d_{bc}d_{ab}} q_{bc}(g)q_{ab}(f) \in \frac{(a,c)(b)}{(b,c)(a,b)} \quad .$$

Corollary 2.7 *Let R be a Bézout domain and $a, b, c \in R^\times$ be three elements such that $a|c$ and $b|c$. Then there exists an isomorphism of $R/\langle c \rangle$ -modules:*

$$\text{Hom}_{R/\langle c \rangle}(R/\langle a \rangle, R/\langle b \rangle) \simeq R/\langle a, b \rangle \quad .$$

Proof. Restriction of scalars along the epimorphism $\pi : R \rightarrow R/\langle c \rangle$ gives a full and faithful functor $\pi^* : \text{Mod}_{R/\langle c \rangle} \rightarrow \text{Mod}_R$. The composition $q_{ab} \circ \pi^*_{R/\langle a \rangle, R/\langle b \rangle} : \text{Hom}_{R/\langle c \rangle}(R/\langle a \rangle, R/\langle b \rangle) \rightarrow R/\langle a, b \rangle$ is the desired isomorphism. \square

Proposition 2.8 *For any $i, j \in \{0, \dots, n\}$, we have an isomorphism of modules:*

$$\text{Hom}_A(V_i, V_j) \simeq_A V_{\min(i,j)} \quad .$$

For any $i \in \{1, \dots, n\}$, we have an isomorphism of rings:

$$\text{End}_A(V_i) \simeq R/\langle p^i \rangle \quad .$$

Proof. Follows immediately from Corollary 2.7. \square

In particular, $\text{End}_\Lambda(V_i)$ is a commutative local ring with maximal ideal $m_i \stackrel{\text{def.}}{=} \langle p \rangle / \langle p^i \rangle$ and residue field equal to \mathbf{k}_p . Consider the field:

$$T(V_i) \stackrel{\text{def.}}{=} \text{End}_\Lambda(V_i, V_i) / m_i \simeq \mathbf{k}_p \quad .$$

Proposition 2.9 *For any $0 \leq j \leq i \leq n$, we have:*

$$V_i / V_j \simeq_\Lambda V_{i-j} \quad .$$

Moreover, the natural surjection $q_{n,i} : V_n \rightarrow V_i$ is a projective cover for all $i \in \{1, \dots, n\}$ and the first syzygy of V_i is given by:

$$\Omega(V_i) = \ker(q_{n,i}) \simeq V_{n-i} \quad .$$

Proof. We have $V_i / V_j = \langle p^{n-i} \rangle / \langle p^{n-j} \rangle \simeq_R R / \langle p^{i-j} \rangle = V_{i-j}$, so similar isomorphisms hold over Λ . Recall that $V_n \simeq \Lambda$ is a projective module. Since each V_k has a single maximal submodule (namely V_{k-1}), we have $\text{rad}(V_k) = V_{k-1}$ for all $k \in \{1, \dots, n\}$. The induced map $\bar{q}_{n,i} : V_n / \text{rad}(V_n) \rightarrow V_i / \text{rad}(V_i)$ is an isomorphism since $V_n / V_{n-1} \simeq_\Lambda R / \langle p \rangle \simeq_\Lambda V_i / V_{i-1}$. This implies that $q_{n,i}$ is a projective cover by [14, Chap I.4, Proposition 4.3, page 13]. It is clear that $\ker(q_{n,i}) \simeq V_{n-i}$. \square

Proposition 2.10 *Let $f : V_i \rightarrow V_j$ be an irreducible morphism in mod_Λ . Then one of the following holds:*

1. *f is injective and $j = i + 1$. In this case, f fits into a short exact sequence:*

$$0 \longrightarrow V_i \xrightarrow{f} V_{i+1} \longrightarrow V_1 \longrightarrow 0 \quad .$$

2. *f is surjective and $j = i - 1$. In this case, f fits into a short exact sequence:*

$$0 \longrightarrow V_1 \longrightarrow V_i \xrightarrow{f} V_{i-1} \longrightarrow 0 \quad .$$

Proof. Recall that an irreducible morphism $f : V_i \rightarrow V_j$ in mod_Λ must be either a monomorphism or an epimorphism [14, Chap. V.5, Lemma 5.1]. Distinguish the cases:

1. If f is a monomorphism, then $\text{im} f = V_k$ for some $k \leq j$. Since $V_i \simeq_\Lambda \text{im} f$, we must have $k = i$ and $\text{im} f = V_i$. Thus $i \leq j$. Moreover, $\text{im} f$ must be a direct summand of any proper submodule of V_j which contains $\text{im} f$. Since no submodule of V_j has a direct summand, we must have $\text{im} f = V_{j-1}$ and hence $j = i + 1$.
2. If f is an epimorphism, then $\ker f = V_k$ for some $k \leq i$. Since $V_j \simeq_\Lambda V_i / \ker f = V_i / V_k \simeq_\Lambda V_{i-k}$, we must have $i \geq j$ and $k = i - j$. Moreover, V_j must be a summand of V_i / M for any non-zero submodule M of V_i which is contained in $\ker f = V_k$, i.e. it must be a summand of $V_i / V_s = V_{i-s}$ for any $s \in \{1, \dots, k\}$. Since none of the modules V_1, \dots, V_n has direct summands, this means that we must have $k = 1$, i.e. $i = j + 1$.

The short exact sequences follow immediately from the above. \square

For any $i \in \{1, \dots, n-1\}$, let $s_{i,i+1} : V_i \rightarrow V_{i+1}$ be the inclusion. For any $i = 2, \dots, n$, let $q_{i,i-1} : V_i \rightarrow V_{i-1}$ be the natural surjection. For any $i \in \{1, \dots, n-1\}$, we have an almost split sequence (see [14, p. 141]):

$$0 \longrightarrow V_i \xrightarrow{g_i} V_{i-1} \oplus V_{i+1} \xrightarrow{f_i} V_i \longrightarrow 0 ,$$

where $g_i = \begin{bmatrix} -q_{i,i-1} \\ s_{i,i+1} \end{bmatrix}$ and $f_i = [s_{i-1,i}, q_{i+1,i}]$. In particular, the morphisms $s_{i,i+1}$ and $q_{i,i-1}$ are irreducible by [14, Chap. V.5., Theorem 5.3, p. 167]. Moreover, the Auslander-Reiten translation $\tau = \text{DTr}$ is given by:

$$\tau(V_i) = V_i , \quad \forall i \in \{1, \dots, n-1\} , \quad \tau(V_n) = 0 .$$

(recall that $\text{DTr}(P) = 0$ iff P is a projective module). It follows that DTr acts trivially on Λ -modules which have no projective direct summands. By [14, page 229], the class $\bar{s}_{i-1,i}$ of $s_{i-1,i}$ generates the $T(V_{i-1})^{\text{opp}}$ -vector space $\text{Irr}(V_{i-1}, V_i)$ while the class $\bar{q}_{i+1,i}$ of $q_{i+1,i}$ generates the $T(V_{i+1})^{\text{opp}}$ -vector space $\text{Irr}(V_{i+1}, V_i)$. Similarly, the class $\bar{s}_{i,i+1}$ of $s_{i,i+1}$ generates the $T(V_{i+1})$ -vector space $\text{Irr}(V_i, V_{i+1})$ and the class $\bar{q}_{i,i-1}$ of $q_{i,i-1}$ generates the $T(V_{i-1})$ -vector space $\text{Irr}(V_i, V_{i-1})$. Thus:

- $\text{Irr}(V_i, V_{i+1})$ is generated by $\bar{s}_{i,i+1}$ over both $T(V_i)^{\text{opp}}$ and $T(V_{i+1})$.
- $\text{Irr}(V_i, V_{i-1})$ is generated by $\bar{q}_{i,i-1}$ over both $T(V_i)^{\text{opp}}$ and $T(V_{i-1})$.

It follows that the arrow $V_i \rightarrow V_{i-1}$ for $i = 2, \dots, n-1$ and the arrows $V_{i-1} \rightarrow V_i$ have trivial valuation $(1, 1)$. The Auslander-Reiten quiver of mod_Λ is shown in Figure 2.1.

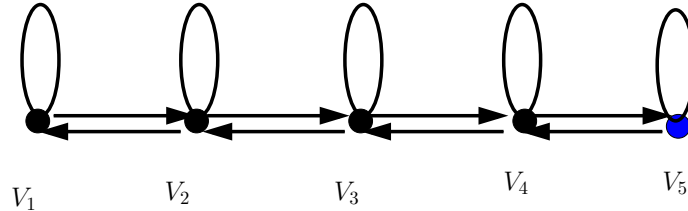


Fig. 2.1. Auslander-Reiten quiver for mod_Λ when $n = 5$. The single projective injective vertex is shown in blue. The Auslander-Reiten translation fixes all non-projective vertices.

2.6. The category $\underline{\text{mod}}_\Lambda$. Let $\underline{\text{mod}}_\Lambda$ denote the projectively-stable category of finitely-generated Λ -modules. Since any projective Λ -module is free, this category has the same objects as mod_Λ and morphisms given by:

$$\underline{\text{Hom}}_\Lambda(M, N) \stackrel{\text{def.}}{=} \text{Hom}_\Lambda(M, N) / \mathcal{P}_\Lambda(M, N) , \quad \forall M, N \in \text{Ob}(\text{mod}_\Lambda) ,$$

where $\mathcal{P}_\Lambda(M, N) \subset \text{Hom}_\Lambda(M, N)$ is the submodule consisting of those morphisms from M to N which factor through a free module of finite rank. Since mod_Λ is a Frobenius category, the stable category $\underline{\text{mod}}_\Lambda$ has a natural triangulated structure.

The first syzygy induces a functor $\Omega : \underline{\text{mod}}_\Lambda \rightarrow \underline{\text{mod}}_\Lambda$ which is an equivalence of categories since Λ is self-injective (see [14, Chap. IV.3]). Since Λ is a symmetric Artin algebra, we also have $D \simeq \text{Hom}_\Lambda(-, \Lambda)$ and $\Omega^2 \simeq \text{DTr} = \tau$. Since DTr acts as the identity functor on indecomposable non-projectives of mod_Λ , we have $\text{DTr} \simeq \text{id}_{\underline{\text{mod}}_\Lambda}$ and hence $\Omega^2 \simeq \text{id}_{\underline{\text{mod}}_\Lambda}$. The functor Ω is the shift functor of the triangulated category $\underline{\text{mod}}_\Lambda$.

For $i, j \in \{0, \dots, n\}$, define:

$$\delta_n(i) \stackrel{\text{def.}}{=} \min(i, n-i) \in \{1, \dots, n-1\}, \quad \mu_n(i, j) \stackrel{\text{def.}}{=} \min[\delta_n(i), \delta_n(j)] = \begin{cases} i & \text{if } i+j \leq n \text{ \& } i \leq j \\ j & \text{if } i+j \leq n \text{ \& } i > j \\ n-i & \text{if } i+j > n \text{ \& } i > j \\ n-j & \text{if } i+j > n \text{ \& } i \leq j \end{cases} . \quad (2.3)$$

Notice the relations $\delta_n(i) = \delta_n(n-i)$ and $\delta_n(n) = 0$ as well as:

$$\mu_n(i, j) = \mu_n(j, i) = \mu_n(n-i, j) = \mu_n(i, n-j), \quad \mu_n(i, n) = 0 . \quad (2.4)$$

Proposition 2.11 *For any $1 \leq i, j \leq n-1$, we have:*

$$\underline{\text{Hom}}_{\Lambda}(V_i, V_j) \simeq_{\Lambda} V_{\mu_n(i, j)} .$$

Proof. A similar statement is proved in [15, Lemma 2.3]. For completeness we sketch the proof. Proposition 2.8 gives an isomorphism of Λ -modules:

$$\text{Hom}_{\Lambda}(V_i, V_j) \simeq_{\Lambda} V_{\min(i, j)} \simeq_{\Lambda} p^{\min(i, j)} \Lambda = p^{\max(n-i, n-j)} \Lambda = (p^{n-i} \Lambda) \cap (p^{n-j} \Lambda) ,$$

where we noticed that $n - \min(i, j) = \max(n-i, n-j)$. The morphism $f \in \text{Hom}_{\Lambda}(p^{n-i} \Lambda, p^{n-j} \Lambda)$ factors through a free module iff³ its image through this isomorphism lies in the ideal $p^{n-i} \Lambda p^{n-j} = p^{2n-i-j} \Lambda$. Thus:

$$\underline{\text{Hom}}(V_i, V_j) \simeq_{\Lambda} \frac{p^{\max(n-i, n-j)} \Lambda}{p^{2n-i-j} \Lambda} \simeq_{\Lambda} \frac{p^{\min(i, j)} \Lambda}{p^{2n-i-j} \Lambda} .$$

The denominator is isomorphic to 0 when $i+j \leq n$. In this case we have:

$$\underline{\text{Hom}}(V_i, V_j) \simeq_{\Lambda} R / \langle p^{\min(i, j)} \rangle = V_{\min(i, j)} .$$

On the other hand, when $i+j > n$, we find:

$$\underline{\text{Hom}}(V_i, V_j) \simeq_{\Lambda} \frac{\langle p^{\max(n-i, n-j)} \rangle}{\langle p^{2n-i-j} \rangle} \simeq_{\Lambda} R / \langle p^{\min(n-i, n-j)} \rangle = V_{\min(n-i, n-j)} ,$$

where we noticed that $2n-i-j = \min(n-i, n-j) + \max(n-i, n-j)$. The conclusion follows upon noticing that:

$$\mu_n(i, j) = \begin{cases} \min(i, j) & \text{if } i+j \leq n \\ \min(n-i, n-j) & \text{if } i+j > n \end{cases} . \quad \square$$

2.7. The Auslander-Reiten quiver of $\underline{\text{mod}}_{\Lambda}$. For any Λ -module M , there exists an injective resolution:

$$M \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \dots$$

whose cohomology in degree one equals $\Omega(M)$. Hence we have natural isomorphisms of Λ -modules $\text{Ext}^1(N, M) \simeq_{\Lambda} \underline{\text{Hom}}_{\Lambda}(N, \Omega(M))$ and any Auslander-Reiten sequence:

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \quad (2.5)$$

³ As in [15, Lemma 2.3], this follows from the fact that the natural morphism of modules from $V_i \simeq p^{n-i} \Lambda$ to $V_i^{\vee} = \text{Hom}_{\Lambda}(p^{n-i} \Lambda, \Lambda) = \text{Hom}_{\Lambda}(V_i, V_n) \simeq V_{\min(i, n)} = V_i$ is an isomorphism by Proposition 2.8.

induces an Auslander-Reiten triangle:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\psi} \Omega(X) \ ,$$

where $\psi \in \underline{\mathrm{Hom}}_{\Lambda}(Z, X[1]) = \mathrm{Ext}^1(Z, X)$ is the extension class defined by the AR sequence (2.5). As a consequence, the category $\underline{\mathrm{mod}}_{\Lambda}$ has Auslander-Reiten triangles which are given by:

$$V_i \xrightarrow{g_i} V_{i-1} \oplus V_{i+1} \xrightarrow{f_i} V_i \longrightarrow \Omega(V_i) \ , \quad \forall i \in \{1, \dots, n-1\} \ . \quad (2.6)$$

In particular, V_1, \dots, V_{n-1} are indecomposable objects of $\underline{\mathrm{mod}}_{\Lambda}$ which have local endomorphism rings. It follows that $\underline{\mathrm{mod}}_{\Lambda}$ is Krull-Schmidt with indecomposables V_1, \dots, V_{n-1} . Moreover, \underline{g}_i are source morphisms and \underline{f}_i are sink morphisms, which implies $\dim_{T(V_i)} \mathrm{Irr}(V_i, V_{i+1}) = \dim_{T(V_{i+1})} \mathrm{Irr}(V_i, V_{i+1}) = 1$ and $\dim_{T(V_i)} \mathrm{Irr}(V_i, V_{i-1}) = \dim_{T(V_{i-1})} \mathrm{Irr}(V_i, V_{i-1}) = 1$ (see [16]). Hence all arrows of the AR quiver of $\underline{\mathrm{mod}}_{\Lambda}$ have trivial valuation $(1, 1)$. The AR translation is given by $\tau(V_i) = V_i$ for all $i \in \{1, \dots, n-1\}$. The AR quiver of $\underline{\mathrm{mod}}_{\Lambda}$ is obtained from that of mod_{Λ} by deleting the projective vertex; an example is shown in Figure 2.2.

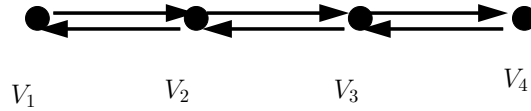


Fig. 2.2. Auslander-Reiten quiver for $\underline{\mathrm{mod}}_{\Lambda}$ when $n = 5$. The translation fixes all vertices.

2.8. The Calabi-Yau property of $\underline{\mathrm{mod}}_{\Lambda}$. Recall that Λ is a self-injective (a.k.a. quasi-Frobenius) commutative ring. This implies that all finitely-generated Λ -modules are reflexive and that the dual $D(M) = \mathrm{Hom}_{\Lambda}(M, \Lambda)$ of any finitely-generated module is finitely-generated [17, Theorem 4.12.21]. Thus D is an involutive auto-equivalence of $\underline{\mathrm{mod}}_{\Lambda}$. Since Λ is self-injective, we have $\underline{\mathrm{mod}}_{\Lambda} \simeq \overline{\mathrm{mod}}_{\Lambda}$ and hence D induces a well-defined involutive autoequivalence of $\underline{\mathrm{mod}}_{\Lambda}$ by [14, Chap. IV.1, Proposition 1.9, page 106], which we denote by the same letter.

Lemma 2.12 *We have:*

$$D(V_i) \simeq_{\Lambda} V_i \ , \quad \forall i \in \{1, \dots, n\} \ .$$

Proof. For any $i \in \{1, \dots, n\}$, we have:

$$D(V_i) = \mathrm{Hom}_{\Lambda}(V_i, V_n) \simeq_{\Lambda} V_{\min(i, n)} = V_i \ ,$$

where we used Proposition 2.8. \square

Recall that an additive autoequivalence S of the R -linear category $\underline{\mathrm{mod}}_{\Lambda}$ is called a *Serre functor* if we have natural isomorphisms of Λ -modules:

$$\underline{\mathrm{Hom}}_{\Lambda}(M, N) \simeq_{\Lambda} D(\underline{\mathrm{Hom}}_{\Lambda}(N, S(M))) \ , \quad \forall M, N \in \mathrm{Ob}[\underline{\mathrm{mod}}_{\Lambda}] \ .$$

This implies that S is a triangulated auto-equivalence. The following proposition shows that the R -linear triangulated category $\underline{\mathrm{mod}}_{\Lambda}$ is “1-Calabi-Yau”:

Proposition 2.13 *The functor $S = \Omega$ is a Serre functor for $\underline{\mathrm{mod}}_{\Lambda}$.*

Proof. Since $\underline{\text{mod}}_\Lambda$ is Krull-Schmidt with indecomposable objects V_1, \dots, V_{n-1} , it suffices to show that we have natural isomorphisms in $\underline{\text{mod}}_\Lambda$:

$$\underline{\text{Hom}}_\Lambda(V_i, V_j) \simeq_{\underline{\text{mod}}_\Lambda} \text{D}(\underline{\text{Hom}}_\Lambda(V_j, \Omega(V_i))) , \quad \forall i, j \in \{1, \dots, n-1\} . \quad (2.7)$$

Since $\Omega V_i \simeq_\Lambda V_{n-i}$, Proposition 2.11 shows that the right hand side of (2.7) is given by:

$$\text{D}(\underline{\text{Hom}}_\Lambda(V_j, \Omega(V_i))) \simeq_\Lambda \text{D}(\underline{\text{Hom}}_\Lambda(V_j, V_{n-i})) \simeq_\Lambda \text{D}(V_{\mu_n(j, n-i)}) = \text{D}(V_{\mu_n(i, j)}) \simeq_\Lambda V_{\mu_n(i, j)} ,$$

where we used relations (2.4) and Lemma 2.12. On the other hand, the left hand side of (2.7) is given by:

$$\underline{\text{Hom}}_\Lambda(V_i, V_j) \simeq_\Lambda V_{\mu_n(i, j)} .$$

Since all isomorphisms above are natural, we conclude that (2.7) holds since any isomorphism in mod_Λ induces an isomorphism in $\underline{\text{mod}}_\Lambda$. \square

2.9. A triangle generator for $\underline{\text{mod}}_\Lambda$. We say that a full subcategory \mathcal{C} of $\underline{\text{mod}}_\Lambda$ is *closed under extensions* (also known as *thick* or *épaisse*) if, given any distinguished triangle:

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Omega(X)$$

of $\underline{\text{mod}}_\Lambda$, we have $Y \in \text{Ob}\mathcal{C}$ provided that X and Z are objects of \mathcal{C} . We say that a full subcategory \mathcal{C} of $\underline{\text{mod}}_\Lambda$ is *isomorphism-closed* (or *strictly full*) if any object of $\underline{\text{mod}}_\Lambda$ which is isomorphic with an object of \mathcal{C} is an object of \mathcal{C} . A full subcategory \mathcal{C} of $\underline{\text{mod}}_\Lambda$ is called *saturated* if it is closed under direct summands. Given an object X of $\underline{\text{mod}}_\Lambda$, let $\langle X \rangle$ denote the smallest triangulated subcategory of $\underline{\text{mod}}_\Lambda$ which contains the object X and is strictly full and saturated. This coincides with the smallest full subcategory of $\underline{\text{mod}}_\Lambda$ which is closed under isomorphisms, direct sums, shifts and direct summands.

Proposition 2.14 *The smallest full subcategory of $\underline{\text{mod}}_\Lambda$ which contains the object $V_1 = \mathbf{k}_p$ and is closed under isomorphisms, direct sums, direct summands and extensions coincides with $\underline{\text{mod}}_\Lambda$. Hence:*

$$\langle V_1 \rangle = \underline{\text{mod}}_\Lambda .$$

Proof. Let $\mathcal{T} = \langle V_1 \rangle$ be the smallest subcategory of $\underline{\text{mod}}_\Lambda$ which is closed under isomorphisms, direct sums, direct summands and shifts and such that any distinguished triangle of $\underline{\text{mod}}_\Lambda$ for which two objects belong to \mathcal{T} lies in \mathcal{T} .

We first show by induction that the modules V_i with $i = 2, \dots, n-1$ belong to \mathcal{T} . Consider the AR triangle (2.6) for $i = 1$:

$$V_1 \xrightarrow{g_0} V_2 \xrightarrow{f_0} V_1 \longrightarrow \Omega(V_0)$$

where we used the fact that $V_0 = 0$. Since $V_1 \in \text{Ob}\mathcal{T}$, we have $V_2 \in \text{Ob}\mathcal{T}$. Suppose now that V_{i-1} and V_i are objects of \mathcal{T} for some $i \in \{2, \dots, n-1\}$. Considering the sequence (2.6) for i , and using the fact that V_i is an object of \mathcal{T} , we conclude similarly that $V_{i-1} \oplus V_{i+1}$ is an object of \mathcal{T} . Thus V_{i+1} is also an object of \mathcal{T} since \mathcal{T} is closed under direct summands. We conclude by induction that V_1, \dots, V_{n-1} belong to \mathcal{T} . This gives the conclusion since \mathcal{T} is closed under direct sums and $\underline{\text{mod}}_\Lambda$ is additively generated by the objects V_1, \dots, V_{n-1} . \square

2.10. *Equivalence between $\underline{\text{mod}}_\Lambda$ and the category of singularities of Λ .* Recall that the category of singularities of Λ is the Verdier quotient:

$$D_{\text{sing}}^b(\Lambda) \stackrel{\text{def.}}{=} D^b(\Lambda)/\text{Perf}(\Lambda) \quad ,$$

where $D^b(\Lambda)$ is the bounded derived category of complexes of finitely generated modules and $\text{Perf}(\Lambda)$ is the triangulated subcategory of perfect complexes. In our case, this category is triangle-equivalent with $\underline{\text{mod}}_\Lambda$, as we explain next.

Recall that the *depth* of a Noetherian Λ -module M is defined through:

$$\text{depth}_\Lambda(M) \stackrel{\text{def.}}{=} \inf_{i \geq 0} \{ \dim \text{Ext}_\Lambda^i(\mathbf{k}, M) > 0 \} \quad .$$

This quantity satisfies the inequality:

$$\text{depth}_\Lambda(M) \leq \text{kdim}(\Lambda/\text{Ann}(M)) \leq \text{kdim} \Lambda \quad .$$

There is another way to formulate this for local rings. Let (R, m) be a local ring. Recall that a sequence $x_1, \dots, x_r \in m$ is called an M -sequence if x_i is a non zero divisor in the quotient $M/\langle x_1, \dots, x_{i-1} \rangle$ for all $1 \leq i \leq r$. The depth of a module over a local ring (R, m) is equal to the length of a maximal M -sequence. A Noetherian Λ -module is called *maximal Cohen-Macaulay* (MCM) if $\text{depth}_\Lambda(M) = \text{kdim}(\Lambda)$. Let $\text{MCM}(\Lambda)$ be the full subcategory of mod_Λ whose objects are the MCM modules.

Lemma 2.15 *Any finitely-generated Λ -module M is maximal Cohen-Macaulay. Thus $\text{MCM}(\Lambda) = \text{mod}_\Lambda$.*

Proof. This is well-known, but we sketch the proof for completeness. Since Λ is an Artinian local ring, it has Krull dimension zero. On the other hand, the depth of any finitely-generated Λ -module is zero since any element of m is nilpotent and hence a divisor of zero. \square

Proposition 2.16 *There exists an equivalence of triangulated categories:*

$$D_{\text{sing}}^b(\Lambda) \simeq \underline{\text{mod}}_\Lambda \quad .$$

Proof. Since Λ is Gorenstein, there exists [7] a natural equivalence of triangulated categories $D_{\text{sing}}^b(\Lambda) \simeq \underline{\text{MCM}}(\Lambda)$, where $\underline{\text{MCM}}(\Lambda)$ is the projective stabilization of $\text{MCM}(\Lambda)$. The conclusion now follows from Lemma 2.15. \square

2.11. *Localization at $U(\Lambda)$.* Since Λ is a local ring with maximal ideal $\langle p \rangle$, the multiplicative set $\Lambda \setminus \langle p \rangle$ coincides with the group of units $U(\Lambda)$.

Proposition 2.17 *Localization at the multiplicative set $U(\Lambda) = \Lambda \setminus \langle p \rangle$ of units of Λ induces an equivalence of triangulated categories:*

$$\text{loc}_p : \underline{\text{mod}}_\Lambda \xrightarrow{\sim} \underline{\text{mod}}_{\Lambda_{(p)}}$$

Proof. Multiplication by any $s \in U(\Lambda)$ gives an isomorphism of the Λ -modules $V_i \simeq_R \Lambda_i$ for each $i \in \{1, \dots, n\}$. Since mod_Λ is additively generated by V_1, \dots, V_n , it follows that s acts as an isomorphism on any finitely-generated Λ -module. In particular, the localization functor loc_p at the multiplicative set $U(\Lambda)$ is an equivalence of categories between mod_Λ and $\text{mod}_{\Lambda_{(p)}}$. Since this functor is exact, it is an equivalence of exact categories. Since mod_Λ is a Frobenius category, it follows that the same is true for $\text{mod}_{\Lambda_{(p)}}$ and that loc_p induces a triangulated equivalence $\underline{\text{loc}}_p$ between the stable categories $\underline{\text{mod}}_\Lambda$ and $\underline{\text{mod}}_{\Lambda_{(p)}}$. \square

Remark 2.3. We have a natural isomorphism of rings:

$$\Lambda_{(p)} \simeq R_{(p)} / \langle p^n \rangle .$$

3. Matrix factorizations over an elementary divisor domain

Let R be an elementary divisor domain and W be a non-zero element of R .

3.1. Isomorphism classes in $\text{zmf}(R, W)$. The Smith normal form theorem over an elementary divisor domain (see Appendix B) allows us to characterize isomorphism classes of objects in the category $\text{zmf}(R, W)$.

Proposition 3.1 *Let $a = (R^{\rho|\rho}, D)$ and $a' = (R^{\rho'|\rho'}, D')$ be two finite rank matrix factorizations of the non-zero element $W \in R^\times$, where $D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$ and $D' = \begin{bmatrix} 0 & v' \\ u' & 0 \end{bmatrix}$. Let $\mathbf{d}_1(v), \dots, \mathbf{d}_\rho(v)$ and $\mathbf{d}_1(v'), \dots, \mathbf{d}_{\rho'}(v')$ be respectively the invariant factors of the matrices $v \in \text{Mat}(\rho, \rho, R)$ and $v' \in \text{Mat}(\rho', \rho', R)$. Then the following statements are equivalent:*

- (a) a and a' are isomorphic in the category $\text{zmf}(R, W)$.
- (b) We have $\rho = \rho'$ and the invariant factors of v and v' are equal:

$$\mathbf{d}_i(v) = \mathbf{d}_i(v') , \quad \forall i \in \{1, \dots, \rho\} .$$

Proof. By [8, Proposition 1.4], the matrix factorizations a and a' are strongly isomorphic iff $\rho = \rho'$ and the matrices v and v' are equivalent. Recall that u, v, u', v' have maximal rank since $W \neq 0$. Since R is an EDD, Proposition B.5 shows that v and v' are equivalent iff $\rho = \rho'$ and their invariant factors satisfy $\mathbf{d}_i(v) = \mathbf{d}_i(v')$ for all $i \in \{1, \dots, \rho\}$. \square

The following result shows that any matrix factorization of W is naturally isomorphic in $\text{zmf}(R, W)$ to a direct sum of elementary factorizations.

Theorem 3.2 *There exists an autoequivalence F of the category $\text{zmf}(R, W)$ such that:*

1. F is isomorphic with the identity functor $\text{id}_{\text{zmf}(R, W)}$.

2. For any matrix factorization $a = (R^{\rho|\rho}, D)$ of W with $D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$, we have:

$$F(a) = e_{d_1(v)} \oplus \dots \oplus e_{d_\rho(v)} ,$$

where $d_1(v), \dots, d_\rho(v) \in R$ are representatives for the invariant factors of v , i.e. $d_i(v) \in \mathbf{d}_i(v)$ for all $i \in \{1, \dots, \rho\}$.

Proof. For any $v \in \text{Mat}(\rho, \rho, R)$, choose invertible matrices $A_v, B_v \in \text{GL}(\rho, R)$ such that $v_0 \stackrel{\text{def.}}{=} A_v v B_v^{-1}$ is in Smith normal form:

$$v_0 = \text{diag}(d_1(v), \dots, d_k(v), 0, \dots, 0) ,$$

where $d_k(v) \in \mathbf{d}_k(v)$. For any matrix factorization $a = (R^{\rho|_{\rho}}, D)$ of W with $D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$, let

$$U_D = \begin{bmatrix} A_v & 0 \\ 0 & B_v \end{bmatrix} \text{ and:}$$

$$D_0 \stackrel{\text{def.}}{=} U_D D U_D^{-1} = \begin{bmatrix} 0 & A_v v B_v^{-1} \\ B_v u A_v^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & v_0 \\ u_0 & 0 \end{bmatrix}, \quad (3.1)$$

where $u_0 = B_v u A_v^{-1}$. Since $uv = vu = W$, we have $u_0 v_0 = v_0 u_0 = W$. This requires that u_0 is diagonal (we have $u_0 = W v_0^{-1}$ in the field of fractions of R), namely we have $u_0 = \text{diag}(u_1, \dots, u_\rho)$, where $u_i = \frac{W}{d_i(v)}$. Since $d_i(v) | d_{i+1}(v)$, we have $u_i | u_{i-1}$ and hence u_0 is the reverse Smith normal form of u :

$$u_0 = \text{diag}(d_\rho(u_0), \dots, d_1(u_0)) \ .$$

Define $F_0 : \text{ObMF}(R, W) \rightarrow \text{ObMF}(R, W)$ through:

$$F_0(R^{\rho|_{\rho}}, D) := a_0 \stackrel{\text{def.}}{=} (R^{\rho|_{\rho}}, D_0) \ . \quad (3.2)$$

Notice that a_0 coincides with the following direct sum of elementary matrix factorizations:

$$a_0 = e_{d_1(v)} \oplus \dots \oplus e_{d_\rho(v)} \ .$$

Moreover, relation (3.1) implies $D_0 U_D = U_D D$, showing that U_D is an isomorphism from a to a_0 in $\text{zmf}(R, W)$:

$$U_D : a \xrightarrow{\sim} a_0 \ . \quad (3.3)$$

For any morphism $f : a \rightarrow a'$ in $\text{zmf}(R, W)$ with $a = (R^{\rho|_{\rho}}, D)$, $a' = (R^{\rho'|_{\rho'}}, D')$ and $D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$, $D' = \begin{bmatrix} 0 & v' \\ u' & 0 \end{bmatrix}$, define a morphism $F_1(f) : a_0 \rightarrow a'_0$ in $\text{zmf}(R, W)$ as follows. Since f is a morphism in $\text{zmf}(R, W)$ it satisfies the condition $D' f = f D$. Define:

$$F_1(f) := f_0 \stackrel{\text{def.}}{=} U_{D'} f U_D^{-1} \ . \quad (3.4)$$

where $U_D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $U_{D'} \stackrel{\text{def.}}{=} U_{v'} = \begin{bmatrix} A' & 0 \\ 0 & B' \end{bmatrix}$, with $A = A_v$, $B = B_v$, $A' = A_{v'}$ and $B' = B_{v'}$. Since $D_0 = U_D D U_D^{-1}$ and $D'_0 = U_{D'} D U_{D'}^{-1}$, the relation $D' f = f D$ implies $D'_0 f_0 = f_0 D_0$, showing that f_0 is a morphism from a_0 to a'_0 in $\text{zmf}(R, W)$. If f is the identity endomorphism, then f_0 is the identity endomorphism. If $g : a' \rightarrow a''$ is another morphism in $\text{zmf}(R, W)$, then we have $(gf)_0 = U_{D''} g f U_D^{-1} = U_{D''} g U_{D'}^{-1} U_{D'} f U_D^{-1} = g_0 f_0$. This shows that $F = (F_0, F_1)$ is an endofunctor of $\text{zmf}(R, W)$. Relation (3.4) shows that the isomorphisms (3.3) satisfy $U_{D'} f = F_1(f) U_D$ and hence give an isomorphism of functors:

$$U : \text{id}_{\text{zmf}(R, W)} \xrightarrow{\sim} F \ .$$

In particular, F is an autoequivalence of $\text{zmf}(R, W)$. \square

The decomposition of a matrix factorization into elementary factorizations is generally non-unique. The ambiguity in this decomposition can be characterized as follows.

Corollary 3.3 *The following statements hold:*

1. *If e_{v_1}, \dots, e_{v_n} are elementary factorizations of W , then we have:*

$$e_{v_1} \oplus \dots \oplus e_{v_n} \simeq_{\text{zmf}(R,W)} e_{d_1} \oplus \dots \oplus e_{d_n} \quad , \quad (3.5)$$

where:

$$d_k \in \frac{\delta_k}{\delta_{k-1}} \quad \forall k \in \{1, \dots, n\} \quad ,$$

with:

$$\delta_k \stackrel{\text{def.}}{=} (\{v_{i_1} \dots v_{i_k} \mid 1 \leq i_1 < \dots < i_k \leq n\}) \quad . \quad (3.6)$$

Moreover, if $v_1 \mid \dots \mid v_n$ then we can take $d_k = v_k$ for all $k \in \{1, \dots, n\}$ while if v_1, \dots, v_n are mutually coprime then we have $e_{v_1} \oplus \dots \oplus e_{v_n} \simeq_{\text{zmf}(R,W)} e_1^{\oplus n-1} \oplus e_{v_1 \dots v_n}$.

2. *If a matrix factorization $a = (R^{\rho}, D)$ of W with $D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$ satisfies:*

$$a \simeq_{\text{zmf}(R,W)} e_{v_1} \oplus \dots \oplus e_{v_n}$$

for some elementary factorizations e_{v_i} such that $v_1 \mid \dots \mid v_n$, then we have $n = \rho$ and $v_i \in \mathbf{d}_i(v)$ for all $i \in \{1, \dots, n\}$. In particular, the strong isomorphism classes of matrix factorizations of W are in bijection with finite ascending sequences of principal ideals $I_n \subset \dots \subset I_1$ such that $W \in I_n$.

Proof.

1. Let $a \stackrel{\text{def.}}{=} e_{v_1} \oplus \dots \oplus e_{v_n}$. Then $a = (R^{\rho}, D)$ with $D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$, where $v = \text{diag}(v_1, \dots, v_n)$.

Since all non-principal minors of a diagonal matrix vanish, the determinantal invariants of v coincide with δ_k , while the invariant factors coincide with \mathbf{d}_k . The first statement now follows from Proposition B.5. If $v_1 \mid \dots \mid v_n$, then we have $\delta_n = (v_1 \dots v_n)$ and $\mathbf{d}_n = (v_n)$. If v_1, \dots, v_n are coprime then we have $\delta_1 = \dots = \delta_{n-1} = (1)$ and $\delta_n = (v_1 \dots v_n)$, thus $\mathbf{d}_1 = \dots = \mathbf{d}_{n-1} = (1)$ and $\mathbf{d}_n = (v_1 \dots v_n)$.

2. Follows immediately from Theorem 3.2 and point 1. above. \square

Remark 3.1. The critical ideal \mathfrak{J}_W defined in (1.3) annihilates the module $\text{Hom}_{\text{hmf}(R,W)}(e_1, e_2)$ for any two elementary matrix factorizations e_1 and e_2 of W (see [8, Remark 2.2.]). Using this fact, Corollary 3.3 implies $\mathfrak{J}_W \text{Hom}_{\text{hmf}(R,W)}(a, b) = 0$ for any two finite rank matrix factorizations a, b of W (notice that an isomorphism in $\text{zmf}(R, W)$ induces an isomorphism in $\text{hmf}(R, W)$). In particular, $\text{hmf}(R, W)$ can be viewed as an R/\mathfrak{J}_W -linear category. Since $W \in \mathfrak{J}_W$, we have a natural epimorphism $R/\langle W \rangle \rightarrow R/\mathfrak{J}_W$. Thus $\text{hmf}(R, W)$ is in particular an $R/\langle W \rangle$ -linear category.

Remark 3.2. Let v_1 and v_2 be two divisors of W . Then $\delta_1(v) = (v_1, v_2)$ and $\delta_2(v) = (v_1 v_2)$ and the quantities (3.6) are $\mathbf{d}_1 = (v_1, v_2)$ and $\mathbf{d}_2 = [v_1, v_2]$. Thus (3.5) takes the form:

$$e_{v_1} \oplus e_{v_2} \simeq_{\text{zmf}(R,W)} e_{d_1} \oplus e_{d_2} \quad (3.7)$$

with $d_1 \in (v_1, v_2)$ and $d_2 \in [v_1, v_2]$. If $v_1 \mid v_2$ and $u_i = W/v_i$, then we have $u_2 \mid u_1$ and:

$$\Sigma(e_{v_1} \oplus e_{v_2}) \simeq_{\text{zmf}(R,W)} e_{u_2} \oplus e_{u_1} \quad ,$$

since $\Sigma e_{v_i} = e_{-u_i} \simeq_{\text{zmf}(R,W)} e_{u_i}$ (see [8, Section 1.7]). Corollary 3.3 shows that the subcategory $\text{zef}(R, W)$ generates $\text{zmf}(R, W)$ under direct sums with the relations (3.7). At the level of isomorphism classes, these relations correspond to the operation $(I_1, I_2) \rightarrow (I_1 \cap I_2, I_1 + I_2)$ on principal ideals I_1, I_2 which contain W , where the RHS is a chain $I_1 \cap I_2 \subset I_1 + I_2$ of principal ideals containing W .

3.2. Direct sum decompositions in $\text{hmf}(R, W)$. The results of the previous subsection imply that elementary matrix factorizations generate the category $\text{hmf}(R, W)$ under direct sums.

Proposition 3.4 *There exists an autoequivalence Ψ of $\text{hmf}(R, W)$ such that:*

1. Ψ is isomorphic with the identity functor $\text{id}_{\text{hmf}(R,W)}$.
2. For any matrix factorization $a = (R^{\rho|}, D)$ of W with $D = \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}$, we have:

$$\Psi(a) = e_{d_1(v)} \oplus \dots \oplus e_{d_\rho(v)} \quad ,$$

where $d_1(v), \dots, d_\rho(v)$ are representatives for the invariant factors of v .

In particular, the subcategory $\text{hef}(R, W)$ generates $\text{hmf}(R, W)$ under direct sum. Thus any matrix factorization $a \in \text{Ob}(\text{MF}(R, W))$ is isomorphic in $\text{hmf}(R, W)$ with a direct sum of a finite collection of elementary factorizations.

Proof. Follows immediately from Theorem 3.2 upon taking Ψ to be the autoequivalence of $\text{hmf}(R, W)$ induced by the autoequivalence F of $\text{zmf}(R, W)$. \square

Notice that the decomposition of an object of $\text{hmf}(R, W)$ as a finite direct sum of elementary factorizations need not be unique up to permutation and isomorphisms in $\text{hmf}(R, W)$. Moreover, an elementary factorization need not be an indecomposable object of $\text{hmf}(R, W)$.

Remark 3.3. For any Bézout domain R , let $\mathbf{hef}(R, W)$ be the subcategory of $\text{hmf}(R, W)$ which is additively generated by elementary factorizations. In [8, Conjecture 3.4] it was conjectured that the inclusion functor:

$$\iota : \mathbf{hef}(R, W) \rightarrow \text{hmf}(R, W)$$

is an equivalence of R -linear categories when W is a critically-finite element. Proposition 3.4 proves this conjecture when R is an elementary divisor domain, under the weaker hypothesis that W is any non-zero element of R . It is an open question whether all Bézout domains are elementary divisor domains.

3.3. Cones over morphisms between elementary factorizations. Let e_{v_1} and e_{v_2} be elementary matrix factorizations of W and set $u_i \stackrel{\text{def.}}{=} W/v_i$. By [8, Proposition 2.2], morphisms $f : e_{v_1} \rightarrow e_{v_2}$ in $\text{hmf}(R, W)$ have the form $f = r \cdot \begin{bmatrix} \frac{v_2}{d} & 0 \\ 0 & \frac{v_1}{d} \end{bmatrix}$, where r is an arbitrary element of R and $d \in (v_1, v_2)$ is a gcd of v_1 and v_2 .

Proposition 3.5 *Let $f : e_{v_1} \rightarrow e_{v_2}$ be a morphism in $\text{hmf}(R, W)$ corresponding to the element $r \in R$. Let:*

$$\xi \in \frac{(v_1, v_2, u_1, u_2, r)(v_1)}{(v_1, v_2)} \quad \text{and} \quad \zeta \stackrel{\text{def.}}{=} -\frac{v_1 u_2}{\xi} . \quad (3.8)$$

Then there exists an isomorphism in $\text{zmf}(R, W)$:

$$C(f) \simeq_{\text{zmf}(R, W)} e_\xi \oplus e_\zeta . \quad (3.9)$$

Proof. Let $d \in (v_1, v_2)$ be a gcd of v_1 and v_2 . Using Definition 1.2, we find that the mapping cone of f is given by:

$$C(f) = \begin{bmatrix} 0 & 0 & -u_1 & 0 \\ 0 & 0 & r \cdot \frac{v_2}{d} & v_2 \\ -v_1 & 0 & 0 & 0 \\ r \cdot \frac{v_1}{d} & u_2 & 0 & 0 \end{bmatrix} .$$

Since R is an elementary divisor domain, the matrices $A \stackrel{\text{def.}}{=} \begin{bmatrix} -v_1 & 0 \\ r \cdot \frac{v_1}{d} & u_2 \end{bmatrix}$ and $B \stackrel{\text{def.}}{=} \begin{bmatrix} -u_1 & 0 \\ r \cdot \frac{v_2}{d} & v_2 \end{bmatrix}$ can be reduced to Smith normal form (see Appendix B). Furthermore, since $AB = W$ we can find invertible matrices P and Q such that PAQ and QBP have normal forms. Let $\xi \in (v_1, u_2, r \cdot \frac{v_1}{d})$. Then $PAQ = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$, where α_i are invariant factors of A . By definition, α_1 is a greatest common divisor of all entries of A , which we can take to equal ξ . On the other hand, we have $\alpha_2 = \frac{\det A}{\xi} = -\frac{v_1 u_2}{\xi} = \zeta$. Hence the Smith normal form of B equals $\begin{bmatrix} \frac{W}{\xi} & 0 \\ 0 & \frac{W}{\zeta} \end{bmatrix}$. We conclude that $C(f)$ is isomorphic in $\text{zmf}(R, W)$ with the matrix:

$$C_0(f) = \begin{bmatrix} 0 & 0 & \frac{W}{\xi} & 0 \\ 0 & 0 & 0 & \frac{W}{\zeta} \\ \xi & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \end{bmatrix} = e_\xi \oplus e_\zeta .$$

Let $s \stackrel{\text{def.}}{=} (v_1, v_2, u_1, u_2) \in R/U(R)$ and $b \stackrel{\text{def.}}{=} \frac{(v_1)}{(v_1, v_2)} \in R/U(R)$. By [8, eqs. (2.4)], we have $(v_1, u_2) = (s)(b)$. Thus:

$$\left(v_1, u_2, r \cdot \frac{v_1}{(v_1, v_2)} \right) = (sb, rb) = (s, r)b , \quad (3.10)$$

which shows that (3.8) holds. \square

Corollary 3.6 *Let $f : e_{v_1} \rightarrow e_{v_2}$ be a morphism in $\text{hmf}(R, W)$ which corresponds to an element $r \in R$ and let ξ and ζ be as in Proposition 3.5. Then f is an isomorphism in $\text{hmf}(R, W)$ if and only if the following relations hold in $R/U(R)$:*

$$(\xi, W/\xi) = (\zeta, W/\zeta) = (1).$$

Proof. The morphism f is an isomorphism in the additive triangulated category $\text{hmf}(R, W)$ iff $C(f)$ is a zero object. By Proposition 3.5, this happens iff both e_ξ and e_ζ are zero objects. By [8, Corollary 2.11], this is the case iff $(\xi, W/\xi) = (\zeta, W/\zeta) = (1)$. \square

3.4. *Primary matrix factorizations.* Recall that an element of R is called primary if it is a power of a prime element.

Definition 3.7 *An elementary factorization e_v of W is called primary if v is a primary divisor of W .*

Let e_v be a primary matrix factorization of W . Then $v = p^i$ for some prime divisor p of W and some integer $i \in \{0, \dots, n\}$, where n is the order of p as a divisor of W . We have $W = p^n W_1$ for some element $W_1 \in R$ such that p does not divide W_1 and $u = p^{n-i} W_1$. Thus $(u, v) = (p^{\min(i, n-i)})$.

Definition 3.8 *The prime divisor p of W is called the prime support of e_v . The order n of p is called the order of e_v while the integer $i \in \{0, \dots, n\}$ is called the size of e_v .*

3.5. *A Krull-Schmidt theorem for $\text{hmf}(R, W)$ when W is critically-finite.* Recall that an object of an additive category is called *indecomposable* if it is not isomorphic with a direct sum of two non-zero objects. A *Krull-Schmidt category* is an additive category for which every object decomposes into a finite direct sum of objects having quasi-local endomorphism rings.

Theorem 3.9 *Let W be a critically-finite element of R . Then $\text{hmf}(R, W)$ is a Krull-Schmidt category whose non-zero indecomposables are the nontrivial primary matrix factorizations of W . In particular, $\text{hmf}(R, W)$ is additively generated by $\text{hef}_0(R, W)$.*

Proof. By [8, Proposition 3.1] and [8, Theorem 3.2], any elementary matrix factorization decomposes into a finite direct sum of primary matrix factorizations. On the other hand, any matrix factorization of W decomposes as a finite direct sum of elementary factorizations and hence also as a finite direct sum of primary factorizations whose prime supports are the prime divisors of W . By [8, Proposition 2.24], every primary matrix factorization has a quasi-local endomorphism ring. \square

Corollary 3.10 *Let $W \in R$ be an element of R which has a finite prime decomposition. Then $\text{hmf}(R, W)$ is a Krull-Schmidt category whose indecomposables are the nontrivial primary matrix factorizations of W .*

Proof. Write $W = W_0 p_1^{n_1} \dots p_N^{n_N}$, where p_j are the critical prime divisors of W , $n_j \geq 2$ and W_0 is the product of the non-critical prime divisors of W . Then W_0 is non-critical and we can apply Theorem 3.9. \square

Remark 3.4. Theorem 3.9 proves [8, Conjecture 3.5] when R is an elementary divisor domain.

3.6. *The category $\text{hmf}_p(R, W)$ and its equivalent descriptions.* Let p be a prime divisor of W of order n . Let $\text{hmf}_p(R, W)$ denote the smallest strictly full⁴ subcategory of $\text{hmf}(R, W)$ which is closed under direct sums and contains all those primary factorizations of W which have prime support p . Propositions 3.4 and [8, Proposition 3.1] imply that $\text{hef}(R, W)$ is additively generated by its strictly full subcategory $\text{hef}_0(R, W)$ whose objects are the primary factorizations of W .

⁴ I.e., full and closed under isomorphisms.

Lemma 3.11 *A matrix factorization a of W is an object of $\mathrm{hmf}_p(R, W)$ iff $\mathrm{Hom}(e_q, a) = 0$ for any prime divisor q of W such that $(q) \neq (p)$.*

Proof. Since $\mathrm{hmf}(R, W)$ is additively generated by $\mathrm{hmf}_0(R, W)$, it suffices to prove the statement when $a = e_v$ is a primary matrix factorization. In this case, we have $v = s^k$ for some prime divisor s of W and:

$$\mathrm{Hom}_{\mathrm{hmf}(R, W)}(e_q, a) = \mathrm{Hom}_{\mathrm{hmf}(R, W)}(e_q, e_{s^k}) \simeq R/\langle q, s^k \rangle \simeq_R \begin{cases} R/(s) & \text{if } (q) = (s) \\ 0 & \text{if } (q) \neq (s) \end{cases}.$$

Hence $\mathrm{Hom}(e_q, a)$ vanishes for any prime divisor q of W such that $(q) \neq (p)$ iff $(s) = (p)$, which is equivalent with the condition that e_v is an object of $\mathrm{hmf}_p(R, W)$. \square

Proposition 3.12 *$\mathrm{hmf}_p(R, W)$ is a triangulated subcategory of $\mathrm{hmf}(R, W)$.*

Proof. The subcategory $\mathrm{hmf}_p(R, W)$ of $\mathrm{hmf}(R, W)$ is strictly full by definition. Since $\mathrm{hmf}_p(R, W)$ is additively generated by primary factorizations of prime support p , [8, Proposition 2.26] implies that $\mathrm{hmf}_p(R, W)$ is closed under suspension. Let $a \rightarrow b \rightarrow c \rightarrow \Sigma a$ be a distinguished triangle of $\mathrm{hmf}(R, W)$ such that a and b are objects of $\mathrm{hmf}_p(R, W)$. For any prime divisor q of W such that $q \neq p$, the homological functor $\mathrm{Hom}_{\mathrm{hmf}(R, W)}(e_q, -)$ takes this triangle into a long exact sequence:

$$\dots \longrightarrow \mathrm{Hom}_{\mathrm{hmf}(R, W)}(e_q, b) \longrightarrow \mathrm{Hom}_{\mathrm{hmf}(R, W)}(e_q, c) \longrightarrow \mathrm{Hom}_{\mathrm{hmf}(R, W)}(e_q, \Sigma a) \longrightarrow \dots \quad (3.11)$$

Since b and Σa are objects of $\mathrm{hmf}_p(R, W)$, we have $\mathrm{Hom}_{\mathrm{hmf}(R, W)}(e_q, b) = \mathrm{Hom}_{\mathrm{hmf}(R, W)}(e_q, \Sigma a) = 0$ by Lemma 3.11 and the sequence (3.11) implies $\mathrm{Hom}_{\mathrm{hmf}(R, W)}(e_q, c) = 0$. Applying Lemma 3.11 once again, we conclude that c is an object of $\mathrm{hmf}_p(R, W)$. Since triangles can be rotated, it follows that any triangle in $\mathrm{hmf}(R, W)$ for which two objects are in $\mathrm{hmf}_p(R, W)$ has all its objects in $\mathrm{hmf}_p(R, W)$. \square

Proposition 3.13 *For any prime element $p \in R$, the ring $R_{(p)}$ is discrete valuation ring. In particular, we have $\mathrm{kdim}R_{(p)} = 1$.*

Proof. The maximal ideal of $R_{(p)}$ is the principal ideal (p) . The powers of this ideal form the strictly descending sequence:

$$R_{(p)} \supseteq \langle p \rangle \supseteq \langle p^2 \rangle \supseteq \dots$$

The same argument as in the proof of Proposition 2.2 (but with R replaced by $R_{(p)}$) shows that these and the zero ideal are all the ideals of $R_{(p)}$. In particular, any strictly ascending sequence of ideals terminates and hence $R_{(p)}$ is Noetherian and thus a PID. Moreover, we have⁵ $\bigcap_{i=1}^{\infty} \langle p^i \rangle = 0$. The zero ideal is prime since $R_{(p)}$ is an integral domain and we have $\langle 0 \rangle \neq \langle p \rangle$. Hence $\mathrm{kdim}R_{(p)} = 1$, which implies that $R_{(p)}$ is not a field. \square

Remark 3.5. Since any discrete valuation ring is a regular local ring, it follows that $R_{(p)}$ is a regular local ring.

Proposition 3.14 *Let p be a prime element of R and $n > 0$ be a positive integer. Then the localization functor $\mathrm{loc}_p : \mathrm{hmf}(R, p^n) \rightarrow \mathrm{hmf}(R_{(p)}, p^n)$ at the multiplicative set $S_p \stackrel{\mathrm{def.}}{=} R \setminus \langle p \rangle$ is a triangulated equivalence.*

⁵ If $x \in \bigcap_{i=1}^{\infty} \langle p^i \rangle$, then $(x) \subset \langle p^i \rangle$ for all i , which requires $x = 0$ since otherwise (x) would equal some $\langle p^j \rangle$.

Proof. Let $W = p^n$. We have:

$$S_{p^n} \stackrel{\text{def.}}{=} \{r \in R \mid (r, p^n) = (1)\} = \{r \in R \mid (r, p) = (1)\} = \{r \in R \mid p \nmid r\} = R \setminus \langle p \rangle = S_p .$$

Hence [8, Proposition 2.15] implies that loc_p is an R -linear equivalence between $\text{hef}(R, p^n)$ and $\text{hef}(R_{(p)}, p^n)$. Since R is Bézout (and hence Prüfer), the localization $R_{(p)}$ is a (possibly non-Noetherian) valuation domain and hence a Bézout domain. Since any local Bézout domain is an EDD [18, Corollary 2.3], it follows that $R_{(p)}$ is an EDD. Since both R and $R_{(p)}$ are EDDs, the categories $\text{hmf}(R, p^n)$ and $\text{hmf}(R_{(p)}, p^n)$ are additively generated by $\text{hef}(R, p^n)$ and $\text{hef}(R_{(p)}, p^n)$. Thus loc_p is an R -linear equivalence between $\text{hmf}(R, p^n)$ and $\text{hmf}(R_{(p)}, p^n)$. This implies the conclusion since loc_p is a triangulated functor by [8, Proposition 2.12]. \square

Proposition 3.15 *Let p be a prime divisor of W of order n . Then the categories $\text{hmf}_p(R, W)$ and $\text{hmf}_p(R_{(p)}, p^n)$ are triangle-equivalent.*

Proof. By [8, Proposition 2.12], localization at the multiplicative set $S_p = R \setminus \langle p \rangle$ gives a triangulated functor $\text{loc}_p : \text{hmf}(R, W) \rightarrow \text{hmf}(R_{(p)}, W_p)$, which restricts to a triangulated functor:

$$\text{loc}_p : \text{hmf}_p(R, W) \rightarrow \text{hmf}(R_{(p)}, W_p) .$$

This restricts to a functor $\Phi : \text{hef}_p(R, W) \rightarrow \text{hef}(R_{(p)}, W_p)$ which maps the elementary factorization e_{p^i} of W to the elementary factorization e'_{p^i} of W_p . It is clear that the functor Φ is essentially surjective. It is also fully faithful, since any element $s \in S_p = R \setminus \langle p \rangle$ acts as an automorphism of each module $\text{Hom}_{\text{hmf}(R, W)}(e_{p^i}, e_{p^j}) \simeq R/\langle p^{\min(i, j)} \rangle$ by [8, Lemma 2.14]. Since $\text{hef}(R, W)$ and $\text{hef}(R_{(p)}, W_p)$ additively generate $\text{hmf}(R, W)$ and $\text{hmf}(R_{(p)}, W_p)$, we conclude that (3.6) is a triangulated equivalence. On the other hand, the localization W_p of W at p is associated in the ring $R_{(p)}$ with the element $p^n \in R_{(p)}$. This gives a triangulated equivalence $\text{hmf}(R_{(p)}, W_p) \simeq \text{hmf}(R_{(p)}, p^n)$ by Proposition 1.4. Composing this with (3.6) gives the conclusion. \square

Composing the triangulated equivalences of Propositions 3.14 and 3.15 gives a triangulated equivalence $\text{hmf}_p(R, W) \simeq \text{hmf}(R, p^n)$. We have a commutative diagram of triangulated categories and triangulated equivalences:

$$\begin{array}{ccc} \text{hmf}_p(R, W) & \longrightarrow & \text{hmf}(R, p^n) \\ \text{loc}_p \downarrow & & \downarrow \text{loc}_p \\ \text{hmf}(R_{(p)}, W_p) & \longrightarrow & \text{hmf}(R_{(p)}, p^n) \end{array}$$

Proposition 3.16 *The restriction to $\text{hmf}_p(R, p^n)$ of the cokernel functor of $\text{hmf}(R, p^n)$:*

$$\text{Cok} : \text{hmf}_p(R, p^n) \rightarrow \underline{\text{mod}}(R/\langle p^n \rangle) = \underline{\text{mod}}_{A_n(p)} \quad (3.12)$$

is a triangulated equivalence.

Proof. Since $R_{(p)}$ is a local ring, the Eisenbud correspondence [19] gives a triangulated equivalence:

$$\text{cok} : \text{hmf}(R_{(p)}, p^n) \xrightarrow{\sim} \underline{\text{mod}}_{R_{(p)}/\langle p^n \rangle} ,$$

where cok is the cokernel functor of $\text{hmf}(R_{(p)}, p^n)$. By Proposition 3.14, localization at the multiplicative set $R \setminus \langle p \rangle$ gives a triangulated equivalence:

$$\text{loc}_p : \text{hmf}(R, p^n) \xrightarrow{\sim} \text{hmf}(R_{(p)}, p^n) \quad .$$

By Proposition 2.17, localization at the multiplicative set $U(R/\langle p^n \rangle)$ gives a triangulated equivalence:

$$\underline{\text{loc}}_p : \underline{\text{mod}}_{R_{(p)}/\langle p^n \rangle} \xrightarrow{\sim} \underline{\text{mod}}_{R/\langle p^n \rangle} \quad .$$

It is easy to see that we have the relation:

$$\underline{\text{loc}}_p \circ \text{Cok} = \text{cok} \circ \text{loc}_p \quad ,$$

which implies that $\text{Cok} = \underline{\text{loc}}_p^{-1} \circ \text{cok} \circ \text{loc}_p$ is a triangulated equivalence. \square

Explicit description of $\text{hmf}(R, p^n)$. Let $p \in R$ be a prime element and $n \geq 2$. By Theorem 3.9, the indecomposable objects of the Krull-Schmidt category $\text{hmf}(R, p^n)$ are the non-zero primary factorizations of the critically-finite element $W = p^n$. For any $1 \leq i \leq n-1$, let:

$$e_i := e_{v_i} = \begin{bmatrix} 0 & p^i \\ p^{n-i} & 0 \end{bmatrix} \quad (3.13)$$

be the non-zero primary matrix factorization of $W = p^n$ corresponding to the primary divisor $v_i = p^i$. For this factorization, we have $u_i = p^{n-i}$ and $(u_i, v_i) = (p^{\min(i, n-i)}) \neq (1)$. Notice that e_i has order $\delta_n(i)$, where $\delta_n(i)$ was defined in (2.3). For any $i, j \in \{1, \dots, n-1\}$, we have $(v_1, v_2, u_1, u_2) = (p^{\mu(i, j)})$, where $\mu(i, j)$ was defined in (2.3). Thus [8, Proposition 2.2] shows that:

$$\text{Hom}_{\text{hmf}(R, p^n)}(e_i, e_j) \simeq_R R/\langle p^{\mu(i, j)} \rangle$$

is a cyclically-presented cyclic module generated by the morphism:

$$\epsilon_{\hat{0}}(v_i, v_j) \stackrel{\text{def.}}{=} \begin{bmatrix} p^{j-\min(i, j)} & 0 \\ 0 & p^{i-\min(i, j)} \end{bmatrix} \quad .$$

On the other hand, [8, Proposition 2.8] shows that the composition of morphisms is given by:

$$f \circ g = p^{\rho(i, j, k)} r s \epsilon_{\hat{0}}(v_i, v_k) \quad ,$$

for all $f = r \epsilon_{\hat{0}}(v_j, v_k) \in \text{Hom}_{\text{hmf}(R, p^n)}(e_j, e_k)$ and $g = s \epsilon_{\hat{0}}(v_i, v_j) \in \text{Hom}_{\text{hmf}(R, p^n)}(e_i, e_j)$, where $r, s \in R$ and:

$$\rho(i, j, n) = \max(i, j, n) - \min(i, j, n) + \min(i, n) - \max(i, n) = \max(i, j, n) - \min(i, j, n) - |i - n| \quad .$$

Since $p^n \in \text{Ann}(\text{Hom}_{\text{hmf}(R, p^n)}(e_i, e_j))$, we can view $\text{hmf}(R, p^n)$ as an $A_n(p)$ -linear category. The triangulated equivalence (3.12) sends the primary matrix factorization e_{v_i} to the cyclic $A_n(p)$ -module $\text{Cok}(v_i) = V_i$. For any $i, j \in \{1, \dots, n-1\}$, we have:

$$\text{Hom}_{\text{hmf}(R, p^n)}(e_i, e_j) \simeq R/\langle p^{\mu(i, j)} \rangle \simeq \underline{\text{Hom}}_{A_n(p)}(V_i, V_j) \quad ,$$

where the last isomorphism follows from Proposition 2.11.

3.7. Proof of the main theorem.

Proposition 3.17 *Let W be a critically-finite element of R with decomposition (1.4). Then we have an orthogonal decomposition:*

$$\mathrm{hmf}(R, W) = \vee_{i=1}^N \mathrm{hmf}_{p_i}(R, W) \quad ,$$

where \vee denotes the orthogonal sum of triangulated categories.

Proof. Theorem 3.9 and [8, Proposition 3.1] imply that $\mathrm{hmf}(R, W)$ is additively generated (and hence also triangle-generated) by the triangulated subcategories $\mathrm{hmf}_{p_i}(R, W)$. These categories are mutually orthogonal by [8, Lemma 2.25]. \square

We are now ready to prove Theorem 0.1.

Proof (of Theorem 0.1). The first equivalence in (0.1) follows from Propositions 3.17 and 3.16. The second equivalence follows from Proposition 2.16. The fact that $A_n(p)$ is Artinian follows from Proposition 2.2. \square

4. Some examples

In this section, we discuss a few classes of examples to which the results of the previous sections apply.

4.1. Holomorphic matrix factorizations over a non-compact Riemann surface. Let Σ be any connected, smooth and borderless non-compact Riemann surface⁶. Then Σ is Stein by a result of [20]. Moreover, any holomorphic vector bundle defined on Σ is holomorphically trivial (see [21, Theorem 30.3]), so in particular Σ has trivial canonical line bundle. The critical set Z_W of any non-constant holomorphic function $W : \Sigma \rightarrow \mathbb{C}$ consists of isolated points, so the total cohomology category $\mathrm{HF}(\Sigma, W)$ of holomorphic factorizations of W defined in [5] can be identified with the total cohomology category $\mathrm{HMF}(\mathcal{O}(\Sigma), W)$ of finite rank matrix factorizations of W over the ring $\mathcal{O}(\Sigma)$ of holomorphic complex-valued functions defined on Σ (see [5, Proposition 7.1]). In particular, the even subcategory $\mathrm{HF}^{\hat{0}}(\Sigma, W)$ can be identified with the homotopy category of matrix factorizations $\mathrm{hmf}(\mathcal{O}(\Sigma), W)$. When the set Z_W is finite, the category $\mathrm{HF}(\Sigma, W) \simeq \mathrm{HMF}(\mathcal{O}(\Sigma), W)$ coincides with the category of D-branes of a B-type open-closed topological Landau-Ginzburg model with finite-dimensional on-shell state spaces (see [2, 3, 4]).

The non-Noetherian ring $\mathcal{O}(\Sigma)$ is an elementary divisor domain [22, 23, 24, 25] whose prime elements are those holomorphic functions having a single simple zero and no other zeros. For each point $z \in \Sigma$, we thus have a prime element $p_z \in \mathcal{O}(\Sigma)$ (a holomorphic function which has a simple zero at z and no other zeroes) which is determined by z up to multiplication with a non-zero complex constant. A critically-finite superpotential is a holomorphic function $W \in \mathcal{O}(\Sigma)$ of the form $W = W_0 W_c$, where $W_0 \in \mathcal{O}(\Sigma)$ has only simple zeros (the number of which may be countably infinite) while $W_c \in \mathcal{O}(\Sigma)$ has a finite number of zeros $z_1, \dots, z_N \in \Sigma$, each of which has multiplicity $n_i \geq 2$ and differs from all zeros of W_0 . The critical set Z_W of such a holomorphic function contains the set $\{z_1, \dots, z_N\}$. In this case, Theorem 0.1 shows that the triangulated category $\mathrm{HF}^{\hat{0}}(\Sigma, W) \simeq \mathrm{hmf}(\mathcal{O}(\Sigma), W)$ is the orthogonal direct sum of

⁶ Notice that such a Riemann surface Σ need not be algebraic. In particular, Σ may have infinite genus as well as an infinite number of Freudenthal ends.

the Krull-Schmidt triangulated categories $\underline{\text{mod}}_{\mathcal{O}(\Sigma)/(p_{z_i}^{n_i})}$ associated with the points z_i , whose Auslander-Reiten quivers are entirely determined by the multiplicities n_i . The Auslander-Reiten quiver of $\underline{\text{mod}}_{\mathcal{O}(\Sigma)/(p_{z_i}^{n_i})}$ has $n_i - 1$ nodes and is of the type shown in Figure 2.2. Notice that only the critical points z_1, \dots, z_N “contribute” to the orthogonal decomposition of the category $\text{hmf}(\mathcal{O}(\Sigma), W)$.

4.2. Valuation domains. Recall that a unital commutative ring is called a *generalized valuation ring* [26] if its elements are linearly preordered by divisibility, i.e. if any two elements $x, y \in R$ satisfy one of the conditions $x|y$ or $y|x$. The following characterizations are well-known [11, 26]:

Proposition 4.1 *Let R be a unital commutative ring. Then the following statements are equivalent:*

- (a) *R is a generalized valuation ring.*
- (b) *The principal ideals of R are linearly ordered by inclusion.*
- (c) *The ideals of R are linearly ordered by inclusion.*
- (d) *R is quasilocal and any finitely-generated ideal of R is principal.*
- (e) *If x_1, \dots, x_n are elements of R , then there exists $j \in \{1, \dots, n\}$ such that $\langle x_1, \dots, x_n \rangle = \langle x_j \rangle$.*

In particular R , is a generalized valuation ring iff R is a quasilocal Bézout ring.

A *valuation domain*⁷ is a generalized valuation ring which is an integral domain. Denote by K the field of fractions of an integral domain R . Then R is a valuation domain iff any $x \in K^\times$ satisfies $x \in R$ or $1/x \in R$. An integral domain R is a valuation domain iff there exists a totally-ordered Abelian group $(G, +, \leq)$ (called the *value group* of R) and a surjective valuation $v : K^\times \rightarrow G$ such that $R = \{x \in K^\times | v(x) \geq 0\} \cup \{0\}$. In this case, $(G, +, \leq)$ is torsion-free [27] and order-isomorphic with the group of divisibility of R (see Subsection 4.4). In fact, a classical result of Krull [28] states that any totally-ordered Abelian group arises as the value group of a valuation domain. By Proposition 4.1, a valuation domain is the same as a quasilocal Bézout domain. Moreover, [18, Corollary 2.3] shows that a valuation domain is an elementary divisor domain and that any finitely-presented module over a valuation domain is a direct sum of cyclic modules.

Proposition 4.2 *Let R be a valuation domain. Then R has prime elements iff the (unique) maximal ideal of R is principal and different from zero. In this case, any two prime elements of R are associated in divisibility.*

Proof. By Proposition 4.1, R is a quasilocal Bézout domain. Thus Lemma 2.1 applies, showing that any prime element $p \in R$ generates a maximal ideal. Since R is quasilocal, this ideal must coincide with the unique maximal ideal of R , which therefore must be principal and different from zero. By the same token, any two prime elements of R must generate the same ideal (namely the maximal ideal of R) and hence they must be associated in divisibility. Conversely, if the maximal ideal of R is principal and different from zero, then any generator of this ideal is a prime element of R since maximal ideals are prime ideals. \square

⁷ In some references, generalized valuation domains are called “valuation rings”, while discrete valuation domains are called “discrete valuation rings”.

Proposition 4.3 *Let R be a valuation domain with a prime element p and $W \in R$ be a non-zero non-unit of R . Then the following statements are equivalent:*

- (a) W is critically-finite.
- (b) We have $W = up^n$ for some $n \geq 2$ and some unit u of R .
- (c) We have $W \in \langle p^n \rangle \setminus \langle p^{n+1} \rangle$ for some $n \geq 2$.

In this case, the category $\text{hmf}(R, W)$ is triangle-equivalent to $\underline{\text{mod}}_{R/\langle p^n \rangle}$.

Proof. By Proposition 4.2, the ideal $m = \langle p \rangle$ coincides with the maximal ideal of R . Since R is quasi-local, we have $U(R) = R \setminus m$.

1. (a) \Rightarrow (b). If W is critically-finite, then $W = W_0W_c$ with $W_c = p^m$ for some $m \geq 2$ and some square-free element $W_0 \in R^\times$. If W_0 is a unit, then we can take $u = W_0$ and $n = m$. If W_0 is not a unit, then $W_0 \in R \setminus U(R) = m$ and hence p divides W_0 . Since W_0 is square-free, it follows that p does not divide $u \stackrel{\text{def.}}{=} W_0/p$, thus u belongs to the complement of m and hence is a unit. In this case, we have $W = up^{m+1}$ and we can take $n = m + 1$.
2. (b) \Rightarrow (c). If $W = up^n$ with $u \in U(R)$ and $n \geq 2$, then $W \in \langle p^n \rangle$. Since $U(R) = R \setminus m$, the prime p cannot divide u , hence $W \notin \langle p^{n+1} \rangle$. Thus $W \in \langle p^n \rangle \setminus \langle p^{n+1} \rangle$.
3. (c) \Rightarrow (a). Suppose that $W \in \langle p^n \rangle \setminus \langle p^{n+1} \rangle$ for some $n \geq 2$. Then $W = up^n$ for some $u \in R \setminus \{0\}$. Since $W \notin \langle p^{n+1} \rangle$, the prime p does not divide u and hence $u \in R \setminus m = U(R)$ is a unit. In particular, u is square-free and hence W is critically-finite.

The remaining statement follows immediately from Theorem 0.1. \square

Example 4.1. We give several examples of non-Noetherian valuation domains.

1. Let $G = \mathbb{Z}^n$ for some $n \geq 2$, totally ordered using the lexicographic order \leq_{lex} . Since G is not cyclic, it is not isomorphic to \mathbb{Z} . Hence the valuation domain associated to $(\mathbb{Z}^n, \leq_{\text{lex}})$ is not Noetherian (see Subsection 4.3). It has exactly one principal prime ideal which is also maximal. Let e_i for $1 \leq i \leq n$ be the canonical basis elements of the free \mathbb{Z} -module \mathbb{Z}^n . The inequality $e_i <_{\text{lex}} e_j$ for $i < j$ implies that the principal filter $\uparrow e_1$ is prime. However, the filters $\uparrow e_i$ for $i > 1$ are not prime. For details on prime filters see Subsection 4.4.
2. Let K be a field and x be an element which is transcendental over K . For any prime number p , consider the tower of integral domains:

$$K[x] \subset K[x^{1/p}] \subset \dots \subset K[x^{1/p^k}] \subset \dots$$

For any $k \geq 0$, let m_k be the maximal ideal of $K[x^{1/p^k}]$ which is generated by the element x^{1/p^k} . The localization $R_k = K[x^{1/p^k}]_{m_k}$ at the multiplicative system given by the complement of m_k is a Noetherian discrete valuation domain. The ring $R \stackrel{\text{def.}}{=} \bigcup_{k \geq 0} R_k$ is a non-Noetherian valuation domain of Krull dimension 1 whose value group is given by $G = \{\frac{m}{p^k} \mid m \in \mathbb{Z}, k \in \mathbb{N}\} \subset \mathbb{Q}$ (endowed with the order induced by the natural order of \mathbb{Q}). The maximal ideal of this valuation domain is the ideal generated by the elements x^{1/p^k} with $k \in \mathbb{N}^*$, which is not principal.

3. Another example of the same type can be obtained by considering the direct limit of all rings of the form $K[x^{1/n}]$ over all non-zero natural numbers $n \in \mathbb{N}^*$. The resulting valuation domain has value group \mathbb{Q} . Therefore, it is not Noetherian. This valuation domain has no non-zero prime element.

4.3. Discrete valuation domains. A *discrete valuation domain* is a Noetherian valuation domain which is not a field, i.e. whose maximal ideal is non-zero. By [6, Chap. II.1, Exercise 1.4], a valuation domain is Noetherian iff its unique maximal ideal m satisfies $\bigcap_{n \geq 1} m^n = 0$. Notice that a valuation domain with non-zero principal maximal ideal need not be a discrete valuation domain (see the Example 4.1). The following characterizations are well-known (see, for example, [29, Proposition 6.3.4]):

Proposition 4.4 *Let R be an integral domain which is not a field and let $K \neq R$ be its field of fractions. Then the following statements are equivalent:*

- (a) R is a discrete valuation domain.
- (b) R is a valuation domain with value group isomorphic to \mathbb{Z} with its natural order.
- (c) Every prime ideal of R is principal [6, Chap. II.1, Exercise 1.3].
- (d) R is a principal ideal domain which has a unique non-zero prime ideal.
- (e) R is a principal ideal domain which has a unique prime element p up to association in divisibility.
- (f) R is Noetherian and local and there is no ring S such that $R \subsetneq S \subsetneq K$.
- (g) R is Noetherian of Krull dimension one and its maximal ideal is principal.
- (h) R is Noetherian of Krull dimension one and integrally closed.
- (i) R is local with principal maximal ideal m and we have $\bigcap_{n \geq 1} m^n = 0$.

In this case, the unique prime ideal of R coincides with the unique maximal ideal m and we have $m = (p)$, where p is the essentially unique prime element (called *uniformizer*) of R . Moreover, the discrete valuation $v : R \rightarrow \mathbb{Z}$ satisfies $v(p) = 1$ and any non-zero ideal of R has the form (p^n) for some $n \geq 0$.

In particular, any valuation domain which is not a field and whose value group is not order-isomorphic to \mathbb{Z} is non-Noetherian. The following result (which follows immediately from Proposition 4.3) recovers a statement which, in this Noetherian situation, also follows from the Buchweitz correspondence [7]:

Proposition 4.5 *Let R be a discrete valuation domain. Fix a \mathbb{Z} -valuation $v : K \rightarrow \mathbb{Z}$ and a uniformizer p of R . Then any critically-finite element of R has the form $W = up^n$, where $n = v(W) \geq 2$ and u is a unit of R . Given such an element of R , the category $\text{hmf}(R, W)$ is triangle-equivalent to $\underline{\text{mod}}_{R/\langle p^n \rangle}$.*

4.4. Constructions through the group of divisibility. Recall that the *group of divisibility* $G(R)$ of an integral domain R is the quotient $K^\times/U(R)$, where K is the quotient field of R and $U(R)$ is the group of units. It is an ordered Abelian group when endowed with the order induced by the divisibility relation. The group of divisibility of a Bézout domain is lattice-ordered. In fact, any lattice-ordered Abelian group G is the group of divisibility of some Bézout domain R which can be obtained explicitly from G by a construction due to Jaffard and Ohm (see [30, 31]). There exists a dictionary between ideals of the Bézout domain R associated to G through the construction given in op. cit. and the set of *positive filters* of G . Given a lattice-ordered Abelian group (G, \leq) and an element $x \in G$, the up and down sets determined by x are defined

via $\uparrow x \stackrel{\text{def.}}{=} \{y \in G \mid x \leq y\}$ and $\downarrow x \stackrel{\text{def.}}{=} \{y \in G \mid y \leq x\}$. A positive filter of (G, \leq) is defined to be a proper subset $F \subset G_+$ such that:

1. F is upward-closed, i.e. $x \in F$ implies $\uparrow x \subset F$.
2. F is closed under finite meets, i.e. $x, y \in F$ implies $\inf(x, y) \in F$.

A positive filter F is called *prime* if $G_+ \setminus F$ is a semigroup; it is called *principal* if it has the form $\uparrow x$ for some $x \in F$. The natural projection $\pi : K^\times \rightarrow G$ induces a one to one correspondence between proper ideals of R and positive filters of (G, \leq) . Thus prime ideals correspond to prime positive filters and non-zero principal ideals correspond to principal positive filters. For more details and precise statements we refer the reader to [8, Section 5.2].

It is an open question whether every Bézout domain is an elementary divisor domain. Here we consider a class of lattice-ordered Abelian groups which correspond to adequate Bézout domains (see Definition B.8), which are special cases of elementary divisor domains (see [22, 23] and Appendix B).

Definition 4.6 *Let (G, \leq) be a lattice-ordered Abelian group and let $G^+ = \{x \in G \mid x \geq 0\}$ denote its positive cone. We say that (G, \leq) is adequate or projectable if for every $a, b \in G^+$ there exist $r, s \in G^+$ satisfying the following conditions:*

1. $a = r + s$.
2. $\inf(r, b) = 0$.
3. If $t \in G$ satisfies $0 < t \leq s$, then we have $\inf(t, b) \neq 0$.

There exists a simple criterion for detecting adequate groups. Let G be a lattice-ordered group. For any $b \in G^+$, define $G_b^+ = \{a \in G^+ \mid \inf(a, b) = 0\}$ and $G_b = \{a_1 - a_2 \mid a_1, a_2 \in G_b^+\}$. It is easy to see that G_b is a lattice subgroup of G . Then [18, Theorem 4.7] states that (G, \leq) is adequate iff G_b is a summand of G for every element $b \in G^+$.

Proposition 4.7 [18] *Let (G, \leq) be an adequate lattice-ordered Abelian group. Then:*

1. *The Bézout domain R associated to (G, \leq) by the Jaffard-Ohm construction is an adequate Bézout domain (and hence also an elementary divisor domain).*
2. *The prime elements of R correspond to the principal prime positive filters of (G, \leq) .*

Proof. The fact that G is adequate was shown in [18]. On the other hand, any adequate Bézout domain is an elementary divisor domain (see [22, 23]). The second statement follows immediately from the discussion above. \square

If R is a Bézout domain with prime elements which is constructed from an adequate lattice-ordered group as in Proposition 4.7 and $W \in R$ is a critically-finite element, then Theorem 0.1 applies to the homotopy category of finite rank matrix factorizations of W over R .

Example 4.2. Let I be a non-empty set and let G be either the direct sum or the direct product of a family of totally ordered groups $(G_i)_{i \in I}$ indexed by I . Then the Bézout domain R associated to G is adequate (see [18, Corollary 4.8]). The prime elements of the corresponding elementary divisor domain R were described in [8, Section 5.2]. Let W be a critically finite element of R . By Proposition 3.17 the category $\text{hmf}(R, W)$ has an orthogonal decomposition indexed by the critical prime divisors of W .

4.5. Constructions through spectral posets. The *spectral poset* of a unital commutative ring R is the prime spectrum $\text{Spec}(R)$ endowed with the order relation \leq given by inclusion. For two elements x, y of a poset (X, \leq) , we write $x \ll y$ if $x < y$ and x is an immediate neighbor of y . The spectral poset of any unital commutative ring satisfies *Kaplansky's conditions* if (see [32]):

- I. Every non-empty totally-ordered subset of $(\text{Spec}(R), \leq)$ has a supremum and an infimum (in particular, \leq is a lattice order).
- II. Given any elements $x, y \in \text{Spec}(R)$ such that $x < y$, there exist distinct elements x_1, y_1 of $\text{Spec}(R)$ such that $x \leq x_1 < y_1 \leq y$ and such that $x_1 \ll y_1$.

A poset (X, \leq) is called a *tree* if for every $x \in X$, the lower set $\downarrow x = \{y \in X \mid y \leq x\}$ is totally ordered. One has the following result due to Lewis:

Theorem 4.8 [33] *Let (X, \leq) be a partially-ordered set. Then the following statements are equivalent:*

- (a) (X, \leq) is a tree which has a unique minimal element $\theta \in X$ and satisfies Kaplansky's conditions I. and II.
- (b) (X, \leq) is isomorphic with the spectral poset of a Bézout domain.

Moreover, R is a valuation domain iff (X, \leq) is a totally-ordered set.

The Bézout domain in Theorem 4.8 is obtained by associating a lattice-ordered group G to the poset (X, \leq) and applying the Jaffard-Ohm construction to G . The following result was proved in [8]:

Proposition 4.9 *Let (X, \leq) be a tree which has a unique minimal element and satisfies Kaplansky's conditions I. and II. and let R be the Bézout domain determined by (X, \leq) as explained above. Then for each maximal element x of X which belongs to the set*

$$X^* \stackrel{\text{def.}}{=} \{x \in X \mid \exists y \in X : y \ll x\} \quad ,$$

the principal positive filter $\uparrow 1_x$ is prime and hence corresponds to a principal prime ideal of R . Moreover, we have:

$$\uparrow 1_x = \{f \in G_+ \mid \text{supp}(f) \cap \downarrow x \neq \emptyset\} \quad (4.1)$$

and:

$$F_x = \{f \in \uparrow 1_x \mid \inf S_f(x) \in S_f(x)\} = \{f \in \uparrow 1_x \mid \exists \min S_f(x)\} \quad , \quad (4.2)$$

where:

$$S_f(x) \stackrel{\text{def.}}{=} \text{supp}(f) \cap \downarrow x \quad .$$

A particularly simple example of elementary divisor domains is provided by those Bézout domains R which are *PM* rings*, i.e. which have the property that any non-zero prime ideal of R is contained in a unique maximal ideal (see Theorem B.7 in Appendix B).

Definition 4.10 *A tree (X, \leq) is called a PM* tree if the following three conditions hold:*

1. X has a unique minimal vertex θ (called the root).
2. X satisfies Kaplansky's conditions I. and II.

3. X is branched only at the root, i.e. for every $x \in X \setminus \{\theta\}$, there exists at most one element $y \in X$ such that $x \ll y$.

Proposition 4.11 *Let X be a PM^* tree and R be the Bézout domain associated to X as explained above. Then R is a PM^* ring and hence an elementary divisor domain.*

Proof. Condition 3. in definition 4.10 implies that every element $x \in X \setminus \{\theta\}$ is bounded from above by a unique maximal element of X . Since the elements of $X \setminus \{\theta\}$ correspond to the non-zero prime ideals of R , this implies that any non-zero prime ideal of R is contained in a unique maximal ideal. Thus R is a PM^* ring. Since R is also a Bézout domain by Theorem 4.8, we conclude by Theorem B.7 that R is an elementary divisor domain. \square

Example 4.3.

- Let X be a tree with a unique minimal element which satisfies Kaplansky's conditions I. and II. Assume that the set of maximal vertices of X is countable. Then it was shown in [34] that the associated Bézout domain R is an elementary divisor domain. As a simple example, consider a countable corolla T as in [8, Example 5.8]. The vertices of T are the elements of the set $\mathbb{N} = \mathbb{Z}_{\geq 0}$, with the partial order given by $0 < x$ for every $x \in \mathbb{N}^* = \mathbb{Z}_{> 0}$ and no further strict inequality. The root of T is the element $0 \in \mathbb{N}$ while every maximal vertex $x \in \mathbb{N}^*$ corresponds to a principal prime ideal of the associated Bézout domain.
- If we replace each edge of the countable corolla T discussed above with some finite tree, then the collection of maximal vertices of the resulting tree T' is still countable and the associated Bézout domain R' is an elementary divisor domain which need not be a PM^* ring.

A. Matrices over a GCD domain

Recall that an integral domain R is called a *GCD domain* if any two elements $f, g \in R$ admit a greatest common divisor (gcd). In this case, any non-empty finite collection of elements $f_1, \dots, f_n \in R$ admits a gcd and an lcm, both of which are determined up to association and whose classes we denote by:

$$(f_1, \dots, f_n) \in R/U(R) \quad \text{and} \quad [f_1, \dots, f_n] \in R/U(R) \quad .$$

The gcd class (f) of a single element $f \in R$ coincides with the equivalence class of f under association in divisibility.

Definition A.1 *Let $A \in \text{Mat}(m, n, R)$ be an m by n matrix with coefficients from a GCD domain R . For any $k \in \{1, \dots, r\}$, the k -th determinantal invariant $\delta_k(A) \in R/U(R)$ of A is defined to be the gcd class of all $k \times k$ minors of A . We also define $\delta_0(A) = (1)$.*

Proposition A.2 [35] *Let R be a GCD domain. For any $A \in \text{Mat}(m, n, R)$, we have:*

$$\delta_{k-1}(A) | \delta_k(A) \quad , \quad \forall k \in \{1, \dots, \text{rk}A\} \quad .$$

Defining the invariant factors $\mathbf{d}_k(A) \in R/U(R)$ by:

$$\mathbf{d}_k(A) \stackrel{\text{def.}}{=} \begin{cases} \frac{\delta_k(A)}{\delta_{k-1}(A)} & \text{if } \delta_{k-1}(A) \neq 0 \\ (1) & \text{if } \delta_{k-1}(A) = 0 \end{cases} \quad , \quad \forall k \in \{1, \dots, \text{rk}A\} \quad ,$$

we have:

$$\mathbf{d}_{k-1}(A) \mid \mathbf{d}_k(A), \quad \forall k \in \{2, \dots, \text{rk}A\} .$$

Proposition A.3 [35] *Let R be a GCD domain and $A, B \in \text{Mat}(m, n, R)$. If A and B are equivalent, then $\text{rk}(A) = \text{rk}(B) = r$ and $\mathbf{d}_k(A) = \mathbf{d}_k(B)$ for all $k \in \{1, \dots, r\}$.*

B. Elementary divisor domains

In this appendix, we collect some facts about elementary divisor domains.

Definition B.1 *An integral domain R is called an elementary divisor domain (EDD) if for any three elements $a, b, c \in R$, there exist $p, q, x, y \in R$ such that $(a, b, c) = pxa + pyb + qyc$ is a GCD of a, b and c .*

B.1. Examples of elementary divisor domains. The following are examples of elementary divisor domains:

- Any Bézout domain which is an F -domain (i.e. for which any non-zero element is contained in at most a finite number of maximal ideals) is an EDD [36, Sec. 4]. In particular, any PID is an EDD.
- The ring \mathbb{A} of algebraic integers is an EDD [37, Theorem 5] which has no prime elements.
- The ring of entire functions defined on the complex plane is an EDD [22, 38]. The prime elements of this ring are the entire functions which have a single simple zero in the complex plane.
- If R is an EDD with quotient field K and J is any integral domain such that $R \subset J \subset K$, then J is an EDD [36, Sec. 4]. When R is a PID, it is known that any domain J of this type is a PID and hence Noetherian.
- Any Kronecker function ring is an EDD [39].
- Any generalized valuation domains is an EDD. If V_1, \dots, V_n are generalized valuation domains with the same quotient field K , then $R \stackrel{\text{def.}}{=} \bigcap_{i=1}^n V_i$ is an EDD [36, Sec. 4].
- The domains formed by Jaffard's pull-back theorems are EDDs [36, Sec. 4].
- Let B be an EDD with quotient field K and let m be the maximal ideal of the power series ring $K[[x]]$ in one variable. Then $R := B + m$ is an EDD [36, Sec. 4].
- Let B be an EDD with quotient field K and X be an indeterminate. Then $R := B + XK[X]$ is an EDD [40].
- Let K be an algebraically closed field of characteristic different from two and let x_1 be an indeterminate over K . Let x_2 be a square root of x_1 , x_3 be a square root of x_2 and so on. Then the ring $R := \bigcup_{n=1}^{\infty} K[x_n, 1/x_n]$ is an EDD [36, Sec. 4].

B.2. Kaplansky's characterization of EDDs.

Definition B.2 *Let R be a commutative ring. We say that R satisfies Kaplansky's condition if for any three elements a, b, c in R such that $(a, b, c) = (1)$, there exist elements $p, q \in R$ such that $(pa, pb + qc) = (1)$.*

Proposition B.3 [41] *An integral domain R is an EDD iff it satisfies the following two conditions:*

1. R is a Bézout domain.
2. R satisfies Kaplansky's condition.

B.3. The Smith normal form theorem over an EDD.

Theorem B.4 [35] *Let R be an EDD. For any matrix $A \in \text{Mat}(m, n, R)$, there exist matrices $U \in \text{GL}(m, R)$ and $V \in \text{GL}(n, R)$ such that:*

$$UAV^{-1} = D ,$$

where $D_{ij} = 0$ for all $i \neq j$ and the diagonal entries $d_i \stackrel{\text{def.}}{=} D_{ii}$ (with $i \in \{1, \dots, r\}$, where $r \stackrel{\text{def.}}{=} \text{rk}A \leq \min(m, n)$) are non-zero elements which satisfy the condition:

$$d_1 | d_2 | \dots | d_r .$$

In this case, the matrix D is called the Smith normal form of A . Moreover, the association classes of d_k coincide with the invariant factors of A :

$$(d_k) = \mathbf{d}_k(A) , \quad \forall k \in \{1, \dots, r\} .$$

Proposition B.5 [35] *Let R be an EDD and $A, B \in \text{Mat}(m, n, R)$. Then A and B are equivalent iff they have the same rank r and their invariant factors coincide:*

$$\mathbf{d}_k(A) = \mathbf{d}_k(B) , \quad \forall k \in \{1, \dots, r\} .$$

B.4. Some special classes of EDDs. It is an unsolved problem (going back at least to [22]) whether any Bézout domain is an EDD. Here we mention a few special classes of Bézout domains which are known to be elementary divisor domains. One special class is provided by those Bézout domains which are PM^* -rings.

Definition B.6 [42] *A PM^* -ring is a unital commutative ring R which has the property that any non-zero prime ideal of R is contained in a unique maximal ideal of R .*

Theorem B.7 [43] *Let R be a Bézout domain which is a PM^* ring. Then R is an EDD.*

It was shown in [44] that a Bézout domain is an EDD iff it has Gelfand range one.

Another special class is that of adequate Bézout domains [18, 22, 45].

Definition B.8 [22] *A Bézout domain R is called adequate if for all $a, b \in R$ with $a \neq 0$, there exist $r, s \in R$ such that $a = rs$, $(r, b) = R$ and such that any non-unit s' which divides s satisfies $(s', b) \neq R$.*

Proposition B.9 [23] *Any adequate Bézout domain is a PM^* ring.*

Corollary B.10 [22] *Any adequate Bézout domain is an EDD.*

Remark B.1. It is known that the inclusions:

$$\{\text{adequate rings}\} \subset \{PM^* \text{ rings}\} \subset \{\text{elementary divisor domains}\}$$

are strict (see [43, 46]).

Theorem B.11 *The ring $O(\Sigma)$ of entire functions on any connected and non-compact borderless Riemann surface is an adequate Bézout domain.*

The case $\Sigma = \mathbb{C}$ of this theorem was established in [22, 23]. This generalizes to any Riemann surface using [24, 25]. Since $O(\Sigma)$ is an adequate Bézout domain, it is also a PM^* ring and hence an EDD.

B.5. The Noetherian case. The following characterizations are well-known.

Proposition B.12 *Let A be a Noetherian integral domain. Then the following statements are equivalent:*

1. A is an EDD.
2. A is a Bézout domain.
3. A is a PID.

In particular, matrices valued in a Noetherian domain A admit a Smith normal form iff A is a PID. It is obvious that every PID is Noetherian.

Proposition B.13 *Let A be an integral domain. Then the following statements are equivalent:*

1. A is a PID.
2. A is a UFD and a Bézout domain.
3. A is a UFD and a Dedekind domain.
4. A is a UFD and has Krull dimension one (equivalently, any non-zero prime ideal is maximal).

Proposition B.14 *Let A be a Noetherian integral domain. Then the following statements are equivalent:*

1. A is a UFD.
2. A is normal and its divisor class group vanishes.
3. Every height one principal ideal of A is principal.

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