# Vanishing via lifting to second Witt vectors and a simple proof of an isotriviality result

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**Abstract**

We give a new proof, based on reduction to finite fields, of Esnault-Viehweg's stronger version of Sommese Vanishing Theorem for $k$-ample line bundles; the strengthening stems from the fact that we make joint use of the positivity expressed by the $k$-ampleness and by the Kodaira-Iitaka dimension of the line bundle. We prove a vector bundle version and a Weak-Lefschetz-type theorem. We apply this vanishing to give a short proof of the isotriviality of fibrations over curves of genus at most one with fibers either curves of genus at least two, or minimal surfaces of general type. This latter result is based on a special case of a certain positivity result of J. Kollár which we prove by mostly elementary means.

## 0 Introduction

In this paper I give a new proof of Esnault-Viehweg's improvement of A. Sommese Vanishing Theorem for $k$-ample line bundles. The proof is reminiscent of M. Raynaud's proof of Akizuki-Kodaira-Nakano Vanishing Theorem; however, it departs from M. Raynaud's and differs from the ones of A. Sommese and Esnault-Viehweg.

Firstly, we deal with a special "geometric" case; then we reduce the general case to the geometric one via cyclic coverings.

The geometric case consists of the characteristic zero data of a surjective morphism $f : X \to Y$ and an ample line bundle $A$ on $Y$; we wish to establish some cohomological vanishing for the line bundle $f^*A$. It is natural to expect some kind of positivity for $f^*A$: the dimensions of $Y$ and of the fibers of $f$ play a role. We "spread out" these data over a suitable $\mathbb{Z}$-algebra of finite type. We "reduce" the data so that they are defined over a finite field $\mathfrak{k}$. This gives a situation defined over a finite field which is not arbitrary: because of this reduction, the variety $X$ lifts automatically to the ring $W_2(\mathfrak{k})$ of Witt vectors of length two of $\mathfrak{k}$ and we can use Deligne-Illusie's decomposition theorem. The vanishing in this special finite field situation follows from the use of Serre Vanishing, an inductive argument using powers of the Frobenius morphism and (here is where we depart again from M. Raynaud's proof) Leray Spectral Sequence and Improved Grauert-Riemenschneider Vanishing Theorem. This last assertion holds in characteristic zero, fails in the finite field case in general, but holds in our special case by the spreading out procedure. The vanishing in characteristic zero follows from semi-continuity.

I prove the vector bundle case and a Weak-Lefschetz-type Theorem which do not appear, to my knowledge, in the literature.

I use the vanishing theorem to give a very short proof of a result of A. Parshin and L. Migliorini on the isotriviality of smooth fibrations over curves of genus at most one with fibers either curves of genus at least two, or minimal surfaces of general type. The proof presented here is simpler than the original ones and than the one of S. Kovács. Moreover, it is algebraic and mostly elementary. The proof is based on a positivity result of J. Kollár which had been originally proved using analytic methods, and then has been proved algebraically by J. Kollár and E. Viehweg. The proof is quite

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articulate. We have simplified it considerably proving also many intermediate steps in a new way, e.g. Proposition 4.4, Proposition 4.5, Lemma 4.6 and Proposition 4.7.

The paper is organized as follows. §1 is preliminary and deals with the spreading out technique. §2 contains the proof of the vanishing in the line bundle case, Theorem 2.1, in the vector bundle case, Corollary 2.3, and of the Weak-Lefschetz-type Theorem, Corollary 2.4; Remark 2.2 discusses how Theorem 2.1 improves upon Sommese's theorem. §3 contains the short proof of A. Parshin and L. Migliorini's result: Theorem 3.2. §4 and its four subsections contain a proof, valid in any dimension, of the positivity result of J. Kollár and E. Viehweg under the additional assumptions of Theorem 3.2.

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# 1 Preliminaries on the "spreading out" technique.

Any reasonable and finite amount of geometry defined over a field of characteristic zero $K$ can be "spread out" over an algebra of finite type over $\mathbb{Z}$, $R$, contained in $K$. We can then try to use the resulting "fibration" to say something about the original situation over $K$.

This very vaguely presented principle is made precise in EGA, IV 8, 10. A concise exposition is given by J.L. Illusie in [4], §6.

Let us exemplify the "spreading out" procedure by listing the properties we need in the sequel of the paper.

Let us fix some notation. Let $Z$ be a scheme, $g : U \to V$ be a $Z$-morphism of $Z$-schemes, $F$ be a $\mathcal{O}_U$-module and $z$ be a point in $Z$. We denote the fibers $U \times_Z Spec\, k(z)$ and $V \times_Z Spec\, k(z)$ simply by $U_z$ and $V_z$, the restriction of $F$ to $U_z$ by $F_z$, and the induced morphism from $U_z \to V_z$ by $g_z$.
A line bundle is said to be *semi-ample* if some positive power of it is generated by its global sections. Note that this does not imply the (false) statement that every sufficiently high power of it is generated by its global sections; e.g. a non-trivial torsion line bundle over an elliptic curve.
Let $b$ be a non-negative integer; a semi-ample line bundle $L$ on a projective variety $X$ is said to be *$b$-ample* if, given a positive integer $N$ such that $L^{\otimes N}$ defines a morphism $\phi_{|NL|} : X \to \mathbb{P}$, then the non-empty fibers of $\phi_{|NL|}$ have dimension at most $b$. This notion does not depend on $N$. Note that the Kodaira-Iitaka dimension $\kappa(L) = \dim \phi_{|NL|}(X)$ for any $N$ as above; see [9].
Given a ring $T$, $W_2(T)$ is the ring of Witt vectors of length two associated with $T$.

**Proposition 1.1** *Let $f : X \to Y$ be a projective $K$-morphism of projective $K$-varieties, $F$, $L$ and $A$, respectively, be a coherent sheaf on $X$, a $b$-ample line bundle on $X$ and an ample line bundle on $Y$, respectively.*

*Then, there exist an integral $\mathbb{Z}$-algebra of finite type, $R$, contained in $K$, projective $Spec\, R =: \mathcal{R}$-schemes $\rho : \mathcal{X} \to \mathcal{R}$ and $\sigma : \mathcal{Y} \to \mathcal{R}$, a projective $\mathcal{R}$-morphism $\mathfrak{f} : \mathcal{X} \to \mathcal{Y}$, coherent sheaves $\mathcal{F}$ and $\mathcal{L}$ on $\mathcal{X}$ and $\mathcal{A}$ on $\mathcal{Y}$, and a Zariski-dense open subset $\mathcal{U} \subset \mathcal{R}$, contained in the locus of $\mathcal{R}$ which is smooth over $Spec\, \mathbb{Z}$, with the properties listed below.*

*(i) The objects $X$, $Y$, $f$, $F$, $L$ and $A$, respectively, are obtained from the corresponding objects $\mathcal{X}$, $\mathcal{Y}$, $\mathfrak{f}$, $\mathcal{F}$, $\mathcal{L}$ and $\mathcal{A}$, respectively, by means of the base change induced by $R \hookrightarrow K$.*

*(ii) The objects $\rho$, $\sigma$, $\mathcal{F}$, $\mathcal{L}$, $\mathcal{A}$ and $R^i\mathfrak{f}_*\mathcal{F}$ are all flat over $\mathcal{U}$. In particular the formation of the sheaves $R^i\mathfrak{f}_*\mathcal{F}$ commutes with taking the fiber over any point $u \in \mathcal{U}$.*

*(iii) The sheaves $\mathcal{L}$ and $\mathcal{A}$ are locally free on $\rho^{-1}(\mathcal{U})$ and $\sigma^{-1}(\mathcal{U})$, respectively, $\mathcal{A}$ is $\rho$-ample and $\mathcal{L}_u$ is $b$-ample for every $u \in \mathcal{U}$.*

*(iv) If $X$ is smooth, then $\mathcal{U}$ can be chosen so that $\rho$ and $\sigma$, respectively, are smooth and flat, respectively, over $\mathcal{U}$.*

*(v) If $s$ is a closed point in $\mathcal{U}$, then $\mathcal{X}_s$ lifts to $W_2(k(s))$. Moreover, we can choose $\mathcal{U}$ so that $char\, k(s) > \dim X$, for every closed point $s \in \mathcal{U}$.*

*Proof.* The first set of properties follows from EGA IV 8. The second one is the theorem on flattening stratifications of Grothendieck, and cohomology and base change as established in EGA III 6.9.10. The third is EGA IV 8; the statement about $b$-ampleness stems from the fact that a line bundle $L$ on $X$ is $b$-ample and not $(b-1)$-ample iff there exists ample divisors $D_1, \ldots, D_b$ such that $L_{|\cap_{i=1}^{b} D_i}$ is ample and $b$ is the minimum number for which this is possible. The fourth one follows from generic smoothness and generic flatness, respectively. The last one follows after shrinking $\mathcal{U}$, if necessary, so that the conclusion on the characteristics is true, and by the fact that $\mathcal{U}$ is smooth (over $\mathbb{Z}$); see [4], page 152-3. $\square$

The following elementary lemma contains basic facts to be used later.

**Lemma 1.2** *Every closed point in $\mathcal{R}$ has finite, and a fortiori perfect, residue field.*
*Every Zariski-dense open subset of $\mathcal{R}$ contains a Zariski-dense set of closed points.*

*Proof.* EGA IV 10.4.6, 10.4.7 $\square$

Let us point out a manifestation of the propagating properties of the spreading out procedure. We shall need the following fact in the sequel when we will need to use the Improved Grauert-Riemenschneider Theorem in the finite field context, where it does not hold in general.

**Lemma 1.3** *Let $f: X \to Y$, $F$, $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ and $\mathcal{F}$ be as in Lemma 1.1 and $\eta$ be the generic point of $\mathcal{R}$. Assume that $R^j f_* F = 0$ for a fixed integer $j$.*
*Then $R^j \mathfrak{f}_{u*} \mathcal{F}_u = 0$ for every point $u$ in a suitable Zariski-dense open subset $\mathcal{U}$ of $\mathcal{R}$.*

*Proof.* The base changes induced by $R \hookrightarrow k(\eta)$ and $k(\eta) \hookrightarrow K$ are both flat, the second one is even faithfully flat. It follows that $[R^j \mathfrak{f}_* \mathcal{F}]_\eta = R^j \mathfrak{f}_{\eta *} \mathcal{F}_\eta = 0$. Since $\sigma$ is proper, we see that, after shrinking $\mathcal{U}$ if necessary, the assertion holds by Lemma 1.1.ii. $\square$

The following result contains the basic fact that we shall need about smooth projective varieties over a perfect field $\mathfrak{k}$ which lift to $W_2(\mathfrak{k})$. It is proved in [2] as a consequence of the beautiful and elementary decomposition Théorème 2.1 (and Corollaire 2.3). We state it in an apparently more general form, more suitable for our needs later on.

**Theorem 1.4** *(Akizuki-Kodaira-Nakano Vanishing Theorem) Let $X$ be a smooth projective variety of dimension $d$ over a perfect field $\mathfrak{k}$ of characteristic $p > d$ which admits a lifting to $W_2(\mathfrak{k})$, $M$ be a line bundle on $X$ and $\nu$ be a non-negative integer.*
*Assume that, for some positive integer $n$,*

$$H^j(X, \Omega_X^i \otimes M^{\otimes p^n}) = \{0\} \qquad \forall\, i + j = \nu.$$

*Then*

$$H^j(X, \Omega_X^i \otimes M) = \{0\} \qquad \forall\, i + j = \nu.$$

*In particular, if $\check{M}$ is ample, then any $\nu < d$ will do.*

*Proof.* See [2] Corollaire 2.3 and Lemme 2.9. Note that in our setting we can conclude that the relevant hypercohomology group $\mathbb{H}^\nu = \{0\}$, which is what is needed. $\square$

If $M$ is ample, then one proves Kodaira-Akizuki-Nakano Vanishing Theorem as an easy consequence of Serre Vanishing (Raynaud). This theorem allows one to re-prove the classical Kodaira-Akizuki-Nakano Vanishing Theorem in characteristic zero by spreading out $X \to K$ to $\mathcal{X} \to \mathcal{R}$, $\Omega^\bullet_{X/K}$ to $\Omega^\bullet_{\mathcal{X}/\mathcal{R}}$, by the upper semi-continuity properties of the dimensions of cohomology groups.

## 2 Esnault-Viehweg's improvement of A. Sommese Vanishing Theorem for $k$-ample vector bundles using $W_2$-lifting.

The following result is a slight improvement of A. Sommese Vanishing Theorem for $k$-ample line bundles; see [11], Theorem 3.36 and Corollary 5.20. This improvement in the line bundle case is due to H. Esnault and E. Viehweg, [3], Theorem 2.4, who use analytical methods.

We offer a new proof which is algebraic and passes through reduction to finite fields, Deligne-Illusie decomposition Theorem and characteristic zero vanishing theorems made valid in the finite field case by "propagation."

The "shifted" version for the vector bundle case follows, as it is nowadays standard, by a theorem of J. Le-Potier's simplified by M. Schneider; see [11], Theorem 5.16.

We first state and prove the result in the line bundle case. We also prove the vector bundle case and a Weak-Lefschetz-type result.

We fix $K$, a field of characteristic zero.

**Theorem 2.1** *Let $X$ be a smooth projective variety of dimension $d$ over $K$, $L$ be a $b$-ample line bundle over $X$ of Kodaira-Iitaka dimension $\kappa(L)$.*
*Then*

$$H^j(X, \Omega^i_X \otimes L^\vee) = \{0\} \qquad \forall\, (i,j) \ s.t. \quad i+j \ < \ \min(\kappa(L), d-b+1). \tag{1}$$

*Equivalently,*

$$H^q(X, \Omega^p_X \otimes L) = \{0\} \qquad \forall\, (p,q) \ s.t. \quad p+q \ > \ 2d - \min(\kappa(L), d-b+1). \tag{2}$$

*Proof.* The two statements are equivalent by virtue of Serre Duality and of the canonical isomorphisms $\Omega^{d-l}_X \simeq K_X \otimes \check{\Omega}^l_X$.

*STEP I.* We first prove the assertion under the additional hypothesis that $mL$ is generated by its global sections for *every* $m \gg 0$.

We prove statement (2).

There are a surjective and projective morphism $g : X \to Y$ with connected fibers onto a normal variety $Y$ and an ample line bundle $A$ on $Y$ such that $L = g^*A$.
By assumption, $Y$ has dimension $\kappa(L)$ and the fibers of $g$ are at most $b$-dimensional.

We have the following two properties:

(a) $R^l g_*$ is the zero functor, for every $l > b$;

(b) $R^l g_* K_X = 0$ for every $l > d - \kappa(L)$; this Improved Grauert-Riemenschneider Vanishing Theorem follows from the improved Kawamata-Viehweg Vanishing Theorem, [11] Corollary 7.50 and from [1], Proposition 8.9. Note that everything is algebraic here and that H. Hironaka's Resolution of Singularities is needed.

We apply Proposition 1.1 to $g : X \to Y$, $L$, $A$ and $F := K_X \simeq \omega_{X/K}$ with the choice $\mathcal{F} \simeq \omega_{\mathcal{X}/\mathcal{R}}$. Let $s \in \mathcal{R}$ be a closed point belonging to the open set $\mathcal{U}$ (recall Lemma 1.2!) over which all the conditions of Proposition 1.1 for $\omega_{\mathcal{X}/\mathcal{R}}$ and all of its direct images via $\mathfrak{g}$ are met.

By virtue of Lemma 1.3, we may shrink $\mathcal{U}$, if necessary, so that the two conditions (a) and (b) above are met for $\mathfrak{g}_s$ as well.

By abuse of notation, we denote $\mathcal{L}_s$, $\mathcal{A}_s$, $\mathcal{X}_s$, $\mathcal{Y}_s$ and $\mathfrak{g}_s$, respectively, by $L_s$, $A_s$, $X_s$. $Y_s$ and $g_s$. In order to apply Theorem 1.4, we need to check that $H^q(X_s, \Omega^p_{X_s} \otimes L_s^{\otimes m}) = \{0\}$, for every $m \gg 0$ in the prescribed range for $p$ and $q$. This is an immediate consequence of the Leray spectral sequence for $g_s$ and conditions (a) and (b) for $g_s$, as we now show.

By virtue of Leray spectral sequence and of Serre Vanishing, there exists $m_s$ such that for every $m \geq m_s$ and for every pair of indices $p$ and $q$ we have that

$$H^q(X_s, \Omega^p_{X_s} \otimes L_s^{\otimes m}) \simeq H^0(Y_s, R^q g_{s*} \Omega^p_{X_s} \otimes A_s^{\otimes m}).$$

If $p$ and $q$ are in the prescribed range, then $R^q g_{s*}\Omega^p_{X_s}$ is zero so that the group on the right vanishes: in fact, $p+q \geq d+b$ and $p \leq d$ so that we can use (a) and (b) above.

Now the standard semi-continuity argument. By virtue of what we have proved and by virtue of Proposition 1.1 we can assert the existence of a Zariski-dense open subset $\mathcal{U} \subset \mathcal{R}$ over which the sheaves $\mathcal{L}$ and $\Omega^\bullet_{\mathcal{X}/\mathcal{R}}$ are locally free and such that, given any closed point $s \in \mathcal{U}$, we have that $char\, k(s) > d$, $X_s$ lifts to $W_2(k(s))$ and there is a certain positive integer $m_s$ such that for every $p$ and $q$ in the prescribed range and for every $m \geq m_s$ we have that

$$H^q(X_s, \Omega^p_{X_s} \otimes L_s^{\otimes m}) = \{0\}.$$

We choose $m := [char\,(k(s)]^{m_s}$ and apply a straightforward descending induction coupled with Theorem 1.4 to deduce that the vanishings above hold with $m = 1$.

The vanishing in characteristic zero follows from the fact that, by the upper-semicontinuity of these dimensions, we have the vanishing over the generic point $\eta \in \mathcal{R}$ and, by the flat base change induced by $k(\eta) \hookrightarrow K$, therefore over $K$.

The proof of *STEP I* is now complete.

*STEP II.* We now remove the additional assumption of *STEP I*: we prove the theorem by induction on $d - \kappa(L)$ using *STEP I* and by means of an easy procedure to construct, on a suitable covering of $X$, $d-\kappa(L)$ sections of the pull-back of $L$ with base locus of dimension $d - \kappa(L)$.

Note that we may assume that $\kappa(L) > 0$, since, if $\kappa(L) = 0$, then there is nothing to prove.

Let $c$ be a positive integer such that $cL$ is generated by its global sections. We get a surjective morphism onto a variety $Y$ of dimension $\kappa(L)$. Choose a general section $\sigma_1$ of $cL$ so that it defines a smooth divisor $D'_1$ on $X$. Consider the corresponding cyclic covering $C_1 : X_1 \to X$ branched along $D'_1$ and ramified along the smooth divisor $D_1 := C_1^{-1}(D'_1)$ which is the zero set of a section of the line bundle $L_1 := C_1^* L$. We also have that $\Omega^l_X$ is a direct summand of $C_{1*}\Omega^l_{X_1}$ for every integer $l$ such that $0 \leq l \leq d$.

By iterating this procedure, we obtain a sequence of cyclic coverings $X_{i+1} \stackrel{C_{i+1}}{\longrightarrow} X_i$ together with line bundles $L_{i+i} = C^*_{i+1} L_i$ such that $\dim Bs|L_{\kappa(L)}| \leq d - \kappa(L)$ and $\Omega^l_X$ is a direct summand of $C_*\Omega^l_{X_{\kappa(L)}}$, where $C : X_{\kappa(L)} \to X$ is the induced morphism. By virtue of Leray Spectral Sequence for finite morphisms, $H^r(X, \Omega^l_X \otimes L)$ is a direct summand of $H^r(X_{\kappa(L)}, \Omega^l_{X_{\kappa(L)}} \otimes L_{\kappa(L)})$.

It is therefore enough to prove the theorem under the additional assumption that $|L| \neq \emptyset$ and that $\dim Bs|L| \leq d - \kappa(L)$.

We work by ascending induction on $\mathfrak{l} := d - \kappa(L)$.

Let $\mathfrak{l} = 0$. Then $\dim Bs|L| = 0$. By Zariski's [16], Theorem 6.2, we obtain that $mL$ is generated by its global sections for *every* $m \gg 0$ and we conclude by virtue of *STEP I.*

Let the contention be true for every $\mathfrak{l}' < \mathfrak{l}$. Let us prove it for $\mathfrak{l}$. We prove statement (1).

Let $H$ be an ample hypersurface of $X$ such that it is smooth, $L_{|H}$ is $(b-1)$-ample and $\kappa(L_{|H}) = \kappa(L)$. Such an $H$ exists by a result of H. Hironaka's, [11], Theorem 3.39, and by the fact that $H$ maps onto $Y$ by the assumption $d - \kappa(L) > 0$. Consider the exact sequences:

$$0 \to \Omega^l_X \otimes L^\vee \otimes H^\vee \to \Omega^l_X \otimes L^\vee \to \Omega^l_X \otimes L^\vee \otimes \mathcal{O}_H \to 0, \tag{3}$$

$$0 \to \Omega^{l-1}_H \otimes L^\vee_{|H} \otimes H^\vee_{|H} \to \Omega^l_X \otimes L^\vee \otimes \mathcal{O}_H \to \Omega^l_H \otimes L^\vee_{|H} \to 0. \tag{4}$$

Since $L + H$ is ample, we can use the standard Akizuki-Kodaira-Nakano Vanishing Theorem on $X$ and on $H$ and get, in the prescribed range for $i$ and $j$, the following two injective maps

$$H^j(X, \Omega^i_X \otimes L^\vee) \hookrightarrow H^j(H, \Omega^i_X \otimes L^\vee \otimes \mathcal{O}_H) \hookrightarrow H^j(H, \Omega^i_H \otimes L^\vee_{|H}).$$

Since

$$\mathfrak{l}_H := \ (d-1) - \kappa(L_{|H}) \ = \ (d-1) - \kappa(L) \ < \ d - \kappa(L) \ = \ \mathfrak{l}$$

and

$$i+j \;<\; \min\left(\kappa(L), d-b+1\right) \;=\; \min\left(\kappa(L_{|H}), (d-1-(b-1)+1\right),$$

we can apply the induction hypothesis and conclude that the last group on the right is trivial. This gives the wanted vanishing result. $\square$

**Remark 2.2** Sommese's theorem is sharp, as stated. However, note that $\kappa(L) \geq d-b$ and that the strict inequality is possible. If we have equality, then $f$ is equidimensional and we get Sommese's statement. If we have strict inequality, then Theorem 2.1 improves Sommese's by one unit.
Consider the case where $K$ is algebraically closed and $f : X \to Y$ is the blowing-up of $\mathbb{P}^3$ at either a (closed) point, or along a line. Let $L$ be the pull-back of the hyperplane bundle. In the former case $\min\left(\kappa(L), d-b+1\right) = \min\left(3,2\right) = 2$; Sommese Vanishing Theorem predicts vanishing for $1+j<1$; Theorem 2.1 predicts vanishing for $i+j<2$; moreover, $H^1(X, \Omega^1_X \otimes L^\vee)$ is one dimensional. In the latter case $\min\left(\kappa(L), d-b+1\right) = \min\left(3,3\right) = 3$; A. Sommese's Theorem predicts vanishing for $i+j<2$ and Theorem 2.1 for $i+j<3$.
This example shows that Theorem 2.1 is sharp and improves upon A. Sommese's. Moreover it shows concretely why it is sharp: for in the case we blow-up a point $p$, we have that $R^1 f_* \Omega^1_X$ is isomorphic to the skyscaper sheaf at $p$ of stalk $K = k(p)$. The second case puts in evidence that, for the purpose of Akizuki-Kodaira-Nakano-type statements, a line bundle which is semi-ample, big and 1-ample is as good as ample; we use this fact in an essential way in the proof of Theorem 3.2.

The following two results follow easily from Theorem 2.1 and they do not need the proof given above. The first one admits a dual formulation which we omit for brevity. The interested reader can easily formulate and prove vanishing results analogous to the first one which involve $K_X \otimes \wedge^l E$ and more generally $\Omega^p_X \otimes \wedge^l E$; see [11], §5.

Recall that a vector bundle $E$ is said to be $b$-ample if the associated tautological line bundle $\xi_E$ is $b$-ample.

**Corollary 2.3** *Let things be as in Theorem 2.1 except that we replace the line bundle $L$ by a rank $r$, $b$-ample vector bundle $E$ and we set $\kappa(E) := \kappa(\xi_E)$.*
*Then*

$$H^q(X, \Omega^p_X \otimes E) = \{0\} \qquad \forall\, (p.q) \;\; s.t. \quad p+q > 2\left[d+(r-1)\right] - \min\left(\kappa(E), d+r-b\right)$$

*Proof.* By virtue of a result of Le Potier's (cf. [11], 5.17, 5.21 and 5.28): $H^q(X, \Omega^p_X \otimes E) \simeq H^q(\mathbb{P}(E), \Omega^p_{\mathbb{P}(E)} \otimes \xi_E)$. Apply Theorem 2.1 to the pair $(\mathbb{P}(E), \xi_E)$. $\square$

**Corollary 2.4** *(Weak Lefschetz Theorem) Let $X$ be a smooth projective variety of dimension $d$ defined over a field of characteristic zero. Let $D$ be an effective smooth divisor on $X$ such that the associated line bundle $L$ is $b$-ample and it is of Kodaira-Iitaka dimension $\kappa(L)$.*
*Then the canonical morphisms of de Rham cohomology $H^l(X,K) \to H^l(D,K)$ are:*
*(i) isomorphisms for $l < \min\left(\kappa(L), d-b+1\right) - 1$,*
*(ii) injective for $l < \min\left(\kappa(L), d-b+1\right)$.*

*Proof.* Note that $\kappa(L_{|D}) = \kappa(L) - 1$ and that $b_D \leq b$. We conclude by means of easy diagram considerations on the long cohomology sequences associated with (3) and (4). $\square$

For more statements in the vein of the corollary above, see [11], Theorem 3.40.

# 3 A short proof of a result of A.N. Parshin and L. Migliorini

We give a short algebraic proof of Theorem 3.2 below. The proof hinges on Theorem 2.1 and on the following positivity result of J. Kollár (which holds in greater generality than the one stated below), whose original proof is Hodge-theoretic and which has been proved again algebraically by J. Kollár and E. Viehweg. we shall consider E. Viehweg's proof in the next section.

In what follows everything is defined over an algebraically closed field of characteristic zero $K$.

**Theorem 3.1** *(Cf. [6]) Let $f : X \to P$ be a surjective morphism with connected fibers of nonsingular projective varieties, where $P$ is a nonsingular curve. Assume that the fibers of $f$ are of general type and are not all birationally isomorphic to each other.*
*Then the vector bundle $f_*\omega_{X/P}^{\otimes m}$ is ample for infinitely many values of the positive integer $m$.*

**Theorem 3.2** *(Cf. [10], [8]) Let $X$ be a nonsingular projective variety of dimension $d$, $P$ be a nonsingular complete curve of genus $g(P) \leq 1$ and $f : X \to P$ be a surjective smooth morphism such that all the fibers are connected varieties of general type with nef canonical bundle.*
*If $d \leq 3$, then all the fibers are isomorphic to each other.*

*Proof.* Since, if necessary, we can take a double cover $P \to \mathbb{P}^1$, $P$ any elliptic curve, we can assume that $g(P) = 1$. Note that, in this case, $K_X = \omega_{X/P} \simeq \Omega_{X/P}^{d-1}$.

Seeking a contradiction, we assume that the fibers of $f$ are not all birationally isomorphic to each other.

By the Base-Point-Free Theorem of Y. Kawamata and V. Shokurov (cf. [1]) applied to the pluricanonical line bundles of the fibers and by neotherian induction on $P$, there exists a positive integer $m_0$ such that: for every $m \geq m_0$, the natural morphism $f^*f_*mK_X \to mK_X$ is surjective and it induces a $P$-morphism $g_m : X \to \mathbb{P}(f_*\omega_{X/P}^{\otimes m})$ which, in turn, induces the birationally isomorphic stable pluricanonical morphisms on the fibers of $f : X \to P$. For $\mu \gg \nu \gg 0$ we have the surjection $S^\mu(f_*\omega_{X/P}^{\otimes\nu}) \to f_*\omega_{X/P}^{\otimes\mu\nu}$.

By virtue of Theorem 3.1, we can choose the integer $m$ above so that $E_m := f_*\omega_{X/P}^{\otimes m}$ is ample. It follows that $mK_X \simeq g_m^*\xi_{E_m}$, being the pull-back of an ample line bundle via $g_m$, is semi-ample and $(d-2)$-ample. This conclusion holds for $K_X$ as well.

The following argument is due to S. Kovács (cf. [7], page 370). Consider the exact sequences:

$$0 \to \Omega_{X/P}^{i-1} \otimes K_X \to \Omega_X^i \otimes K_X \to \Omega_{X/P}^i \otimes K_X \to 0.$$

For every $1 \leq p \leq d-1$, we get short exact sequences

$$H^{d-p}(X, \Omega_{X/P}^p \otimes K_X) \xrightarrow{\alpha_p} H^{d-(p-1)}(X, \Omega_{X/P}^{p-1} \otimes K_X) \longrightarrow H^{d-(p-1)}(X, \Omega_X^p \otimes K_X) = \{0\},$$

where, when $d \leq 3$, the maps $\alpha_p$ are all surjective by Theorem 2.1. We compose all these surjective maps $\alpha_p$ and get a surjection

$$\{0\} = H^1(X, K_X \otimes K_X) \longrightarrow\!\!\!\!\!\to H^d(X, K_X) \simeq K,$$

the first isomorphism on the left being Kawamata-Viehweg Vanishing Theorem. This is a contradiction.

The fibers are therefore birationally isomorphic to each other. The result follows from the uniqueness of minimal models for curves and surfaces. □

**Remark 3.3** The case $d = 2$ is due to A.N. Parshin; see [10]. The case $d = 3$ is due to L. Migliorini; see [8]; his proof uses analytic techniques.
The result is false without the restriction on the genus of the base; see [5].
Since the automorphism group of the fibers is finite and the fibers are all isomorphic to each other, the fibration is isotrivial, i.e. it becomes trivial after a finite base change $P' \to P$.
If we drops the nefness assumption, then a birationally isomorphic statement still holds; see [8].
A similar but weaker statement holds for any value of $d$ and it is due to S. Kovács [7].

# 4 Proof of Theorem 3.1 in arbitrary dimension, but with the assumptions of Theorem 3.2.

Theorem 3.1 is proved in [6] using Hodge Theory. If $d \leq 3$, then a simple algebraic proof can be given using moduli spaces for curves and surfaces. However, since Theorem 3.1 holds in every dimension $d$, it is desirable to prove it algebraically for every value of $d$. Such a proof is given by J. Kollár and E. Viehweg; see [15], Theorem 4.34 and [13], Theorem 3.5.

We want to offer a simple proof of Theorem 3.1 for any dimension $d$ of $X$ under the additional assumptions of Theorem 3.2. It will not be a mere repetition of the proofs mentioned above, since, given the fact that we shall work on an elliptic curve, we are able to simplify those proofs considerably and offer new proofs of the key facts based on elementary properties of elliptic curves.

It is enough to prove, under the assumptions of Theorem 3.2, but in any dimension:

1) the nefness (or triviality) of the sheaves $f_*\omega_{X/P}^{\otimes k}$, for $k \geq 1$.

2) The fact that det $f_*\omega_{X/P}^{\otimes k}$ is ample for some value $k$.

3) If det $f_*\omega_{X/P}^{\otimes k}$ is ample for some value $k$, then $f_*\omega_{X/P}^{\otimes m}$ is ample for infinitely many values of $m$.

The assumptions of Theorem 3.2 are needed only for 2); 1) and 3) are general phenomena.

Statement 1) is due due to T. Fujita and Y. Kawamata. An algebraic proof can be found in [14]. We give an even simpler proof in §4.2 based on J. Kollár Vanishing Theorem and on multiplication maps on elliptic curves. Note that in our context the proof of the case $k \geq 2$ is independent of the case $k = 1$. The usual proofs reduce the $k \geq 2$ case to the case $k = 1$.

Statement 2) is proved in [15] Theorem 4.34. There is nothing that we can add to the nice proof given there. However, we would like to offer, in §4.3, a short proof based of some now standard results of higher dimensional geometry, namely Kodaira-Iitaka Subadditivity and Abundance. In this proof, the role of the birational geometry of the fibers plays a very direct and transparent role by means of the Iitaka fibration.

Statement 3) is E. Viehweg's [12], 5.4. The proof given there is rather articulate: one needs [13], 3.5, 3.4, [12], 5.4 and 5.1 which ultimately depends on 1). We have simplified it, in the case at hand, in §4.4. In the case of an elliptic curve it can be further simplified by choosing, in the proof, $\nu = 1$, for $M$ will be generated by its global sections.

## 4.1 Further notation and results.

We need to fix some notation and state some known results.

Let $X$ be a nonsingular projective variety of dimension $d$, $P$ be a nonsingular complete curve and $f : X \to P$ be a surjective morphism with connected fibers, i.e a *fibration*. Let $\Sigma_f$ be the finite set of points on $P$ over which $f$ is not smooth.

If $\alpha : C \to P$ is a morphism, then we have the following cartesian diagram:

$$\begin{array}{ccc} X^C & \xrightarrow{\alpha'} & X \\ \downarrow f' & \square & \downarrow f \\ C & \xrightarrow{\alpha} & P \end{array}$$

We will always choose surjective morphisms $\alpha : C \to P$ with $C$ a nonsingular curve and $\alpha$ branched outside $\Sigma_f$ so that $X^C$ is nonsingular. These morphisms $\alpha$ are automatically flat; this is important for "cohomology and base change."

Under these conditions we have, for every nonnegative integer $k$:

$$\omega_{X/P}^{\otimes k} \simeq k(K_X - f^*K_P) \qquad \textit{and} \qquad k(K_{X^C} - f'^*K_C) \simeq \omega_{X^C/C}^{\otimes k} \simeq \alpha^*\omega_{X/P}^{\otimes k}.$$

Note that since $\alpha$ is a finite morphism, the ampleness (nefness, respectively) of $\omega_{X/P}^{\otimes k}$ is equivalent to the ampleness (nefness, respectively) of $\omega_{X^C/C}^{\otimes k}$.

Recall that a vector bundle is said to be nef if the associated tautological line bundle (in Grothendieck's notation) is nef. By abuse of notation we say that the zero sheaf is nef as well; this abuse will be harmless in the sequel.

We shall need the following two results.

**Theorem 4.1** *(J. Kollár Vanishing Theorem; cf. [14].) Let $X$ be a nonsingular projective variety, $L$ be line bundle on $X$, $B$ be an effective divisor on $X$ and assume that there exists a positive integer $N$ such that $\Gamma(X, NL - B) \neq 0$. Then the natural maps $H^q(X, K_X + L) \to H^q(X, K_X + L + B)$ are injective for every $q$.*

**Theorem 4.2** *(Extracting roots out of sections; cf. [12], Lemma 2.3.) Let $X$ be a nonsingular projective variety, $L$ be a line bundle on $X$, $\sum_t m_t E_t$ be an effective divisor in normal crossings, where all the $E_t$ are distinct prime divisors, and $N$ be an integer such that $NL \simeq \sum_t m_t E_t$. For every integer $i = 0, \dots N-1$, define $L^{(i)} := iL - (\sum_t \lfloor \frac{im_t}{N} \rfloor E_t)$. Consider the following diagram:*

$$Z \xrightarrow{d} \hat{Y} \xrightarrow{\nu} Y \xrightarrow{c} X$$

*where $c$ is the degree $N$ cyclic covering of $X$ associated with the section $\sum_t m_t E_t$, $\nu$ is the normalization of $Y$ and $d$ is any desingularization of $\hat{Y}$. Denote by $\gamma : Z \to X$ the natural map. Then*

$$\gamma_* \omega_Z \simeq \bigoplus_{i=0}^{N-1} L^{(i)} \otimes \omega_X.$$

We also need the following elementary fact.

**Lemma 4.3** *Let $E$ be a vector bundle on a nonsingular complete curve such that it is generated at the generic point by its global sections. Then $E$ is nef.*

*Proof.* We consider the following push-out commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & H^0(E) \otimes \mathcal{O}_P & \longrightarrow & \mathcal{E} & \to & \mathcal{O}_P^{\oplus l} & \to & 0 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
0 & \to & H^0(E) \otimes \mathcal{O}_P & \xrightarrow{evaluation} & E & \to & \tau (= skyscraper) & \to & 0,
\end{array}$$

where the rows are exact and all vertical morphisms are surjective. The top row defines an extension, $\mathcal{E}$, of locally free coherent sheaves which are nef. This extension is therefore locally free and nef. This shows that $E$, being the quotient of a nef vector bundle, is nef. $\square$

## 4.2 Nefness of $f_* \omega_{X/P}^{\otimes k}$, $k \geq 1$, using multiplication maps on elliptic curves.

We prove 1) of the introduction to §4.

**Proposition 4.4** *Let $f : X \to P$ be a fibration onto a nonsingular curve. Then $f_* \omega_{X/P}$ is a nef vector bundle (possibly the zero-sheaf).*

*Proof when $g(P) \leq 1$.* The assumptions and conclusions are compatible with a finite flat base change $P' \to P$ for which $X^{P'}$ is nonsingular. Therefore we may assume that $g(P) = 1$.

The sheaf in question is torsion free and therefore locally free. There is a Zariski-dense open subset $U$ of $P$ over which the formation of it commutes with an arbitrary base change. Let $q \in U$ be any point and fix any line bundle $H$ of degree two.

*CLAIM* : $H \otimes f_*\omega_{X/P}$ *is nef.*
The natural map $a$

$$H^0(P, H \otimes f_*\omega_{X/P}) \simeq H^0(X, K_X \otimes f^*H) \xrightarrow{a} H^0(\omega_{f^{-1}(q)}) \simeq H \otimes f_*\omega_{X/P} \otimes k(q)$$

is surjective by virtue of Theorem 4.1. It follows that $H^{\otimes 2} \otimes f_*\omega_{X/P}$ is generated by its global sections at $q$. Now use Lemma 4.3.

We consider the self-map $\mu : P' := P \to P$ given by multiplication by the positive integer $m$. Note that $\mu^*H \simeq H^{\otimes m}$. We easily see that $S^m(H \otimes f'_*\omega_{X^{P'}/P'}) \simeq \mu^*[S^m(f_*\omega_{X^P/P}) \otimes H]$. By virtue of *CLAIM* , applied to $f' : X^{P'} \to P'$, and since $\mu$ is finite, we see that $S^m(f_*\omega_{X/P}) \otimes H$ is nef for every integer $m$.

The theorem follows easily by projectivizing $f_*\omega_{X/P}$, for the line bundles $m\xi + \pi^*H$, where $\xi$ is the tautological line bundle and $\pi : \mathbb{P}(f_*\omega_{X/P}) \to P$ is the natural projection, are all nef so that $\xi$ and, by definition, $f_*\omega_{X/P}$ are nef. $\square$

**Proposition 4.5** *Let $f : X \to P$ be a fibration over a nonsingular curve such that $\omega_{X/P}$ is $f$-nef and $f$-big. Then for every positive integer $m \geq 2$, $f_*\omega_{X/P}^{\otimes k}$ is a nef vector bundle (possibly the zero-sheaf).*

*Proof when $g(P) \leq 1$.* The assumptions and the statement are compatible with finite flat base changes $\pi : P' \to P$ for which $X^{P'}$ is nonsingular. Therefore, we may assume that $g(P) = 1$. Note that, as a consequence we may also assume that $K_X$ is nef: for if $K_X$ where not nef, then, by a theorem of S. Mori, $X$ would contain a rational curve dominating $P$, a contradiction.

*CLAIM : Let $H$ be a line bundle of degree at least 2. Then $H \otimes f_*\omega_{X/P}^{\otimes k}$ is generated by its global sections. In particular it is nef.*
By virtue of Lemma 4.3, the second statement follows from the first. For every $k \geq 2$ the set $U$ over which the formation of $f_*\omega_{X/P}^{\otimes k}$ commutes with arbitrary base change coincides with $P$, by virtue of the relative version of Kawamata-Viehweg Vanishing Theorem. Let $q$ be any point on $P$. It is enough to show that $H^1(X, K_X + (k-1)K_X + f^*(H - (k-1)K_P - q)) = \{0\}$. Note that $K_P \simeq \mathcal{O}_P$. By the choice of the degree of $H$, the line bundle $H - q$ is ample. Since $K_X$ is nef and $f$-big, the line bundle $(k-1)K_X + f^*(H - q)$ is nef and big. The vanishing follows from Kawamata-Viehweg Vanishing Theorem. The claim is proved.

We now apply the same argument, involving multiplication maps, as in the last part of the proof of Theorem 4.4. $\square$

## 4.3 Ampleness of $\det f_*\omega_{X/P}^{\otimes m}$, for some $m \gg 0$.

We prove 2) of the introduction to §4. We need the following result.

**Lemma 4.6** *Let $X$ be a nonsingular projective variety of dimension $d$, $P$ be a nonsingular complete curve of any genus and $f : X \to P$ be a surjective smooth morphism such that all the fibers are connected varieties of general type with nef canonical bundle. Assume that the fibers of $f$ are not birationally isomorphic to each other.*
*Then $\omega_{X/P}$ is nef and big.*

*Proof.* The assumptions and the conclusions are compatible with finite flat base changes $\pi : P' \to P$ for which $X^{P'}$ is nonsingular. Therefore we may assume that $g(P) \geq 1$.

By virtue of Y. Kawamata's result on sub-additivity over curves [9], 6.2.e, we have that $\kappa(X) \geq \kappa(P) + d - 1$. If $g(P) \geq 2$, then we would be done. Assume that $g(P) = 1$. So that we have $K_X \simeq \omega_{X/P}$ and $\kappa(X) \geq d - 1$. By virtue of Y. Kawamata's results on Abundance (cf. [9], page 314) $K_X$ is semi-ample.

Consider the Iitaka fibration $I : X \to Y$, $I^*A = lK_X$, where $A$ is an ample line bundle on $Y$, $Y$ is a normal variety, $I$ has connected fibers and $l$ is some positive integer.

Seeking a contradiction, assume that $K_X$ is not big, i.e. that $\dim Y = d-1$.
Let $\Gamma$ be a general fiber of $I$. $\Gamma$ is an elliptic curve which dominates the elliptic curve $P$. The corresponding morphism $\Gamma \to P$ is étale. The morphism $f' : X^\Gamma \to \Gamma$ has a natural section $\Gamma'$. The line bundle $K_{X^\Gamma}$ is nef, but not big. Consider the Stein factorization $I' : X^\Gamma \to Y'$ of the morphism $X^\Gamma \to Y$. By construction this morphism is the Iitaka fibration for $X^\Gamma$ and the section $\Gamma'$ is a smooth fiber of $I'$. Clearly $\Gamma'$ intersects each fiber of $f'$ in exactly one point. In particular each fiber maps birationally onto $Y'$. This is the seeked contradiction. $\Box$.

Now we use the fact that $\omega_{X/P}$ is nef and big to easily prove that $\det f_*\omega_{X/P}^{\otimes k}$ is ample for some positive integer $k$. Clearly, it is enough to prove that $\delta_k := \deg f_*\omega_{X/P}^{\otimes k} > 0$ for some $k$.

**Proposition 4.7** *Let $f : X \to P$ be as in Theorem 3.2. Then $\delta_k > 0$ for some $k$.*

*Proof.* As before, we reduce ourselves to the case $g(P) = 1$. Seeking a contradiction, assume that $\delta_m = 0$ for all $m \gg 0$. Note that, since $K_X = \omega_{X/P}$ is big by Lemma 4.6 and by virtue of Kawamata-Viehweg Vanishing and the Leray spectral sequence, $H^1(P, f_*\omega_{X/P}^{\otimes m}) = \{0\}$, for all $m \geq 2$. By Riemann-Roch on the elliptic curve we have that $H^0(P, f_*\omega_{X/P}^{\otimes m}) = \{0\}$, for all $m \gg 0$. The dimension $H^0(P, f_*\omega_{X/P}^{\otimes m}) \simeq H^0(X, mK_X)$ grows like $m^d$, and this is the seeked contradiction. $\Box$

## 4.4 A construction of E. Viehweg's.

We prove statement 3) of the introduction to §4.

**Lemma 4.8** *Let $f : X \to P$ be a fibration onto a nonsingular curve. Assume that the natural maps $f^*f_*\omega_{X/P}^{\otimes l} \to \omega_{X/P}^{\otimes l}$ are surjective for every $l \geq l_0$, for some fixed integer $l_0$. Assume that there are a positive integer $k$, an ample line bundle $H$ on $P$ and an injection $H \to f^*\omega_{X/P}^{\otimes k}$.*
*Then there exist $k_0 > 0$ such that for every $k_1 > 0$ the vector bundle $f_*\omega_{X/P}^{\otimes k_0 k_1 k}$ is ample. In particular, there are infinitely many values of $m$ for which $f_*\omega_{X/P}^{\otimes m}$ is ample.*
*Moreover, assume that $\det f_*\omega_{X/P}^{\otimes t}$ is ample for some value $t > 0$ and let $s$ be the rank of $f_*\omega_{X/P}^{\otimes t}$, then we can choose $H = \det f_*\omega_{X/P}^{\otimes t}$ and $k = s \cdot t$.*

*Proof.* The assumptions and the statements are compatible with finite flat base changes $\pi : P' \to P$ which preserve the nonsingularity of $X^{P'}$. Therefore we may assume that $g(P) \geq 1$. By looking at the natural maps
$$H^{\otimes t} \to \otimes^t f_*\omega_{X/P}^{\otimes k} \to S^t(f_*\omega_{X/P}^{\otimes k}) \to f_*\omega_{X/P}^{\otimes kt},$$
we obtain an injection $H^{\otimes t} \to f_*\omega_{X/P}^{\otimes kt}$ so that we may assume that $k > 1$, $k_0 = k_1 = 1$, $k \geq l_0$ and $\deg H \geq 2g(P)$.

The injection $H \to f_*\omega_{X/P}^{\otimes k}$ corresponds to the equation
$$\omega_{X/P}^{\otimes k} \simeq f^*H + \sum a_i A_i, \qquad a_i > 0, \quad A_i \, distinct\, prime\, divisors, \forall i.$$

If necessary, blow-up $X$, $\beta : \tilde{X} \to X$, and call $\tilde{f} : \tilde{X} \to P$ the natural morphism, so that $\tilde{X}$ is nonsingular and one has
$$\omega_{\tilde{X}/P}^{\otimes k} \simeq \beta^*\omega_{X/P}^{\otimes k} + \sum \mu_j F_j \simeq \tilde{f}^*H + \sum \mu_j F_j + \sum a_i A_i' + \sum c_j F_j,$$
where the divisors $F_j$ are the $\beta$-exceptional divisors, all $\mu_j$ are positive, all $c_j$ are nonnegative, the $A_i'$ are the prime divisors on $\tilde{X}$ which map surjectively onto the $A_i$ and the divisor $\sum F_j + \sum A_i'$ is in normal crossing.

Choose $b$ to be an integer such that $b > a_i$ and $b > c_j, \forall i, j$.

Choose $\pi : P' \to P$ étale with $P'$ integral so that $\pi^* H = H''^{\otimes bk+1}$ for some $H''$ ample on $P'$ with $\deg H' \geq 2g(P')/bk + 1$.

Everything pulls-back via $\pi$. The ampleness of $f_* \omega_{X/P}^{\otimes \kappa}$ is equivalent to the ampleness of $\tilde{f}'_* \omega_{\tilde{X^{P'}}/P'}^{\otimes \kappa} \simeq \pi^* f_* \omega_{X/P}^{\otimes \kappa}$. Therefore, we may assume that $H = (bk+1)H''$ for some $H''$ of degree $\deg H'' \geq 2g(P)/bk + 1$.

Define

$$N := bk, \quad \nu := bk + 1, \quad L := \omega_{\tilde{X}/P} - \tilde{f}^* H'', \quad M := (b-1)\beta^* \omega_{X/P}^{\otimes k} + \tilde{f}^* H''.$$

Since $f_* \omega_{X/P}^{\otimes k}$ is nef by Theorem 4.5, so is $S^{\nu(b-1)}(f_* \omega_{X/P}^{\otimes k})$. Since $\deg H'' \geq 2g(P)/\nu$, standard vanishing techniques show that $S^{\nu(b-1)}(f_* \omega_{X/P}^{\otimes k}) \otimes H''^{\otimes \nu}$ is generated by its global sections. Since $k \geq l_0$ we have that also $\nu M$ is generated by its global sections:

$$\beta^* f^* S^{\nu(b-1)}(f_* \omega_{X/P}^{\otimes k}) \otimes \tilde{f}^* H''^{\otimes \nu} \twoheadrightarrow \nu[(b-1)\beta^* \omega_{X/P}^{\otimes k} + \tilde{f}^* H''] = \nu M.$$

By Bertini Theorem, we can choose a section $B$ of $\nu M$ such that $B + \sum F_j + \sum A'_i$ has support in normal crossings.

We have $\nu N L \simeq B + \sum \nu b \mu_j F_j + \sum \nu a_i A'_i + \sum \nu c_j F_j$.

With the notation of Theorem 4.2, we have that:

$$L^{(k-1)} \otimes \omega_{\tilde{X}/P} = \omega_{\tilde{X}/P}^{\otimes k} - (k-1)\tilde{f}^* H'' - \sum \mu'_j F_j = \beta^* \omega_{X/P}^{\otimes k} - (k-1)\tilde{f}^* H'' + \sum (\mu_j - \mu'_j) F_j,$$

where $0 \leq \mu'_j \leq \mu_j$. By Theorem 4.2, applied to $Z \to \tilde{X}$ and by the projection formula, we see that $\tilde{f}_* L^{(k-1)} \otimes \omega_{\tilde{X}/P} \simeq f_* \omega_{X/P}^{\otimes k} \otimes H''^{\otimes -(k-1)}$ is nef so that $f_* \omega_{X/P}^{\otimes k}$ is ample.

Finally, we have natural maps:

$$\det f_* \omega_{X/P}^{\otimes t} \to \bigotimes^s f_* \omega_{X/P}^{\otimes t} \to S^s(f_* \omega_{X/P}^{\otimes t}) \to f_* \omega_{X/P}^{\otimes s \cdot t},$$

whose composition is injective because pure tensors do not map to zero; see [13], proof of 3.5. □

Under the assumptions of Theorem 3.2, an integer $l_0$ as in Lemma 4.8 exists by Kawamata-Shokurov Base-Point-Free Theorem. The proof of 3) is complete.

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