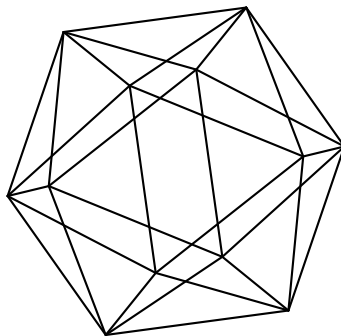# Max-Planck-Institut für Mathematik Bonn

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by

Jonathan Bowden
Diarmuid Crowley
András I. Stipsicz

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Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Mathematisches Institut
Universität Augsburg
Universitätsstr. 14
86159 Augsburg
Germany

Rény Institute of Mathematics
Reáltanoda u. 13-15
1053 Budapest
Hungary

THE TOPOLOGY OF STEIN FILLABLE MANIFOLDS IN HIGH DIMENSIONS I

JONATHAN BOWDEN, DIARMUID CROWLEY, AND ANDRÁS I. STIPSICZ

ABSTRACT. We give a bordism-theoretic characterisation of those closed almost contact $(2q + 1)$-manifolds (with $q \geq 2$) which admit a Stein fillable contact structure. Our method is to apply Eliashberg's h-principle for Stein manifolds in the setting of Kreck's modified surgery. As an application, we show that any simply connected almost contact 7-manifold with torsion free second homotopy group is Stein fillable. We also discuss the Stein fillability of exotic spheres and examine subcritical Stein fillability.

1. INTRODUCTION

There have been several recent breakthroughs concerning the existence of contact structures in higher dimensions. In [Bou, GS, HW] contact structures on certain product manifolds were constructed, and Casals-Pancholi-Presas [CPP] and Etnyre [Etn] have shown that every almost contact 5-manifold is contact. The general existence question on higher dimensional manifolds is, however, still open. (In the following we will assume that all almost contact manifolds are closed — for open manifolds the existence question has been settled by using Gromov's h-principle.)

Motivated by their 3-dimensional analogues, various notions of fillability and overtwistedness of contact structures on higher dimensional manifolds have been extensively studied, cf. [MNW, MNPS]. The class of contact structures satisfying an appropriate h-principle (as overtwisted 3-dimensional contact manifolds do), however, has not yet been identified.

In view of this one is led to consider the existence of contact structures with special properties, the most natural of which is perhaps Stein fillability. Recall that a contact manifold is Stein fillable if it can be realised as the boundary of a *Stein domain*, which is a compact, complex manifold with boundary admitting a strictly plurisubharmonic function for which the boundary is a regular level set. One of the motivating problems which concerns us here is the following:

Problem 1.1 (Stein Realisation Problem). *Determine the almost contact structures which are realised by Stein fillable contact structures.*

By Eliashberg's characterisation of Stein manifolds [E1, CE], the existence of Stein fillings in higher dimensions is reduced to a topological question about whether a given manifold admits a nullbordism containing only handles up to the middle dimension and whose tangent bundle admits an almost complex structure. For a given $2n$-manifold (with $n > 2$), a direct argument can decide whether it admits a Stein structure, but it is more delicate to see whether an odd dimensional manifold can be presented as the boundary of a manifold carrying a Stein structure.

This question can be naturally studied within the framework of (the appropriate) cobordism theory. The main goal of this article is to elucidate the algebro-topological consequences of the above characterisation of Stein fillability by Eliashberg. This approach for constructing contact structures on manifolds was initiated by Geiges [Ge2], and here we pursue the same ideas, using the general setting of Kreck's modified surgery [Kr2].

Let M be a closed oriented manifold of dimension $2q + 1 \geq 5$ and let φ be an almost contact structure on M: φ gives rise to a complex structure on the stable normal bundle of M which we regard as a map $\zeta \colon M \to BU$, where BU denotes the classifying space of the stable unitary group U. The qth Postnikov factorisation of ζ consists of a space B_ζ^{q-1} and maps

$$M \xrightarrow{\bar{\zeta}} B_\zeta^{q-1} \xrightarrow{\eta_\zeta^{q-1}} BU,$$

such that η_ζ^{q-1} is a fibration and $\zeta = \eta_\zeta^{q-1} \circ \bar{\zeta}$. The fibre homotopy class of the fibration η_ζ^{q-1} is a well-defined invariant of the stably complex manifold (M, ζ), called the *complex normal* $(q-1)$-*type of* (M, ζ); see Definition 2.4. In addition, there is a canonical bundle isomorphism $\bar{\zeta}^*(\eta_\zeta^{q-1}) \cong \nu_M$, where ν_M is the stable normal bundle of M (and η_ζ^{q-1} is regarded as a stable oriented vector bundle over B_ζ^{q-1}). It follows that $(M, \bar{\zeta})$ defines a bordism class,

$$[M, \bar{\zeta}] \in \Omega_{2q+1}(B_\zeta^{q-1}; \eta_\zeta^{q-1}),$$

in the bordism theory defined by the complex bundle $(B_\zeta^{q-1}, \eta_\zeta^{q-1})$; see Section 2.1 and Definition 2.3. With these notions in hand we have the following:

Theorem 1.2. *A closed almost contact manifold* (M, φ) *of dimension* $(2q+1) \geq 5$ *admits a Stein filling if and only if* $[M, \bar{\zeta}] = 0 \in \Omega_{2q+1}(B_\zeta^{q-1}; \eta_\zeta^{q-1})$.

(An expanded version of the result is given in Theorem 3.7). The bordism groups appearing in Theorem 1.2 are isomorphic, via the Pontrjagin-Thom isomorphism, to the stable homotopy groups of the Thom spectrum of η_ζ^{q-1}. Hence the entire apparatus of stable homotopy theory is available to compute these groups, and if one can show that $\Omega_{2q+1}(B_\zeta^{q-1}; \eta_\zeta^{q-1}) = 0$, a general existence result follows. We will show that this is the case for simply connected 7-manifolds with torsion free second homotopy groups.

Theorem 1.3. *Let* M *be a closed simply connected 7-manifold with* $\pi_2(M)$ *torsion free. Then* M *admits an almost contact structure, and every almost contact structure on* M *can be represented by a Stein fillable contact structure.*

(Theorem 1.3 can be interpreted as an extension of existence results for contact structures on 1-connected almost contact 5-manifolds and 2-connected 7-manifolds [Ge1, Ge2]).

As expected, the existence of a Stein fillable contact structure on a manifold depends on the smooth structure it carries, and not simply on the underlying homeomorphism type. This fact can be most transparently demonstrated by showing that certain exotic spheres (i.e. smooth manifolds homeomorphic but not diffeomorphic to the sphere of the same dimension) do not carry any Stein fillable contact structures. Using the obstruction class of Theorem 1.2, we prove the following theorem which answers a question raised by Eliashberg, see [E2, 3.8], in roughly three-quarters of all dimensions.

Theorem 1.4. *Let Σ^{2q+1} be a homotopy sphere which admits no framing bounding a parallellizable manifold. If $q \not\equiv 1, 3, 7 \mod 8$ or if $q \equiv 1 \mod 8$ and $q > 9$ or if $q = 7$ or 15, then Σ admits no Stein fillable contact structure.*

(A more precise version of the result is given in Theorem 5.4.)

The obstruction for manifolds to carry Stein fillable contact structures can also be used to establish the following extension of a 3-dimensional result found in [Bow] to higher dimensions. (The construction is based on non-connected examples of exactly fillable manifolds of [MNW] which are not Stein fillable.)

Theorem 1.5. *There exist connected, exactly fillable, contact manifolds that are not Stein fillable in all dimensions greater than three.*

Further results (involving more elaborate homotopy theoretic arguments in determining bordism groups and elements in them) are deferred to a continuation of the present work in [BCS2] — in the present paper we emphasize the basic features of the method and restrict ourselves to the applications listed above.

The paper is organized as follows. In Section 2 we give a review of the formulation of Kreck's surgery theory, with the necessary adaptations to the setting of contact and Stein geometry. In Section 3 we set up notations, identify the obstruction for an almost contact structure to be representable by a Stein fillable contact structure and prove the topological characterization of Stein fillability of Theorem 1.2. In Section 4 we provide the proof of Theorem 1.3. Section 5 concentrates on highly connected manifolds, and (among other results) we prove Theorem 1.4. In Section 6 we discuss further obstructions for Stein fillability, and prove Theorem 1.5. Finally in Section 7, we formulate Theorem 7.1, a version of the Filling Theorem which provides an obstruction for subcritical Stein fillability. We also examine the Stein fillability of the product of a contact manifold with a 2-dimensional surface.

2. Complex modified surgery

In this section we develop the theory of modified surgery from [Kr2] in the setting where all the stable vector bundles under consideration have complex structures. We start with a brief overview of modified surgery Section 2.1. To be consistent with the existing literature, we formulate the set-up using stable normal maps, although in our applications we will need almost complex structures on even dimensional manifolds. In Section 2.2 we discuss the connection between the stable normal setting and the stable tangential setting and formulate the basic concepts and definitions of "complex modified surgery". In Section 2.3 we handle the transition from stable complex structures to almost complex structures and almost contact structures. After proving our main surgery lemmas in Section 2.4 (which lead to the identification of the obstruction class of Theorem 1.2 in Section 3), we discuss some explicit constructions and computations in Section 2.5.

2.1. **The surgery setting: stable normal bundles.** In this subsection we briefly recall the definition of a "(B,μ)-manifold" which is a manifold with extra topological structure on its stable normal bundle. The theory of (B,μ)-manifolds goes back to [La1] and was used systematically in the modified surgery setting of [Kr2]. For a detailed treatment of (B,μ)-manifolds we refer the reader to [Stn, Chapter II] and [Kr2, §2]. While we briefly recall some fundamental definitions for (B,μ)-manifolds, we shall assume that the reader is familiar with these ideas.

Since we shall be working with stably complex manifolds, we start with a fibration

$$\mu\colon B \to BSO$$

where BSO is the classifying space of the stable special orthogonal group SO and B has the homotopy type of a CW complex with a finite number of cells in each dimension. Since BSO classifies oriented stable vector bundles, we regard (B,μ) as an oriented stable vector bundle over B. Given a compact oriented n-manifold X, let

$$\nu\colon X \to BSO$$

denote the stable normal Gauss map of X. The stable Gauss map is a somewhat subtle concept: ν is defined by the classifying map of the normal bundle of an embedding $X \to \mathbb{R}^{n+k}$ for $k \gg n$. Letting k tend to infinity, the space of such embeddings is contractible and hence ν is a well-defined stable vector bundle over X.

A (B,μ)-structure on X is an equivalence class of maps $\bar{\nu}\colon X \to B$ which lift ν over μ; we spell out the equivalence below. In particular, there is a commutative diagram:

$$
\begin{array}{ccc}
 & & B \\
 & \nearrow{\scriptstyle \bar{\nu}} & \downarrow{\scriptstyle \mu} \\
X & \xrightarrow{\ \nu\ } & BSO.
\end{array}
$$

A (B,μ)-structure on $\bar{\nu}\colon X \times [0,1]$ defines (B,μ)-structures on $\bar{\nu}_0$ and $\bar{\nu}_1$ on $X \times \{0\}$ and $X \times \{1\}$ and two (B,μ)-structures on X are called equivalent if they are so related. A *normal (B,μ)-manifold* is a pair $(X, \bar{\nu})$ as above. For later use we record the following

Definition 2.1 (Normal k-smoothing)**.** *A normal k-smoothing in (B,μ) is a normal (B,μ)-manifold $(X, \bar{\nu})$, where $\bar{\nu}\colon X \to B$ is a $(k+1)$-equivalence; i.e. $\bar{\nu}$ induces an isomorphism on homotopy groups π_i for $i \le k$ and a surjection on π_{k+1}.*

Given a (B,μ)-structure $\bar{\nu}_1\colon X_1 \to B$ and a diffeomorphism $f\colon X_0 \cong X_1$, there is a canonical pull-back B-structure $f^*(\bar{\nu}_1)$ on X_0. If $(X_0, \bar{\nu}_0)$ and $(X_1, \bar{\nu}_1)$ are B-manifolds, a *(B,μ)-diffeomorphism*

$$f\colon (X_0, \bar{\nu}_0) \cong (X_1, \bar{\nu}_1)$$

is a diffeomorphism $f\colon X_0 \to X_1$ such that $f^*(\bar{\nu}_1)$ and $\bar{\nu}_0$ define equivalent B-structures on X_0.

We now turn to the relation of (B,μ)-bordism. A (B,μ)-structure $\bar{\nu}$ on X defines a canonical (B,μ)-structure on $X \times [0,1]$ via pull-back, denoted $\pi^*(\bar{\nu})$. If $\pi^*(\bar{\nu})_i := \pi^*(\bar{\nu})|_{X\times\{i\}}$, $i = 0, 1$, denotes the restriction of $\pi^*(\bar{\nu})$ to each end of $X \times [0,1]$, then $\bar{\nu} = \pi^*(\bar{\nu})_0$ and

$$-\bar{\nu} := (\pi^*\bar{\nu})_1 = \pi^*(\bar{\nu}|_{X\times\{1\}})$$

is the (B, μ)-structure defined on the other end of $X \times [0,1]$ via $\pi^*(\bar{\nu})$. More generally, if W is an $(n+1)$-manifold with boundary $\partial W = X_0 \cup X_1$, the disjoint union of two closed n-manifolds, and if $\bar{\nu} \colon W \to B$ is a (B, μ)-structure on W, then $\bar{\nu}$ restricts to give B-structures $\bar{\nu}_0 \colon X_0 \to B$ and $\bar{\nu}_1 \colon X_1 \to B$. In this case $(X_0, \bar{\nu}_0)$ and $(-X_1, -\bar{\nu}_1)$ are (B, μ)-bordant. For example, if $f \colon (X_0, \nu_0) \cong (X_1, \nu_1)$ is a (B, μ)-diffeomorphism between closed manifolds then the s-cobordism

$$(X_0 \times [0,1]) \cup_f (X_1 \times [1,2])$$

admits the structure of a (B, μ)-bordism between $(X_0, \bar{\nu}_0)$ and $(X_1, \bar{\nu}_1)$. In particular, (B, μ)-diffeomorphic closed manifolds are (B, μ)-bordant.

The *n-dimensional (B, μ)-bordism group* is the group of (B, μ)-bordism classes of closed n-dimensional (B, μ)-manifolds with addition given by disjoint union and additive inverse given by $-[M, \bar{\nu}] = [-M, -\bar{\nu}]$:

$$\Omega_n(B; \mu) := \{(X, \bar{\nu}) | \ (X, \bar{\nu}) \text{ is closed } n\text{-dimensional } (B, \mu)\text{-manifold}\}/(B, \mu)\text{-bordism}.$$

2.2. Stable complex structures.
An example of (B, μ)-manifolds of primary interest in this paper is given by

$$(B, \mu) = (BU, F)$$

where $F \colon BU \to BO$ is the canonical forgetful map between classifying spaces. A (BU, F)-manifold is nothing but a stably complex manifold. Notice that an almost complex structure J on a $2q$-manifold X (that is, a reduction of the structure group of the tangent bundle of X from $SO(2q)$ to $U(q)$) naturally induces a stable complex structure on τ_X, the stable tangent bundle of X. As there is a canonical bundle isomorphism,

$$\tau_X \oplus \nu_X \cong \varepsilon,$$

where ν_X is the stable normal bundle of X and ε denotes the trivial stable bundle, a stable complex structure on τ_X induces a stable complex structure on ν_X: choose the unique stable complex structure on ν_X so that the sum with the given stable complex structure on τ_X is the trivial stable complex structure on ε. We shall denote the stable normal complex structure associated to (W, J) by SJ or sometimes ζ_X.

As in the even-dimensional case, an almost contact structure φ on a closed $(2q+1)$-manifold M (that is, the reduction of the structure group from $SO(2q+1)$ to $U(q)$) induces a stable complex structure $S\varphi = \zeta$ on the stable normal bundle of M. (We will also call the stabilized structures *complex* rather than *contact* in the odd-dimensional case.) Since stable tangential complex and stable normal complex structures determine each other, we will focus on the normal picture (although in the applications we will need results for the tangential structures).

Building on the discussion of (B, μ)-manifolds from Section 2.1, we now establish the basic notions in stable complex surgery which we shall use throughout this paper. Let

$$\eta \colon B \to BU$$

be a fibration, where, as before, B has the homotopy type of a CW complex with a finite number of cells in each dimension. We regard (B, η) as a stable complex vector bundle over B with underlying oriented bundle $F \circ \eta \colon B \to BSO$. We shall be interested in the situation

described by the following commutative diagram:

(1)

$$
\begin{array}{ccc}
 & B & \\
{\scriptstyle\bar\zeta}\nearrow & \downarrow{\scriptstyle\eta} & \searrow{\scriptstyle F\circ\eta} \\
X \xrightarrow{\ \zeta\ } & BU \xrightarrow{\ F\ } & BSO.
\end{array}
$$

Here X is an oriented manifold with stable normal bundle $\nu = F \circ \zeta$, (X, ζ) is a compatibly oriented stably complex manifold and $(X, \bar\zeta)$ is a $(B, F\circ\eta)$-manifold. Since F is fixed, we shall call $(X, \bar\zeta)$ a (B, η)-manifold for short: this simply means that $(X, \bar\zeta)$ is a $(B, F\circ\eta)$-manifold.

Definition 2.2 (ζ-compatible (B, η)-manifold)**.** *In the situation of the commutative diagram (1) above, we say that $\bar\zeta\colon X \to B$ is a ζ-compatible (B, η)-manifold; i.e., $(X, \bar\zeta)$ is a $(B, F\circ\eta)$-manifold with underlying stably complex manifold (X, ζ).*

It follows from the definitions that $(-X, -\bar\zeta)$ is a $(-\zeta)$-compatible (B, η)-manifold and that a $(B, F \circ \eta)$-diffeomorphism $f\colon (X_0, \bar\zeta_0) \cong (X_1, \bar\zeta_1)$ is also a stably complex diffeomorphism $f\colon (X_0, \eta\circ\zeta_0) \cong (X_1, \eta\circ\zeta_1)$. Of fundamental importance in this work are the (B, η)-bordism groups.

Definition 2.3 ((B, η)-bordism)**.** *We define*

$$
\Omega_n(B, \eta) := \Omega_n(B, F \circ \eta)
$$

to be the bordism group of $(B, F \circ \eta)$-bordism classes of closed n-dimensional $(B, F \circ \eta)$-manifolds as defined at the end of Section 2.1.

For the purposes of understanding Stein fillings, we shall be interested in the case where $\bar\zeta$ is a k-smoothing for certain k: recall that this means that $\bar\zeta\colon X \to B$ is a $(k+1)$-equivalence. One may ask whether for some stably complex manifold (X, ζ) there are any ζ-compatible normal k-smoothings $\bar\zeta\colon X \to B$ at all. In fact this is always the case because the map $\zeta\colon X \to BU$ can be factorised up to homotopy as a composition

$$
X \xrightarrow{\ \bar\zeta\ } B_\zeta^k \xrightarrow{\ \eta_\zeta^k\ } BU,
$$

where $\bar\zeta$ is a $(k+1)$-equivalence and η_ζ^k is a fibration. The space B_ζ^k and the maps $\bar\zeta$ and η_ζ^k make up the kth Postnikov factorisation of ζ. The existence of the kth Postnikov factoriation is proven in [Ba, Theorem 5.3.1], its defining properties are identified in Definition 2.4 below and we discuss some examples in Section 2.5. In general, for any $k \geq 0$, a map $f\colon X \to Y$ between CW-complexes, has a kth Postnikov factorisation $f \simeq \eta_f^k \circ \bar f$ by maps $\bar f\colon X \to Y_f^k$ and $\eta^k\colon Y_f^k \to Y$. Such factorisations are built by first converting f into a fibration and then working inductively so that there are fibrations $Y_f^k \to Y_f^{k-1}$ with fibre $K(\pi_k(F), k)$ where F is the homotopy fibre of $f\colon X \to Y$.

Definition 2.4 (Complex normal k-type)**.** *Let (X, ζ) be a stably complex manifold. The complex normal k-type of (X, ζ), denoted $(B_\zeta^k, \eta_\zeta^k)$, is defined to be the fibre homotopy type of the fibration η_ζ^k in the following diagram:*

$$
\begin{array}{cc}
 & B_\zeta^k \\
{\scriptstyle\bar\zeta}\nearrow & \downarrow{\scriptstyle\eta_\zeta^k} \\
X \xrightarrow{\ \zeta\ } & BU.
\end{array}
$$

The fibration η_ζ^k is uniquely defined up to fibre homotopy type by the following properties:

   (1) *the map $\bar\zeta$ is a $(k+1)$-equivalence,*
   (2) *the map η_ζ^k is a $(k+1)$-coequivalence, i.e. $(\eta_\zeta^k)_* \colon \pi_j(B_\zeta^k) \to \pi_j(BU)$ is injective when $j = k+1$ and an isomorphism if $j > k+1$.*

We conclude this subsection by considering the role of the choice of normal $(q-1)$-smoothing $\bar\zeta \colon X \to B$ on the bordism class $[X, \bar\zeta] \in \Omega_{2q+1}(B, \eta)$. Our method is to adapt the key point of the proof of [Kr1, Proposition 7.4] to the complex setting. Given a stable complex vector bundle $\eta \colon B \to BU$, let $\mathrm{Aut}(B, \eta)$ be the group of fibre homotopy classes of fibre self-homotopy equivalences of η. That is, $\mathrm{Aut}(B, \eta)$ consists of fibre homotopy equivalence classes of maps $\alpha \colon B \simeq B$ which make the following diagram commute:

$$
\begin{array}{ccc}
B & \xrightarrow{\;\alpha\;} & B \\
\downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} \\
BU & \xrightarrow{\;\mathrm{Id}\;} & BU.
\end{array}
$$

The group $\mathrm{Aut}(B, \eta)$ acts on the set of (B, η)-diffeomorphism classes of complex normal k-smoothings in (B, η) by mapping a complex normal k-smoothing $\bar\zeta \colon X \to B$ to the complex normal k-smoothing $\alpha \circ \bar\zeta \colon X \to B$.

Lemma 2.5 (cf. [Kr1, Proposition 7.4]). *Suppose that (X_0, ζ_0) and (X_1, ζ_1) are stably complex manifolds and that for $i = 0, 1$, $\bar\zeta_i \colon X_i \to B_{\zeta_0}^k$ is a ζ_i-compatible normal k-smoothing in $(B_{\zeta_0}^k, \eta_{\zeta_0}^k)$, the complex normal k-type of (X_0, ζ_0). If $f \colon (X_0, \zeta_0) \cong (X_1, \zeta_1)$ is a stably complex diffeomorphism, then there is a fibre homotopy self-equivalence $\alpha \in \mathrm{Aut}(B_{\zeta_0}^k, \eta_{\zeta_0}^k)$ such that f is a $(B_{\zeta_0}^k, \eta_{\zeta_0}^k)$-diffeomorphism from $(X_0, \alpha \circ \bar\zeta_0)$ to $(X_1, \bar\zeta_1)$.*

Proof. The maps

$$
\bar\zeta_0, \; f^*(\bar\zeta_1) \colon X_0 \to B_{\zeta_0}^k
$$

determine two complex normal k-smoothings on X_0. Now the universal properties of Postnikov stages of maps [Ba, Corollary 5.3.8] ensure that there is a fibre homotopy equivalence $\alpha \colon (B_{\zeta_0}^k, \eta_{\zeta_0}^k) \simeq (B_{\zeta_0}^k, \eta_{\zeta_0}^k)$ such that $\alpha \circ \bar\zeta_0$ and $f^*(\bar\zeta_1)$ are equivalent $(B_{\zeta_0}^k, \eta_{\zeta_0}^k)$-structures on X_0. By definition, this means that f is a $(B_{\zeta_0}^k, \eta_{\zeta_0}^k)$-diffeomorphism from $(X_0, \alpha \circ \bar\zeta_0)$ to $(X_1, \bar\zeta_1)$. $\qquad\square$

The following corollary is an important consequence of Lemma 2.5 which arises from the fact that the induced action of $\mathrm{Aut}(B_\zeta^k, \eta_\zeta^k)$ on $\Omega_n(B_\zeta^k; \eta_\zeta^k)$ is by group automorphisms.

Corollary 2.6. *Let $(B_\zeta^k, \eta_\zeta^k)$ be the normal k-type of (X, ζ). If $\bar\zeta \colon X \to B_\zeta^k$ is a closed ζ-compatible normal k-smoothing such that $[X, \bar\zeta] = 0 \in \Omega_n(B_\zeta^k; \eta_\zeta^k)$, then for all ζ-compatible normal k-smoothings $\hat\zeta \colon X \to B_\zeta^k$ we have that $[X, \hat\zeta] = 0 \in \Omega_n(B_\zeta^k; \eta_\zeta^k)$.*

Proof. Applying Lemma 2.5 to $(X, \bar\zeta)$ and $(X, \hat\zeta)$ we deduce that there is a fibre homotopy equivalence $\alpha \colon (B_\zeta^k, \eta_\zeta^k) \simeq (B_\zeta^k, \eta_\zeta^k)$ such that $(X, \hat\zeta)$ is $(B_\zeta^k, \eta_\zeta^k)$-diffeomorphic to $(X, \alpha \circ \bar\zeta)$. Hence $[X, \hat\zeta] = \alpha_*([X, \bar\zeta]) = 0$. $\qquad\square$

2.3. **Stable and unstable surgery.** Propagating contact structures over Weinstein handles requires information about almost contact structures of M. On the other hand, computations in (B, η)-bordism require the use of the stable normal bundle of M. In this subsection we prove two lemmas which allow us to move between these two settings.

Recall that $h_{k+1} := D^{k+1} \times D^{n-k}$ is an $(n+1)$-dimensional $(k+1)$-handle. Let (M, φ) be a closed $(2q+1)$-dimensional almost contact manifold and set $n := 2q+1$. For an *almost complex k-surgery* on (M, φ) we require the following data:

(1) An embedding $\phi \colon S^k \times D^{n-k} \to M$;
(2) An almost complex structure J on

$$W_\phi := (M \times I) \cup_\phi h_{k+1},$$

extending the natural almost complex structure $\varphi \times I$ on $M \times I \subset W_\phi$ induced by φ.

The result of this surgery is the other boundary component of W, denoted M_ϕ. It is an almost contact manifold with almost contact structure $\varphi_\phi := J|_{M_\phi}$.

Definition 2.7 (Almost complex surgery). *In the situation above, we shall say that the almost contact manifold (M_ϕ, φ_ϕ) is obtained from (M, φ) by a k-dimensional almost complex surgery.*

When we work with stably complex manifolds, we have the analogous situation, where almost contact structures and almost complex structures on the tangent bundle are replaced first by stable complex structures on the tangent bundle and then by stable complex structures on the normal bundle. Thus to perform stable complex k-surgery on a stably complex n-manifold (M, ζ), we require an embedding $\phi \colon S^k \times D^{n-k} \to M$ along with an extension of the stable complex structure ζ to a stable complex structure on the trace of the surgery $W_\phi = (M \times I) \cup_\phi h_{k+1}$. In this case we shall say that (M_ϕ, ζ_ϕ) is obtained from (M, ζ) via stable complex surgery.

Given an almost contact manifold (M, φ), let (M, ζ) denote the stably complex manifold defined by φ.

Lemma 2.8. *Let (M, φ) be a $(2q+1)$-dimensional almost contact manifold and $(W, \zeta_W; M, M_\phi)$ the trace of a stable complex k-surgery on (M, ζ) with $k \leq 2q$. Then there is an almost complex structure J on W with $SJ = \zeta_W$ and which restricts to $\varphi \times I$ on $M \times I$.*

(Notice that the above lemma is stated for $k \leq 2q$ — in our applications, however, we will use the statement only in the range $k \leq q+1$.)

Proof. First we notice that the stable normal complex structure on W can be converted to a stable tangential complex structure on W which stabilies $\varphi \times I$ when restricted to $M \times I$. Our problem is to reduce structure group of the tangent bundle of W to $U(q+1)$, and we must do this relative to the chosen reduction corresponding to $\varphi \times I$ on $M \times I$. We encounter the unstable lifting problem which maps to the stable lifting problem:

$$
\begin{array}{ccccccc}
M \times I & \xrightarrow{\varphi \times I} & BU(q) & \longrightarrow & BU(q+1) & \longrightarrow & BU \\
\downarrow & \nearrow & & & \downarrow & & \downarrow \\
W & \xrightarrow{\quad \tau_W \quad} & & BSO(2q+2) & \longrightarrow & BSO,
\end{array}
$$

where τ_W classifies the tangent bundle of W. The lifting obstructions for these problems lie in the groups

$$H^{k+1}(W, M; \pi_k(SO(2q+2)/U(q+1))) \quad \text{and} \quad H^{k+1}(W, M; \pi_k(SO/U)),$$

and the unstable lifting obstruction maps to the stable lifting obstruction under the coefficient homomorphism $S_*\colon \pi_k(SO(2q+2)/U(q+1)) \to \pi_k(SO/U)$. But the map S_* is an isomorphism for $k \leq 2q$ by [Gr, p. 432]. Hence the vanishing of the stable obstruction ensures the vanishing of the unstable obstruction. It follows therefore that there is an almost complex structure J on W compatible with $(M \times I, \varphi \times I)$. $\square$

Note that in the setting of Lemma 2.8 there may be several homotopy classes of almost contact structures φ' on M_ϕ such that $S\varphi'$ is homotopic to ζ_ϕ. The almost complex structure J will induce one such structure $J|_{M_\phi}$ on M_ϕ. To obtain an almost complex bordism $(W, J; (M, \varphi), (M_\phi, \varphi'))$ for a specific almost contact structure φ' we may need to find an alternative bordism.

Lemma 2.9. *Suppose that the two almost contact structures φ and φ' on a manifold M^{2q+1} are stably equivalent. Then there is an almost contact structure φ_d on the sphere S^{2q+1} with the property that $(M, \varphi)\#(S^{2q+1}, \varphi_d)$ and (M, φ') are equivalent as almost contact manifolds, and furthermore (S^{2q+1}, φ_d) bounds an almost complex $(2q+2)$-manifold with a handle decomposition of handles of indices $\leq q + 1$.*

Proof. The homotopy classes of almost contact structures in a given stable class can all be obtained from a given almost contact representative by connected sum with almost contact structures on spheres. Homotopy classes of almost contact structures on spheres are in turn parametrised by elements in the group $\pi_{2q+1}(SO(2q + 1)/U(q))$. Furthermore, as noted in [Ge2, p. 1201], results of Sato that build on work of Morita show that for q even one can realise all actual homotopy classes of almost contact structures via contact structures on various standard Brieskorn spheres, which are in particular Stein fillable. The Stein fillings then provide the required almost complex $(2q + 2)$-manifolds.

A similar observation holds when q is odd, by utilising the calculations of [DG]. One simply takes the product of even dimensional spheres $W = S^{q+1} \times S^{q+1}$. The stable tangent bundle of W is trivial, so we choose a stable trivialisation. After removing a ball we obtain a stably complex filling of S^{2q+1} which is built of two $(q + 1)$-handles and a zero-handle. Thus the stable almost complex structure determines a unique complex structure $J^\bullet$ on $W^\bullet = W \setminus \text{int}(D^{2q+1})$. Using the notation of [DG] we have $\mathfrak{o}(W^\bullet, J^\bullet) = 2$ and the formulae of [DG, p. 3831] allow one to realise all possible *unstable* almost contact structures in any equivalence class of *stable* almost contact structures via connect sums with $\partial W^\bullet$. Applying boundary connect sums with $W^\bullet$ allows us to move through all the homotopy classes of contact structures on S^{2q+1} which are stably trivial and this proves the lemma. $\square$

2.4. **The surgery lemmas.** In this subsection $\eta\colon B \to BU$ is again a stable complex vector bundle. The following two lemmas are consequences of a theorem of Wall [Wa3, Theorem 3].

Lemma 2.10 (Filling lemma). *Let (M, φ) be an almost contact $(2q+1)$-manifold with induced stable complex structure ζ and complex normal $(q - 1)$-type $(B_\zeta^{q-1}, \eta_\zeta^{q-1})$. If $q \geq 2$, then the following are equivalent:*

(1) *(M, φ) is the boundary of a compact almost complex $(2q + 2)$-manifold (W, J) with handles only of index $q + 1$ and smaller.*

(2) *For any ζ-compatible normal $(q-1)$-smoothing $\bar{\zeta} \colon M \to B_\zeta^{q-1}$, we have*

$$[M, \bar{\zeta}] = 0 \in \Omega_{2q+1}(B_\zeta^{q-1}; \eta_\zeta^{q-1}).$$

(3) *For some stable complex bundle (B, η) and some ζ-compatible normal $(q-1)$-smoothing $\bar{\zeta} \colon M \to B$, we have*

$$[M, \bar{\zeta}] = 0 \in \Omega_{2q+1}(B; \eta).$$

Proof. (1) $\Rightarrow$ (2): Suppose that (W, J) is as in the statement of the lemma. The almost complex structure J defines a stable complex structure $\zeta_W \colon W \to BU$. Let $(B_{\zeta_W}^{q-1}, \eta_{\zeta_W}^{q-1})$ be the complex normal $(q-1)$-type of (W, ζ_W) and let $\bar{\zeta}_W \colon W \to B_{\zeta_W}^{q-1}$ be a $(q-1)$-smoothing in $B_{\zeta_W}^{q-1}$. Let $i \colon M \to W$ be the inclusion. Then the map

$$\psi := \bar{\zeta}_W \circ i \colon M \to B_{\zeta_W}^{q-1}$$

defines a $(B_{\zeta_W}^{q-1}, \eta_{\zeta_W}^{q-1})$-structure on M which is compatible with $\zeta = S\varphi$ since $J|_{\partial W} = \varphi$. Since the smooth manifold W admits a handle decomposition with handles only of index $(q+1)$ or less, by turning such a decomposition upside down we obtain that W has a handle decomposition starting from M and adding handles of dimension $(q+1)$ and higher. It follows that $i \colon M \to W$ is a q-equivalence and hence the map $\xi \colon M \to B_{\zeta_W}^{q-1}$ is a q-equivalence. Since $(B_{\zeta_W}^{q-1}, \eta_{\zeta_W}^{q-1})$ is the complex normal $(q-1)$-type of W, the map $\eta_{\zeta_W}^{q-1} \colon B_{\zeta_W}^{q-1} \to BU$ is a q-coequivalence. It follows that $(B_{\zeta_W}^{q-1}, \eta_{\zeta_W}^{q-1})$ is a model for the complex normal $(q-1)$-type of (M, ζ) and so we identify $(B_\zeta^{q-1}, \eta_\zeta^{q-1}) = (B_{\zeta_W}^{q-1}, \eta_{\zeta_W}^{q-1})$. By construction $\psi \colon M \to B_\zeta^{q-1}$ is a complex normal $(q-1)$-smoothing and $(W, \bar{\zeta}_W)$ is a $(B_\zeta^{q-1}, \eta_\zeta^{q-1})$-null bordism of (M, ψ). It follows that $[M, \psi] = 0 \in \Omega_{2q+1}(B_\zeta^{q-1}; \eta_\zeta^{q-1})$. Now by Lemma 2.5, $[M, \bar{\zeta}] = 0 \in \Omega_{2q+1}(B_\zeta^{q-1}, \eta_\zeta^{q-1})$ for any complex normal $(q-1)$-smoothing $\bar{\zeta} \colon M \to B_\zeta^{q-1}$.

(2) $\Rightarrow$ (3): Take $(B, \eta) = (B_\zeta^{q-1}, \eta_\zeta^{q-1})$.

(3) $\Rightarrow$ (1): Let $(W, \bar{\zeta}_W)$ be a B-nullbordism of $(M, \bar{\zeta})$. Using surgery below the middle dimension as in [Kr2, Proposition 4], we may assume that $\bar{\zeta}_W \colon W \to B$ is a $(q+1)$-equivalence and in particular there are isomorphisms of fundmental groups $\pi = \pi_1(M) \cong \pi_1(B) \cong \pi_1(W)$. If $i \colon M \to W$ denotes, the inclusion, the commutative diagram

$$\begin{array}{ccc} \pi_i(M) & \xrightarrow{\ i_* \ } & \pi_i(W) \\ & \searrow{\scriptstyle \bar{\zeta}_*} \quad \swarrow{\scriptstyle (\bar{\zeta}_W)_*} & \\ & \pi_i(B), & \end{array}$$

and the facts that $\bar{\zeta} \colon M \to B$ is a q-equivalence and $\bar{\zeta}_W \colon W \to B$ is a $(q+1)$-equivalence show that the inclusion $i \colon M \to W$ is a q-equivalence. By a theorem of Wall [Wa3, Theorem 3], it follows that W is diffeomorphic to a manifold obtained from M by attaching handles in dimension $(q+1)$ and higher.

Turning the above handle decomposition upside down, we see that W has a handle decomposition consisting of k-handles with $k \le q+1$. Moreover, the (B, η)-strucure on W defines a stable complex structure ζ_W on W. Applying Lemma 2.8 to the handlebody decomposition of W we deduce that W admits an almost complex structure J such that $SJ = \zeta_W$. The almost complex structure J induces some almost contact structure $J|_{\partial W}$ on M such that $SJ|_{\partial W} = S\varphi$. It follows that $\varphi = J|_{\partial W} + \varphi_0$ where $\varphi_0 \in \pi_{2q+1}(SO(2q+1)/U(q))$ is

a stably trivial almost contact structure on S^{2q+1}. By Lemma 2.9 the almost contact manifold (S^{2q+1}, φ_0) admits a Stein filling (W_0, σ_0) and in particular an almost complex filling (W_0, J_0). It follows that the boundary connected sum $(W \natural W_0; J \natural J_0)$ is an almost complex filling of $J|_{\partial W} + \varphi_0 = \varphi$. $\qquad \square$

The above result admits a 'relative' version, where now we examine bordisms between two smoothings:

Lemma 2.11 (Stable surgery Lemma). *Let $(W, \bar{\zeta}_W; M_0, M_1)$ be a (B, η)-bordism between normal $(q-1)$-smoothings $(M_0, \bar{\zeta}_0)$ and $(M_1, \bar{\zeta}_1)$ of dimension $2q+1 \geq 5$. Then for $j = 0, 1$ the bordism W admits a handlebody decomposition relative to M_j consisting of handles of index $k \leq q+1$.*

Proof. Let $i_j \colon M_j \to W$, $j = 0, 1$ denote the inclusion maps. Using surgery below the middle dimension as in [Kr2, Proposition 4], we may assume that $\bar{\zeta}_W \colon W \to B$ is a $(q+1)$-equivalence. Now consider the following commutative diagram

$$\pi_i(M_0) \xrightarrow{(i_0)_*} \pi_i(W) \xleftarrow{(i_1)_*} \pi_i(M_1)$$
$$\begin{array}{ccc} & (\bar{\zeta}_0)_* \searrow & \downarrow (\bar{\zeta}_W)_* & \swarrow (\bar{\zeta}_1)_* \\ & & \pi_i(B). \end{array}$$

Since the maps $\bar{\zeta}_i \colon M_i \to B$ are q-equivalences and $\bar{\zeta}_W \colon W \to B$ is a $(q+1)$-equivalence, it follows that each inclusion $i_j \colon M_j \to W$ is a q-equivalence. By [Wa3, Theorem 3], W admits a handlebody decomposition relative to M_{j+1} consisting of handles of index $k' \geq q+1$. If we turn this handbody decomposition upside down we obtain a handlebody decomposition of W relative to M_j consisting of handles of index $k \leq q+1$. $\qquad \square$

We next give the unstable version of the previous lemma:

Lemma 2.12 (Unstable surgery Lemma). *Let (M_0, φ_0) and (M_1, φ_1) be almost contact manifolds of dimension $2q+1 \geq 5$ with associated stable complex structures ζ_0 and ζ_1. Suppose for $i = 0, 1$, that $\bar{\zeta}_i \colon M_i \to B$, are ζ_i-compatible normal $(q-1)$-smoothings in a stable complex bundle (B, η) which are (B, η)-bordant. Then there is an almost complex bordism $(W, J; (M_0, \varphi_0), (M_1, \varphi_1))$ between (M_0, φ_0) and (M_1, φ_1) such that for $j = 0, 1$ the manifold W admits a handlebody decomposition relative to M_j consisting of handles of index $k \leq q+1$.*

Proof. Let us give the proof for $j = 0$, the proof for $j = 1$ is similar. By Lemmas 2.8 and 2.11 there is an almost complex bordism $(W, J; (M_0, \varphi_0), (M_1, \varphi_1))$ where W is obtained from M_0 by attaching handles of index $(q+1)$ or less and where the almost contact structure φ_1' satisfies $S\varphi_1' = S\varphi_1$. It follows that $\varphi_1 = \varphi_1' + \varphi_0$ where $\varphi_0 \in \pi_{2q+1}(SO(2q+1)/U(q))$ is a stably trivial almost contact structure on S^{2q+1}. By Lemma 2.9 the almost contact manifold (S^{2q+1}, φ_0) admits an almost complex filling (W_0, J_0) with handles of indices $\leq q+1$. Taking the boundary connected sum of W and W_0 at the M_1 boundary component of W we obtain an almost complex bordism $(W \natural W_0, J \natural J_0; (M_0, \varphi_0), (M_1, \varphi_1))$ where $W \natural W_1$ has a handlebody decomposition relative to M_0 consisting of handles of index $(q+1)$ or less. $\qquad \square$

2.5. **Complex normal k-types.** In this subsection, we identify the complex normal k-type $(B_\zeta^k, \eta_\zeta^k)$, of a general stably complex manifold (X, ζ) under certain assumptions for $k = 1$ and $k = 2$. These computations will play crucial roles in our applications (cf. Section 4). We

shall use the following notation. Since we do not distinguish between stable complex bundles and their classifying maps, we shall write $f \oplus g \colon X \times Y \to BU$ for the exterior Whitney sum of stable complex bundles classified by maps $f \colon X \to BU$ and $g \colon Y \to BU$. Also, we let $\pi_{SU} \colon BSU \to BU$ be the map of classifying spaces induced by the inclusion $SU \subset U$.

Lemma 2.13. *Let (X, ζ) be a stably complex manifold with $\pi = \pi_1(X)$.*

(1) *If $\zeta_* \colon \pi_2(X) \to \pi_2(BU)$ is onto then*

$$(B_\zeta^1, \eta_\zeta^1) = \big(K(\pi, 1) \times BU, \mathrm{pr}_{BU} \big).$$

(2) *If $c_1(\zeta) = 0 \in H^2(X)$ then*

$$(B_\zeta^1, \eta_\zeta^1) = \big(K(\pi, 1) \times BSU, \pi_{SU} \circ \mathrm{pr}_{BU} \big).$$

Proof. Both pr_{BU} and $\pi_{SU} \circ \mathrm{pr}_{BU}$ are 2-coequivalences. Thus, from the defining properties of the second Postnikov approximation of $\zeta \colon X \to BU$, it suffices to find maps $\bar{\zeta} \colon X \to B_\zeta^1$ which are 2-equivalences and which factor ζ over η_ζ^1.

(1) Let $u \colon X \to K(\pi, 1)$ classify the universal covering of X and define $\bar{\zeta}$ by

$$\bar{\zeta} := (u \times \zeta) \colon X \to K(\pi, 1) \times BU.$$

The assumption that ζ_* in onto on π_2 ensures that ζ is a 2-equivalence and clearly $\mathrm{pr}_{BU} \circ \bar{\zeta} = \zeta$.

(2) Since $c_1(\zeta) = 0$, there is a lift of ζ to $\zeta' \colon X \to BSU$. Define $\bar{\zeta}$ by

$$\bar{\zeta} := (u \times \zeta') \colon X \to K(\pi, 1) \times BSU.$$

Since $\pi_2(BSU) = 0$, ζ is a 2-equivalence and clearly $\pi_{SU} \circ \mathrm{pr}_{BU} \circ \bar{\zeta} = \zeta$. □

Now we consider the complex normal 2-type of (X, ζ). Let $p_2 \colon X \to P_2(X)$ denote a 3-equivalence from X to its second Postnikov stage, $P_2(X)$.

Lemma 2.14. *Let (X, ζ) be a stably complex manifold and let γ_ζ by the unique complex line bundle over $P_2(X)$ such that $c_1(p_2^*(\gamma_\zeta)) = -c_1(\zeta)$. Then*

$$(B_\zeta^2, \eta_\zeta^2) = \big(P_2(X) \times BSU, \gamma_\zeta \oplus \pi_{SU} \big).$$

Proof. By definition, the map on second cohomology induced by p_2 is an isomorphism: $p_2^* \colon H^2(P_2(X)) \cong H^2(X)$. Hence there is a (unique isomorphism class of) line bundle γ_ζ over $P_2(X)$ such that $p_2^*(\gamma_X) = -c_1(\zeta)$. The stable complex bundle $\xi := \zeta \oplus p_2^*(\gamma_\zeta)$ satisfies

$$c_1(\xi) = c_1(\zeta) - c_1(\zeta) = 0 \in H^2(X),$$

and so ξ admits an SU-structure classified by a map $\xi' \colon X \to BSU$. We define $\bar{\zeta}$ by

$$\bar{\zeta} := (p_2 \times \xi') \colon X \to P_2(X) \times BSU.$$

Since BSU is 3-connected and $\pi_3(P_2(X)) = 0$, $\bar{\zeta}$ is a 3-equivalence. By construction we have $(\gamma_\zeta \oplus \pi_{SU}) \circ \bar{\zeta} = \zeta$ and clearly $\gamma_\zeta \oplus \pi_{SU}$ is a 3-coequivalence. It follows that $(B_\zeta^2, \eta_\zeta^2)$ is the complex normal 2-type of (X, ζ). □

3. Contact structures and complex normal bordism

After recalling the necessary definitions and the statement of Eliashberg's h-principle, we state our main surgery theorems, Theorems 3.7 and 3.8. The proofs of these theorems rest on the discussion presented in Section 2.4.

3.1. **Symplectic fillability and contact surgery.** Recall that a symplectic manifold (W, ω) is a $(2q + 2)$-dimensional manifold W with a closed 2-form ω such that $\omega^{q+1} \neq 0$ at every point in W. In particular, a symplectic manifold carries a canonical orientation. Recall, furthermore, that a cooriented, codimension-1 distribution ξ on a $(2q + 1)$-manifold M is a *contact structure* if there is a 1-form α such that $\ker(\alpha) = \xi$ and

$$\alpha \wedge (d\alpha)^q \neq 0.$$

Note that this then also determines an orientation of M. Two contact manifolds (M_0, ξ_0) and (M_1, ξ_1) are *contactomorphic* if there is a diffeomorphism $\phi : M_0 \to M_1$ such that

$$\phi_*(\xi_0) = \xi_1.$$

We now recall the various notions of fillability for contact structures.

Definition 3.1 (Strongly symplectically fillable and exactly fillable)**.** *A contact manifold* (M, ξ) *is called strongly symplectically fillable if it bounds a compact symplectic manifold* (W, ω) *and there is an outward pointing vector field V near ∂X such that the Lie derivative satisfies $L_V\omega = \omega$, and $\lambda = \iota_V\omega$ is a defining 1-form for ξ. If the symplectic form ω is exact then we say that (M, ξ) is exactly fillable.*

A further specialisation of the notion of fillability is that of Stein fillability. Recall that a Stein domain is a compact, complex manifold (W, J) with boundary that admits a function $\phi : W \to [0, 1]$ so that $\omega = -dd^{\mathbb{C}}\phi$ is a symplectic form and $\phi^{-1}(1) = \partial W$ is a regular level.

Definition 3.2 (Stein fillable)**.** *A contact manifold* (M, ξ) *is called Stein fillable if it bounds a Stein domain* (W, J) *such that* $\xi = J(TM) \cap TM$.

These notions of fillability fit into the following sequence of inclusions of contactomorphism classes of contact manifolds:

(2)   {Stein fillable} $\subseteq$ {exactly fillable} $\subseteq$ {strongly fillable}.

Surgery on an isotropic sphere S^k in a contact manifold (M^{2q+1}, ξ) can be performed in a way that is compatible with the contact structure. If $k \leq q$ and $2q + 1 \geq 5$ then any embedded sphere can be realised by an isotropic sphere and such surgeries can be realised by the attachment of a symplectic or "Weinstein" $(k + 1)$-handle $h_{k+1} := D^{k+1} \times D^{2q+1-k}$, provided that the associated almost complex structure on the product manifold $M^{2q+1} \times [0, 1]$ extends over the trace $(M^{2q+1} \times [0, 1]) \cup h_{k+1}$ of the surgery (cf. [E1, CE]) . Furthermore, the symplectic nature of the handle attachment shows that the symplectic fillability of a contact structure is preserved under such contact surgeries. In addition, Eliashberg showed that when attaching a Weinstein handle to a Stein manifold, the Stein structure also extends. (For more details concerning contact surgery and Weinstein handles we refer to [CE] or [Ge4] or [We].)

Theorem 3.3. *Let* (M^{2q+1}, ξ) *be a contact manifold of dimension $2q+1 \geq 5$ with associated almost contact structure φ. Suppose that $k \leq q$ and that (M', φ') is obtatined from (M, φ) via a k-dimensional almost complex surgery with trace $(M \times I) \cup h_{k+1}$ as in Defintion 2.7. Then M' admits a contact structure ξ'. If is (M^{2q+1}, ξ) symplectically or exactly fillable, then so is (M', ξ'). Moreover, if (W, J) is a Stein filling of (M^{2q+1}, ξ) then there is J' on $W \cup h_{k+1}$ such that it is a Stein filling of (M', ξ').* $\qquad\square$

Remark 3.4. Although [Ge4, Theorem 6.3.1] is not stated explicitly for exact fillability, the proof also holds in the case of exact fillability, since attaching Weinstein handles does not affect the exactness of the symplectic form on the filling.

Applying Theorem 3.3 inductively over a handle decomposition, one obtains the following (cf. [CE], Theorem 8.15):

Corollary 3.5 (Eliashberg's *h*-principle)**.** *Let (W, J) be a compact $(2q + 2)$-dimensional almost complex manifold with handles only in dimensions $q+1$ or less. Then J is homotopic to an almost complex structure $\tilde{J}$ so that $(W, \tilde{J})$ is a Stein filling of $M = \partial W$ and in particular, M is Stein fillable.* □

3.2. **Surgery theorems.** In this subsection we state our main theorems concerning Stein fillings and contact surgery. The results will be mainly translations of the surgery theoretic results from Section 2.4. We begin with the result corresponding to Lemma 2.9.

Lemma 3.6. *Suppose that the almost contact manifold (M, φ) can be realised a contact structure ξ. Then every homotopy class of almost contact structure which is stably equivalent to φ admits a contact structure obtained from ξ by connected sum with a Stein fillable contact structure on S^{2q+1}.*

Proof. The proof is a simple combination of the proof of Lemma 2.9 and Corollary 3.5: the almost contact structures found on S^{2q+1} in the proof of Lemma 2.9 are Stein fillable contact structures, and the boundary connect sum of two Stein fillings is a Stein filling. □

Using the notation and terminology of Section 2, we obtain the following bordism characterisation of Stein fillability, proving (an expanded version of) Theorem 1.2:

Theorem 3.7 (Filling Theorem)**.** *Let (M, φ) be a closed almost contact $(2q + 1)$-manifold with induced stable complex structure ζ and complex normal $(q-1)$-type $(B_\zeta^{q-1}, \eta_\zeta^{q-1})$. If $q \geq 2$, then the following are equivalent:*

(1) *(M, φ) admits a Stein-fillable contact structure.*
(2) *For any ζ-compatible normal $(q-1)$-smoothing $\bar{\zeta} \colon M \to B_\zeta^{q-1}$, we have*
$$[M, \bar{\zeta}] = 0 \in \Omega_{2q+1}(B_\zeta^{q-1}; \eta_\zeta^{q-1}).$$
(3) *For some stable complex bundle (B, η) and some ζ-compatible normal $(q-1)$-smoothing $\bar{\zeta} \colon M \to B$, we have*
$$[M, \bar{\zeta}] = 0 \in \Omega_{2q+1}(B; \eta).$$

Proof. For stable almost contact structures this is just a combination of Lemma 2.10 and Eliashberg's *h*-principle (cf. Corollary 3.5). Lemma 3.6 then implies that any almost contact structure in a given stable class can be realised as a Stein fillable contact structure, as soon as one can. □

Similar arguments provide

Theorem 3.8 (Surgery Theorem)**.** *Let (M_0, φ_0) and (M_1, φ_1) be almost contact manifolds of dimension $2q+1 \geq 5$ with associated stable complex structures ζ_0 and ζ_1. Suppose for $i = 0, 1$, that $\bar{\zeta}_i \colon M_i \to B$ are ζ_i-compatible normal $(q - 1)$-smoothings in a stable complex bundle (B, η) such that*
$$[M_0, \bar{\zeta}_0] = [M_1, \bar{\zeta}_1] \in \Omega_{2q+1}(B; \eta).$$

Then (M_0, φ_0) admits a contact structure if and only if (M_1, φ_1) does. Moreover, (M_0, φ_0) admits a fillable contact structure in any sense (cf. display (2) above) if and only if (M_1, φ_1) does.

Proof. By Lemma 2.11, there is a (B, η)-bordism $(W, \bar{\zeta}_W)$ between $(M_0, \bar{\zeta}_0)$ and $(M_1, \bar{\zeta}_1)$ such that $(W, \bar{\zeta}_W)$ is obtained from $(M_i, \bar{\zeta}_i) \times [0, 1]$, $i = 0, 1$, by attaching k-handles, $k \leq q+1$, over which the almost complex structure extends. The result now follows from Theorem 3.3 above. This then gives contact structures in the desired stable class of almost contact structures. However, by Lemma 3.6 one can then realise all almost contact structures via connected sum with certain contact structures on spheres. As all these contact structures are Stein fillable, this does not affect the fillability of the contact structures. $\square$

Remark 3.9. The idea of constructing contact structures via surgery techniques is not new, and Geiges and Thomas, in particular, have employed such methods to prove the existence of contact structures under various topological assumptions. Indeed, using the explicit description of normal 1-types given in Lemma 2.13, one can deduce the Bordism Theorem of [Ge3] as a special case of Theorem 3.8. The main benefit of Theorem 3.8 is that it provides a unified approach to this point of view without making any assumptions on the almost contact structures involved.

In the following sections we will use the Filling Theorem above to produce Stein fillable contact structures and obstructions to Stein fillability. The Surgery Theorem, on the other hand, is useful for finding contact structures on manifolds which cannot carry Stein fillable structures as we now explain. Let β denote a class of contact structures which is closed under Weinstein handle attachment and which includes Stein fillable contact structures; for example β could be the class of symplectically fillable contact structures. We define

$$\Omega_{2q+1}^{\beta}(B; \eta) \subset \Omega_{2q+1}(B, \eta)$$

to be the set of bordism classes with representatives $\bar{\zeta} \colon N \to B$ such that $\bar{\zeta}$ is ζ-compatible and such that (N, ζ) admits a contact structure ξ in the class β. We emphasise that here we make no connectivity assumption on the map $\bar{\zeta} \colon N \to B$.

Corollary 3.10. *Let (M, φ) be an almost contact $(2q + 1)$-manifold with associated stable complex structure ζ and let (B, η) be a stable complex bundle. If $q \geq 2$, the map $\bar{\zeta} \colon M \to B$ is a ζ-compatible normal $(q - 1)$-smoothing and*

$$[M, \bar{\zeta}] \in \Omega_{2q+1}^{\beta}(B, \eta),$$

then (M, ζ) admits a contact structure in the class β.

Proof. By assumption, there is a contact manifold (N, ξ) with associated stable complex structure ζ_N and with a ζ_N-compatibe (B, η)-structure $\bar{\zeta}_N \colon N \to B$ such that $[N, \bar{\zeta}_N] = [M, \bar{\zeta}] \in \Omega_{2q+1}(B, \eta)$. By [Kr2, Proposition 4], we may perform (B, η)-surgeries of dimension q or less on $\bar{\zeta} \colon N \to B$ to obtain a $(q - 1)$-smoothing $\bar{\zeta}_{N'} \colon N' \to B$, with induced stable complex structure ζ' say. By Lemmas 2.8 and 2.9 and Theorem 3.3, (N', φ') admits a contact structure ξ' in the class β and with associated almost contact structure φ' which stabilises to ζ'. Applying Theorem 3.8 to (N', φ') and (M, φ), we deduce that (M, φ) admits a contact structure in the class β. $\square$

The line of reasoning from the proof of Corollary 3.10 was pursued in [BCS1] for β the class of all contact structures. There contact structures on manifolds of the form $M \times S^2$

with M contact were shown to exist by finding Stein cobordisms from $M \times T^2$ and applying a result of Bourgeois [Bou] which provides a contact structure for this latter manifold once M is contact. More generally, Corollary 3.10 gives a framework for approaching the symplectic version of the Stein Realisation Problem 1.1:

Problem 3.11 (Symplectic Relaization Problem)**.** *Determine which almost contact structures on a given manifold can be realised by strongly/exactly fillable contact structures. Does the answer depend on whether one considers strong or exact fillings?*

4. Simply connected 7-manifolds

As an application of the methods developed in Section 3, we now give a proof of Theorem 1.3. The proof will show that the Stein fillability obstruction of Theorem 3.7 vanishes by showing that the relevant bordism group is itself trivial.

Before turning to the computation of the bordism group, however, we show that every 7-manifold considered in Theorem 1.3 admits an almost contact structure.

To start the argument, recall that a manifold M admits a spin^c structure, that is, a lift of the structure group of TM from $SO(n)$ to the group $Spin^{\mathbb{C}}(n)$, if and only if the second Stiefel-Whitney class $w_2(M) \in H^2(M; \mathbb{Z}_2)$ admits an integral lift. (The Lie group $Spin^{\mathbb{C}}(n)$ can be defined as the extension of $SO(n)$ by S^1 with the property that $Spin^{\mathbb{C}}(n) \to SO(n)$ is the unique nontrivial principal S^1-bundle over $SO(n)$.) Since each manifold M in Theorem 1.3 is simply connected and has torsion free $\pi_2(M) \cong H_2(M)$, the mod 2 reduction map $H^2(M) \to H^2(M; \mathbb{Z}_2)$ is onto, and so M admits a spin^c structure.

Notice that $U(n) \subset SO(2n)$, and since any S^1-bundle over $U(n)$ is trivial (by the fact that $H^2(U(n)) = 0$), we have that the restriction of the bundle $Spin^{\mathbb{C}}(2n) \to SO(2n)$ over $U(n)$ is trivial. Consequently $Spin^{\mathbb{C}}(2n)$ contains $U(n) \times S^1$, so in particular $U(n)$ embeds into $Spin^{\mathbb{C}}(2n)$. This embedding provides a homomorphism of topological groups $U \to Spin^{\mathbb{C}}$. Similarly, $SU(n)$ embeds into $SO(2n)$, and since $SU(n)$ is simply connected, this embedding lifts to an embedding $SU(n) \to Spin(2n)$ (recall that $Spin(2n) \to SO(2n)$ is the nontrivial double cover of $SO(2n)$). This construction then provides a homomorphism of topological groups $SU \to Spin$.

We first show that every spin^c structure on a 7-manifold is induced by an almost contact structure.

Lemma 4.1. *A compact oriented 7-manifold X admits an almost contact structure if and only if it admits a spin^c structure. Moreover, any spin^c structure on X is induced from some almost contact structure on X.*

Proof. Elementary obstruction theory applied to the fibration sequence $U(3) \to U(4) \to S^7$ shows that any stable complex structure ζ on X can be destabilised to an almost contact structure φ: the argument is analogous to the proof of Lemma 2.8. Hence it is enough to show that X admits a stable complex structure if and only if X admits a stable spin^c structure, that is, a map into $BSpin^{\mathbb{C}}$ covering the map $X \to BSO$ given by the stable tangent bundle. Since a stable complex structure on X induces a spin^c structure on X, we only need to show that any spin^c structure on X can be lifted to a stable complex structure.

The homomorphism of topological groups $U \to Spin^{\mathbb{C}}$ induces a map of classifying spaces which gives a fibre bundle

$$(3) \qquad\qquad Spin^{\mathbb{C}}/U \xrightarrow{i} BU \to BSpin^{\mathbb{C}}.$$

By Bott periodicity the quotient $Spin^{\mathbb{C}}/U$ is 5-connected and $\pi_6(Spin^{\mathbb{C}}/U) \cong \mathbb{Z}$. Suppose that $\theta\colon X \to BSpin^{\mathbb{C}}$ is a spin^c structure on X. We must show that the following lifting problem has a solution:

$$
\begin{array}{ccc}
 & & BU \\
 & \nearrow & \downarrow \\
X & \xrightarrow{\;\theta\;} & BSpin^{\mathbb{C}}.
\end{array}
$$

Since $Spin^{\mathbb{C}}/U$ is 5-connected, the primary obstruction to lifting θ is a cohomology class $\theta^*(\alpha) \in H^7(X)$, where we have identified $\pi_6\,(Spin^{\mathbb{C}}/U)$ with $\mathbb{Z}$ and the universal obstruction class $\alpha \in H^7(BSpin^{\mathbb{C}})$ is defined below. We shall show that $2\alpha = 0$. Since $H^7(X;\mathbb{Z})$ is torsion free, it follows that $\theta^*(\alpha) = 0$ and hence θ lifts to a stable complex structure on X.

It remains to define α and to prove that $2\alpha = 0$. Let $x \in \pi_6(Spin^{\mathbb{C}}/U) \cong \mathbb{Z}$ be a generator. Since $Spin^{\mathbb{C}}/U$ is 5-connected, there is a generator $\widehat{x} \in H^6(Spin^{\mathbb{C}}/U)$ such that $\langle \widehat{x}, \rho(x) \rangle = 1$ where $\rho\colon \pi_6(Spin^{\mathbb{C}}/U) \to H_6(Spin^{\mathbb{C}}/U)$ is the Hurewicz homomorphism. The class $\widehat{x}$ is transgressive in the Leray-Serre cohomology spectral sequence of the fibration (3), and we define

$$\alpha := \tau(\widehat{x}) \in H^7(BSpin^{\mathbb{C}}),$$

where $\tau\colon H^6(Spin^{\mathbb{C}}/U) \to H^7(BSpin^{\mathbb{C}})$ is the transgression homomorphism. Since the kernel of τ is the image of the homomoprhism $i^*\colon H^6(BU) \to H^6(Spin^{\mathbb{C}}/U) \cong \mathbb{Z}$, it suffices to show that the image of i^* is the subgroup of index two. Now $H^*(BU) = \mathbb{Z}[c_1, c_2, c_3, \dots]$ is the polynomial algebra on the Chern classes and the composition $S^6 \xrightarrow{x} Spin^{\mathbb{C}}/U \xrightarrow{i} BU$ determines the stable complex vector bundle $x^*i^*(EU)$ over S^6 where $EU \to BU$ is the universal bundle. By [Hu, Corollary 9.8], every complex bundle E over S^6 is such that $c_3(E) \in 2 \cdot H^6(S^6)$ and moreover there is a complex bundle E_0 over S^6 where $c_3(E_0)$ is twice a generator of $H^6(S^6)$. It follows that $i^*(H^6(BU)) = 2 \cdot H^6(Spin^{\mathbb{C}}/U)$ and the lemma follows. $\qquad\square$

We now reduce the proof of Theorem 1.3 to the calculation of certain bordism groups. Let φ be an almost contact structure on M, with associated stable complex structure ζ, let $H = H_2(M)$ and let γ be the complex line bundle over the Eilenberg-MacLane space $K(H, 2)$ with $c_1(\gamma) = c_1(\zeta) \in H^2(K(H, 2); \mathbb{Z}) \cong H^2(M; \mathbb{Z})$. By Lemma 2.14, the complex normal 2-type of the stably complex manifold (M, ζ) is

$$(B^2_\zeta; \eta^2_\zeta) = (K(H, 2) \times BSU; \gamma \oplus \pi_{SU}),$$

where $\pi_{SU}\colon BSU \to BU$ is map induced by the inclusion $SU \to U$ and $\oplus$ denotes the exterior Whitney sum of complex bundles: for further details, see Section 2.5. It follows that there is an isomorphism of bordism groups

$$(4) \qquad\qquad \Omega_7(B^2_\zeta; \eta^2_\zeta) \cong \Omega^{SU}_7(K(H, 2); \gamma),$$

where is the latter group is a certain γ-twisted SU-bordism group of $K(H, 2)$. This is the bordism group of triples (N, f, α) where N is a closed smooth manifold, $f\colon N \to K(H, 2)$ is a map and α is an SU structure on the Whitney sum of $f^*(\gamma)$ and the stable normal bundle of N.

The remainder of this subsection gives the proof of the following proposition.

Proposition 4.2. *For any finitely generated free abelian group H and for any complex line bundle γ over $K(H, 2)$, we have $\Omega^{SU}_7(K(H, 2); \gamma) = 0$.*

We shall need the following result on SU-bordism groups.

Lemma 4.3. *If k is not divisible by 4 then $\Omega^{SU}_{2k+1} = 0$.*

Proof. By [Stn, p. 117], we know that $\Omega^{U}_* = 0$ for any odd dimension $* = 2k + 1$. Also by [Stn, p. 238], the kernel of the forgetful homomorphism $\Omega^{SU}_* \to \Omega^{U}_*$ is the torsion subgroup of Ω^{SU}_*. But by [Stn, p. 248], the torsion subgroup of Ω^{SU}_{2k+1} vanishes if k is not divisible by 4, concluding the proof. $\qquad\square$

For the case $H = 0$ in Proposition 4.2, by Lemma 4.3 implies that $\Omega^{SU}_7 = 0$ as required. Hence we assume that H is not the zero group. We wish to compute the γ-twisted SU-bordism of $K(H, 2)$. A very similar situation is discussed in [KS, Section 6] where Kreck and Stolz compute certain twisted spin bordism groups of $K(\mathbb{Z}, 2)$. Since the Thom space of the exterior Whitney sum of bundles is homotopy equivalent to the smash product of the individual Thom spaces, just as in [KS, §6], the Pontrjagin-Thom construction gives an isomorphism

$$(5) \qquad \Omega^{SU}_*(K(H, 2); \gamma) \cong \pi_*(MSU \wedge T(\gamma)) \cong \widetilde{\Omega}^{SU}_*(T(\gamma)).$$

Here MSU is the Thom spectrum defined by special unitary boridsm, $T(\gamma)$ is the Thom space of the bundle γ over $K(H, 2)$, $\wedge$ denotes the smash product of spectra and $\widetilde{\Omega}^{SU}_*$ denotes reduced special unitary bordism groups. As a consequence of (5), there is an Atiyah-Hirzebruch spectral sequence (AHSS),

$$E^2_{p,q} = H_{p+2}(T(\gamma); \Omega^{SU}_q) \Longrightarrow \Omega^{SU}_{p+q}(K(H, 2); \gamma),$$

which converges to the associated graded object of a filtration on $\Omega^{SU}_{p+q}(K(H, 2); \gamma)$. By the Thom isomorphism, $\widetilde{H}^*(T(\gamma))$ is a free module over $H^*(K(H, 2))$ with generator the Thom class $U \in H^2(T(\gamma))$ of γ. As a consequence, $H_*(T(\gamma))$ vanishes in odd degrees. Now, by Lemma 4.3, $\Omega^{SU}_{2k+1} = 0$ for $k = 1, 2, 3$ and $\Omega^{SU}_1 \cong \mathbb{Z}_2$. It follows that the 7-line of the E^2-page of the AHSS above has only one non-vanishing term and that is

$$E^2_{6,1} = H_8(T(\gamma); \Omega^{SU}_1) \cong H_6(K(H, 2); \Omega^{SU}_1).$$

We claim that $E^3_{6,1} = 0$, which proves Proposition 4.2. To see that $E^3_{6,1} = 0$, we need to understand the following differentials in the AHSS:

$$(6) \qquad d^2_{8,0} \colon H_{10}(T(\gamma)) \to H_8(T(\gamma); \mathbb{Z}_2) \text{ and } d^2_{6,1} \colon H_8(T(\gamma); \mathbb{Z}_2) \to H_6(T(\gamma); \mathbb{Z}_2).$$

Since the map $SU \to \mathrm{Spin}$ is a 6-equivalence, these differentials for SU-bordism will coincide with the corresponding differentials for spin bordism. The differentials in the spin case have been computed by Teichner [Te, Lemma 2.3.2]. Hence we have the following lemma.

Lemma 4.4 ([Te, Lemma 2.3.2]). *Let $\rho_2 \colon H_*(T(\gamma)) \to H_*(T(\gamma); \mathbb{Z}_2)$ be the homomorphism induced by reduction mod 2 and let $(Sq^2)^* \colon H_{*+2}(T(\gamma); \mathbb{Z}_2) \to H_*(T(\gamma); \mathbb{Z}_2)$ be the dual of the Steenrod squaring operation $Sq^2 \colon H^*(T(\gamma); \mathbb{Z}_2) \to H^{*+2}(T(\gamma); \mathbb{Z}_2)$. Then the differentials in (6) above are given by*

$$d^2_{8,0} = \rho_2 \circ (Sq^2)^* \quad and \quad d^2_{6,1} = (Sq^2)^*.$$

$\qquad\square$

The following lemma is equivalent to the claim that $E^3_{6,1} = 0$ in the AHSS and hence completes the proof of Proposition 4.2.

Lemma 4.5. *For all finitely generated free abelian groups H and for all complex line bundles γ over $K(H,2)$ we have*

$$\mathrm{Ker}(d_{6,1}^2) = \mathrm{Im}(d_{8,0}^2).$$

Proof. The lemma is trivial if $H = 0$, so we assume that H in non-zero. We give the proof by viewing the situation from the point of view of homological algebra over the field $\mathbb{Z}_2$. Recall that $T(\gamma)$ denotes the Thom space of γ. We define a chain complex $(C_*(H,\gamma), d)$ by setting

$$C_i(H,\gamma) := H_{2i+2}(T(\gamma); \mathbb{Z}_2), \text{ for } i \geq 0$$

and defining the differential d by

$$d_{i+1} := (Sq^2)^* \colon H_{2i+4}(T(\gamma); \mathbb{Z}_2)) \to H_{2i+2}(T(\gamma); \mathbb{Z}_2).$$

To see that the differential satisfies $d^2 = 0$, we first recall the Adem relation $Sq^2 Sq^2 = Sq^3 Sq^1$, which entails that $Sq^2 Sq^2 = 0$ on $H^*(T(\gamma); \mathbb{Z}_2)$ since the non-zero mod 2 cohomology groups of $T(\gamma)$ are concentrated in even degrees. It follows that $(Sq^2)^*(Sq^2)^*$, which is the dual of $Sq^2 Sq^2$, vanishes.

Since the homomorphism $\rho_2 \colon H_{10}(T(\gamma)) \to H_{10}(T(\gamma); \mathbb{Z}_2)$ is onto, to prove the lemma it suffices to show that the third homology group of the chain complex $(C_*(H,\gamma), d)$ vanishes:

$$H_3(C_*(H,\gamma), d) = 0.$$

In the case where $H \cong \mathbb{Z}$, it is a simple exercise using the Thom isomorphism to check that the homology of $(C_*(\mathbb{Z},\gamma), d)$ is trivial if $w_2(\gamma) \neq 0$, and if $w_2(\gamma) = 0$ then

$$H_*(C_*(\mathbb{Z},\gamma), d) \cong \begin{cases} \mathbb{Z}_2 & * = 0 \\ 0 & * > 0. \end{cases}$$

We shall prove the general case by induction from these two cases. Let γ be a complex line bundle over $K(H,2)$. When H has rank greater than one, let $H = H_0 \oplus \mathbb{Z}$ with the property that $c_1(\gamma)|_{H_0} = 0$. (If $c_1(\gamma) = 0$, then any decomposition of H will do, if $c_1(\gamma) \neq 0$ then take $H_0 := \mathrm{Ker}(c_1(\gamma) \colon H \to \mathbb{Z})$.) Observe that there is a split short exact sequence $H_0 \to H \xrightarrow{\pi} \mathbb{Z}$, such that $\gamma \cong \pi^* \gamma'$ for the map $\pi \colon K(H,2) \to K(\mathbb{Z},2)$ and for some complex line bundle γ' over $K(\mathbb{Z},2)$. If $\mathbb{C}_X$ denotes the trivial line bundle over a space X, then there is an isomorphism of complex vector bundles

$$\mathbb{C}_{K(H,2)} \oplus \gamma \cong \mathbb{C}_{K(H_0,2)} \oplus \gamma'.$$

where the first $\oplus$ denotes the usual Whitney sum over $K(H,2)$ and the second $\oplus$ the exterior Whitney sum over $K(H_0,2) \times K(\mathbb{Z},2) = K(H,2)$. Since the Thom space of the exterior Whitney sum of bundles is homotopy equivalent to the smash product of the Thom spaces of each bundle,

$$T(\gamma) \wedge S^2 \simeq M(\mathbb{C}_{K(H_0,2)}) \wedge T(\gamma').$$

If $x \in H^*(M(\mathbb{C}_{K(H_0,2)}); \mathbb{Z}_2)$ and $y \in H^*(T(\gamma'); \mathbb{Z}_2)$ and $x \wedge y$ denotes their exterior cup product in $H^*(M(\mathbb{C}_{K(H_0,2)}) \wedge T(\gamma'); \mathbb{Z}_2)$, then the Cartan formula for Sq^2 gives

$$Sq^2(x \wedge y) = Sq^2 x \wedge y + Sq^1 x \wedge Sq^1 y + x \wedge Sq^2 y = Sq^2 x \wedge y + x \wedge Sq^2 y$$

since $Sq^1 x$ and $Sq^1 y$ have odd degree and are thus zero. Thus there is an isomorphism of chain complexes

$$(C_*(H_0 \oplus \mathbb{Z}, \gamma), d) \cong (C_*(H_0, 0), d) \otimes (C_*(\mathbb{Z}, \gamma'), d),$$

where $\otimes$ denotes the tensor product of chain complexes. Applying the Kunneth theorem for the homology groups of a tensor product of chain complexes over $\mathbb{Z}_2$ inductively gives us that $H_3(C_*(H, \gamma), d) = 0$ for all groups H and all complex line bundles γ. This completes the proof of the lemma. $\square$

Proof of Theorem 1.3. Since a closed, oriented, simply connected manifold M with torsion free $\pi_2(M)$ admits a spin^c structure, Lemma 4.1 implies the existence of an almost contact structure φ on M. By Equation (4), Proposition 4.2 implies $\Omega_7(B_\zeta^2; \eta_\zeta^2) = 0$ where $\zeta = S\varphi$. It follows that for any ζ-compatible normal 2-smoothing $\bar{\zeta} \colon M \to B_\zeta^2$, we have $[M, \zeta] = 0 \in \Omega_7(B_\zeta^2, \eta_\zeta^2)$. By Theorem 3.7 the almost contact manifold (M, φ) is then Stein fillable. $\square$

5. Stein fillings of homotopy spheres

Recall that an n-dimensional homotopy sphere is a closed, smooth, oriented manifold Σ which is homotopy equivalent to S^n. The set of oriented diffeomorphism classes of homotopy n-spheres forms an abelian group Θ_n under the operation of connected sum:

$$\Theta_n := \{ [\Sigma] \,|\, \Sigma \simeq S^n \}.$$

For $n \geq 5$, every homotopy n-sphere Σ is homeomorphic to S^n [Sm], hence Θ_n may be regarded as the group of oriented diffeomorphism classes of smooth structures on the n-sphere.

We now recall some fundamental facts about the group Θ_n proved by Kervaire and Milnor. For further information, we refer the reader to [KM, Le] and [Lü, 6.6]. Let O denote the stable orthogonal group, π_n^S the nth stable homotopy group of spheres and recall the J-homomorphism

$$J_n \colon \pi_n(O) \to \pi_n^S.$$

Since π_n^S is a finite group, the cokernel of J_n, $\mathrm{Coker}(J_n)$, is also finite. We state the following theorem of Kervaire and Milnor only for the case of interest to us where $n = 2q + 1 \geq 5$.

Theorem 5.1 ([KM, Section 4], [KM, Theorem 6.6]). *For $2q + 1 \geq 5$ the abelian group Θ_{2q+1} lies in a short exact sequence*

$$0 \longrightarrow bP_{2q+2} \longrightarrow \Theta_{2q+1} \overset{\eta}{\longrightarrow} \mathrm{Coker}(J_{2q+1}) \longrightarrow 0$$

where bP_{2q+2} denotes the finite cyclic group of homotopy $(2q + 1)$-spheres which bound parallelisable manifolds. $\square$

When we move to the stable complex setting, we have the following

Example 5.2. Every homotopy $(2q + 1)$-sphere Σ is stably parallelisable by [KM, Theorem 3.1] and hence admits an almost contact structure φ with stabilisation $\zeta := S\varphi$. The complex normal $(q - 1)$-type of (Σ, ζ) is independent of the choice of φ and is given by

$$(B_\zeta^{q-1}, \eta_\zeta^{q-1}) = (BU\langle q+1 \rangle; \pi_{q+1}),$$

where $\pi_{q+1} \colon BU\langle q+1 \rangle \to BU$ is the qth connective cover of BU, i.e. π_{q+1} is the universal map such that $\pi_i(BU\langle q+1 \rangle) = 0$ for $i \leq q$ and $(\pi_{q+1})_* \colon \pi_i(BU\langle q+1 \rangle) \cong \pi_i(BU)$ for $i \geq q + 1$.

For homotopy spheres bounding parallelisable manifolds we have the following well-known proposition.

Proposition 5.3. *Every homotopy sphere $\Sigma \in bP_{2q+2}$ is Stein fillable.*

Proof. In order to exhibit an explicit Stein filling for Σ, we use the fact that every $\Sigma \in bP_{2q+2}$ is diffeomorphic to a 'Brieskorn sphere', [Br, Korollar 2]. That is, $\Sigma \cong \Sigma(a_1, a_2, \ldots, a_{q+2})$ is realised as the intersection of the singular hypersurface

$$H_0 = \{(z_1, \ldots, z_{2q+2}) \,|\, z_1^{a_1} + z_2^{a_2} + \ldots + z_{q+2}^{a_{q+2}} = 0\} \subset \mathbb{C}^{q+2}$$

with the unit sphere $S^{2q+3} \subset \mathbb{C}^{q+2}$ for suitable $a_i \in \mathbb{N}$. A Stein filling is then given by considering the part of a regular hypersurface

$$H_\epsilon = \{(z_1, \ldots, z_{2q+2}) \,|\, z_1^{a_1} + z_2^{a_2} + \ldots + z_{q+2}^{a_{q+2}} = \epsilon\}$$

that intersects the unit ball $B^{2q+4} \subset \mathbb{C}^{q+2}$ for any small $\epsilon \neq 0$ and the strictly plurisubharmonic function is given by $||z||^2$. $\qquad\square$

When we move to homotopy $(2q+1)$-spheres mapping non-trivially to $\mathrm{Coker}(J_{2q+1})$, *there is no known example admitting a Stein filling.* The following proposition, which is a more precise version of Theorem 1.4 from the introduction, is a consequence of Theorem 3.3 as well as results of Wall and Schultz about homotopy spheres bounding highly-connected manifolds.

Theorem 5.4. *Let Σ^{2q+1} be a homotopy sphere which maps non-trivially into $\mathrm{Coker}(J_{2q+1})$.*

(1) *If $q \not\equiv 1, 3, 7 \bmod 8$ or if $q \equiv 1 \bmod 8$ and $q > 9$ or if $q = 7$ or 15, then Σ is not Stein fillable.*
(2) *If $q = 9$ or if $q \equiv 3, 7 \bmod 8$, then there is a cyclic subgroup $C_q^U \subset \mathrm{Coker}(J_{2q+1})$ such that Σ is Stein fillable if and only if Σ maps to zero in $\mathrm{Coker}(J_{2q+1})/C_q^U$.*
   (a) *For $q = 9$, we have $C_9^U \cong 0$ or $\mathbb{Z}_2$.*
   (b) *For $q \equiv 7 \bmod 8$, we have $C_{8k-1}^U \subset 4 \cdot \mathrm{Coker}(J_{16k-1})$.*

There are many cases where the group $\mathrm{Coker}(J_{2q+1})/C_q^U$ is non-zero: we discuss some examples in Corollary 5.6 and Lemma 5.8 below. By the Generalized Poincaré Conjecture, Theorem 5.1 and Theorem 5.4 imply the following

Corollary 5.5. *In general, the existence of a Stein fillable contact structure depends on the smooth structure of M and not simply the underlying homeomorphism type of M.* $\qquad\square$

Proof of Theorem 5.4. Let (W, J) be a Stein filling of Σ^{2q+1}. Since W has handles only in dimension $(q+1)$ or less, it follows that W is obtained from Σ by attaching handles of dimension $(q+1)$ or greater. Hence W is q-connected and so Σ bounds a q-connected smooth manifold W with a stable complex structure $\bar{\zeta}_W$. This constrains the diffeomorphism type of Σ as recorded in the statement of the proposition, as we now explain.

The classification of oriented q-connected $(2q+2)$-manifolds with boundary a homotopy sphere is given in [Wa2]. Such manifolds are homotopy equivalent to a finite wedge of $(q+1)$-spheres and are classified by triples

$$(H, \lambda, \alpha) = (H_{q+1}(W), \lambda_W, \alpha_W)$$

where $(H_{q+1}(W), \lambda_W)$ is the usual intersection form of W, which is a unimodular bilinear form over the integers, and $\alpha_W \colon H_{q+1}(W) \to \pi_q(SO(q+1))$ is a quadratic refinement of λ_W as explained in [Wa2, Lemma 2]. The stabilisation of α_W is a homomorphism

$$S\alpha_W \colon H_{q+1}(W) \to \pi_q(SO),$$

which describes the stable tangent bundle of W along each $(q+1)$-sphere in the homotopy type of W. In particular, W admits a complex structure if and only if

$$(7) \qquad\qquad \mathrm{Im}(S\alpha_W) \subset \mathrm{Im}\big(\pi_q(U) \to \pi_q(SO)\big).$$

To study the diffeomorphism type of the homotopy sphere $\Sigma = \partial W$, Wall [Wa1, Theorems 2 & 3] defined the bordism group

$$A_{2q+2}^{\langle q+1\rangle} := \{[W] \,|\, W \text{ is } q\text{-connected and } \partial W \cong \Sigma\},$$

the *rel. boundary bordism group* of smooth oriented q-connected $(2q+2)$-manifolds with boundary a homotopy sphere. (The notation is from [Sto] and a similar notation appears in [Wa2, §17].) In analogy, we define the bordism group

$$A_{2q+2}^{BU\langle q+1\rangle} := \{[W, J] \,|\, W \text{ is } q\text{-connected and } \partial W \cong \Sigma\},$$

to be the *rel. boundary bordism group of almost complex q-connected $(2q+2)$-manifolds* with boundary a stably complex homotopy sphere. We consider the homomorphisms

$$A_{2q+2}^{BU\langle q+1\rangle} \xrightarrow{F} A_{2q+2}^{\langle q+1\rangle} \xrightarrow{\partial} \Theta_{2q+1} \xrightarrow{\eta} \mathrm{Coker}(J_{2q+1})$$

where F remembers only the orientation underlying an almost complex structure, ∂ is defined by taking the diffeomorphism type of the bounding homotopy sphere, and η is the homomorphism from Theorem 5.1. The above discussion shows that the group

$$C_q^U := \mathrm{Im}(\eta \circ \partial \circ F) \subset \mathrm{Coker}(J_{2q+1})$$

is isomorphic to the group of Stein fillable homotopy spheres modulo bP_{2q+2}. Wall [Wa2, Theorem 2] computed the bordism group $A_{2q+2}^{\langle q+1\rangle}$ by proving that it is isomorphic to a certain Witt group of quadratic forms (H, λ, α) as above. We do not go into the details but summarise the facts relevant for our proof. By [Wa2, Theorem 4] $\eta \circ \partial = 0$ for $3 \le q \le 7$ and hence $C_q^U = 0$ in these dimensions. For $q \ge 8$, there is an isomorphism

$$\Phi \colon A_{2q+2}^{\langle q+1\rangle} \cong L_{1-q}(e) \oplus \big(\pi_q(SO) \otimes \pi_q(SO)\big), \quad [H, \lambda, \alpha] \mapsto \big(\sigma(H, \lambda, \alpha), \widehat{\alpha}^2\big).$$

Here $L_{1-q}(e)$ denotes the 4-periodic surgery obstruction group, $\widehat{\alpha} \in H$ is an element such that $\lambda(x, \widehat{\alpha}) = \alpha(x)$ for all $x \in H$, intrepreted mod 2 if $\pi_q(SO) \cong \mathbb{Z}_2$, and $\widehat{\alpha}^2 = \lambda(\widehat{\alpha}, \widehat{\alpha})$. By definition, the summand $L_{1-q}(e) \subset A_{2q+2}^{\langle q+1\rangle}$ corresponds to the subgroup generated by the stably parallelisable spheres, and hence $\partial(\Phi^{-1}(L_{1-q}(e))) \in bP_{2q+2}$. We therefore concentrate on the other summand.

Let $F_* \colon \pi_q(U) \to \pi_q(SO)$ be the homomorphism induced by $U \to SO$. From (7) above, we see that for $q \ge 8$ there is an isomorphism

$$\Phi^U \colon F\big(A_{2q+2}^{BU\langle q+1\rangle}\big) \cong L_{1-q}(e) \oplus \big(\mathrm{Im}(F_*) \otimes \mathrm{Im}(F_*)\big).$$

It follows that C_q^U is the zero group if $F_* = 0$. Given our knowledge of the homomorphism $\pi_q(U) \to \pi_q(SO)$ this occurs unless $q \equiv 1, 3, 7 \mod 8$. When $q \equiv 1, 3, 7 \mod 8$, we see that C_q^U is the cyclic group generated by the element

$$\big(\eta \circ \partial \circ (\Phi^U)^{-1}\big)\big(0, F_*(1) \otimes F_*(1)\big) \in \mathrm{Coker}(J_{2q+1}),$$

where $1 \in \pi_q(U)$ is a generator. For $q \equiv 1, 3 \mod 8$, F_* is onto. However, if $q = 8k + 1 > 9$ Schultz [Sc, Corollary 3.2] states that $\eta \circ \partial = 0$, proving that $C_{8k+1}^U = 0$ if $8k+1 > 9$. Finally, for $q \equiv 7 \mod 8$, $F_*(\pi_q(U)) \subset \pi_q(SO)$ is a subgroup of index two and so the bilinearity of

the tensor product ensures that $C_q^U = 4 \cdot \mathrm{Im}(\eta \circ \partial)$. When $q = 15$, we have $\mathrm{Coker}(J_{31}) \cong \mathbb{Z}_2^2$, [R, Table A3.3], and hence $C_{15}^U = 0$. $\qquad\square$

We now give some examples of exotic spheres which are not Stein fillable. Since every homotopy sphere has a unique spin structure, there is a homomorphism $\omega^{Spin} \colon \Theta_n \to \Omega_n^{Spin}$, given by mapping a homotopy sphere to it spin bordism class. Recall now the α-invariant

$$\alpha \colon \Omega_*^{Spin} \to KO_*$$

which is a ring homomorphism from spin bordism to real K-theory defined by taking the KO-valued index of the Dirac operator on a spin manifold, [Hi, §4.2]. Composing α with ω^{Spin} we obtain the α-invariant for homotopy spheres

$$\alpha \colon \Theta_n \to \Omega_n^{Spin} \to KO_n.$$

It is known that in all dimensions $8k + 1$, $k \geq 1$, there are exotic spheres with non-trivial α-invariant in $KO_{8k+1} \cong \mathbb{Z}_2$. The existence of such spheres follows from theorems of Milnor and Adams as is explained in [Hi, p. 44]. If $\alpha(\Sigma) = 1$ then Σ does not bound a spin manifold. On the other hand, a Stein filling of Σ is $4k$-connected and in particular admits a unique spin structure. Hence we obtain an alternative proof of the following special case of Theorem 5.4.

Lemma 5.6. *If $\Sigma \in \Theta_{8k+1}$ has $\alpha(\Sigma) = 1 \in KO_{8k+1}$ then Σ is not Stein fillable.* $\qquad\square$

Next we show that taking connected sum with α-invariant-1 homotopy spheres can often destroy the Stein fillability of more general manifolds. Since $\pi_{8k+1}(SO/U) = 0$, it follows that every homotopy $(8k + 1)$-sphere has a unique stable complex structure ζ_Σ. Given a stably complex manifold (M, ζ), we shall write $(M\sharp\Sigma, \zeta\sharp\zeta_\Sigma)$ for the stably complex manifold obtained by taking the connected sum of the stably complex manifolds (M, ζ) and (Σ, ζ_Σ).

Proposition 5.7. *Let (M, φ) be a Stein fillable almost contact manifold of dimension $8k+1$ with $\zeta := S\varphi$ and $c_1(\zeta) = 0$. If Σ is a homotopy $(8k+1)$-sphere with $\alpha(\Sigma) = 1 \in KO_{8k+1}$, then the stably complex manifold $(M\sharp\Sigma, \zeta\sharp\zeta_\Sigma)$ is not Stein fillable.*

Proof. Since $c_1(\zeta) = 0$, there is a lift of the normal complex structure $\zeta \colon M \to BU$ to BSU. It follows that there is a map of stable complex bundles $F \colon (B_\zeta^{4k-1}, \eta_\zeta^{4k-1}) \to (BSU, \pi_{SU})$. The bundle map F induces a homomorphism of bordism groups

$$F_* \colon \Omega_{8k+1}(B_\zeta^{4k-1}; \eta_\zeta^{4k-1}) \to \Omega_{8k+1}^{SU}.$$

Since (M, φ) is Stein fillable by Theorem 3.7, every $(4k-1)$-smoothing $\bar\zeta \colon M \to B_\zeta^{4k-1}$ is null-bordant. Moreover $\pi_{8k+1}(SO/U) = 0$ and thus Σ admits a unique $(B_\zeta^{4k-1}, \eta_\zeta^{4k-1})$-structure $\bar\zeta_\Sigma$. The connected sum $(M\sharp\Sigma, \bar\zeta\sharp\bar\zeta_\Sigma)$ is a $(\zeta\sharp\zeta_\Sigma)$-compatible normal $(4k - 1)$-smoothing in $(B_\zeta^{4k-1}, \eta_\zeta^{4k-1})$. Now we have

$$F_*([M\sharp\Sigma, \bar\zeta\sharp\bar\zeta_\Sigma]) = [\Sigma, \zeta_\Sigma] \neq 0 \in \Omega_{8k+1}^{SU},$$

where the last inequality holds since the homomorphism $SU \to Spin$ induces a homomorphism $\Omega_*^{SU} \to \Omega_*^{Spin}$. Since $\alpha(\Sigma) = 1$, it follows that $[\Sigma, \zeta_\Sigma] \neq 0 \in \Omega_{8k+1}^{SU}$. The above argument therefore shows that $[M\sharp\Sigma, \bar\zeta\sharp\bar\zeta_\Sigma] \neq 0 \in \Omega_{8k+1}(B_\zeta^{4k-1}; \eta_\zeta^{4k-1})$, and so by Theorem 3.7, $(M\sharp\Sigma, \zeta\sharp\zeta_\Sigma)$ is not Stein-fillable. $\qquad\square$

We next construct an exotic 9-sphere Σ which lies the kernel of the α invariant $\alpha \colon \Theta_9 \to KO_9$, but which does not bound a parallelisable manifold. By Theorem 5.4, this homotopy sphere is not Stein fillable, but from a topological point of view, one can argue that it is one of the "least exotic" homotopy spheres which is not Stein fillable.

By Theorem 5.1 above and results of Toda [To, p. 189], there is a short exact sequence

$$0 \to bP_{10} \to \mathrm{Ker}(\alpha) \to \mathbb{Z}_2 \to 0.$$

We shall given a explicit description of a homotopy sphere Σ where $[\Sigma]$ generates $\mathrm{Ker}\,\alpha/bP_{10}$. We first recall the well-known plumbing pairing

$$\sigma_{p,q} \colon \pi_p(SO(q)) \times \pi_q(SO(p)) \longmapsto \Theta_{p+q+1}, \quad (\beta, \gamma) \longmapsto \partial W(S(\beta), S(\gamma)),$$

where $S \colon \pi_p(SO(q)) \to \pi_p(SO(q+1))$ is the the stabilisation homomorphism and

$$W(S(\beta), S(\gamma)) := (D^{q+1} \tilde{\times}_{S(\beta)} S^{p+1}) \cup_{D^{q+1} \times D^{p+1}} (D^{p+1} \tilde{\times}_{S(\gamma)} S^{q+1})$$

is the compact smooth $(p + q + 2)$-manifold obtained by plumbing the disc bundles of $S(\beta)$ and $S(\gamma)$ together: see for example [Sc, Remark p. 741]. We let $\beta_5 \in \pi_3(SO(5)) \cong \mathbb{Z}$ and $\gamma_3 \in \pi_5(SO(3)) \cong \mathbb{Z}_2$ be generators and define the homotopy 9-sphere

$$\Sigma^9_{\beta_5, \gamma_3} := \sigma_{3,5}(\beta_5, \gamma_3).$$

Notice that there is a homotopy equivalence $W(S(\beta_5), S(\gamma_3)) \simeq S^4 \vee S^6$, so that the manifold $W(S(\beta_5), S(\gamma_3))$ cannot admit a Stein structure, but from the point of view of the dimensions of the handles, $W(S(\beta_5), S(\gamma_3))$ is as close as possible to admitting a Stein structure.

Lemma 5.8. *The homotopy 9-sphere $\Sigma_{\beta_5, \gamma_3}$ maps to a generator of $\mathrm{Ker}(\alpha)/bP_{10} \cong \mathbb{Z}_2$.*

Proof. The proof starts with the exotic 8-sphere $\Sigma^8 \in \Theta_8 \cong \mathbb{Z}_2$. By [Sto, Satz 12.1] and [Hu, Proposition 12.20], there is a diffeomorphism $\Sigma^8 \cong \partial W(\beta_5, \delta_4)$ where $\beta_5 \in \pi_3(SO(5))$ is as above and $\delta_4 \in \pi_4(SO(4))$ is given by the composition $\tau_{S^4} \circ \eta_3 \colon S^4 \to S^3 \to SO(4)$, where τ_{S^4} is the characteristic map of the tangent bundle of the 4-sphere and η_3 is essential. We claim that $\delta_4 = S(\delta_3)$ where $\delta_3 \in \pi_4(SO(3)) \cong \mathbb{Z}_2$ is a generator. To see this, we use the commutative diagram of exact sequences

$$\begin{array}{ccccc}
\pi_3(SO(3)) & \xrightarrow{\ S\ } & \pi_3(SO(4)) & \xrightarrow{\ E_3\ } & \pi_3(S^3) \\
\downarrow{\scriptstyle \circ\eta_3} & & \downarrow{\scriptstyle \circ\eta_3} & & \downarrow{\scriptstyle \circ\eta_3} \\
\pi_4(SO(3)) & \xrightarrow{\ S\ } & \pi_4(SO(4)) & \xrightarrow{\ E_4\ } & \pi_4(S^3),
\end{array}$$

where the horizontal sequences are part of the homotopy long exact sequence of the fibration $SO(3) \to SO(4) \to S^3$, the vertical maps are given by pre-composition with η_3, and the map E_3 takes the Euler class of the corresponding bundle. Since $E_3(\tau_{S^4}) = \pm 2 \in \pi_3(S^3) \cong \mathbb{Z}$, it follows that $E_3(\tau_{S^4}) \circ \eta_3 = 0$ and so $E_4(\tau_{S^4} \circ \eta_3) = 0$. Hence $\tau_{S^4} \circ \eta_3 \in \mathrm{Im}(S)$. Since Σ^8 is non-standard, $\tau_{S^4} \circ \eta_3$ is non-zero and this proves the claim. It follows that $\Sigma^8 \cong \sigma_{3,4}(\beta_4, \delta_3)$, where $\beta_4 \in \pi_3(SO(4))$ stabilises to β_5.

To relate Σ^8 to $\Sigma_{\beta_5, \gamma_3}$ we shall use the Milnor-Munkres-Novikov pairing [La2, p. 583],

$$\tau_{p,q} \colon \pi_p(SO_q) \times \Theta_q \longmapsto \Theta_{p+q}, \quad (\alpha, \Sigma) \longmapsto \partial W(\alpha, \Sigma),$$

where $W(\alpha, \Sigma)$ is the plumbing manifold

$$(D^q \tilde{\times}_\alpha S^{p+1}) \cup_{D^{p+1} \times D^q} (D^{p+1} \times \Sigma^q)$$

obtained by plumbing the disc bundle of α with the trivial $(p+1)$-disc bundle over the homotopy sphere Σ. By [Sc, Theorem 2.5], if $\mu_n \in \pi_1(SO(n)) \cong \mathbb{Z}_2$ is a generator for $n \geq 3$, then

$$\tau_{1,8}(\mu_8, \sigma_{3,4}(\beta_4, \delta_3))) = \sigma_{3,5}(S\beta_4, \delta_3 \circ \eta_4),$$

so long as the Samelson product $S(\beta_4) * S(\mu_4) \in \pi_4(SO(5))$ is trivial: we assume this for now and complete the proof. Since $\gamma_3 = \delta_3 \circ \eta_4$, it follows that $\Sigma_{\beta_5, \gamma_3} \cong \tau_{1,8}(\mu_8, \Sigma^8)$. But it is clear from the definition of the pairing $\tau_{p,q}$ that $\eta(\tau_{1,8}(\mu_8, \Sigma_8)) = [\eta(\Sigma^8) \circ \eta_8] \in \text{Coker}(J_9)$. But by [To, p. 189], $[\eta(\Sigma^8) \circ \eta_8] \neq 0 \in \text{Coker}(J_9)$ and so $\Sigma_{\beta_5, \gamma_3}$ does not belong to bP_{10}. On the other hand, $\Sigma_{\beta_5, \gamma_3}$ bounds a spin manifold by construction and so $\Sigma_{\beta_5, \beta_3} \in \text{Ker}(\alpha)$.

To complete the proof, we must show that the Samelson product $\beta_5 * \mu_5$ vanishes. It suffices to show that $\beta_4 * \mu_4$ vanishes. Recall that the Samelson product $\beta_4 * \mu_4 \colon S^4 \to SO(4)$ is defined to be the homotopy class of the map induced on S^4 by the following map

$$S^3 \times S^1 \to SO(4), \quad (x, \lambda) \mapsto \beta_4(x)\mu_4(y)\beta_4^{-1}(x)\mu_4^{-1}(y).$$

Now, we represent β_4 and μ_4 by the following maps:

$$\beta_4(x)(y) = x \cdot y \quad \text{and} \quad \mu_4(\lambda)(y) = y \cdot \lambda,$$

where $y \in \mathbb{H}$ is a quaternion, $x \in S^3$ a unit quaternion and $\lambda \in S^1 \subset S^3$ a unit complex number. Evidently $\beta_4(x), \mu_4(\lambda) \in SO(4)$ commute for all values of (x, λ) and hence the Samelson product $\beta_4 * \mu_4$ vanishes. $\square$

By Lemma 5.3 homotopy spheres in $bP_{2q+2} \subset \Theta_{2q+1}$ (i.e. the ones mapping trivially to $\text{Coker}(J_{2q+1})$) are all Stein fillable, while Theorem 5.4 shows that many homotopy spheres with non-trivial image in $\text{Coker}(J_{2q+1})$ do not admit Stein fillings. This observation naturally leads us to the following

Conjecture 5.9. *A homotopy sphere Σ^{2q+1} is Stein fillable if and only $\Sigma^{2q+1} \in bP_{2q+2}$. That is, in the notation of Theorem 5.4, $C_q^U = 0$ for all q.*

Notice that while Theorem 5.4 shows that many exotic spheres are not Stein fillable, those same homotopy spheres might admit symplectically fillable contact structures.

Problem 5.10 (Symplectic fillability of homotopy spheres)**.** *Do all homotopy spheres admit symplectically fillable contact structures? If not, then determine all those that do.*

The positive resolution of this problem would imply that symplectic fillability is invariant under the action of the group of exotic spheres under connect sum. Notice that although our Filling Theorem 3.7 is not useful in searching for symplectic fillings which are not also Stein fillings, Corollary 3.10 may be helpful in finding symplectically fillable contact structures on homotopy spheres which do not admit Stein fillings.

6. Further properties of Stein fillable manifolds

In this section we discuss several topological properties of Stein fillable manifolds.

6.1. **(Co)homological obstructions to Stein fillability.** In this subsection we discuss topological obstructions to Stein fillability, which are not present in dimension 3, and some of their consequences. (See also [PP] and [EKP] for similar obstructions.) As usual, let (M, φ) be an almost contact manifold with associated stable complex structure ζ, let $(B_\zeta^{q-1}, \eta_\zeta^{q-1})$ be the complex normal $(q-1)$-type of (M, ζ) and let $\bar{\zeta} \colon M \to B_\zeta^{q-1}$ be a ζ-compatible normal $(q-1)$-smoothing. We begin by observing that there is a commutative diagram,

$$
\begin{array}{ccc}
 & B_\zeta^{q-1} & \\
\overset{\bar{\zeta}}{\nearrow} & \downarrow{\scriptstyle p_B \times \eta_\zeta^{q-1}} & \overset{\eta_\zeta^{q-1}}{\searrow} \\
M \xrightarrow[p_M \times \zeta]{} & P_{q-1}(M) \times BU \xrightarrow[\mathrm{pr}_{BU}]{} & BU,
\end{array}
$$

where $P_{q-1}(B) \simeq P_{q-1}(M)$ is the $(q-1)^{st}$ Postnikov stage of M and B, and the maps $p_M \colon M \to P_{q-1}(M)$ and $p_B \colon B \to P_{q-1}(M)$ are q-equivalences. We see that the induced homomorphism

$$(p_B \times \eta_\zeta^{q-1})_* \colon \Omega_{2q+1}(B_\zeta^{q-1}) \to \Omega_{2q+1}^U(P_{q-1}(M))$$

is such that $(p_B \times \eta_\zeta^{q-1})_*([M, \bar{\zeta}]) = [(M, \zeta), p_M]$. Applying Theorem 3.7 we obtain

Lemma 6.1. *If $[(M, \zeta), p_M] \neq 0 \in \Omega_{2q+1}^U(P_{q-1}(M))$, then (M, ζ) does not admit a Stein fillable contact structure.* $\qquad\square$

The following proposition combines Lemma 6.1 with other elementary observations to give obstructions to Stein fillability. Let $\pi = \pi_1(M)$ denote the fundamental group of M.

Proposition 6.2. *Suppose that (M, φ) is an almost contact manifold of dimension $2q+1 \geq 5$ that admits a Stein fillable contact structure and let $u \colon M \to K(\pi, 1)$ be the classifying map of the universal cover of M. Then the following hold:*

(1) *The homomorphism $u_* \colon H_i(M; \mathbb{Z}) \to H_i(K(\pi, 1))$ vanishes for $q+2 \leq i \leq 2q+1$. In particular $u_*([M]) = 0 \in H_{2q+1}(K(\pi, 1))$, where $[M]$ denotes the fundamental class of M.*

(2) *M is not aspherical.*

(3) *For any $\beta \in H^j(P_{q-1}(M))$ and for any k-tuple $\{i_1, \ldots, i_k\}$ of positive integers with $2q + 1 - j = 2\left(\Sigma_{n=1}^k i_n\right)$, all products of the form $p_M^*(\beta) \cup c_{i_1}(\zeta) \cup \cdots \cup c_{i_k}(\zeta) \in H^{2q+1}(M)$ vanish.*

(4) *For any k-tuple $\{i_1, \ldots, i_k\}$ of positive integers with $\Sigma_{n=1}^k i_n = q$, all products of Chern classes $c_{i_1}(\zeta) \cup \cdots \cup c_{i_k}(\zeta) \in H^{2q}(M)$ vanish.*

Proof. (1) Let $(W, \bar{\zeta}_W)$ be a B-nullbordism of $(M, \bar{\zeta})$. After surgery we may assume that W has no handles in dimension greater than $q + 1$ and hence $H_i(W; \mathbb{Z}) = 0$ for $i > q + 1$. Now $\bar{\nu} \colon M \to B$ factors over W and $u \colon M \to K(\pi, 1)$ can be factored as $u_B \circ \bar{\nu} \colon M \to B \to K(\pi, 1)$, where $u_B \colon B \to K(\pi, 1)$ classifies the universal covering of B.

(2) If M is aspherical then $M \simeq K(\pi, 1)$ and so $u_*([M])$ is a generator of the group $H_{2q+1}(K(\pi, 1)) \cong \mathbb{Z}$. Now apply part (1).

(3) The integer

$$\langle p_M^*(\beta) \cup c_{i_1}(\zeta) \cup \cdots \cup c_{i_k}(\zeta), [M]\rangle$$

is an invariant of unitary bordism of $P_{q-1}(M)$. By Lemma 6.1 this integer vanishes for the unitary $P_{q-1}(M)$-manifold $((M, \zeta), p_M)$. Since $H^{2q+1}(M) \cong \mathbb{Z}$, this finishes the proof.

(4) We apply part (3) with $\beta \in H^1(P_{q-1}(M)) = H^1(M)$ and then use a version of part (1) with mod $\mathbb{Z}/p$ coefficients to conclude that $c_{i_1}(\zeta) \cup \cdots \cup c_{i_k}(\zeta)$ vanishes in $H^{2q}(M; \mathbb{Q})$ and also in $H^{2q}(M; \mathbb{Z}/p)$ for all primes p. It follows that $c_{i_1}(\zeta) \cup \cdots \cup c_{i_k}(\zeta) = 0 \in H^{2q}(M)$. $\square$

Proposition 6.2 allows us to prove the following

Corollary 6.3. *In general, the Stein fillability of an almost contact manifold (M, φ) depends on the choice φ and not just the underlying diffeomorphism type of M.*

Proof. The manifold $M = S^1 \times S^6$ clearly admits a Stein fillable almost contact structure φ_0 since $M = \partial(S^1 \times D^7)$. On the other hand, S^6 admits an almost complex structure J with $c_3(J) = 2 \in H^6(S^6)$. For the induced almost contact structure φ_1 on M, Proposition 6.2 (4) implies that (M, φ_1) is not Stein fillable. $\square$

As a consequence of Proposition 6.2, we obtain obstructions to the Stein fillability of certain Boothby-Wang contact structures.

Example 6.4 (Boothby-Wang contact structures). A *Boothby-Wang contact structure* on a (nontrivial) principal S^1-bundle

$$S^1 \longrightarrow E \xrightarrow{\pi} B$$

over a symplectic base (B, ω) of dimension $2q$ with $c_1(E) = [\frac{\omega}{2\pi}]$ is given as the kernel of an S^1-invariant 1-form α which is non-vanishing on the fibers and satisfies $d\alpha = \pi^*(\omega)$ for some integral symplectic form ω.

Note that the associated disc bundle of the principal S^1-bundle E is a strong symplectic filling (see, e.g., [GS], Lemma 3), which is not Stein since it is homotopy equivalent to the $2q$-dimensional base B. (However, if the base is $\mathbb{C}P^2$ and the Euler class of the bundle is a generator of $H^2(\mathbb{C}P^2; \mathbb{Z})$, then the total space is the 5-sphere which is of course Stein fillable.) On the other hand we do have the following example:

Example 6.5 (Lens spaces). Let L_k^5 be the standard 5-dimensional lens space with cyclic fundamental group of order k. That is, L_k^5 is the quotient of

$$S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\},$$

with the action of a generator of $\mathbb{Z}_k$ defined by

$$(z_1, z_2, z_3) \mapsto (\mu z_1, \mu z_2, \mu z_3)$$

for $\mu \in \mathbb{C}$ a kth root of unity. The resulting manifold inherits an S^1-bundle projection $\pi \colon L_k^5 \to \mathbb{C}P^2$. Since the classifying map of the universal cover u induces a non-trivial map

$$u_* \colon H_i(L_k^5) \to H_i(K(\mathbb{Z}_k, 1)),$$

we conclude that although the lens spaces L_k^5 are symplectically fillable (by the Boothby-Wang construction), by Lemma 6.2 (1) they are not Stein fillable for all $k \geq 2$. (Obstructions for Stein fillability of these manifolds were already noticed in [EKP], cf. also [PP].)

In conclusion, we see both examples of Boothby-Wang contact structures which are Stein fillable, and others which are not. This observation leads to the following question:

Problem 6.6 (Fillability of Boothby-Wang manifolds)**.** *Determine which Boothby-Wang manifolds are Stein/exactly fillable.*

By recent work of Massot, Niederkrüger and Wendl [MNW], Proposition 6.2 also gives examples of exactly fillable contact structures that are not Stein fillable in all dimensions. Such examples were discussed in [Bow] for 3-dimensional manifolds, although in this case the non-fillability only applied to certain contact structures rather than to the manifolds themselves. We are now in the position to provide the proof of Theorem 1.5 from the Introduction:

Proof of Theorem 1.5. By [MNW, Theorem C], there are exact symplectic fillings of the form $M \times [0,1]$ such that both ends are convex in all dimensions $2q+2$. The manifolds M are quotients of contractible Lie groups and are consequently aspherical. After attaching a Weinstein 1-handle to $M \times [0,1]$, we obtain an exact filling of $N = -M \# M$. Assuming that $q > 1$, $\pi_1(N)$ is the free product two copies of $\pi_1(M)$ and so there is a homotopy equivalence $K(\pi_1(N), 1) \simeq K(\pi_1(M), 1) \vee K(\pi_1(M), 1)$. Since M is aspherical, we see that the classifying map of the universal cover of N maps non-trivially on $H_{2q+1}(N)$. Hence by Proposition 6.2 (1), N is not Stein fillable if $q > 1$. □

6.2. **Stein fillability and orientations.** A cooriented contact structure $\xi = \ker(\alpha)$ determines an orientation of the underlying $(2q+1)$-manifold M, since the form $\alpha \wedge (d\alpha)^q$ is nowhere vanishing. When we speak of an oriented manifold admitting a contact structure, we mean that the orientation determined by the contact structure is the given one. Moreover, if the dimension of M is of the form $4k+1$, and hence the dimension of the Stein filling of M is of the form $4k+2$, then taking the conjugate complex structure on W reverses orientations. The resulting Stein fillable contact structure then gives the opposite coorientation of ξ, i.e. replaces α by $-\alpha$, which in turn swaps the orientation determined by the contact structure. So in these dimensions it is clear that M is Stein fillable if and only if $-M$ is.

However, if the dimension of M is $4k+3$, then this is not immediately clear and it is indeed false in dimension 3, with many examples given by Seifert fibred spaces, the most famous of which is the Poincaré homology sphere [Li]. On the other hand, Eliashberg's h-principle implies the following

Proposition 6.7. *Let (M, φ) be an almost contact $(2q+1)$-dimensional manifold with $q \geq 2$ and associated stable complex structure ζ. Then (M, ζ) is Stein fillable if and only if $(-M, -\zeta)$ is.*

Proof. The fact that any Stein filling of (M, ζ) is a manifold with boundary means that TW admits a nonvanishing section and thus as complex bundles

$$(TW, J) \cong (E, J|_E) \oplus \underline{\mathbb{C}}.$$

We then define an almost complex structure $\bar{J}$ by taking $J|_E$ on E and the conjugate complex structure on $\underline{\mathbb{C}}$. The almost complex structure $\bar{J}$ then induces the orientation $-W$, and applying Eliashberg's h-principle gives a Stein fillable contact structure on $-M$ with associated stable complex structure $-\zeta$. □

## 7. Subcritical Stein fillings and Stein fillings of products

We fix a closed almost contact $(2q+1)$-manifold (M, φ) and as usual we let $\zeta = S\varphi$ denote the stable complex structure induced by the almost contact structure φ. A *subcritical Stein filling* of (M, φ) is a Stein filling (W, J) of (M, φ) where W admits a handle decomposition with handles of dimension q and less. Subcritical Stein fillings have special properties; see [CE].

Another filling question is the following: suppose that (F, J_F) is an almost complex structure on a closed, orientable surface F. Then we can ask if the product almost contact manifold $(M \times F, \varphi \times J_F)$ admits a Stein filling. It is easy to see that if (M, φ) has a subcritical Stein filling, then $(M \times F, \varphi \times J_F)$ is Stein fillable: if (W, J_W) is the subcritical filling of (M, φ) then $(W \times F, J_W \times J_F)$ is an almost complex manifold with boundary $(M \times F, \varphi \times J_F)$ which admits a handle decomposition with handles of dimension $q + 2$ and less (and the dimension of $W \times F$ is $2q + 4$), therefore Eliashberg's h-principle implies the result.

We shall further relate the two questions about Stein fillings to the bordism theory of $(B_\zeta^q, \eta_\zeta^q)$, the complex normal q-type of (M, ζ). We pose five related questions:

(A) When does (M, φ) admit a subcritical Stein filling?
(B) When does $(M \times F, \varphi \times J_F)$ admit a Stein filling?
(C) When does $[M, \bar{\zeta}] = 0 \in \Omega_{2q+1}(B_\zeta^q, \eta_\zeta^q)$ hold?
(D) When does $\bar{\zeta}_*([M]) = 0 \in H_{2q+1}(B_\zeta^q)$ hold?
(E) When does $TH_q(M)$, the torsion subgroup of $H_q(M)$, vanish?

We next graphically summarise the relationship between positive answers to the questions above, writing $g(F) > 0$ for the case where F has positive genus; see Theorem 7.1 below.

$$(A) \Longrightarrow (B) \Longrightarrow (D) \Longrightarrow (E)$$

$$(C) \xleftarrow{\quad \text{if } g(F) > 0 \quad}$$
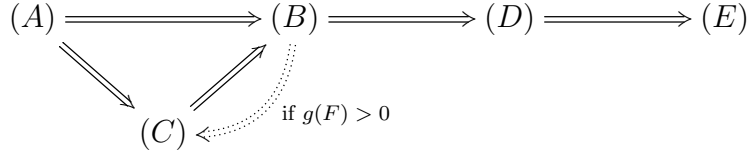
Theorem 7.1 (Subcritical Filling Theorem). *Let $(B_\zeta^q, \eta_\zeta^q)$ be the complex normal q-type of (M, ζ) and let $\bar{\zeta} \colon M \to B_\zeta^q$ be any ζ-compatible normal q-smoothing. If $q \geq 2$, then the following hold.*

(1) *If (M, φ) admits a subcritical filling then $[M, \bar{\zeta}] = 0 \in \Omega_{2q+1}(B_\zeta^q, \eta_\zeta^q)$.*
(2) *If $[M, \bar{\zeta}] = 0 \in \Omega_{2q+1}(B_\zeta^q, \eta_\zeta^q)$ then $(M \times F, \varphi \times J_F)$ admits a Stein filling.*
(3) *If $(M \times F, \varphi \times J_F)$ admits a Stein filling and $g(F) > 0$, then the bordism class $[M, \bar{\zeta}]$ satisfies $[M, \bar{\zeta}] = 0 \in \Omega_{2q+1}(B_\zeta^q, \eta_\zeta^q)$. In particular, (M, ζ) is Stein fillable.*
(4) *If $(M \times F, \varphi \times J_F)$ admits a Stein filling then $\bar{\zeta}_*([M]) = 0 \in H_{2q+1}(B_\zeta^q)$.*
(5) *If $\bar{\zeta}_*([M]) = 0 \in H_{2q+1}(B_\zeta^q)$ then $TH_q(M) = 0$.*

Proof. (1) The proof is similar to the proof of part (1) of Lemma 2.10. Let (W, ζ_W) denote the subcritical filling with its induced stable complex structure; it is built from (M, ζ) by adding handles with stable complex structure of dimension $q + 2$ and higher. Therefore the complex normal q-type of M can be identified that of W and the claim follows.

(2) Let ζ_F be the stable normal complex structure defined by J_F, let $P_q(F)$ be the qth Postnikov stage of F and let $p_F \colon F \to P_q(F)$ be a $(q+1)$-equivalence (if $g(F) > 0$, then $P_q(F) = K(\pi_1(F), 1)$), and let L_{ζ_F} be the unique complex line bundle over $P_q(F)$ such that $c_1(\zeta_F) = p_F^*(c_1(L_{\zeta_F}))$. The complex normal q-type of $(M \times F, \zeta \times \zeta_F)$ is given by

$$(B_{\zeta \times \zeta_F}^q, \eta_{\zeta \times \zeta_F}^q) = (B_\zeta^q \times P_q(F), \eta_\zeta^q \oplus L_{\zeta_F}),$$

where, as in Section 2.5, $\eta_\zeta^q \oplus L_{\zeta_F}$ denotes the exterior Whitney sum of stable complex bundles. By assumption there is a $(B_\zeta^q, \eta_\zeta^q)$-null bordism $(W, \bar\zeta_W)$ of $(M, \bar\zeta)$. We observe that

$$\bar\zeta_W \times p_F \colon W \times F \to B_\zeta^q \times P_q(F)$$

is a $(B_{\zeta \times \zeta_F}^q, \eta_\zeta^q \times L_{\zeta_F})$-nullbordism of $(M \times F, \zeta \times \zeta_F)$. By Theorem 3.7, $(M \times F, \varphi \times J_F)$ is Stein fillable.

(3) If $g(F) > 0$, then F is a $K(\pi, 1)$ manifold and $P_q(F) = F$. It follows that the complex normal q-type of $(M \times F, \zeta \times \zeta_F)$ is given by

$$(B_{\zeta \times \zeta_F}^q, \eta_{\zeta \times J_F}^q) = (B_\zeta^q \times F, \eta_\zeta^q \oplus L_{\zeta_F}),$$

where L_{ζ_F} is defined as in the proof of (2). There is a canonical isomorphism of bordism groups

$$\theta \colon \Omega_*(B_\zeta^q \times F, \eta_\zeta^q \oplus \zeta_F) \;\cong\; \Omega_*^{(B_\zeta^q, \eta_\zeta^q)}(F; L_{\zeta_F})$$

with range the ζ_F-twisted $(B_\zeta^q, \eta_\zeta^q)$-bordism group of F. Taking the transverse inverse image of a point $x \in F$ defines a homomorphism

$$\pitchfork \colon \Omega_*^{(B_\zeta^q, \eta_\zeta^q)}(F; L_{\zeta_F}) \to \Omega_{*-2}(B_\zeta^q, \eta_\zeta^q).$$

On the other hand, taking the product with (F, ζ_F) defines a homomorphism

$$\Pi \colon \Omega_*(B_\zeta^q, \eta_\zeta^q) \to \Omega_{*+2}(B_\zeta^q \times F, \eta_\zeta^q \oplus \zeta_F), \quad [X, \zeta_X] \mapsto [X \times F, \zeta_X \times \zeta_F].$$

From the definitions of the above homomorphisms, we see that there is a commutative diagram

$$\begin{array}{ccc}
\Omega_*(B_\zeta^q, \eta_\zeta^q) & \xrightarrow{\quad\text{id}\quad} & \\
\Big\downarrow{\scriptstyle\Pi} & & \\
\Omega_{*+2}(B_\zeta^q \times F, \eta_\zeta^q \oplus \zeta_F) \xrightarrow{\;\theta\;} \Omega_{*+2}^{(B_\zeta^q, \eta_\zeta^q)}(F; L_{\zeta_F}) \xrightarrow{\;\pitchfork\;} & \Omega_*(B_\zeta^q, \eta_\zeta^q).
\end{array}$$

If $(M \times F, \varphi \times J_F)$ is Stein fillable, then by Theorem 3.7, $[M \times F, \bar\zeta \times \bar\zeta_F] = \Pi([M, \zeta]) = 0 \in \Omega_{2q+3}(B_\zeta^q \times F, \eta_\zeta^q \oplus \zeta_F)$. The diagram then shows that that $[M, \bar\zeta] = 0 \in \Omega_{2q+1}(B_\zeta^q, \eta_\zeta^q)$.

(4) If $(M \times F, \zeta \times \zeta_F)$ is Stein fillable then by Theorem 3.7 all $(\zeta \times \zeta_F)$-compatible normal q-smoothings of $(M \times F, \zeta \times \zeta_F)$ bound over $(B_\zeta^q \times P_q(F), \eta_\zeta^q \times L_{\zeta_F})$. As a consequence,

$$(\bar\zeta \times p_F)_*([M \times F]) = 0 \in H_{2q+3}(B_\zeta^q \times P_q(F)).$$

Since $(p_F)_*([F]) \in H_2(P_q(F)) \cong \mathbb{Z}$ is a generator, the result now follows from the Kunneth theorem.

(5) Recall that the linking form of M is a nonsingular bilinear pairing

$$TH_q(M) \times TH_q(M) \to \mathbb{Q}/\mathbb{Z}.$$

We will show that the assumption $\bar\zeta_*([M]) = 0$ ensures that the linking form of M vanishes, and this can only happen if $TH_q(M)$ vanishes.

Let $p \colon H_q(M) \to TH_q(M)$ be a splitting and let $p \colon K(H_q(M), q) \to K(TH_q(M), q)$ also denote the induced map of Eilenberg-MacLane spaces. The map $q_M \colon M \to K(H_q(M), q)$ inducing the identity on H_q is such that the composition

$$M \xrightarrow{\;q_M\;} K(H_q(M), q) \xrightarrow{\;p\;} K(TH_q(M), q)$$

satisfies $(p \circ q_M)_*([M]) \neq 0 \in H_{2q+1}(K(TH_q(M)), q)$ if $TH_q(M) \neq 0$: This follows from the cohomological definition of the linking form and its nonsingularity. But since the map $\bar{\zeta} \colon M \to B_\zeta^q$ is a $(q+1)$-equivalence, it follows that q_M can be factored through $\bar{\zeta}$. Hence if $\bar{\zeta}_*([M]) = 0$ then $(p \circ q_M)_*([M]) = 0$, the linking form of M vanishes, and $TH_q(M) = 0$. $\square$

Example 7.2. The converse of (1) in Theorem 7.1 (that $[M, \bar{\zeta}] = 0 \in \Omega_{2q+1}(B_\zeta^q, \eta_\zeta^q)$ implies that (M, φ) is subcritically Stein fillable) does not hold. Notice first that the adaptation of the proof breaks down, since the surgery method of [Kr2] works only up to the middle dimension. Indeed, if $\Sigma \in bP_{2q+2}$ is exotic, then Σ admits an almost contact structure with stabilisation ζ such that $[\Sigma, \zeta] = 0 \in \Omega_{2q+1}(B_\zeta^q, \eta_\zeta^q)$, but Σ does not admit a subcritical Stein filling: a subcritical Stein filling of a homotopy sphere must be contractible, implying that the filling is diffeomorphic to the disk and that the homotopy sphere is standard.

Example 7.3. A simple example of a Stein fillable manifold M with the property that $M \times S^2$ is not Stein fillable is provided by $M = S^1 \times S^2 \times S^2$: by the fact that $M = \partial(S^1 \times S^2 \times D^3)$ we see that it is Stein fillable, while Proposition 6.2 implies that $S^1 \times S^2 \times S^2 \times S^2$ is not Stein fillable.

Theorem 7.1 shows that the existence of a subcritical filling of (M, φ) places strong constraints on the topology of M. We next pursue this point further for simply connected manifolds in dimensions 5 and 7. Let $S^3 \widetilde{\times} S^2$ and $S^5 \widetilde{\times} S^2$ be the total spaces of the nontrivial linear n-sphere bundle over the 2-sphere, $n = 3, 5$.

Proposition 7.4. *Suppose that (M, φ) is a simply connected almost contact manifold of dimension 5 or 7 and that (M, φ) admits a subcritical Stein filling.*

(1) *If M has dimension 5, then there are nonnegative integers r, s such that M is diffeomorphic to one of the connected sums*

$$\sharp_r(S^3 \times S^2) \quad or \quad (S^3 \widetilde{\times} S^2) \sharp_r(S^3 \times S^2),$$

depending on whether M is spin or not.

(2) *If M has dimension 7 and $\pi_2(M)$ is torsion free, then there are integers r, s such that M is diffeomorphic to one of the connected sums*

$$\sharp_r(S^5 \times S^2) \sharp_s(S^4 \times S^3) \quad or \quad (S^5 \widetilde{\times} S^2) \sharp_r(S^5 \times S^2) \sharp_s(S^4 \times S^3),$$

depending on whether M is spin or not.

Proof. If (W, J) is a subcritical filling of (M, φ), then W is obtained from M by attaching $(q+2)$-handles and higher. It follows that the map $M \to W$ is a $(q+1)$-equivalence. Now by Theorem 7.1 (1), (3), (4) and (5), $TH_q(M) = 0$ and so $TH_q(W) = 0$. Since W is also a simply connected manifold consisting only of handles of dimension q or less, we conclude the following: if $q = 2$, it follows the W is homotopy equivalent to a wedge of 2-spheres and if $q = 3$, then W is homotopy equivalent to wedge of 2-spheres and 3-spheres. Note that for the case $q = 3$ we use the assumption that $\pi_2(M) \cong H_2(M)$ is torsion free. It follows, using the terminology of [Wa2] that the manifold W is then a stable thickening of a wedge of spheres. By [Wa2, Propsition 5.1] stable thickenings are classified up to diffeomorphism by their homotopy type and the map classifying their stable tangent bundle. Now for the W we consider, $[W, BSO] \cong H^2(W; \mathbb{Z}_2)$, the bijection being given by the second Stiefel-Whitney class. If $\natural$ denotes the boundary connected sum of manifolds with boundary and $D^4 \widetilde{\times} S^2$ and

$D^6 \widetilde{\times} S^2$ denote the non-trivial linear disc bundles over S^2, we deduce that W is diffeomorphic to one of the following manifolds:

Dimension 5: $\natural_r(D^4 \times S^2)$ or $(D^4 \widetilde{\times} S^2)\natural_r(D^4 \times S^2)$,

Dimension 7: $\natural_r(D^6 \times S^2)\sharp_s(D^5 \times S^3)$ or $(D^6 \widetilde{\times} S^2)\sharp_r(D^6 \times S^2)\sharp_s(D^5 \times S^3)$,

and the proposition follows. $\square$

We conclude this section by viewing Example 7.2 in a more general framework. Let $L_{2q+2}^{s,\tau}(\pi)$ denote the group of units in the surgery obstruction monoid $l_{2q+2}(\pi)$, which was defined in [Kr2, §6]. (The notation is from [Kr1, §4] and differs from [Kr2]. In addition, $L_{2q+2}^{s,\tau}(\pi)$ may be identified with the group obstruction group $L_{2q+2}^C(\pi)$ of [Wa4, 17D].) The group $L_{2q+2}^{s,\tau}(\pi)$ acts on the set of $(B_\zeta^q, \eta_\zeta^q)$-diffeomorphism classes of complex normal q-smoothings $\bar{\zeta} \colon M \to B_\zeta^q$ without changing the $(B_\zeta^q, \eta_\zeta^q)$-bordism class. That is, writing $(M+\rho, \bar{\zeta}_{+\rho})$ for the action of $\rho \in L_{2q+2}^{s,\tau}(\pi)$ on $(M, \bar{\zeta})$, we have

$$[M, \bar{\zeta}] = [M+\rho, \bar{\zeta}_{+\rho}] \in \Omega_{2q+1}(B_\zeta^q, \eta_\zeta^q).$$

For example, if M is simply connected, then $L_{2q+2}^{s,\tau}(e) \cong \mathbb{Z}$ or $\mathbb{Z}_2$ as q is odd or even, and the action of $L_{2q+2}^{s,\tau}(e)$ is via connected sum with $(\Sigma, \bar{\zeta}_\Sigma)$, where Σ is a generator of bP_{2q+2} and $\bar{\zeta}_\Sigma$ is a certain a $(B_\zeta^q, \eta_\zeta^q)$-structure on Σ.

Question 7.5. *Suppose that $\bar{\zeta} \colon M \to B_\zeta^q$ is a normal q-smoothing such that $[M, \bar{\zeta}] = 0 \in \Omega_{2q+1}(B_\zeta^q, \eta_\zeta^q)$. Under what conditions on M can we deduce that there is an element $\rho \in L_{2q+2}^{s,\tau}(\pi)$ such that $(M+\rho, \bar{\zeta}_{+\rho})$ admits a subcritical Stein filling? For example, if M is simply connected, is there a homtopy sphere $\Sigma \in bP_{2q+2}$ such that $(M\sharp\Sigma, \bar{\zeta}\sharp\bar{\zeta}_\Sigma)$ admits a subcritical Stein filling?*

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MATH. INSTITUT, UNIVERSITÄT AUGSBURG, UNIVERSITÄTSTR 14, 86159 AUGSBURG, GERMANY
E-mail address: jonathan.bowden@math.uni-augsburg.de

MAX PLANCK INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, D-53111 BONN, GERMANY
E-mail address: diarmuidc23@gmail.com

RÉNYI INSTITUTE OF MATHEMATICS, REÁLTANODA U. 13-15., BUDAPEST, HUNGARY H-1053
E-mail address: stipsicz@renyi.hu