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A SYMPLECTICALLY NON-SQUEEZABLE SMALL SET AND THE REGULAR COISOTROPIC CAPACITY

JAN SWOBODA AND FABIAN ZILTENER

Contents

Abstract. We prove that there exists a compact subset X of the closed ball in $\mathbb{R}^{2n}$ of radius $\sqrt{2}$, such that X has Hausdorff dimension n and does not symplectically embed into the standard open symplectic cylinder. The second main result is a lower bound on the d-th regular coisotropic capacity, which is sharp up to a factor of 3. For an open subset of a geometrically bounded, aspherical symplectic manifold, this capacity is a lower bound on its displacement energy. The proofs of the results involve a certain Lagrangian submanifold of linear space, which was considered by M. Audin and L. Polterovich.

1. Motivation and results

Continuing our previous work [SZ1, SZ2], in the present article we study the following question.

Question. *How much symplectic geometry can a small subset of a symplectic manifold carry?*

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More concretely, we are concerned with the problem of finding a small subset of $\mathbb{R}^{2n}$ that cannot be squeezed symplectically. To explain this, let (M,ω) and (M',ω') be symplectic manifolds, and $X \subseteq M$ a subset. We say that X *(symplectically) embeds into* M' iff there exists an open neighborhood $U \subseteq M$ of X and a symplectic embedding $\varphi\colon U \to M'$. Let $n \in \mathbb{N}$. For $a > 0$ we denote by $B^{2n}(a)$ and $\overline{B}^{2n}(a)$ the open and closed balls in $\mathbb{R}^{2n}$, of radius $\sqrt{a/\pi}$, around 0. (These balls have Gromov-width a.) We denote

$$\begin{aligned} &B^{2n} := B^{2n}(\pi), \quad \overline{B}^{2n} := \overline{B}^{2n}(\pi), \quad \mathbb{D} := \overline{B}^2 \\ &Z^{2n}(a) := B^2(a) \times \mathbb{R}^{2n-2}, \quad Z^{2n} := Z^{2n}(\pi), \\ &\overline{Z}^{2n}(a) := \overline{B}^2(a) \times \mathbb{R}^{2n-2}, \quad \overline{Z}^{2n} := \overline{Z}^{2n}(\pi). \end{aligned}$$

Let $d \in [0,2n]$.

Question. *What is*

$$a(n,d) := \inf a,$$

where the infimum runs over all numbers $a > 0$, for which there exists a compact subset X of $B^{2n}(a)$ of Hausdorff dimension at most d, such that X does not symplectically embed into Z^{2n}?

The collection of numbers $a(n,d)$ ($d \in [0,2n]$) measures how small a subset of $\mathbb{R}^{2n}$ can be and still carry interesting symplectic non-embedding information. Here we interpret smallness in two ways: in the sense of Hausdorff dimension and in terms of the size of a ball containing the subset. Note that we always have $a(n,d) \geq \pi$, and $a(n,d)$ is decreasing in d. Furthermore, if $d < 2$ then $a(n,d) = \infty$. This is a consequence of the following result.

1. **Proposition** (Two-dimensional squeezing). *For all $n \in \mathbb{N}$ and every $a > 0$, every compact subset X of $\mathbb{R}^{2n}$ with vanishing 2-dimensional Hausdorff measure symplectically embeds into $Z^{2n}(a)$.*

In contrast with this, a straight-forward argument shows that $a(1,2) = \pi$. Hence in the case $n = 1$, the values $a(1,d)$ are all known.

Consider now the case $n \geq 2$. We are interested in finding an upper bound on $a(n,d)$. Gromov's non-squeezing result (cf. [Gr]) implies that $a(n,2n) = \pi$. This can be strengthened to the equality $a(n,2n-1) = \pi$, which follows from [SZ1, Theorem 6]. The first main result is the following. We define

$$\overline{P}_n := \begin{cases} \mathbb{D}^n, & \text{if } n \text{ is even,} \\ \mathbb{D}^{n-1} \times \mathbb{R}^2, & \text{if } n \text{ is odd.} \end{cases}$$

2. **Theorem** (Non-squeezable small set)**.** *For every $n \geq 2$ there exists a compact subset*

$$X \subseteq \overline{P}_n \cap \overline{B}^{2n}(2\pi)$$

of Hausdorff dimension n, which does not symplectically embed into Z^{2n}.

It follows that $a(n,d) \leq 2\pi$, for every $d \in [n, 2n]$. The set X in this result is almost "minimal": If $z \in S^1 = \partial\mathbb{D}$ then the statement of Theorem 2 is wrong, if $\overline{P}_n$ is replaced by $(\mathbb{D} \setminus \{z\}) \times \mathbb{D}^{n-1}$ (case n even), or $(\mathbb{D} \setminus \{z\}) \times \mathbb{D}^{n-2} \times \mathbb{R}^2$ (case n odd), respectively. This follows from an elementary argument, using compactness of X and Moser isotopy in two dimensions. Furthermore, the condition $X \subseteq \overline{B}^{2n}(2\pi)$ is "sharp up to a factor of 2". In fact, the following holds.

3. **Proposition.** *For $n \in \mathbb{N}$ every compact subset of $\overline{B}^{2n}$ with vanishing $(2n-1)$-dimensional Hausdorff measure symplectically embeds into Z^{2n}.*

In the case $n \geq 2$ the condition on the Hausdorff measure in this result is necessary, since then the unit sphere does not symplectically embed into Z^{2n}. (See [SZ1, Corollary 5].)

The idea of the proof of Theorem 2 is to construct X out of a certain closed Lagrangian submanifold $L \subseteq \mathbb{R}^{2n}$ that is contained in $\overline{B}^{2n}$ and has minimal symplectic area equal to $\frac{\pi}{2}$. This submanifold was studied by A. Weinstein [We], M. Audin [Au], and L. Polterovich [Po]. In order to achieve the properties stated in Theorem 2, we need to suitably rotate and rescale L, and glue a disk to it, so that the resulting space is simply-connected. That this space cannot be squeezed into Z^{2n} will be a consequence of a result by Y. Chekanov about the displacement energy of a Lagrangian submanifold.

We may ask whether the condition in Theorem 2 on the Hausdorff dimension of X is optimal:

Question. *Does every compact set $X \subseteq \mathbb{R}^{2n}$ of Hausdorff dimension less than n symplectically embed into an arbitrarily small symplectic cylinder or ball? Is this even true for any compact set X with vanishing n-dimensional Hausdorff measure?*

To our knowledge these questions are open.

An important class of "small" subsets of a given symplectic manifold are coisotropic submanifolds. Based on these submanifolds, in [SZ1] we defined a collection of capacities, one for each $d \in \{n, \ldots, 2n-1\}$. Our second main result implies that the n-th capacity is normalized up to a factor of 2, and that for $d > n$ the d-th capacity is normalized up to a factor of 3.

To explain this, we call a symplectic manifold (M,ω) *(symplectically) aspherical* iff for every $u\in C^\infty(S^2,M)$ we have $\int_{S^2}u^*\omega=0$. Let $d\in\{n,\ldots,2n-1\}$. We define the *d-th regular coisotropic capacity* to be the map

$$A^d_{\mathrm{coiso}}\colon \big\{\text{aspherical symplectic manifold, }\dim M=2n\big\}\to[0,\infty],$$
$$A^d_{\mathrm{coiso}}(M,\omega):=\sup A(N),$$

where $N\subseteq M$ runs over all non-empty closed regular coisotropic submanifolds of dimension d, satisfying the following condition:

(1) $\forall$ isotropic leaf F of N, $\forall x\in C(S^1,F)\colon$ x is contractible in M.

Here $A(N)=A(M,\omega,N)$ denotes the minimal (symplectic) area (or action) of the coisotropic submanifold N. (For explanations see Subsection 2.1.) By [SZ1, Theorem 4] the map A^d_{coiso} is a (not necessarily normalized) symplectic capacity. The d-th regular coisotropic capacity of an open subset of an aspherical symplectic manifold (M,ω) is a lower bound on its displacement energy, if (M,ω) is geometrically bounded. (This follows from [Zi, Theorem 1.1].) For $d=n$ we abbreviate

$$A_{\mathrm{Lag}}:=A^n_{\mathrm{coiso}}.$$

Since every Lagrangian submanifold is regular, $A_{\mathrm{Lag}}(M,\omega)$ equals the supremum of all minimal areas $A(L)$, where L runs over all those non-empty closed Lagrangian submanifolds of M, for which every continuous loop in L is contractible in M. Our second main result is the following. We denote by ω_0 the standard symplectic form on $\mathbb{R}^{2n}$. Let $n\geq 2$.

4. **Theorem** (Regular coisotropic capacity)**.** *We have*

$$A_{\mathrm{Lag}}(B^{2n}):=A_{\mathrm{Lag}}(B^{2n},\omega_0)\geq\tfrac{\pi}{2},\tag{2}$$
$$A^d_{\mathrm{coiso}}(B^{2n})\geq\tfrac{\pi}{3},\quad\forall d\in\{n+1,\ldots,2n-3\}.\tag{3}$$

In [SZ1, Theorem 4] we proved the inequalities

$$A^d_{\mathrm{coiso}}(Z^{2n})\leq\pi,\ \forall d\in\{n,\ldots,2n-1\},$$
$$A^{2n-1}_{\mathrm{coiso}}(B^{2n})=\pi,$$
$$A^{2n-2}_{\mathrm{coiso}}(B^{2n})\geq\tfrac{\pi}{2}.$$

Combining this with Theorem 4, it follows that the capacity A^d_{coiso} is normalized for $d=2n-1$, normalized up to a factor of 2 for $d=n$ and $2n-2$, and up to a factor of 3, otherwise. (In the case $n=1$ the Lagrangian capacity A_{Lag} is also normalized.)

Remark. *Consider the* oriented Lagrangian capacity A^+_{Lag}*, which we define like* A_{Lag}*, by requiring additionally that the Lagrangian submanifold* L *is orientable. Then we have*

$$A^+_{\mathrm{Lag}}(B^4) = \frac{\pi}{2}.$$

To see this, we denote by $\mathbb{T}^2 = (S^1)^2$ *the standard torus in* $\mathbb{R}^4$*. For every* $r < \frac{1}{\sqrt{2}}$ *the rescaled torus* $r\mathbb{T}^2$ *is a Lagrangian submanifold of* B^4*, with minimal area* πr^2*. It follows that* $A^+_{\mathrm{Lag}}(B^4) \geq \frac{\pi}{2}$*. To see the opposite inequality, note that every orientable closed connected Lagrangian submanifold* $L \subseteq B^4$ *is diffeomorphic to the torus* $\mathbb{T}^2$*, since its Euler characteristic vanishes. For such an* L*, K. Cieliebak and K. Mohnke proved* [CM] *that* $A(L) < \frac{\pi}{2}$*. The statement follows.*

Related work.

Work related to Theorem 2. M. Gromov's famous non-squeezing result [Gr] says that the ball $B^{2n}(a)$ does not symplectically embed into the cylinder Z^{2n}, if $a > \pi$. Sikorav [Si] proved that there does not exist a symplectomorphism of $\mathbb{R}^{2n}$ which maps $\mathbb{T}^n$ into Z^{2n}. F. Schlenk noted in [Schl] (p. 8), that combining this result with the Extension after Restriction Principle implies the "Symplectic Hedgehog Theorem": For every $n \geq 2$, no starshaped domain in $\mathbb{R}^{2n}$ containing the torus $\mathbb{T}^n$ symplectically embeds into the cylinder Z^{2n}. It follows that no neighborhood of the set

$$\{ax \,|\, a \in [0,1],\, x \in \mathbb{T}^n\}$$

can be squeezed into Z^{2n}. This set has Hausdorff dimension $n+1$ and is contained in the ball $\overline{B}^{2n}(n\pi)$. This shows that $a(n, n+1) \leq n\pi$. Theorem 2 improves this statement in two ways: The set X in this result has Hausdorff dimension only n and is contained in the ball of radius only $\sqrt{2}$.

In [SZ1, Theorem 6] the authors proved that $a(n,d)$ is bounded above by π times some integer, which is a combinatorial expression in n and d. For $n = d$ this integer behaves asymptotically like $\sqrt{n}$, as $n \to \infty$.

Work related to the regular coisotropic capacity and Theorem 4. Let $n \in \mathbb{N}$. We denote

$$\mathcal{M} := \{(M,\omega) \text{ symplectic manifold } | \; \dim M = 2n,\, \pi_i(M) \text{ trivial },\, i = 1,2\}.$$

In [CM] K. Cieliebak and K. Mohnke defined the *Lagrangian capacity* to be the map $c_L\colon \mathcal{M} \to [0,\infty)$, given by

$$c_L(M,\omega) := \sup\big\{A(M,\omega,L)\,\big|\,L \subseteq M \text{ embedded Lagrangian torus}\big\}.$$

(See also [CHLS], Sec. 2.4, p. 11.) The authors proved that

$$c_L(B^{2n},\omega_0) = \frac{\pi}{n}. \tag{4}$$

The capacity c_L is bounded above by A_{Lag}. For $n \geq 3$, it is strictly smaller than A_{Lag}, when applied to (B^{2n},ω_0). This follows from inequality (2) and equality (4).

Organization. In Section 2 we give some background on coisotropic submanifolds, and we prove Propositions 1,3, and Theorems 2,4.

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2. Background and proofs of the results of section 1

2.1. **Background.** Let (M,ω) be a symplectic manifold and $N \subseteq M$ a submanifold. Then N is called *coisotropic* iff for every $x \in N$ the subspace

$$T_xN^\omega = \big\{v \in T_xM\,\big|\,\omega(v,w) = 0,\ \forall w \in T_xN\big\}$$

of T_xM is contained in T_xN. Examples include $N = M$, hypersurfaces, and Lagrangian submanifolds of M. Let $N \subseteq M$ be a coisotropic submanifold. Then ω gives rise to the isotropic (or characteristic) foliation on N. We define the *isotropy relation on* N to be the subset

$$R^{N,\omega} := \big\{(x(0),x(1))\,\big|\,x \in C^\infty([0,1],N)\colon \dot{x}(t) \in (T_{x(t)}N)^\omega,\ \forall t\big\}$$

of $N \times N$. This is an equivalence relation on N. For a point $x_0 \in N$ we call the $R^{N,\omega}$-equivalence class of x_0 the *isotropic leaf* through x_0. (This is the leaf of the isotropic foliation, which contains x_0.) We call N *regular* if $R^{N,\omega}$ is a closed subset and a submanifold of $N \times N$. This holds if and only if there exists a manifold structure on the set of isotropic leaves of N, such that the canonical projection π_N from N to the set of leaves is a submersion, cf. [Zi, Lemma 15]. If N is closed then by C. Ehresmann's theorem this implies that π_N is a smooth (locally trivial) fiber bundle. (See the proposition on p. 31 in [Eh].)

We define the *(symplectic) area (or action) spectrum* and the *minimal area* of N as

$$S(M,\omega,N) := \left\{ \int_{\mathbb{D}} u^*\omega \;\middle|\; u \in C^\infty(\mathbb{D},M)\colon\ \exists \text{ isotropic leaf } F \text{ of } N\colon\ u(S^1) \subseteq F \right\}, \tag{5}$$

$$A(N) = A(M,\omega,N) := \inf\big(S(M,\omega,N)\cap(0,\infty)\big) \in [0,\infty].$$

2.2. **Proof of Proposition 1 (Two-dimensional squeezing).**

Proof of Proposition 1. Let $n \in \mathbb{N}$ and $a > 0$. We denote by $\pi\colon \mathbb{R}^{2n} = \mathbb{R}^2 \times \mathbb{R}^{2n-2} \to \mathbb{R}^2$ the canonical projection, and $Y := \pi(X)$.

Claim. *There exists a compact neighborhood $K \subseteq \mathbb{R}^2$ of Y of area at most a, with smooth boundary.*

Proof of the claim. Since, by hypothesis, X has vanishing 2-dimensional Hausdorff measure, the same holds for Y. (This follows from a standard result, see e.g. [Fe, p. 176]).) It follows that there exists a countable collection of open balls in $\mathbb{R}^2$ covering the set Y, such that the sum of the areas of the balls is bounded above by a. By compactness of Y we can choose a finite subcollection, still covering Y. By shrinking these balls slightly if necessary, we may assume without loss of generality that their boundaries intersect transversally, and there are no triple intersections. We denote by K the closure of the union of these shrunk balls. It has C^0-boundary that is smooth away from finitely many points. Shrinking K slightly, we may assume that it has smooth boundary. This proves the claim. □

We choose a neighborhood $K \subseteq \mathbb{R}^2$ as in the claim, and denote by U the interior of K. Consider first the case in which K is connected. We denote by b its area. It follows from Lemma 10 below that there exists a finite subset $S \subseteq B^2(b)$ and a diffeomorphism $\varphi\colon U \to V := B^2(b)\setminus S$. We set $\omega := \varphi_*\omega_0$. We have

$$\int_V \omega = \int_U \omega_0 = b = \int_V \omega_0.$$

Hence a theorem by Greene and Shiohama ([GS, Theorem 1], which is based on Moser isotopy) implies that there exists a diffeomorphism $\psi\colon V \to V$ such that $\omega = \psi^*\omega_0$. The map $(\psi\circ\varphi)\times\mathrm{id}\colon U\times\mathbb{R}^{2n-2} \to \mathbb{R}^{2n}$ is a symplectic embedding (with respect to ω_0). Furthermore, since $b \le a$, its image is contained in $Z^{2n}(a)$. Moreover, its domain is an open neighborhood of X. Hence it has the required properties.

Consider now the general situation (in which K may be disconnected). Since K is compact, it has finitely many connected components $K_1, \ldots, K_N$. We denote by b_i the area of K_i, and $c_i := \sum_{j=1}^i b_j$. The cylinder $Z^{2n}(b_i)$ is symplectomorphic to $\Omega_i := (0,1) \times (c_{i-1}, c_i) \times \mathbb{R}^{2n-2}$. (This follows from Greene and Shiohama's result in two dimensions.) Therefore, by what we proved above, there exist symplectic embeddings $\varphi_i : U_i \times \mathbb{R}^{2n-2} \to \Omega_i$, where U_i denotes the interior of K_i. We denote by U the interior of K. We define $\varphi : U \times \mathbb{R}^{2n-2} \to \Omega := (0,1) \times (0, c_N) \times \mathbb{R}^{2n-2}$ to be the map that restricts to φ_i on $U_i \times \mathbb{R}^{2n-2}$. This map is a symplectic embedding. Note that c_N is the area of U, and this is bounded above by a. It follows that there exists a symplectic embedding of Ω into $Z^{2n}(a)$. Composing this embedding with φ, we obtain a map with the required properties. This proves Proposition 1. □

2.3. **Proof of Theorem 2 (Non-squeezable small set).** The proof of Theorem 2 is based on the following result.

5. **Proposition.** *Let $n \geq 2$, and $L \subseteq \mathbb{R}^{2n}$ be a non-empty closed Lagrangian submanifold. Then there exists a compact subset X of the set*

$$[0,1] \cdot L := \big\{ cx \,\big|\, c \in [0,1], x \in L \big\},$$

such that X has Hausdorff dimension n and does not symplectically embed into the cylinder $Z^{2n}(A(\mathbb{R}^{2n}, \omega_0, L))$.

The proof of this proposition follows the lines of the proof of [SZ1, Proposition 21]. It is based on the following result, which is due to Y. Chekanov. Let (M, ω) be a symplectic manifold. We denote by $\mathcal{H}(M, \omega)$ the set of all functions $H \in C^\infty\big([0,1] \times M, \mathbb{R}\big)$ whose Hamiltonian time-t flow $\varphi_H^t \colon M \to M$ exists and is surjective, for every $t \in [0,1]$. We define the *Hofer norm*

$$\|\cdot\| \colon \mathcal{H}(M, \omega) \to [0, \infty]$$

by

$$\|H\| := \int_0^1 \big(\sup_M H^t - \inf_M H^t \big) dt.$$

We define the *displacement energy* of a subset $X \subseteq M$ to be

$$e(X, M, \omega) := \inf \big\{ \|H\| \,\big|\, H \in \mathcal{H}(M, \omega) \colon \varphi_H^1(X) \cap X = \emptyset \big\}.$$

6. **Theorem.** *Let $L \subseteq M$ be a closed Lagrangian submanifold. Assume that (M, ω) is geometrically bounded (see* [Ch]*). Then we have*

$$e(L, M, \omega) \geq A(M, \omega, L).$$

Proof of Theorem 6. This follows from the Main Theorem in [Ch] by an elementary argument. $\square$

For the proof of Proposition 5, we also need the following.

7. **Lemma.** *Let (M,ω) and (M',ω') be symplectic manifolds of the same dimension, $N \subseteq M$ a coisotropic submanifold, and $\varphi\colon M \to M'$ a symplectic embedding. Assume that (M',ω') is aspherical, and every continuous loop in a leaf of N is contractible in M. Then we have*

$$A\big(M',\omega',\varphi(N)\big) = A(M,\omega,N).$$

Proof of Lemma 7. This follows from [SZ1, Remark 32 and Lemma 33]. $\square$

Proof of Proposition 5. Without loss of generality we may assume that L is connected. We choose a point $x_0 \in L$. Since L is a closed manifold, its fundamental group $\pi_1(L,x_0)$ is finitely generated. Therefore, there exists a finite set $\mathcal{L}$ of smooth loops $x\colon S^1 \subseteq \mathbb{C} \to L$ satisfying $x(1) = x_0$, whose continuous homotopy classes with fixed base point generate $\pi_1(L,x_0)$. We define

$$X := L \cup \bigcup_{c\in[0,1],\,x\in\mathcal{L},\,z\in S^1} cx(z) \subseteq \mathbb{R}^{2n}.$$

This set is contained in $[0,1]\cdot L$. Furthermore, a standard result (cf. [Fe, p. 176]) implies that the set $\bigcup_{c\in[0,1],\,x\in\mathcal{L},\,z\in S^1} cx(z)$ has Hausdorff dimension at most 2. Since $n \geq 2$, it follows that X has Hausdorff dimension n. Let U be an open neighborhood of X, and $\varphi\colon U \to \mathbb{R}^{2n}$ a symplectic embedding. The statement of the proposition is a consequence of the following claim.

Claim. *The image $\varphi(U)$ is not contained in $Z^{2n}\big(A(\mathbb{R}^{2n},\omega_0,L)\big)$.*

Proof of the claim. In order to apply Lemma 7, we check that every loop in L is contractible in U. Let $x\colon S^1 \to L$ be a continuous loop. It follows from our choice of the set $\mathcal{L}$ that there exist $\ell \in \mathbb{N}\cup\{0\}$, $x_1,\ldots,x_\ell \in \mathcal{L}$, and $\epsilon_1,\ldots,\epsilon_\ell \in \{1,-1\}$, such that x is continuously homotopic inside L to $x_1^{\epsilon_1}\#\cdots\#x_\ell^{\epsilon_\ell}$. Here $\#$ denotes concatenation of loops based at x_0, and $x_i^{-1}(z) := x_i(\bar z)$. Since X contains the image of the map $[0,1]\times S^1 \ni (c,z) \mapsto cx_i(z) \in \mathbb{R}^{2n}$, for every $i = 1,\ldots,\ell$, it follows that x is contractible in X, and hence in U. Therefore, the hypotheses of Lemma 7 are satisfied with $\big(M,\omega,M',\omega',N\big) := \big(U,\omega_0|U,\mathbb{R}^{2n},\omega_0,L\big)$. (Here $\omega_0|U$ denotes the restriction of ω_0 to U.) Applying this result, it follows that

$$A\big(U,\omega_0|U,L\big) = A\big(\mathbb{R}^{2n},\omega_0,\varphi(L)\big). \tag{6}$$

Similarly, applying Lemma 7 with φ replaced by the inclusion map of U into $\mathbb{R}^{2n}$, we have

$$A(\mathbb{R}^{2n},\omega_0,L)=A\big(U,\omega_0|U,L\big). \tag{7}$$

By Theorem 6, we have

$$A\big(\mathbb{R}^{2n},\omega_0,\varphi(L)\big)\leq e\big(\varphi(L),\mathbb{R}^{2n},\omega_0\big). \tag{8}$$

An elementary argument shows that

$$e\big(Z^{2n}(a),\mathbb{R}^{2n},\omega_0\big)\leq a,\quad \forall a>0.$$

Combining this with (6,7,8), it follows that

$$A(\mathbb{R}^{2n},\omega_0,L)\leq a,\quad \forall a>0 \text{ such that } \varphi(L)\subseteq Z^{2n}(a). \tag{9}$$

Assume by contradiction that $\varphi(U)\subseteq Z^{2n}\big(A(\mathbb{R}^{2n},\omega_0,L)\big)$. Since L is compact and contained in U, it follows that $\varphi(L)\subseteq Z^{2n}(a)$ for some number $a<A(\mathbb{R}^{2n},\omega_0,L)$. This contradicts (9). The statement of the claim follows. This proves Proposition 5. □

In the proof of Theorem 2 we will apply Proposition 5 with a rotated and rescaled version of the Lagrangian submanifold

$$L:=\big\{zq\,\big|\,z\in S^1\subseteq\mathbb{C},\ q\in S^{n-1}\subseteq\mathbb{R}^n\big\}\subseteq\mathbb{C}^n. \tag{10}$$

This submanifold was used by L. Polterovich in [Po, Section 3] as an example of a monotone Lagrangian in $\mathbb{C}^n$ with minimal Maslov number n. Previously, it was considered by A. Weinstein in [We, Lecture 3] and M. Audin in [Au, p. 620].

8. **Lemma.** *For $n\geq 2$ the minimal symplectic area of the Lagrangian L in $\mathbb{R}^{2n}$ equals $\frac{\pi}{2}$.*

Proof of Lemma 8. Let $n\geq 2$. We write a point in $\mathbb{R}^{2n}$ as (q,p), and denote by $\alpha:=q\cdot dp$ the Liouville one-form. Since $d\alpha=\omega_0$, Stokes' theorem implies that the area spectrum of L in $\mathbb{R}^{2n}$ is given by

$$S(L)=\widetilde{S}(L):=\left\{\int_{S^1}x^*\alpha\,\Big|\,x\in C^\infty(S^1,L)\right\}. \tag{11}$$

To calculate $\widetilde{S}(L)$, we need the following.

Claim. *If $x:S^1\to L$, $\varphi:[0,1]\to\mathbb{R}$, and $q:[0,1]\to S^{n-1}$ are smooth maps, such that*

$$x(e^{2\pi it})=e^{i\varphi(t)}q(t),\quad \forall t\in[0,1], \tag{12}$$

then we have

$$\int_{S^1}x^*\alpha=\frac{\varphi(1)-\varphi(0)}{2}. \tag{13}$$

Proof of the claim. We have $|q|^2 = 1$ and $q \cdot \dot{q} = 0$, and therefore,

$$\begin{aligned}\int_{S^1} x^*\alpha &= \int_0^1 \mathrm{Re}\left(e^{i\varphi}q\right) \cdot \mathrm{Im}\left(e^{i\varphi}(i\dot{\varphi}q + \dot{q})\right) dt \\ &= \int_0^1 \cos(\varphi)^2 \dot{\varphi}\, dt \\ &= \left(\frac{1}{4}\sin(2\varphi(t)) + \frac{\varphi(t)}{2}\right)\bigg|_{t=0}^1 . \qquad (14)\end{aligned}$$

On the other hand, equality (12) implies that $\varphi(1) - \varphi(0) \in \pi\mathbb{Z}$, and therefore, the first term in (14) vanishes. Equality (13) follows. This proves the claim. □

We show that $\widetilde{S}(L) \subseteq \frac{\pi}{2}\mathbb{Z}$: Let $x \in C^\infty(S^1, L)$. The map $\mathbb{R} \times S^{n-1} \ni (\varphi, q) \mapsto e^{i\varphi}q \in L \subseteq \mathbb{C}^n$ is a smooth covering map. Therefore, there exist smooth paths $\varphi\colon [0,1] \to \mathbb{R}$ and $q\colon [0,1] \to S^{n-1}$ such that equality (12) holds. It follows that $\varphi(1) - \varphi(0) \in \pi\mathbb{Z}$. Combining this with the claim, we obtain $\int_{S^1} x^*\alpha \in \frac{\pi}{2}\mathbb{Z}$. This shows that $\widetilde{S}(L) \subseteq \frac{\pi}{2}\mathbb{Z}$.

To prove the opposite inclusion, we choose a path $q \in C^\infty([0,1], S^{n-1})$ that is constant near the ends and satisfies $q(1) = -q(0)$. (Here we use that $n \geq 2$, and therefore, S^{n-1} is connected.) We define $x : S^1 \to L$ by $x(e^{2\pi i t}) := e^{\pi i t}q(t)$, for $t \in [0,1)$. This is a smooth loop. By the above claim we have $\int_{S^1} x^*\alpha = \pi/2$. By considering multiple covers of x, it follows that $\widetilde{S}(L) \supseteq \frac{\pi}{2}\mathbb{Z}$.

Hence the equality $\widetilde{S}(L) = \frac{\pi}{2}\mathbb{Z}$ holds. Combining this with equality (11), it follows that $A(L) = \pi/2$. This proves Lemma 8. □

Proof of Theorem 2. Let $n \geq 2$. We define L as in (10), and

$$\widetilde{L} := \left\{\sqrt{2}zw \,\middle|\, z \in S^1 \subseteq \mathbb{C},\, w \in S^{2n-1} \subseteq \mathbb{C}^n : w_{n+1-j} = \overline{w}_j,\, \forall j = 1, \dots, n\right\}.$$

Claim. *There exists a unitary transformation U of $\mathbb{C}^n$, such that $\widetilde{L} = \sqrt{2}UL$.*

Proof of the claim. The set

$$W := \left\{w \in \mathbb{C}^n \,\middle|\, w_{n+1-j} = \overline{w}_j,\, \forall j = 1, \dots, n\right\}$$

is a Lagrangian subspace of $\mathbb{C}^n$. Therefore, there exists a unitary transformation U of $\mathbb{C}^n$, such that $W = U\mathbb{R}^n$. The statement of the claim holds for every such U. □

We choose U as in the claim. Since U is a symplectic linear map, the set $\widetilde{L}$ is a Lagrangian submanifold of $\mathbb{C}^n$, and satisfies

$$A(\mathbb{C}^n,\omega_0,\widetilde{L}) = 2A(\mathbb{C}^n,\omega_0,L).$$

By Lemma 8 the right hand side equals π. Therefore, applying Proposition 5, it follows that there exists a compact subset $X \subseteq [0,1]\cdot\widetilde{L}$ of Hausdorff dimension n, such that X does not symplectically embed into Z^{2n}. (Here we use the hypothesis $n \geq 2$.) Since L is contained in $\overline{B}^{2n}$ and U is an orthogonal transformation of $\mathbb{R}^{2n}$, the Lagrangian $\widetilde{L}$ and therefore X is contained in $\overline{B}^{2n}(2\pi)$.

Let $\widetilde{w} \in \widetilde{L}$. We choose $z \in S^1$ and $w \in S^{2n-1}$, such that $w_{n+1-j} = \overline{w}_j$, for all j, and $\widetilde{w} = \sqrt{2}zw$. If $j \in \{1,\dots,n\}$ is an index such that $j \neq \frac{n+1}{2}$, then we have

$$|\widetilde{w}_j|^2 = 2|w_j|^2 = |w_j|^2 + |w_{n+1-j}|^2 \leq |w|^2 = 1.$$

Therefore if n is even, then $\widetilde{L}$, and hence X is contained in $\mathbb{D}^n$. It follows that X has all the required properties in this case. Consider the case in which n is odd. We denote $n =: 2k+1$ and define

$$T:\mathbb{C}^n \to \mathbb{C}^n, \quad Tw := \big(w_1,\dots,w_k,w_{k+2},\dots,w_n,w_{k+1}\big).$$

It follows that $T\widetilde{L}$ is contained in $\mathbb{D}^{n-1}\times\mathbb{C}$, and hence the same holds for TX. Therefore, TX has the required properties. This proves Theorem 2. □

2.4. **Proof of Proposition 3.**

Proof of Proposition 3. Let $n \in \mathbb{N}$ and X be a compact subset of $\overline{B}^{2n}$ with vanishing $(2n-1)$-dimensional Hausdorff measure. Then X does not contain S^{2n-1}, and hence there exists an orthogonal linear symplectic map $\varphi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, such that $(1,0,\dots,0) \notin \varphi(X)$. Since $\varphi(X)$ is compact and contained in $\overline{B}^{2n}$, an elementary argument shows that there exists $c < 1$, such that

$$\varphi(X) \subseteq \big\{(q,p)\in\mathbb{D} \mid q < c\big\} \times \mathbb{R}^{2n-2}. \tag{15}$$

It follows from a result by Greene and Shiohama ([GS, Theorem 1]) that some open neighborhood of $\{(q,p)\in\mathbb{D} \mid q<c\}$ symplectically embeds into B^2. Using (15), it follows that $\varphi(X)$ symplectically embeds into Z^{2n}. Hence the same holds for X. This proves Proposition 3. □

2.5. **Proof of Theorem 4 (Regular coisotropic capacity).** The idea of the proof of this result is to consider the Lagrangian submanifold L defined in (10) and a product of it with a sphere. We need the following result. Recall the definition of the area spectrum (5).

9. **Lemma.** *Let (M,ω) and (M',ω') be symplectic manifolds, and $N \subseteq M$ and $N' \subseteq M'$ coisotropic submanifolds. Then*

$$S(M \times M', \omega \oplus \omega', N \times N') = S(M,\omega,N) + S(M',\omega',N').$$

Proof. For a proof we refer to [SZ1, Remark 31]. □

Proof of Theorem 4. To prove **inequality** (2), we define L as in (10). Let $r < 1$. Then rL is a closed Lagrangian submanifold of B^{2n}. Furthermore, condition (1) is satisfied with $(M,\omega) := (B^{2n},\omega_0)$, since B^{2n} is contractible. An elementary argument using Lemmas 8 and 7, shows that $A(B^{2n},\omega_0,rL) = \frac{\pi}{2}r^2$. Therefore, for every $r<1$ we have $A_{\mathrm{Lag}}(B^{2n},\omega_0) \geq \frac{\pi}{2}r^2$. Inequality (2) follows.

We prove **inequality** (3). Let $d \in \{n+1,\ldots,2n-3\}$. We define L as in (10) with n replaced by $2n-d-1$. We denote by $S^{k-1}_r \subseteq \mathbb{R}^k$ the sphere of radius $r>0$, around 0. Let $r<1$. The set

$$N := \sqrt{\frac{2}{3}}rL \times S^{2d-2n+1}_{\sqrt{1/3}r} \tag{16}$$

is a closed regular coisotropic submanifold of B^{2n}, of dimension d. Each factor has area spectrum in linear space given by $\frac{\pi r^2}{3}\mathbb{Z}$. (For the second factor this follows e.g. from the proof of [Zi, Proposition 1.3].) Hence Lemma 9 implies that $A(\mathbb{R}^{2n},\omega_0,N) = \frac{\pi r^2}{3}$. Lemma 7 implies that this number equals $A(B^{2n},\omega_0,N)$. It follows that $A^d_{\mathrm{coiso}}(B^{2n},\omega_0) \geq \frac{\pi r^2}{3}$, for every $r<1$. Inequality (3) follows. This proves Theorem 4. □

Remark. *The ratio of the scaling factors used in the definition (16) above is optimal. Namely, for $r,r'>0$ consider the coisotropic submanifold $N := rL \times S^{2d-2n+1}_{r'}$ of $\mathbb{R}^{2n}$. It follows from Lemma 9 that*

$$A(\mathbb{R}^{2n},\omega_0,N) = \pi \gcd\left\{\frac{r^2}{2}, r'^2\right\}. \tag{17}$$

Here we define the greatest common divisor of two real numbers a,b to be

$$\gcd\{a,b\} := \sup\left\{c \in (0,\infty) \,\middle|\, a,b \in c\mathbb{Z}\right\}.$$

(Our convention is that the supremum over the empty set equals 0.) In order for N to be contained in B^{2n}, we need $r^2 + r'^2 < 1$. For a given $c<1$, the expression (17) is largest (namely equal to $\frac{c\pi}{3}$) under the

restriction $r^2 + r'^2 = c$, provided that $\frac{r^2}{2} = r'^2$. This corresponds to the choice in (16).

Appendix A. An auxiliary lemma

In the proof of Proposition 1, we used the following.

10. **Lemma.** *Let $U \subseteq S^2$ be a connected open subset with compact closure and smooth boundary. Then U is diffeomorphic to S^2 with finitely many points removed.*

Proof of Lemma 10. For $k \in \mathbb{N} \cup \{0\}$ consider the following statement:

Statement $A(k)$. *Let $U \subseteq S^2$ be a connected open subset with compact closure and smooth boundary consisting of k connected components. Furthermore, let $X \subseteq U$ be a finite set. Then $U \setminus X$ is diffeomorphic to S^2 with $k + |X|$ points removed.*

We prove by induction that $A(k)$ holds for every $k \in \mathbb{N} \cup \{0\}$: $A(0)$ holds, since in the case $k = 0$, we have $U = S^2$. Let $k \in \mathbb{N}$ and assume that we have proved $A(k-1)$. We show that $A(k)$ holds: Let U and X be as above. We choose a connected component γ of ∂U, a point $x_0 \in U$, and a diffeomorphism $\varphi : S^2 \setminus \{x_0\} \to \mathbb{R}^2$. By the smooth Schoenflies theorem there exists a smooth embedding $\psi_0 : \mathbb{D} \to \mathbb{R}^2$, such that $\psi_0(S^1) = \varphi(\gamma)$. (Such an embedding can be constructed using a decomposition of $\mathbb{R}^2$ into horizontal strips, similarly to the proof of [Ha, Theorem 1.1].) Using that U is connected, $\psi_0(S^1) \cap \varphi(U \setminus \{x_0\}) = \emptyset$, and $x_0 \in U$, an elementary argument shows that $\psi_0(B^2) \cap \varphi(U \setminus \{x_0\}) = \emptyset$. We define

$$\widetilde{U} := U \cup \varphi^{-1} \circ \psi_0(\mathbb{D}), \quad \widetilde{X} := X \cup \varphi^{-1} \circ \psi(0) \in S^2.$$

The set $\widetilde{U}$ is connected and open, contains $\widetilde{X}$, and has compact closure and smooth boundary equal to $\partial U \setminus \gamma$. Hence by the induction hypothesis, $\widetilde{U} \setminus \widetilde{X}$ is diffeomorphic to S^2 with $k - 1 + |\widetilde{X}| = k + |X|$ points removed. The induction step is a consequence of the following claim.

Claim. *The open set $U \setminus X$ is diffeomorphic to $\widetilde{U} \setminus \widetilde{X}$.*

Proof of the claim. The embedding ψ_0 extends to an embedding $\psi : \mathbb{R}^2 \to \mathbb{R}^2$, such that $\psi^{-1}(\varphi(U \setminus \{x_0\})) = \mathbb{R}^2 \setminus \mathbb{D}$ and $\varphi^{-1} \circ \psi(\mathbb{R}^2) \cap X = \emptyset$. We choose a diffeomorphism $\zeta : \mathbb{R}^2 \setminus \mathbb{D} \to \mathbb{R}^2 \setminus \{0\}$ that is the identity outside some ball. We define the map $\chi : U \to S^2$ by

$$\chi(x) := \begin{cases} \varphi^{-1} \circ \psi \circ \zeta \circ \psi^{-1} \circ \varphi(x), & \text{if } x \in \varphi^{-1} \circ \psi(\mathbb{R}^2 \setminus \mathbb{D}), \\ x, & \text{otherwise.} \end{cases}$$

Since $\varphi^{-1} \circ \psi(\mathbb{R}^2) \cap X = \emptyset$, the map χ restricts to a diffeomorphism between $U \setminus X$ and $\widetilde{U} \setminus \widetilde{X}$. This proves the claim, terminates the induction, and hence concludes the proof of Lemma 10. □

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