Algebraic equivalence of cycles and algebraic models of smooth manifolds

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1 Introduction and results

Throughout this paper the term *real algebraic variety* designates a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some n, endowed with the Zariski topology and the sheaf of $\mathbb{R}$-valued regular functions. Morphisms between real algebraic varieties will be called *regular maps*. Basic facts on real algebraic varieties and regular maps can be found in [7]. Every real algebraic variety carries also the Euclidean topology, which is determined by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the Euclidean topology.

Let $\mathcal{X}$ be a reduced quasiprojective scheme over $\mathbb{R}$. The set $\mathcal{X}(\mathbb{R})$ of $\mathbb{R}$-rational points of $\mathcal{X}$ is contained in an affine open subset of $\mathcal{X}$. Thus if $\mathcal{X}(\mathbb{R})$ is dense in $\mathcal{X}$, we can regard $\mathcal{X}(\mathbb{R})$ as a real algebraic variety whose structure sheaf is the restriction of the structure sheaf of $\mathcal{X}$; up to isomorphism, each real algebraic variety is of this form.

Given a compact nonsingular real algebraic variety X (as in [3, 7], nonsingular means that the irreducible components of X are pairwise disjoint, nonsingular, and of the same dimension), we can find a nonsingular quasiprojective scheme $\mathcal{X}$ over $\mathbb{R}$ with $\mathcal{X}(\mathbb{R}) = X$ dense in $\mathcal{X}$. Then we have the cycle homomorphism

$$c\ell_{\mathbb{R}} : Z^k(\mathcal{X}) \to H^k(X, \mathbb{Z}/2)$$

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defined on the group $Z^k(\mathcal{X})$ of algebraic cycles on $\mathcal{X}$ of codimension k : for any integral subscheme $\mathcal{V}$ of $\mathcal{X}$ of codimension k, the cohomology class $c\ell_{\mathbb{R}}(\mathcal{V})$ is Poincaré dual to the homology class represented by the subvariety $\mathcal{V}(\mathbb{R})$ of X assuming $\mathcal{V}(\mathbb{R})$ has codimension k in X, and otherwise $c\ell_{\mathbb{R}}(\mathcal{V}) = 0$ [10]. The subgroup

$$H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) = c\ell_{\mathbb{R}}(Z^k(\mathcal{X}))$$

of $H^k(X, \mathbb{Z}/2)$ plays a fundamental role in real algebraic geometry (cf. [9] for a short survey of its properties and applications). We define

$$\mathrm{Alg}^k(X),$$

the main object of our investigation here, to be the image under $c\ell_{\mathbb{R}}$ of the subgroup of $Z^k(\mathcal{X})$ consisting of the cycles algebraically equivalent to 0 (we refer to [13, Chapter 10] for the theory of algebraic equivalence). Thus, by definition, $\mathrm{Alg}^k(X)$ is a subgroup of $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$. It readily follows that $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$ and $\mathrm{Alg}^k(X)$ do not depend on the choice of $\mathcal{X}$. Note that $\mathrm{Alg}^k(X)$ can also be described as follows. An element u of $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$ belongs to $\mathrm{Alg}^k(X)$ if and only if there exist a compact nonsingular irreducible real algebraic variety T, two points t_0 and t_1 in T, and a cohomology class z in $H^k_{\mathrm{alg}}(X \times T, \mathbb{Z}/2)$ such that $u = i^*_{t_1}(z) - i^*_{t_0}(z)$, where given t in T, we let $i_t : X \to X \times T$ denote the map defined by $i_t(x) = (x, t)$ for all x in X, while

$$i^*_t : H^*(X \times T, \mathbb{Z}/2) \to H^*(X, \mathbb{Z}/2)$$

is the induced homomorphism (this does not force $u = 0$, the parameter space T being possibly disconnected).

Why is the group $\mathrm{Alg}^k(X)$ of interest? It was R. Silhol who first demonstrated that $\mathrm{Alg}^1(X)$ is important for understanding of $H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$ [19]. In [17] it is proved, among other things, that $\mathrm{Alg}^1(-)$ is a birational invariant. The group $\mathrm{Alg}^k(X)$ strongly influences the behavior of $H^{n-k}_{\mathrm{alg}}(X, \mathbb{Z}/2)$, where $n = \dim X$ [16, 17]. Substantial constructions of [18], at the borderline between real algebraic geometry and differential topology, depend on $\mathrm{Alg}^k(-)$. For some remarkable properties of $\mathrm{Alg}^k(X)$ contained in [1, 16] see also Theorem

2.1 in Section 2. It is in general very difficult to compute $\mathrm{Alg}^k(X)$, except the cases $k = 0$ or $k = \dim X$ (cf. for example [1] to see how these trivial cases are settled). In this paper we investigate the groups $\mathrm{Alg}^k(X)$ as X runs through the class of varieties diffeomorphic to a fixed variety. Below we make this precise.

All smooth (of class $\mathcal{C}^\infty$) manifolds that appear here are paracompact and without boundary. By Tognoli's theorem [21, 7], any compact smooth manifold M has an algebraic model, that is, there exists a nonsingular real algebraic variety X diffeomorphic to M. We study how the groups $\mathrm{Alg}^k(X)$ vary as X run through the class of algebraic models of M. The kth Stiefel-Whitney class of M will be denoted by $w_k(M)$, while $[M]$ will stand for the fundamental class of M in $H_m(M, \mathbb{Z}/2)$, $m = \dim M$. As usual, we use $\cup$ and $<,>$ to denote the cup product and scalar (Kronecker) product.

Theorem 1.1 *Let M be a compact smooth manifold of dimension m with $m \geq 2$. Given a subgroup G of $H^1(M, \mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exist an algebraic model X of M and a smooth diffeomorphism $\varphi : M \to X$ such that $\varphi^*(\mathrm{Alg}^1(X)) = G$,*

(b) *G is contained in the image of the reduction modulo 2 homomorphism $H^1(M, \mathbb{Z}) \to H^1(M, \mathbb{Z}/2)$ and for each integer ℓ, $1 \leq \ell \leq m$, and all $u_1, \dots, u_\ell$ in G, one has $< u_1 \cup \ldots \cup u_\ell \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M), [M] >= 0$ for all nonnegative integers $i_1, \dots, i_r$ with $i_1 + \cdots + i_r = m - \ell$.*

Furthermore, if (b) *holds, then X in* (a) *can be chosen irreducible.*

Our second result is the following.

Theorem 1.2 *Let M be a compact connected smooth manifold of dimension m with $m \geq 3$. Given a subgroup G of $H^{m-1}(M, \mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exist an algebraic model X of M and a smooth diffeomorphism $\varphi : M \to X$ such that $\varphi^*(\mathrm{Alg}^{m-1}(X)) = G$,*

(b) $< u \cup w_1(M), [M] > 0$ *for all* u *in* G.

Theorems 1.1 and 1.2 are proved in Section 2. We also have another result of the same type, Theorem 2.5 in Section 2, dealing with certain subgroups G of $H^k(M, \mathbb{Z}/2)$ for other values of k. Example 2.7 at the end of the paper shows how Theorems 1.1, 1.2, and 2.5 work in a special case.

2 Proofs

The groups $H^k_{\mathrm{alg}}(-, \mathbb{Z}/2)$ and $\mathrm{Alg}^k(-)$ have the expected functorial properties. If $f : X \to Y$ is a regular map between compact nonsingular real algebraic varieties, then the induced homomorphism

$$f^* : H^*(Y, \mathbb{Z}/2) \to H^*(X, \mathbb{Z}/2)$$

satisfies

$$f^*(H^*_{\mathrm{alg}}(Y, \mathbb{Z}/2)) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2) \text{ and } f^*(\mathrm{Alg}^*(Y)) \subseteq \mathrm{Alg}^k(X).$$

Furthermore,

$$H^*_{\mathrm{alg}}(X, \mathbb{Z}/2) = \oplus_{q \geq 0} H^q_{\mathrm{alg}}(X, \mathbb{Z}/2)$$

is a subring of the cohomology ring $H^*(X, \mathbb{Z}/2)$, whereas

$$\mathrm{Alg}^*(X) = \oplus_{q \geq 0} \mathrm{Alg}^q(X)$$

is an ideal of $H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$. These assertions concerning $H^k_{\mathrm{alg}}(-, \mathbb{Z}/2)$ are proved in [10, 6] and they immediately imply the corresponding assertions about $\mathrm{Alg}^k(-)$.

Recall that if M is a smooth manifold, then a cohomology class u in $H^k(M, \mathbb{Z}/2)$, $k \geq 1$, is said to be *spherical*, provided that $u = f^*(c)$, where $f : M \to S^k$ is a continuous (or equivalently smooth) map from M into the unit k-sphere S^k and c is the unique generator of the group $H^k(S^k, \mathbb{Z}/2) \simeq \mathbb{Z}/2$.

We shall make use of the following result.

Theorem 2.1 *Let X be a compact nonsingular real algebraic variety. Then:*

(i) $< u \cup v, [X] >= 0$ *for all u in* $\mathrm{Alg}^k(X)$ *and v in* $H^{\ell}_{\mathrm{alg}}(X, \mathbb{Z}/2)$, *where* $k + \ell = \dim X$,

(ii) $< u \cup w_{i_1}(X) \cup \ldots \cup w_{i_r}(X), [X] >= 0$ *for all u in* $\mathrm{Alg}^k(X)$ *and all nonnegative integers* $i_1, \ldots, i_r$ *with* $i_1 + \cdots + i_r = \dim X - k$,

(iii) *If $k = 1$ or if $k = \dim X - 1$ and X is connected, then every cohomology class in* $\mathrm{Alg}^k(X)$ *is spherical.*

Reference for the proof. [16, Theorem 2.1] and [1, Theorem 1.1].

The next fact will also be very useful. Let B^k be a nonsingular irreducible real algebraic variety with precisely two connected components B^k_0 and B^k_1, each diffeomorphic to S^k, $k \geq 1$. For example, one can take

$$B^k = \{(x_0, \ldots, x_k) \in \mathbb{R}^{k+1} \mid x_0^4 - 4x_0^2 + 1 + x_1^2 + \cdots + x_k^2 = 0\}.$$

Let $B = B^k \times \cdots \times B^k$ and $B_0 = B^k_0 \times \cdots \times B^k_0$ be the d-fold products, and let $\delta : B_0 \hookrightarrow B$ be the inclusion map. It is known [18, Example 4.5] that

$$H^q(B_0, \mathbb{Z}/2) = \delta^*(H^q(B, \mathbb{Z}/2)) = \delta^*(\mathrm{Alg}^q(B)) \text{ for all } q \geq 0. \tag{2.2}$$

We now recall an important result from differential topology.

Theorem 2.3 *Let P be a smooth manifold. Two smooth maps $f : M \to P$ and $g : M \to P$, where M and N are compact smooth manifolds of dimension d, represent the same bordism class in the unoriented bordism group $\mathcal{N}_*(P)$ if and only if for every nonnegative integer q and every cohomology class v in $H^q(P, \mathbb{Z}/2)$, one has*

$$< f^*(v) \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M),\ [M] >=< g^*(v) \cup w_{i_1}(N) \cup \ldots \cup w_{i_r}(N), [N] >$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = d - q$.

Reference for the proof. [11, (17.3)].

If Y is a nonsingular real algebraic variety, then a bordism class in $\mathcal{N}_*(Y)$ is said to be *algebraic* provided that it can be represented by a regular map $f : X \to Y$ of a compact nonsingular real algebraic variety X into Y, cf. [2, 3, 4, 5, 6].

A topological real vector bundle on a real algebraic variety Y is said to admit an *algebraic structure* if it is topologically isomorphic to an algebraic subbundle of the trivial vector bundle with total space $Y \times \mathbb{R}^p$ for some p (cf. [7] for various characterizations of such vector bundles and for their basic properties).

Given smooth manifolds N and P, we endow the set $\mathcal{C}^\infty(N, P)$ of all smooth maps from N into P with the $\mathcal{C}^\infty$ topology [14] (in our applications N is always compact so it does not matter whether we take the weak $\mathcal{C}^\infty$ topology or the strong one).

Our basic tools include the following approximation theorem.

Theorem 2.4 *Let M be a compact smooth submanifold of $\mathbb{R}^n$ and let W be a nonsingular real algebraic variety. Let $f : M \to W$ be a smooth map whose bordism class in $\mathcal{N}_*(W)$ is algebraic. Suppose that M contains a (possibly empty) Zariski closed nonsingular subvariety L of $\mathbb{R}^n$, the restriction $f|L : L \to W$ is a regular map, and the restriction to L of the tangent bundle of M admits an algebraic structure. If $2 \dim M + 1 \leq n$, then there exist a smooth embedding $e : M \to \mathbb{R}^n$, a Zariski closed nonsingular subvariety X of $\mathbb{R}^n$, and a regular map $g : X \to W$ such that $L \subseteq X = e(M)$, $e|L : L \to \mathbb{R}^n$ is the inclusion map, $g|L = f|L$, and $g \circ \bar{e}$ (where $\bar{e} : M \to e(M)$ is the smooth diffeomorphism defined by $\bar{e}(x) = e(x)$ for all x in M) is homotopic to f. Furthermore, given a neighborhood $\mathcal{U}$ in $\mathcal{C}^\infty(M, \mathbb{R}^n)$ of the inclusion map $M \hookrightarrow \mathbb{R}^n$ and a neighborhood $\mathcal{V}$ of f in $\mathcal{C}^\infty(M, W)$, the objects e, X, and g can be chosen in such a way that e is in $\mathcal{U}$ and $g \circ \bar{e}$ is in $\mathcal{V}$.*

Reference for the proof. Precisely this formulation is in [18, Theorem 4.2]. It is based on very similar results of [2, 3, 4, 5].

After these preparations we return to the main topic of our paper.

Theorem 2.5 *Let M be a compact smooth manifold of dimension m. Let G be a subgroup of $H^k(M, \mathbb{Z}/2)$, where $k \geq 1$. Assume that G is generated by spherical cohomology classes. If $2k+1 \leq m$, then the following conditions are equivalent:*

(a) *There exist an algebraic model X of M and a smooth diffeomorphism $\varphi : M \to X$ such that $\varphi^*(\mathrm{Alg}^k(X)) = G$,*

(b) *For every integer ℓ satisfying $\ell \geq 1$ and $\ell k \leq m$, one has*

$$< u_1 \cup \ldots \cup u_\ell \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M), [M] >= 0$$

for all $u_1, \ldots, u_\ell$ in G and all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = m - \ell k$.

Furthermore, if (b) *holds, then X in* (a) *can be chosen irreducible.*

Proof. It follows from Theorem 2.1(ii) that (a) implies (b). Suppose then that (b) holds. We prove below that (a), with X irreducible, is satisfied. We assume that M is a smooth submanifold of $\mathbb{R}^{2m+1}$.

Let us set

$$\Gamma = \{v \in H^{m-k}(M, \mathbb{Z}/2) | < u \cup v, [M] >= 0 \text{ for every } u \text{ in } G\},$$
$$\Gamma = \{v_1, \ldots, v_s\}.$$

Since $2k+1 \leq m$, the homology class in $H_k(M, \mathbb{Z}/2)$ Poincaré dual to v_i can be represented by a compact smooth submanifold N_i of M [20, Théorème II.26]. Thus we have

$$e_{i*}([N_i]) = v_i \cap [M],$$

where $e_i : N_i \hookrightarrow M$ is the inclusion map and $\cap$ stands for the cap product. We may assume that $N_1, \ldots, N_s$ are pairwise disjoint. Note that

(1) $$< e_i^*(u), [N_i] >=< u \cup v_i, [M] > \text{ for all } u \text{ in } H^k(M, \mathbb{Z}/2).$$

Indeed standard properties of $\cup$, $\cap$, $<$, $>$ (cf. for example [12]) yield

$$\begin{aligned} < e_i^*(u), [N_i] > &=< u, e_{i*}([N_i]) > \\ &=< u, v_i \cap [M] > \\ &=< u \cup v_i, [M] > . \end{aligned}$$

The bilinear map

$$H^k(M, \mathbb{Z}/2) \times H^{m-k}(M, \mathbb{Z}/2) \to \mathbb{Z}/2, (u, v) \to < u \cup v, [M] >$$

is a dual pairing [12, p. 300, Proposition 8.13] and hence

$$G = \{u \in H^k(M, \mathbb{Z}/2) | < u \cup v, [M] >= 0 \text{ for every } v \text{ in } \Gamma\}.$$

By applying (1), we obtain

$$G = \{u \in H^k(M, \mathbb{Z}/2) | < e_i^*(u), [N_i] >= 0 \text{ for } 1 \leq i \leq s\}. \tag{2}$$

We shall now successively modify M and $N_1, \dots, N_s$ to ensure that they satisfy some additional desirable conditions.

Let $\gamma_{n,m}$ denote the universal vector bundle on the Grassmannian $\mathbb{G}_{n,m}$ of m-dimensional vector subspaces of $\mathbb{R}^n$. Assuming that n is large enough, we can find a smooth classifying map $h_i : N_i \to \mathbb{G}_{n,m}$ for the restriction $\tau(M)|N_i$ of the tangent bundle $\tau(M)$ of M (this means that the vector bundles $\tau(M)|N_i$ and $h_i^* \gamma_{n,m}$ are isomorphic). Recall that $\mathbb{G}_{n,m}$ is endowed with a canonical structure sheaf which makes it into a real algebraic variety in the sense of this paper [7, Theorem 3.4.4] ($\mathbb{G}_{n,m}$ is an affine real algebraic variety according to the terminology used in [7]). Moreover, $\mathbb{G}_{n,m}$ is nonsingular [7, Proposition 3.4.3] and every bordism class in $\mathcal{N}_*(\mathbb{G}_{n,m})$ is algebraic [7, Proposition 11.3.3] and [3, Lemma 2.7.1]. It follows that Theorem 2.4 can be applied to $h_i : N_i \to \mathbb{G}_{n,m}$ (with L empty) and hence modifying M, we may assume that N_i is a Zariski closed nonsingular subvariety of $\mathbb{R}^{2m+1}$ and $h_i : N_i \to \mathbb{G}_{n,m}$ is a regular map for $1 \leq i \leq s$.

Let $u_1, \dots, u_d$ be spherical cohomology classes generating G. Using the same notation as in (2.2), choose a smooth map $f_j : M \to B^k$ such that $f_j(M) \subseteq B_0^k$ and $f_j^*(H^1(B^k, \mathbb{Z}/2))$ is the subgroup of G generated by u_j. By (2.2), we have

(3)
$$G = f^*(H^k(B, \mathbb{Z}/2)) = f^*(\mathrm{Alg}^k(B)),$$

where $f = (f_1, \dots, f_d) : M \to B = B^k \times \dots \times B^k$.

We assert that the maps $(f|N_i, h_i) : N_i \to B \times \mathbb{G}_{n,m}$ and $(c_i, h_i) : N_i \to B \times \mathbb{G}_{n,m}$, where $c_i : N_i \to B$ is a constant map sending N_i to a point in B_0, represent the same class in the bordism group $\mathcal{N}_*(B \times \mathbb{G}_{n,m})$. By Theorem 2.3 and Künneth's theorem in cohomology, in order to prove the assertion it suffices to show that given cohomology classes ξ in $H^p(B, \mathbb{Z}/2)$ and η in $H^q(\mathbb{G}_{n,m}, \mathbb{Z}/2)$, we have

$$\begin{aligned} &< (f|N_i, h_i)^*(\xi \times \eta) \cup w_{j_1}(N_i) \cup \ldots \cup w_{j_r}(N_i), [N_i] >= \\ &\qquad\qquad < (c_i, h_i)^*(\xi \times \eta) \cup w_{j_1}(N_i) \cup \ldots \cup w_{j_r}(N_i), [N_i] > \end{aligned}$$

for all nonnegative integers $j_1, \dots, j_r$ satisfying $j_1 + \dots + j_r = k - (p + q)$. Since $(f|N_i, h_i)^*$ $(\xi \times \eta) = (f|N_i)^*(\xi) \cup h_i^*(\eta)$ and $(c_i, h_i)^*(\xi \times \eta) = c_i^*(\xi) \cup h_i^*(\eta)$, the last displayed equality is equivalent to

(4)
$$\begin{aligned} &< (f|N_i)^*(\xi) \cup h_i^*(\eta) \cup w_{j_1}(N_i) \cup \ldots \cup w_{j_r}(N_i), [N_i] >= \\ &\qquad\qquad < c_i^*(\xi) \cup h_i^*(\eta) \cup w_{j_1}(N_i) \cup \ldots \cup w_{j_r}(N_i), [N_i] > . \end{aligned}$$

We now justify (4). If p is not a multiple of k, then $\xi = 0$ and hence (4) holds. It remains to consider two cases: $(p, q) = (0, k)$ and $(p, q) = (k, 0)$. If $(p, q) = (0, k)$, then $(f|N_i)^*(\xi) = c_i^*(\xi)$, which implies (4). If $(p, q) = (k, 0)$, then $c_i^*(\xi) = 0$ and (4) is reduced to

(5)
$$< (f|N_i)^*(\xi) \cup h_i^*(\eta), [N_i] >= 0.$$

Since $f|N_i = f \circ e_i$, we have

$$(f|N_i)^*(\xi) \cup h_i^*(\eta) = e_i^*(f^*(\xi)) \cup h_i^*(\eta) = \lambda e_i^*(f^*(\xi)),$$

where $\lambda = 0$ or $\lambda = 1$. Hence (5) follows from (2) and (3). This means that (4) always holds and therefore the proof of the assertion is complete.

Since $(c_i, h_i) : N_i \to B \times \mathbb{G}_{n,m}$ is a regular map, the assertion implies that the bordism class of $(f|N_i, h_i) : N_i \to B \times \mathbb{G}_{n,m}$ in $\mathcal{N}_*(B \times \mathbb{G}_{n,m})$ is algebraic. Theorem 2.4 can be applied to $(f|N_i, h_i) : N_i \to B \times \mathbb{G}_{n,m}$ (with L empty) and therefore modifying M and f, we may assume that N_i is still a Zariski closed nonsingular subvariety of $\mathbb{R}^{2m+1}$ and $(f|N_i, h_i) : N_i \to B \times \mathbb{G}_{n,m}$ is a regular map for $1 \le i \le s$. By construction, $\tau(M)|N_i$ admits an algebraic structure (being isomorphic to $h_i^* \gamma_{n,m}$). Note that $N = N_1 \cup \ldots \cup N_s$ is a Zariski closed nonsingular subvariety of $\mathbb{R}^{2m+1}$ and

(6) $$f|N : N \to B \text{ is a regular map,}$$

(7) $$\tau(M)|N \text{ admits an algebraic structure.}$$

We can further modify f so that it is constant on some open subset U of M which is disjoint from N and has a nonempty intersection with each connected component of M. Let P be a compact k-dimensional smooth submanifold of U such that each connected component of M contains a connected component of P, each connected component of P is diffeomorphic to S^k, and the restriction $\tau(M)|P$ is a trivial vector bundle. There is a smooth diffeomorphism σ of $\mathbb{R}^{2m+1}$ such that $\sigma(x) = x$ for x in N and $\sigma(P)$ is a Zariski closed nonsingular irreducible subvariety of $\mathbb{R}^{2m+1}$. Replacing M by $\sigma(M)$, we may assume that P itself is a Zariski closed nonsingular irreducible subvariety of $\mathbb{R}^{2m+1}$.

Note that $N \cup P$ is a Zariski closed nonsingular subvariety of $\mathbb{R}^{2m+1}$. Since f is constant on P, it follows from (6) that

(8) $$f|(N \cup P) : N \cup P \to B \text{ is a regular map.}$$

Furthermore, in view of (7), we get

(9) $$\tau(M)|(N \cup P) \text{ admits an algebraic structure.}$$

We claim that $f : M \to B$ and a constant map $M \to B$ sending M to a point in B_0 represent the same class in the bordism group $\mathcal{N}_*(B)$. We verify the claim via Theorem 2.3. It suffices to show that given a positive integer q and a cohomology class ξ in $H^q(B, \mathbb{Z}/2)$, we have

(10)
$$< f^*(\xi) \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M), [M] >= 0$$

for all nonnegative integers $i_1, \ldots , i_r$ with $i_1 + \cdots + i_r = m - q$. If q is not a multiple of k, then $\xi = 0$ and (10) holds. If $q = \ell k \leq m$, then ξ is a linear combination of cohomology classes of the form $\xi_1 \cup \ldots \cup \xi_\ell$, where $\xi_1, \ldots , \xi_\ell$ are in $H^k(B, \mathbb{Z}/2)$. By (3), the cohomology classes $f^*(\xi_1), \ldots , f^*(\xi_\ell)$ are in G and hence (10) follows from (b). Thus the claim is proved.

The claim implies that the class of $f : M \to B$ in the bordism group $\mathcal{N}_*(B)$ is algebraic. In view of (8) and (9) we can apply Theorem 2.4 to $f : M \to B$ (with $L = N \cup P$). Hence there exist a Zariski closed nonsingular subvariety X of $\mathbb{R}^{2m+1}$, a smooth diffeomorphism $\varphi : M \to X$, and a regular map $g : X \to B$ such that $X = \varphi(M)$, $\varphi(x) = x$ for all x in $N \cup P$, and f is homotopic to $g \circ \varphi$. Clearly, X is irreducible, the variety P being irreducible. In order to complete the proof it remains to show $\varphi^*(\mathrm{Alg}^k(X)) = G$. We argue as follows. Since $g : X \to B$ is a regular map, we have $g^*(\mathrm{Alg}^k(B)) \subseteq \mathrm{Alg}^k(X)$. Hence using (3) and $f^* = (g \circ \varphi)^* = \varphi^* \circ g^*$, we obtain

$$G = f^*(\mathrm{Alg}^k(B)) = \varphi^*(g^*(\mathrm{Alg}^k(B))) \subseteq \varphi^*(\mathrm{Alg}^k(X)).$$

Suppose there is an element w in $\mathrm{Alg}^k(X)$ such that $\varphi^*(w)$ is not in G. By (2), one can find i, $1 \leq i \leq s$, for which

$$< e_i^*(\varphi^*(w)), [N_i] >\neq 0.$$

If $\epsilon_i : N_i \hookrightarrow X$ is the inclusion map, then $\epsilon_i = \varphi \circ e_i$ and hence $\epsilon_i^*(w) = e_i^*(\varphi^*(w))$. It follows that

$$< \epsilon_i^*(w), [N_i] >\neq 0.$$

This contradicts Theorem 2.1(i) since $\epsilon_i^*(w)$ is in $\mathrm{Alg}^k(N_i)$, the map ϵ_i being regular. Thus $\varphi^*(\mathrm{Alg}^k(X)) = G$ and the proof is complete. $\square$

Proposition 2.6 *Let M be a compact smooth manifold of dimension m with $m \geq 2$. Let G be a subgroup of $H^{m-1}(M, \mathbb{Z}/2)$. Assume that G is generated by spherical cohomology classes. Then the following conditions are equivalent:*

(a) *There exist an algebraic model X of M and a smooth diffeomorphism $\varphi : M \to X$ such that $\varphi^*(\mathrm{Alg}^{m-1}(X)) = G$,*

(b) *$< u \cup w_1(M), [M] >= 0$ for all u in G, and when $m = 2$, then in addition $< u_1 \cup u_2, [M] >= 0$ for all u_1 and u_2 in G.*

Futhermore, if (b) *holds, then X in* (a) *can be chosen irreducible.*

Proof. It follows from Theorem 2.1(ii) that (a) implies (b). Suppose then that (b) holds. We prove below that (a), with X irreducible, is satisfied. In the proof we assume that M is a smooth submanifold of $\mathbb{R}^{2m+1}$.

Let us set

$$\Gamma = \{v \in H^1(M, \mathbb{Z}/2) | < u \cup v, [M] >= 0 \text{ for all } u \text{ in } G\},\ \Gamma = \{u_1, \dots, u_s\}.$$

If n is sufficiently large and $A = \mathbb{P}^n(\mathbb{R}) \times \cdots \times \mathbb{P}^n(\mathbb{R})$ is the product of s copies of real projective n-space $\mathbb{P}^n(\mathbb{R})$, then there exists a smooth map $f : M \to A$ for which

$$\Gamma = f^*(H^1(A, \mathbb{Z}/2)). \tag{1}$$

Let $u_1, \dots, u_d$ be spherical cohomology classes generating G. Using the same notation as in (2.2), with $k = m-1$, choose a smooth map $g_j : M \to B^{m-1}$ such that $g_j(M) \subset B_0^{m-1}$ and $g_j^*(H^{m-1}(B^{m-1}, \mathbb{Z}/2))$ is the subgroup of G generated by u_j. Note that (2.2) implies

$$G = g^*(H^{m-1}(B, \mathbb{Z}/2)) = g^*(\mathrm{Alg}^{m-1}(B)), \tag{2}$$

where $g = (g_1, \dots, g_d) : M \to B = B^{m-1} \times \cdots \times B^{m-1}$.

We can choose f and g so that the map $(f, g) : M \to A \times B$ is constant on some open subset U of M which has a nonempty intersection with each connected component of M. Let C be a compact smooth curve in U such that each connected component of M contains a connected component of C and the restriction $\tau(M)|C$ is trivial. There is a smooth diffeomorphism σ of $\mathbb{R}^{2m+1}$ such that $\sigma(C)$ is a Zariski closed nonsingular irreducible curve in $\mathbb{R}^{2m+1}$. Replacing M by $\sigma(M)$, we may assume that C itself is a Zariski closed nonsingular irreducible curve in $\mathbb{R}^{2m+1}$.

We assert that the maps $(f, g) : M \to A \times B$ and $(f, c) : M \to A \times B$, where $c : M \to B$ is a constant map sending M to a point in B_0, represent the same class in the bordism group $\mathcal{N}_*(A \times B)$. By Theorem 2.3 and Künneth's theorem in cohomology, in order to prove the assertion it suffices to show that given cohomology classes ξ in $H^p(A, \mathbb{Z}/2)$ and η in $H^q(B, \mathbb{Z}/2)$, we have

$$\begin{aligned} &< (f,g)^*(\xi \times \eta) \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M), [M] >= \\ &\qquad < (f,c)^*(\xi \times \eta) \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M), [M] > \end{aligned}$$

for all nonnegative integers $i_1, \dots, i_r$ satisfying $i_1 + \cdots + i_r = m - (p+q)$. Since $(f,g)^*(\xi \times \eta) = f^*(\xi) \cup g^*(\eta)$ and $(f,c)^*(\xi \times \eta) = f^*(\xi) \cup c^*(\eta)$, the last displayed equality is equivalent to

$$\begin{aligned} &< f^*(\xi) \cup g^*(\eta) \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M), [M] >= \\ &\qquad < f^*(\xi) \cup c^*(\eta) \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M), [M] > . \end{aligned} \tag{3}$$

If q is not a multiple of $m-1$, then $\eta = 0$ and hence (3) holds. If $q = 0$, then $g^*(\eta) = c^*(\eta)$ and (3) is also satisfied. It remains to consider the following three cases: $(p,q) = (1, m-1)$, $(p,q) = (0, m-1)$, and $(p,q) = (0,2)$ with $m = 2$. In each of these cases $c^*(\eta) = 0$. If $(p,q) = (1, m-1)$, then (3) is reduced to

$$< f^*(\xi) \cup g^*(\eta), [M] >= 0,$$

which holds in view of (1), (2), and the definition of Γ. If $(p,q) = (0, m-1)$, then (3) is equivalent to

$$< f^*(\xi) \cup g^*(\eta) \cup w_1(M), [M] >= 0,$$

which follows from (2) and (b) (note that $f^*(\xi) \cup g^*(\eta) = \lambda g^*(\eta)$, where $\lambda = 0$ or $\lambda = 1$). If $(p,q) = (0,2)$, $m = 2$, then $f^*(\xi) \cup g^*(\eta) = \lambda g^*(\eta)$, where $\lambda = 0$ or $\lambda = 1$, and (3) is equivalent to

$$< \lambda g^*(\eta), [M] >= 0.$$

The last equality follows from (b) since η is a linear combination of cohomology classes of the form $\eta_1 \cup \eta_2$, where η_1, η_2 are in $H^1(B, \mathbb{Z}/2)$, and in view of (2), $g^*(\eta_1)$, $g^*(\eta_2)$ are in G. This completes the proof of (3) and hence the assertion holds.

We shall now prove

(4) the bordism class of $(f,g) : M \to A \times B$ in $\mathcal{N}_*(A \times B)$ is algebraic.

Since every bordism class in $\mathcal{N}_*(A)$ is algebraic [3, Lemma 2.7.1], in view of Theorem 2.4, there exist a Zariski closed nonsingular subvariety Y of $\mathbb{R}^{2m+1}$, a smooth diffeomorphism $\psi : M \to Y$, and a regular map $\bar{f} : Y \to A$ such that f is homotopic to $\bar{f} \circ \psi$. Clearly, $(f,c) : M \to A \times B$ and $(\bar{f}, c \circ \psi^{-1}) : Y \to A \times B$ represent the same bordism class in $\mathcal{N}_*(A \times B)$. Note that $(\bar{f}, c \circ \psi^{-1}) : Y \to A \times B$ is a regular map, $c \circ \psi^{-1} : Y \to B$ being constant. Hence (4) follows from the assertion proved above.

By construction, $(f,g) : M \to A \times B$ is constant on C and $\tau(M)|C$ is a trivial vector bundle. Thus (4) allows us to apply Theorem 2.4 to $(f,g) : M \to A \times B$ (with $L = C$). Therefore there exist a Zariski closed nonsingular subvariety X of $\mathbb{R}^{2m+1}$, a smooth diffeomorphism $\varphi : M \to X$, and a regular map $(\alpha, \beta) : X \to A \times B$ such that $X = \varphi(M)$, $\varphi(x) = x$ for all x in C, and (f,g) is homotopic to $(\alpha, \beta) \circ \varphi = (\alpha \circ \varphi, \beta \circ \varphi)$. Obviously, X is irreducible, the curve C being irreducible.

It remains to prove $\varphi^*(\mathrm{Alg}^{m-1}(X)) = G$. Since $\beta : X \to B$ is a regular map, we have

$\beta^*(\mathrm{Alg}^{m-1}(B)) \subseteq \mathrm{Alg}^{m-1}(X)$. Making use of $g^* = (\beta \circ \varphi)^* = \varphi^* \circ \beta^*$ and (2), we get

$$G = g^*(\mathrm{Alg}^{m-1}(B)) = \varphi^*(\beta^*(\mathrm{Alg}^{m-1}(B))) \subseteq \varphi^*(\mathrm{Alg}^{m-1}(X)).$$

Suppose there exists w in $\mathrm{Alg}^{m-1}(X)$ such that $\varphi^*(w)$ is not in G. We obtain a contradiction as follows. The bilinear map

$$H^{m-1}(M, \mathbb{Z}/2) \times H^1(M, \mathbb{Z}/2) \to \mathbb{Z}/2, \quad (u, v) \to < u \cup v, [M] >$$

is a dual pairing [12, p.300, Proposition 8.13] and hence one can find an element v in Γ with $< \varphi^*(w) \cup v, [M] > \neq 0$. By (1), we have $v = f^*(z)$ for some z in $H^1(A, \mathbb{Z}/2)$. Since $f^* = (\alpha \circ \varphi)^* = \varphi^* \circ \alpha^*$, we get $v = \varphi^*(\alpha^*(z))$. Thus $< \varphi^*(w) \cup \varphi^*(\alpha^*(z)), [M] > \neq 0$, which yields

(5)
$$< w \cup \alpha^*(z), [X] > \neq 0.$$

Note that $\alpha^*(z)$ is in $H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$, the map $\alpha : X \to A$ being regular and $H^1(A, \mathbb{Z}/2) = H^1_{\mathrm{alg}}(A, \mathbb{Z}/2)$. Hence (5) contradicts Theorem 2.1(i). The proof is complete. □

Proof of Theorem 1.1. Obviously, every spherical cohomology class with coefficients in $\mathbb{Z}/2$ is the reduction modulo 2 of a cohomology class with coefficients in $\mathbb{Z}$. Therefore it follows from Theorem 2.1(ii), (iii) that (a) implies (b). Suppose (b) holds. The first part of (b) guarantees that every cohomology class in G is spherical [15, p. 49, Theorem 7.1]. Thus (a), with X irreducible, holds by virtue of Theorem 2.5 and Proposition 2.6 (Proposition 2.6 is required only when $m = 2$). □

Proof of Theorem 1.2. We already know that, by Theorem 2.1(ii), (a) implies (b). Suppose (b) is satisfied. Since M is connected, given u in $H^{m-1}(M, \mathbb{Z}/2)$ with $< u \cup w_1(M), [M] >= 0$, we get $u \cup w_1(M) = 0$. The last equality implies that the homology class in $H_1(M, \mathbb{Z}/2)$ Poincaré dual to u can be represented by a compact smooth curve in M with trivial normal vector bundle, cf. for example [8, p. 599]. This in turn implies that u is spherical [20, Théorème II.1]. Hence every cohomology class in G is spherical. In view of Proposition 2.6, condition (a), with X irreducible, holds. $\square$

We conclude the paper with an example.

Example 2.7. Let $T^m = S^1 \times \cdots \times S^1$ be the m-fold product with $m \geq 2$. Clearly, $H^\ell_{\mathrm{alg}}(T^m, \mathbb{Z}/2) = H^\ell(T^m, \mathbb{Z}/2)$ for all $\ell \geq 0$ and hence, by Theorem 2.1(i),

$$\mathrm{Alg}^k(T^m) = 0 \text{ for all } k \geq 0.$$

On the other hand, let G be a subgroup of $H^k(T^m, \mathbb{Z}/2)$ and suppose that one of the following conditions is satisfied:

(i) $k = 1$ and $G \neq H^1(T^m, \mathbb{Z}/2)$,

(ii) $k = m - 1$ and $m \geq 3$,

(iii) $k \geq 1$, $2k+1 \leq m$, m is not divisible by k, and G is generated by spherical cohomology classes,

(iv) $m = k\ell$, where ℓ is an integer satisfying $\ell \geq \max\{2m/(m-1), \dim_{\mathbb{Z}/2} G + 1\}$, and G is generated by spherical cohomology classes.

Then there exist an algebraic model X of T^m and a smooth diffeomorphism $\varphi : T^m \to X$ such that

$$\varphi^*(\mathrm{Alg}^k(X)) = G.$$

Indeed, since the tangent bundle to T^m is trivial, we have $w_i(T^m) = 0$ for all $i \geq 1$. Furthermore, if either (i) or (iv) is satisfied, $m = k\ell$, and $u_1, \ldots, u_\ell$ are in G, then $u_1, \cup \ldots \cup u_\ell = 0$. Thus X and φ with the required property exist in view of Theorems 1.1, 1.2, and 2.5.

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