COMPARISON OF THE REFINED ANALYTIC AND THE BURGHELEA-HALLER TORSIONS

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Abstract. We express the Burghelea-Haller complex Ray-Singer torsion in terms of the square of the refined analytic torsion and the η-invariant. As an application we obtain new results about the Burghelea-Haller torsion. In particular, we prove a weak version of the Burghelea-Haller conjecture relating their torsion with the square of the Turaev combinatorial torsion.

1. Introduction

1.1. **The refined analytic torsion.** Let M be a closed oriented manifold of odd dimension $d = 2r - 1$ and let E be a complex vector bundle over M endowed with a flat connection ∇. In a series of papers [4, 5, 3], we defined and studied the non-zero element

$$\rho_{\mathrm{an}} \;=\; \rho_{\mathrm{an}}(\nabla) \;\in\; \mathrm{Det}\,\big(H^\bullet(M,E)\big)$$

of the determinant line $\mathrm{Det}\,\big(H^\bullet(M,E)\big)$ of the cohomology $H^\bullet(M,E)$ of M with coefficients in E. This element, called the *refined analytic torsion*, can be viewed as an analogue of the refinement of the Reidemeister torsion due to Turaev [22, 23, 24] and, in a more general context, to Farber-Turaev [14, 15]. The refined analytic torsion carries information about the Ray-Singer metric and about the η-invariant of the odd signature operator associated to ∇ and a Riemannian metric on M. In particular, if ∇ is a hermitian connection, then the Ray-Singer norm of $\rho_{\mathrm{an}}(\nabla)$ is equal to 1. One of the main properties of the refined analytic torsion is that it depends holomorphically on ∇. Using this property we computed the ratio between the refined analytic torsion and the Turaev-Farber torsion, cf. Th. 14.5 of [4] and Th. 5.11 of [3]. This result extends the classical Cheeger-Müller theorem about the equality between the Ray-Singer and the Reidemeister torsions [21, 12, 19, 20, 2].

1.2. **The Burghelea-Haller complex Ray-Singer torsion.** On a different line of thoughts, Burghelea and Haller [11, 9] have introduced a refinement of the square of the Ray-Singer torsion for a closed manifold of arbitrary dimension, provided that the complex vector bundle E admits

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a non-degenerate complex valued symmetric bilinear form b. They defined a complex valued quadratic form

$$\tau \;=\; \tau_{b,\nabla} \tag{1.1}$$

on the determinant line $\operatorname{Det}\big(H^\bullet(M,E)\big)$, which depends holomorphically on the flat connection ∇ and is closely related to (the square of) the Ray-Singer torsion. Burghelea and Haller then defined a complex valued quadratic form, referred to as *complex Ray-Singer torsion*. In the case of a closed manifold M of odd dimension it is given by

$$\tau^{\rm BH}_{b,\alpha,\nabla} \;:=\; \tau_{b,\nabla}\cdot e^{-2\int_M\,\omega_{\nabla,b}\wedge\alpha}, \tag{1.2}$$

where $\alpha\in\Omega^{d-1}(M)$ is an arbitrary closed $(d-1)$-form and $\omega_{\nabla,b}\in\Omega^1(M)$ is the Kamber-Tondeur form, cf. [9, §2] – see the discussion at the end of Section 5 of [9] for the reasons to introduce this extra factor. Burghelea and Haller conjectured that, for a suitable choice of α the form $\tau^{\rm BH}_{b,\alpha,\nabla}$ is roughly speaking equal to the square of the Turaev torsion, cf. [9, Conjecture 5.1] and Conjecture 1.9 below.

Note that τ seems not to be related to the η-invariant, whereas the refined analytic torsion is closely related to it. In fact, our study of $\rho_{\rm an}$ leads to new results about η, cf. [4, Th. 14.10, 14.12] and [3, Prop. 6.2, Cor. 6.4].

1.3. **The comparison theorem.** The main result of this paper is the following theorem establishing a relationship between the refined analytic torsion and the Burghelea-Haller quadratic form.

Theorem 1.4. *Suppose M is a closed oriented manifold of odd dimension $d=2r-1$ and let E be a complex vector bundle over M endowed with a flat connection ∇. Assume that there exists a symmetric bilinear form b on E so that the quadratic form* (1.1) *on* $\operatorname{Det}\big(H^\bullet(M,E)\big)$ *is defined. Then*

$$\tau(\rho_{\rm an}) \;=\; \pm\, e^{-\,2\pi i\,\big(\,\eta(\nabla)-\operatorname{rank}E\cdot\eta_{\rm trivial}\,\big)}, \tag{1.3}$$

where $\eta(\nabla)$ stands for the η-invariant of the odd signature operator, associated to the flat vector bundle (E,∇) and a Riemannian metric on M (cf. Definition 2.5) and $\eta_{\rm trivial}$ is the η-invariant of the trivial line bundle over M.

The proof is given in Section 4.

Remark 1.5. For mere convenience of notation we use a slightly modified version of the torsion introduced in [5] - see Definition 2.13 and Remark 2.14 below for details. The difference $\eta(\nabla)-\operatorname{rank}E\cdot\eta_{\rm trivial}$ in (1.3) is called the ρ-invariant of (E,∇) and its reduction modulo $\mathbb{Z}$ is independent of the Riemannian metric.

The above theorem takes an especially simple form, when the bundle (E,∇) is acyclic, i.e., when $H^\bullet(M,E)=0$. Then the determinant line bundle $\operatorname{Det}\big(H^\bullet(M,E)\big)$ is canonically isomorphic to $\mathbb{C}$ and both, τ and $\rho_{\rm an}$, can be viewed as non-zero complex numbers. We then obtain the following

Corollary 1.6. *If in addition to the assumptions of Theorem 1.4 one has $H^\bullet(M,E)=0$, then*

$$\tau \;=\; \pm\Big(\rho_{\rm an}\cdot e^{\pi i\,(\,\eta(\nabla)-\operatorname{rank}E\cdot\eta_{\rm trivial})}\Big)^{-2}. \tag{1.4}$$

In general, τ does not admit a square root which is holomorphic in ∇, cf. Remark 5.12 and the discussion after it in [9]. In particular, the product $\rho_{\rm an}\cdot e^{\pi i(\eta(\nabla)-\operatorname{rank}E\cdot\eta_{\rm trivial})}$ is not a holomorphic function of ∇, since $e^{\pi i(\eta(\nabla)-\operatorname{rank}E\cdot\eta_{\rm trivial})}$ is not even continuous in ∇. Thus the refined analytic torsion can be viewed as a modified version of the inverse square root of $\pm\tau$, which is holomorphic.

1.7. **Applications.** As an application of Theorem 1.4 and the results of our previous papers [4, 5, 3] we obtain new properties and new proofs of some known ones of the Burghelea-Haller form τ. In particular, we give a new proof of the Burghelea-Haller theorem [9, Th. 4.2] stating that $\tau_{b,\nabla}$ is independent of the Riemannian metric and is locally constant in b, cf. Theorem 5.2. We also slightly improve the latter result, cf. Theorem 5.3.

Next we discuss our main application of Theorem 1.4.

1.8. **Comparison between the Turaev and the Burghelea-Haller torsions.** In [9], Burghelea and Haller made a conjecture that the quadratic form (1.2) is roughly speaking equal to the square of the Turaev torsion, cf. [9, Conjecture 5.1]. More precisely, recall that the Turaev torsion depends on the Euler structure ε and a choice of a cohomological orientation, i.e, an orientation $\mathfrak{o}$ of the determinant line of the cohomology $H^\bullet(M,\mathbb{R})$ of M. The set of Euler structures $\operatorname{Eul}(M)$, introduced by Turaev, is an affine version of the integer homology $H_1(M,\mathbb{Z})$ of M. It has several equivalent descriptions [22, 23, 7, 10]. For our purposes, it is convenient to adopt the definition from Section 6 of [23], where an Euler structure is defined as an equivalence class of nowhere vanishing vector fields on M – see [23, §5] for the description of the equivalence relation. The definition of the Turaev torsion was reformulated by Farber and Turaev [14, 15]. The Farber-Turaev torsion, depending on ε, $\mathfrak{o}$, and ∇, is an element of the determinant line $\operatorname{Det}\big(H^\bullet(M,E)\big)$, which we denote by $\rho_{\varepsilon,\mathfrak{o}}(\nabla)$.

Recall that the quadratic form $\tau^{\rm BH}_{b,\alpha,\nabla}$ is defined in (1.2). Burghelea and Haller made a conjecture, [9, Conjecture 5.1], relating the quadratic form $\tau^{\rm BH}_{b,\alpha,\nabla}$ and $\rho_{\varepsilon,\mathfrak{o}}(\nabla)$, which extends the Bismut-Zhang theorem [2]. They have proved their conjecture modulo sign in the case when the dimension of the manifold M is even and the bundle E admits a parallel symmetric bilinear form ([9, Th. 5.7]) and in some other cases as well - see [9, §5]. Though Burghelea and Haller stated their conjecture for manifolds of arbitrary dimensions, we restrict our formulation to the odd dimensional case.

Conjecture 1.9. **[Burghelea-Haller]** *Suppose M is a closed oriented odd dimensional manifold. Let $\varepsilon\in\operatorname{Eul}(M)$ be an Euler structure on M represented by a non-vanishing vector field X. Fix a Riemannian metric g^M on M and let $\Psi(g^M)\in\Omega^{d-1}(TM\backslash\{0\})$ denote the Mathai-Quillen form,* [18, §7]*,* [2, pp. 40-44]*. Set*

$$\alpha_\varepsilon \;=\; \alpha_\varepsilon(g^M) \;:=\; X^*\Psi(g^M).$$

Assume that (E,∇) is a flat vector bundle over M which admits a non-degenerate symmetric bilinear form b. Then

$$\tau^{\mathrm{BH}}_{b,\alpha_\varepsilon,\nabla}\big(\,\rho_{\varepsilon,\mathfrak{o}}(\nabla)\,\big) \;=\; 1. \tag{1.5}$$

In Theorem 5.1 of [3] we computed the ratio of the refined analytic and the Turaev torsions. Using this result and Theorem 1.4 we establish the following weak version of Conjecture 1.9.

Theorem 1.10. *Under the same assumptions as in Conjecture 1.9, for each connected component $\mathcal{C}$ of the set* $\mathrm{Flat}(E)$ *of flat connections on E there exists a constant $R_{\mathcal{C}}$ with $|R_{\mathcal{C}}|=1$, such that*

$$\tau^{\mathrm{BH}}_{b,\alpha_\varepsilon,\nabla}\big(\,\rho_{\varepsilon,\mathfrak{o}}(\nabla)\,\big) \;=\; R_{\mathcal{C}}, \qquad \textit{for all} \quad \nabla\in\mathcal{C}. \tag{1.6}$$

The proof is given in Subsection 5.4.

Remark 1.11. It was brought to our attention by Stefan Haller that one can modify the arguments of the proofs of Theorem 1.4 and of [3, Th. 5.1] so that they can be applied directly to the Burghelea-Haller torsion. In this way, one can obtain a more direct proof of Theorem 1.10 and Theorems 5.2–5.2. Moreover it might lead to a proof of the statements of these theorems for even dimensional manifolds as well.

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2. The Refined Analytic Torsion

In this section we recall the definition of the refined analytic torsion from [5]. The refined analytic torsion is constructed in 3 steps: first, we define the notion of refined torsion of a finite dimensional complex endowed with a chirality operator, cf. Definition 2.3. Then we fix a Riemannian metric g^M on M and consider the odd signature operator $\mathcal{B}=\mathcal{B}(\nabla,g^M)$ associated to a flat vector bundle (E,∇), cf. Definition 2.5. Using the *graded determinant* of $\mathcal{B}$ and the definition of the refined torsion of a finite dimensional complex with a chirality operator we construct an element $\rho=\rho(\nabla,g^M)$ in the determinant line of the cohomology, cf. (2.14). The element ρ is almost the refined analytic torsion. However, it might depend on the Riemannian metric g^M (though it does not if $\dim M\equiv 1 \pmod 4$ or if $\mathrm{rank}(E)$ is divisible by 4). Finally we "correct" ρ by multiplying it by an explicit factor, the metric anomaly of ρ, to obtain a diffeomorphism invariant $\rho_{\mathrm{an}}(\nabla)$ of the triple (M,E,∇), cf. Definition 2.13.

2.1. **The determinant line of a complex.**

Given a complex vector space V of dimension $\dim V=n$, the *determinant line* of V is the line $\mathrm{Det}(V):=\Lambda^nV$, where Λ^nV denotes the n-th exterior power of V. By definition, we set $\mathrm{Det}(0):=\mathbb{C}$. Further, we denote by $\mathrm{Det}(V)^{-1}$ the dual line of $\mathrm{Det}(V)$. Let

$$(C^\bullet,\partial):\quad 0\;\to C^0\;\xrightarrow{\;\;\partial\;\;}\;C^1\;\xrightarrow{\;\;\partial\;\;}\;\cdots\;\xrightarrow{\;\;\partial\;\;}\;C^d\;\to\;0 \tag{2.1}$$

be a complex of finite dimensional complex vector spaces. We call the integer d the *length* of the complex $(C^\bullet,\partial)$ and denote by $H^\bullet(\partial)=\bigoplus_{i=0}^d H^i(\partial)$ the cohomology of $(C^\bullet,\partial)$. Set

$$\mathrm{Det}(C^\bullet) \;:=\; \bigotimes_{j=0}^{d}\mathrm{Det}(C^j)^{(-1)^j}, \qquad \mathrm{Det}(H^\bullet(\partial)) \;:=\; \bigotimes_{j=0}^{d}\mathrm{Det}(H^j(\partial))^{(-1)^j}. \tag{2.2}$$

The lines $\mathrm{Det}(C^\bullet)$ and $\mathrm{Det}(H^\bullet(\partial))$ are referred to as the *determinant line of the complex* $C^\bullet$ and the *determinant line of its cohomology*, respectively. There is a canonical isomorphism

$$\phi_{C^\bullet} \;=\; \phi_{(C^\bullet,\partial)}:\mathrm{Det}(C^\bullet) \;\longrightarrow\; \mathrm{Det}(H^\bullet(\partial)), \tag{2.3}$$

cf., for example, §2.4 of [5].

2.2. **The refined torsion of a finite dimensional complex with a chirality operator.** Let $d=2r-1$ be an odd integer and let $(C^\bullet,\partial)$ be a length d complex of finite dimensional complex vector spaces. A *chirality operator* is an involution $\Gamma: C^\bullet\to C^\bullet$ such that $\Gamma(C^j)=C^{d-j}$, $j=0,\dots,d$. For $c_j\in\mathrm{Det}(C^j)$ $(j=0,\dots,d)$ we denote by $\Gamma c_j\in\mathrm{Det}(C^{d-j})$ the image of c_j under the isomorphism $\mathrm{Det}(C^j)\to\mathrm{Det}(C^{d-j})$ induced by Γ. Fix non-zero elements $c_j\in\mathrm{Det}(C^j)$, $j=0,\dots,r-1$ and denote by c_j^{-1} the unique element of $\mathrm{Det}(C^j)^{-1}$ such that $c_j^{-1}(c_j)=1$. Consider the element

$$c_\Gamma \;:=\; (-1)^{\mathcal{R}(C^\bullet)}\cdot c_0\otimes c_1^{-1}\otimes\cdots\otimes c_{r-1}^{(-1)^{r-1}}\otimes(\Gamma c_{r-1})^{(-1)^r}\otimes(\Gamma c_{r-2})^{(-1)^{r-1}}\otimes\cdots\otimes(\Gamma c_0)^{-1} \tag{2.4}$$

of $\mathrm{Det}(C^\bullet)$, where

$$\mathcal{R}(C^\bullet) \;:=\; \frac12\sum_{j=0}^{r-1}\dim C^j\cdot\big(\dim C^j+(-1)^{r+j}\big). \tag{2.5}$$

It follows from the definition of c_j^{-1} that c_Γ is independent of the choice of c_j $(j=0,\dots,r-1)$.

Definition 2.3. *The* refined torsion *of the pair* $(C^\bullet,\Gamma)$ *is the element*

$$\rho_\Gamma \;=\; \rho_{C^\bullet,\Gamma} \;:=\; \phi_{C^\bullet}(c_\Gamma)\;\in\mathrm{Det}\big(H^\bullet(\partial)\big), \tag{2.6}$$

where $\phi_{C^\bullet}$ *is the canonical map* (2.3).

2.4. **The odd signature operator.** Let M be a smooth closed oriented manifold of odd dimension $d=2r-1$ and let (E,∇) be a flat vector bundle over M. We denote by $\Omega^k(M,E)$ the space of smooth differential forms on M of degree k with values in E and by

$$\nabla:\Omega^\bullet(M,E)\;\longrightarrow\Omega^{\bullet+1}(M,E)$$

the covariant differential induced by the flat connection on E. Fix a Riemannian metric g^M on M and let $*:\Omega^\bullet(M,E)\to\Omega^{d-\bullet}(M,E)$ denote the Hodge $*$-operator. Define the *chirality operator* $\Gamma=\Gamma(g^M):\Omega^\bullet(M,E)\to\Omega^\bullet(M,E)$ by the formula

$$\Gamma\,\omega \;:=\; i^r\,(-1)^{\frac{k(k+1)}{2}}\;*\,\omega, \qquad \omega\in\Omega^k(M,E), \tag{2.7}$$

with r given as above by $r=\frac{d+1}{2}$. The numerical factor in (2.7) has been chosen so that $\Gamma^2=1$, cf. Proposition 3.58 of [1].

Definition 2.5. *The* odd signature operator *is the operator*

$$\mathcal{B} \;=\; \mathcal{B}(\nabla, g^M) \;:=\; \Gamma\nabla \;+\; \nabla\Gamma : \Omega^\bullet(M,E) \;\longrightarrow\; \Omega^\bullet(M,E). \tag{2.8}$$

We denote by $\mathcal{B}_k$ the restriction of $\mathcal{B}$ to the space $\Omega^k(M,E)$.

2.6. **The graded determinant of the odd signature operator.** Note that for each $k = 0,\dots,d$, the operator $\mathcal{B}^2$ maps $\Omega^k(M,E)$ into itself. Suppose $\mathcal{I}$ is an interval of the form $[0,\lambda]$, $(\lambda,\mu]$, or (λ,∞) $(\mu>\lambda\geq 0)$. Denote by $\Pi_{\mathcal{B}^2,\mathcal{I}}$ the spectral projection of $\mathcal{B}^2$ corresponding to the set of eigenvalues, whose absolute values lie in $\mathcal{I}$. Set

$$\Omega^\bullet_{\mathcal{I}}(M,E) \;:=\; \Pi_{\mathcal{B}^2,\mathcal{I}}\big(\Omega^\bullet(M,E)\big) \;\subset\; \Omega^\bullet(M,E).$$

If the interval $\mathcal{I}$ is bounded, then, cf. Section 6.10 of [5], the space $\Omega^\bullet_{\mathcal{I}}(M,E)$ is finite dimensional.

For each $k=0,\dots,d$, set

$$\begin{aligned}\Omega^k_{+,\mathcal{I}}(M,E) \;&:=\; \mathrm{Ker}\,(\nabla\Gamma)\cap\Omega^k_{\mathcal{I}}(M,E) \;=\; \big(\Gamma\,(\mathrm{Ker}\,\nabla)\big)\cap\Omega^k_{\mathcal{I}}(M,E);\\ \Omega^k_{-,\mathcal{I}}(M,E) \;&:=\; \mathrm{Ker}\,(\Gamma\nabla)\cap\Omega^k_{\mathcal{I}}(M,E) \;=\; \mathrm{Ker}\,\nabla\cap\Omega^k_{\mathcal{I}}(M,E).\end{aligned} \tag{2.9}$$

Then

$$\Omega^k_{\mathcal{I}}(M,E) \;=\; \Omega^k_{+,\mathcal{I}}(M,E)\oplus\Omega^k_{-,\mathcal{I}}(M,E) \qquad \text{if}\quad 0\notin\mathcal{I}. \tag{2.10}$$

We consider the decomposition (2.10) as a *grading* [1] of the space $\Omega^\bullet_{\mathcal{I}}(M,E)$, and refer to $\Omega^k_{+,\mathcal{I}}(M,E)$ and $\Omega^k_{-,\mathcal{I}}(M,E)$ as the positive and negative subspaces of $\Omega^k_{\mathcal{I}}(M,E)$.

Set

$$\Omega^{\mathrm{even}}_{\pm,\mathcal{I}}(M,E) \;=\; \bigoplus_{p=0}^{r-1}\Omega^{2p}_{\pm,\mathcal{I}}(M,E)$$

and let $\mathcal{B}^{\mathcal{I}}$ and $\mathcal{B}^{\mathcal{I}}_{\mathrm{even}}$ denote the restrictions of $\mathcal{B}$ to the subspaces $\Omega^\bullet_{\mathcal{I}}(M,E)$ and $\Omega^{\mathrm{even}}_{\mathcal{I}}(M,E)$ respectively. Then $\mathcal{B}^{\mathcal{I}}_{\mathrm{even}}$ maps $\Omega^{\mathrm{even}}_{\pm,\mathcal{I}}(M,E)$ to itself. Let $\mathcal{B}^{\pm,\mathcal{I}}_{\mathrm{even}}$ denote the restriction of $\mathcal{B}^{\mathcal{I}}_{\mathrm{even}}$ to the space $\Omega^{\mathrm{even}}_{\pm,\mathcal{I}}(M,E)$. Clearly, the operators $\mathcal{B}^{\pm,\mathcal{I}}_{\mathrm{even}}$ are bijective whenever $0\notin\mathcal{I}$.

Definition 2.7. *Suppose $0\notin\mathcal{I}$. The* graded determinant *of the operator $\mathcal{B}^{\mathcal{I}}_{\mathrm{even}}$ is defined by*

$$\mathrm{Det}_{\mathrm{gr},\theta}(\mathcal{B}^{\mathcal{I}}_{\mathrm{even}}) \;:=\; \frac{\mathrm{Det}_\theta(\mathcal{B}^{+,\mathcal{I}}_{\mathrm{even}})}{\mathrm{Det}_\theta(-\mathcal{B}^{-,\mathcal{I}}_{\mathrm{even}})} \;\in\; \mathbb{C}\backslash\{0\}, \tag{2.11}$$

where Det_θ denotes the ζ-regularized determinant associated to the Agmon angle $\theta\in(-\pi,0)$, cf., for example, §6 of [5].

It follows from formula (6.17) of [5] that (2.11) is independent of the choice of $\theta\in(-\pi,0)$.

[1] Note, that our grading is opposite to the one considered in [8, §2].

2.8. **The canonical element of the determinant line.** Since the covariant differentiation ∇ commutes with $\mathcal{B}$, the subspace $\Omega^\bullet_{\mathcal{I}}(M,E)$ is a subcomplex of the twisted de Rham complex $(\Omega^\bullet(M,E),\nabla)$. Clearly, for each $\lambda \geq 0$, the complex $\Omega^\bullet_{(\lambda,\infty)}(M,E)$ is acyclic. Since

$$\Omega^\bullet(M,E) \;=\; \Omega^\bullet_{[0,\lambda]}(M,E) \oplus \Omega^\bullet_{(\lambda,\infty)}(M,E), \tag{2.12}$$

the cohomology $H^\bullet_{[0,\lambda]}(M,E)$ of the complex $\Omega^\bullet_{[0,\lambda]}(M,E)$ is naturally isomorphic to the cohomology $H^\bullet(M,E)$. Let $\Gamma_{\mathcal{I}}$ denote the restriction of Γ to $\Omega^\bullet_{\mathcal{I}}(M,E)$. For each $\lambda \geq 0$, let

$$\rho_{\Gamma_{[0,\lambda]}} \;=\; \rho_{\Gamma_{[0,\lambda]}}(\nabla, g^M) \in \operatorname{Det}(H^\bullet_{[0,\lambda]}(M,E)) \tag{2.13}$$

denote the refined torsion of the finite dimensional complex $(\Omega^\bullet_{[0,\lambda]}(M,E),\nabla)$ corresponding to the chirality operator $\Gamma_{[0,\lambda]}$, cf. Definition 2.3. We view $\rho_{\Gamma_{[0,\lambda]}}$ as an element of $\operatorname{Det}(H^\bullet(M,E))$ via the canonical isomorphism between $H^\bullet_{[0,\lambda]}(M,E)$ and $H^\bullet(M,E)$.

It is shown in Proposition 7.8 of [5] that the nonzero element

$$\rho(\nabla) \;=\; \rho(\nabla, g^M) \;:=\; \operatorname{Det}_{\mathrm{gr},\theta}(\mathcal{B}^{(\lambda,\infty)}_{\mathrm{even}}) \cdot \rho_{\Gamma_{[0,\lambda]}} \;\in\; \operatorname{Det}(H^\bullet(M,E)) \tag{2.14}$$

is independent of the choice of $\lambda \geq 0$. Further, $\rho(\nabla)$ is independent of the choice of the Agmon angle $\theta \in (-\pi, 0)$ of $\mathcal{B}_{\mathrm{even}}$. However, in general, $\rho(\nabla)$ might depend on the Riemannian metric g^M (it is independent of g^M if $\dim M \equiv 3 \pmod 4$). The refined analytic torsion, cf. Definition 2.13, is a slight modification of $\rho(\nabla)$, which is independent of g^M.

2.9. **The η-invariant.** First, we recall the definition of the η-function of a non-self-adjoint elliptic operator D, cf. [16]. Let $D : C^\infty(M,E) \to C^\infty(M,E)$ be an elliptic differential operator of order $m \geq 1$ whose leading symbol is self-adjoint with respect to some given Hermitian metric on E. Assume that θ is an Agmon angle for D (cf., for example, Definition 3.3 of [4]). Let $\Pi_>$ (resp. $\Pi_<$) be the spectral projection whose image contains the span of all generalized eigenvectors of D corresponding to eigenvalues λ with $\operatorname{Re}\lambda > 0$ (resp. with $\operatorname{Re}\lambda < 0$) and whose kernel contains the span of all generalized eigenvectors of D corresponding to eigenvalues λ with $\operatorname{Re}\lambda \leq 0$ (resp. with $\operatorname{Re}\lambda \geq 0$). For all complex s with $\operatorname{Re}s < -d/m$, we define the η-function of D by the formula

$$\eta_\theta(s,D) \;=\; \zeta_\theta(s,\Pi_>,D) \;-\; \zeta_\theta(s,\Pi_<,-D), \tag{2.15}$$

where $\zeta_\theta(s,\Pi_>,D) := \operatorname{Tr}(\Pi_> D^s)$ and, similarly, $\zeta_\theta(s,\Pi_<,D) := \operatorname{Tr}(\Pi_< D^s)$. Note that, by the above definition, the purely imaginary eigenvalues of D do not contribute to $\eta_\theta(s,D)$.

It was shown by Gilkey, [16], that $\eta_\theta(s,D)$ has a meromorphic extension to the whole complex plane $\mathbb{C}$ with isolated simple poles, and that it is regular at 0. Moreover, the number $\eta_\theta(0,D)$ is independent of the Agmon angle θ.

Since the leading symbol of D is self-adjoint, the angles $\pm\pi/2$ are principal angles for D. Hence, there are at most finitely many eigenvalues of D on the imaginary axis. Let $m_+(D)$ (resp., $m_-(D)$) denote the number of eigenvalues of D, counted with their algebraic multiplicities, on the positive (resp., negative) part of the imaginary axis. Let $m_0(D)$ denote the algebraic multiplicity of 0 as an eigenvalue of D.

Definition 2.10. *The η-invariant $\eta(D)$ of D is defined by the formula*

$$\eta(D) \;=\; \frac{\eta_\theta(0,D)+m_+(D)-m_-(D)+m_0(D)}{2}. \tag{2.16}$$

As $\eta_\theta(0,D)$ is independent of the choice of the Agmon angle θ for D, cf. [16], so is $\eta(D)$.

Remark 2.11. Note that our definition of $\eta(D)$ is slightly different from the one proposed by Gilkey in [16]. In fact, in our notation, Gilkey's η-invariant is given by $\eta(D)+m_-(D)$. Hence, reduced modulo integers, the two definitions coincide. However, the number $e^{i\pi\eta(D)}$ will be multiplied by $(-1)^{m_-(D)}$ if we replace one definition by the other. In this sense, Definition 2.10 can be viewed as a *sign refinement* of the definition given in [16].

Let ∇ be a flat connection on a complex vector bundle $E\to M$. Fix a Riemannian metric g^M on M and denote by

$$\eta(\nabla) \;=\; \eta\big(\mathcal{B}_{\rm even}(\nabla,g^M)\big) \tag{2.17}$$

the η-invariant of the corresponding odd signature operator $\mathcal{B}(\nabla,g^M)$, cf. Definition 2.5.

2.12. **The refined analytic torsion.** Let $\eta_{\rm trivial}=\eta_{\rm trivial}(g^M)$ denote the η-invariant of the operator $\mathcal{B}_{\rm trivial}=\Gamma\,d+d\,\Gamma:\Omega^\bullet(M)\to\Omega^\bullet(M)$. In other words, $\eta_{\rm trivial}$ is the η-invariant corresponding to the trivial line bundle $M\times\mathbb{C}\to M$ over M.

Definition 2.13. *Let (E,∇) be a flat vector bundle on M. The* refined analytic torsion *is the element*

$$\rho_{\rm an} \;=\; \rho_{\rm an}(\nabla) \;:=\; \rho(\nabla,g^M)\cdot\exp\Big(i\pi\cdot{\rm rank}\,E\cdot\eta_{\rm trivial}(g^M)\Big) \;\in\; {\rm Det}(H^\bullet(M,E)), \tag{2.18}$$

where g^M is any Riemannian metric on M and $\rho(\nabla,g^M)\in{\rm Det}(H^\bullet(M,E))$ is defined by (2.14)*.*

It is shown in Theorem 9.6 of [5] that $\rho_{\rm an}(\nabla)$ is independent of g^M.

Remark 2.14. For convenience we use a slightly different definition of the refined analytic torsion than in [4, 5, 3]. In these papers we considered an oriented manifold N whose oriented boundary is the disjoint union of two copies of M and instead of the exponential factor in (2.18) used the term

$$\exp\Big(\frac{i\pi\cdot{\rm rank}\,E}{2}\int_N L(p,g^M)\Big),$$

where $L(p,g^M)$ is the Hirzebruch L-polynomial in the Pontrjagin forms of any Riemannian metric on N which near M is the product of g^M and the standard metric on the half-line. The advantage of this definition is that the latter factor is simpler to calculate than $e^{i\pi\eta_{\rm trivial}}$. In addition, if $\dim M\equiv 3 \pmod 4$, then $\int_M L(p,g^M)=0$ and, hence, the refined analytic torsion then coincides with $\rho(\nabla,g^M)$. However, in general, the refined analytic torsion as defined in our previous papers depends on the choice of N (though only up to a multiplication by $i^{k\cdot{\rm rank}(E)}$ ($k\in\mathbb{Z}$)). For this reason we decided to use an alternative definition in this paper, cf. also [17].

2.15. **Relationship with the η-invariant.** To simplify the notation set

$$T_\lambda \;=\; T_\lambda(\nabla, g^M, \theta) \;=\; \prod_{j=0}^{d} \Big(\operatorname{Det}_{2\theta} \Big[\big((\Gamma\nabla)^2 + (\nabla\Gamma)^2 \big)\big|_{\Omega^j_{(\lambda,\infty)}(M,E)} \Big] \Big)^{(-1)^{j+1} j}. \tag{2.19}$$

where $\theta \in (-\pi/2, 0)$ and both, θ and $\theta+\pi$, are Agmon angles for $\mathcal{B}_{\text{even}}$ (hence, 2θ is an Agmon angle for $\mathcal{B}^2_{\text{even}}$). We shall use the following proposition, cf. [5, Prop. 8.1]:

Proposition 2.16. *Let ∇ be a flat connection on a vector bundle E over a closed Riemannian manifold (M, g^M) of odd dimension $d = 2r-1$. Assume $\theta \in (-\pi/2, 0)$ is such that both θ and $\theta+\pi$ are Agmon angles for the odd signature operator $\mathcal{B} = \mathcal{B}(\nabla, g^M)$. Then, for every $\lambda \geq 0$,*

$$\Big(\operatorname{Det}_{\text{gr},2\theta}(\mathcal{B}^{(\lambda,\infty)}_{\text{even}}) \Big)^2 \;=\; T_\lambda \cdot e^{-2\pi i\, \eta(\nabla, g^M)}. \tag{2.20}$$

Note that Proposition 8.1 of [5] gives a similar formula for the logarithm of $\operatorname{Det}_{\text{gr},2\theta}(\mathcal{B}^{(\lambda,\infty)}_{\text{even}})$, thus providing a sign refined version of (2.20). In the present paper we won't need this refinement.

Proof. Set

$$\eta_\lambda \;=\; \eta_\lambda(\nabla, g^M) \;:=\; \eta(\mathcal{B}^{(\lambda,\infty)}_{\text{even}}). \tag{2.21}$$

From Proposition 8.1 and equality (10.20) of [5] we obtain

$$\operatorname{Det}_{\text{gr},2\theta}(\mathcal{B}^{(\lambda,\infty)}_{\text{even}})^2 \;=\; T_\lambda \cdot e^{-2\pi i\, \eta_\lambda} \cdot e^{-i\pi\, \dim \Omega^{\text{even}}_{[0,\lambda]}(M,E)}. \tag{2.22}$$

The operator $\mathcal{B}^{[0,\lambda]}_{\text{even}}$ acts on finite dimensional vector space $\Omega^{\text{even}}_{[0,\lambda]}(M,E)$. Hence, $2\eta\big(\mathcal{B}^{[0,\lambda]}_{\text{even}}\big) \in \mathbb{Z}$ and

$$2\eta\big(\mathcal{B}^{[0,\lambda]}_{\text{even}}\big) \;\equiv\; \dim \Omega^{\text{even}}_{[0,\lambda]}(M,E) \mod 2. \tag{2.23}$$

Since $\eta_\lambda = \eta\big(\mathcal{B}_{\text{even}}\big) - \eta\big(\mathcal{B}^{[0,\lambda]}_{\text{even}}\big)$, we obtain from (2.23) that

$$e^{-i\pi\, \left(2\eta_\lambda + \dim \Omega^\bullet_{[0,\lambda]}(M,E) \right)} \;=\; e^{-2i\pi\, \eta\left(\mathcal{B}_{\text{even}}\right)}.$$

The equality (2.20) follows now from (2.22). $\square$

3. The Burghelea-Haller Quadratic Form

In this section we recall the construction of the quadratic form on the determinant line $\operatorname{Det}\big(H^\bullet(M,E)\big)$ due to Burghelea and Haller, [9]. Throughout the section we assume that the vector bundle $E \to M$ admits a non-degenerate symmetric bilinear form b. Such a form, required for the construction of τ, might not exist on E, but there always exists an integer N such that on the direct sum $E^N = E \oplus \cdots \oplus E$ of N copies of E such a form exists, cf. Remark 4.6 of [9].

3.1. **A quadratic form on the determinant line of the cohomology of a finite dimensional complex.** Consider the complex (2.1) and assume that each vector space C^j ($j = 0, \dots, d$) is endowed with a non-degenerate symmetric bilinear form $b_j : C^j \times C^j \to \mathbb{C}$. Set $b = \oplus b_j$. Then b_j induces a bilinear form on the determinant line $\mathrm{Det}(C^j)$ and, hence, one obtains a bilinear form on the determinant line $\mathrm{Det}(C^\bullet)$. Using the isomorphism (2.3) we thus obtain a bilinear form on $\mathrm{Det}(H^\bullet(\partial))$. This bilinear form induces a quadratic form on $\mathrm{Det}(H^\bullet(\partial))$, which we denote by $\tau_{C^\bullet,b}$.

The following lemma establishes a relationship between $\tau_{C^\bullet,b}$ and the construction of Subsection 2.2 and is an immediate consequence of the definitions.

Lemma 3.2. *Suppose that d is odd and that the complex $(C^\bullet, \partial)$ is endowed with a chirality operator Γ, cf. Subsection 2.2. Assume further that Γ preserves the bilinear form b, i.e. $b(\Gamma x, \Gamma y) = b(x, y)$, for all $x, y \in C^\bullet$. Then*

$$\tau_{C^\bullet,b}(\rho_\Gamma) \; = \; 1. \tag{3.1}$$

where ρ_Γ is given by (2.6).

3.3. **Determinant of the generalized Laplacian.** Assume now that M is a compact oriented manifold and E is a flat vector bundle over M endowed with a non-degenerate symmetric bilinear form b. Then b together with the Riemannian metric g^M on M define a bilinear form

$$\mathfrak{b} : \Omega^\bullet(M, E) \times \Omega^\bullet(M, E) \; \to \; \mathbb{C} \tag{3.2}$$

in a natural way.

Let $\nabla : \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E)$ denote the flat connection on E and let $\nabla^\# : \Omega^\bullet(M, E) \to \Omega^{\bullet-1}(M, E)$ denote the formal transpose of ∇ with respect to $\mathfrak{b}$. Following Burghelea and Haller we define a (generalized) Laplacian

$$\Delta \; = \; \Delta_{g^M,b} \; := \; \nabla^\# \nabla \; + \; \nabla \nabla^\#. \tag{3.3}$$

Given a Hermitian metric on E, Δ is not self-adjoint, but has a self-adjoint positive definite leading symbol, which is the same as the leading symbol of the usual Laplacian. In particular, Δ has a discrete spectrum, cf. [9, §4].

Suppose $\mathcal{I}$ is an interval of the form $[0, \lambda]$ or (λ, ∞) and let $\Pi_{\Delta_k,\mathcal{I}}$ be the spectral projection of Δ corresponding to $\mathcal{I}$. Set

$$\check{\Omega}^k_{\mathcal{I}}(M, E) \; := \; \Pi_{\Delta_k,\mathcal{I}}\big(\,\Omega^k(M, E)\big) \; \subset \; \check{\Omega}^k(M, E), \qquad k = 0, \dots, d.$$

For each $\lambda \geq 0$, the space $\check{\Omega}^\bullet_{[0,\lambda]}(M, E)$ is a finite dimensional subcomplex of the de Rham complex $(\Omega^\bullet(M, E), \nabla)$, whose cohomology is isomorphic to $H^\bullet(M, E)$. Thus, according to Subsection 3.1, the bilinear form (3.2) restricted to $\check{\Omega}^\bullet(M, E)$ defines a quadratic form on the determinant line $\mathrm{Det}(H^\bullet(M, E))$, which we denote by $\tau_{[0,\lambda]} = \tau_{b,\nabla,[0,\lambda]}$.

Let $\Delta^{\mathcal{I}}_k$ denote the restriction of Δ_k to $\check{\Omega}^k_{\mathcal{I}}(M, E)$. Since the leading symbol of Δ is positive definite the ζ-regularized determinant $\mathrm{Det}'_\theta(\Delta^{\mathcal{I}}_k)$ does not depend on the choice of the Agmon angle θ. Set

$$\tau_{(\lambda,\infty)} \;=\; \tau_{b,\nabla,(\lambda,\infty)} \;:=\; \prod_{j=0}^{d} \big(\operatorname{Det}'_\theta(\Delta_j^{(\lambda,\infty)})\big)^{(-1)^j j} \;\in\; \mathbb{C}\backslash\{0\}. \tag{3.4}$$

Note that both, $\tau_{b,\nabla,[0,\lambda]}$ and $\tau_{b,\nabla,(\lambda,\infty)}$, depend on the choice of the Riemannian metric g^M.

Definition 3.4. *The Burghelea-Haller quadratic form $\tau = \tau_{b,\nabla}$ on $\operatorname{Det}\big(H^\bullet(M,E)\big)$ is defined by the formula*

$$\tau \;=\; \tau_{b,\nabla} \;:=\; \tau_{b,\nabla,[0,\lambda]}\cdot\tau_{b,\nabla,(\lambda,\infty)}. \tag{3.5}$$

It is easy to see, cf. [9, Prop. 4.7], that (3.5) is independent of the choice of $\lambda \geq 0$. Theorem 4.2 of [9] states that τ is independent of g^M and locally constant in b. Since we are not going to use this result in the proof of Theorem 1.4, the latter theorem provides a new proof of Theorem 4.2 of [9] in the case when the dimension of M is odd, cf. Subsection 5.1.

4. Proof of the Comparison Theorem

In this section we prove Theorem 1.4 adopting the arguments which we used in Section 11 of [5] to compute the Ray-Singer norm of the refined analytic torsion.

4.1. **The dual connection.** Suppose M is a closed oriented manifold of odd dimension $d = 2r-1$. Let $E \to M$ be a complex vector bundle over M and let ∇ be a flat connection on E. Assume that there exists a non-degenerate bilinear form b on E. The *dual connection* ∇' to ∇ with respect to the form b is defined by the formula

$$d\,b(u,v) \;=\; b(\nabla u, v) \;+\; b(u, \nabla' v), \qquad u,v \in C^\infty(M,E).$$

We denote by E' the flat vector bundle (E,∇').

4.2. **Choices of the metric and the spectral cut.** Till the end of this section we fix a Riemannian metric g^M on M and set $\mathcal{B} = \mathcal{B}(\nabla, g^M)$ and $\mathcal{B}' = \mathcal{B}(\nabla', g^M)$. We also fix $\theta \in (-\pi/2, 0)$ such that both θ and $\theta+\pi$ are Agmon angles for the odd signature operator $\mathcal{B}$. Recall that for an operator A we denote by $A^\#$ its formal transpose with respect to the bilinear form (3.2) defined by g^M and b. One easily checks that

$$\nabla^\# \;=\; \Gamma\,\nabla'\,\Gamma, \qquad (\nabla')^\# \;=\; \Gamma\,\nabla\,\Gamma, \qquad \text{and} \quad \mathcal{B}^\# \;=\; \mathcal{B}', \tag{4.1}$$

cf. the proof of similar statements when b is replaced by a Hermitian form in Section 10.4 of [5]. As $\mathcal{B}$ and $\mathcal{B}^\#$ have the same spectrum it then follows that

$$\eta(\mathcal{B}') \;=\; \eta(\mathcal{B}) \qquad \text{and} \qquad \operatorname{Det}_{\mathrm{gr},\theta}(\mathcal{B}') \;=\; \operatorname{Det}_{\mathrm{gr},\theta}(\mathcal{B}). \tag{4.2}$$

4.3. **The duality theorem for the refined analytic torsion.** The pairing (3.2) induces a non-degenerate bilinear form

$$H^j(M,E')\otimes H^{d-j}(M,E) \ \longrightarrow\ \mathbb{C}, \qquad j=0,\dots,d,$$

and, hence, identifies $H^j(M,E')$ with the dual space of $H^{d-j}(M,E)$. Using the construction of Subsection 3.4 of [5] (with $\tau:\mathbb{C}\to\mathbb{C}$ being the identity map) we thus obtain a linear isomorphism

$$\alpha:\ \mathrm{Det}\big(H^\bullet(M,E)\big) \ \longrightarrow\ \mathrm{Det}\big(H^\bullet(M,E')\big). \tag{4.3}$$

We have the following analogue of Theorem 10.3 from [5]

Theorem 4.4. *Let $E\to M$ be a complex vector bundle over a closed oriented odd-dimensional manifold M endowed with a non-degenerate bilinear form b and let ∇ be a flat connection on E. Let ∇' denote the connection dual to ∇ with respect to b. Then*

$$\alpha\big(\,\rho_{\mathrm{an}}(\nabla)\,\big) \ =\ \rho_{\mathrm{an}}(\nabla'). \tag{4.4}$$

The proof is the same as the proof of Theorem 10.3 from [5] (actually, it is simple, since $\mathcal{B}$ and $\mathcal{B}'$ have the same spectrum and, hence, there is no complex conjugation involved) and will be omitted.

4.5. **The Burghelea-Haller quadratic form and the dual connection.** Let

$$\Delta' \ =\ (\nabla')^{\#}\,\nabla' \ +\ \nabla'\,(\nabla')^{\#},$$

denote the Laplacian of the connection ∇'. From (4.1) we conclude that

$$\Delta' \ =\ \Gamma\circ\Delta\circ\Gamma.$$

Hence, a verbatim repetition of the arguments in Subsection 11.6 of [5] implies that we have

$$\tau_{b,\nabla,(\lambda,\infty)} \ =\ \tau_{b,\nabla',(\lambda,\infty)}, \tag{4.5}$$

and, for each $h\in\mathrm{Det}\big(H^\bullet(M,E)\big)$,

$$\tau_{b,\nabla}(h) \ =\ \tau_{b,\nabla'}\big(\,\alpha(h)\,\big) \tag{4.6}$$

with α being the duality isomorphism (4.3).

From (4.4) and (4.6) we get

$$\tau_{b,\nabla}\big(\,\rho_{\mathrm{an}}(\nabla)\,\big) \ =\ \tau_{b,\nabla'}\big(\,\rho_{\mathrm{an}}(\nabla')\,\big). \tag{4.7}$$

4.6. **Direct sum of a connection and its dual.** Let

$$\tilde\nabla \ =\ \begin{pmatrix}\nabla & 0\\ 0 & \nabla'\end{pmatrix} \tag{4.8}$$

denote the flat connection on $E\oplus E$ obtained as a direct sum of the connections ∇ and ∇'. The bilinear form b induces a bilinear form $b\oplus b$ on $E\oplus E$. To simplify the notations we shall denote this form by b. For each $\lambda\ge 0$, one easily checks, cf. Subsection 11.7 of [5], that

$$\tau_{b,\tilde\nabla,(\lambda,\infty)} \ =\ \tau_{b,\nabla,(\lambda,\infty)}\cdot\tau_{b,\nabla',(\lambda,\infty)} \tag{4.9}$$

and

$$\tau_{b,\tilde{\nabla}}\big(\,\rho_{\rm an}(\tilde{\nabla})\,\big) \;=\; \tau_{b,\nabla}\big(\,\rho_{\rm an}(\nabla)\,\big)\cdot\tau_{b,\nabla'}\big(\,\rho_{\rm an}(\nabla')\,\big). \tag{4.10}$$

Combining the later equality with (4.7), we get

$$\tau_{b,\tilde{\nabla}}\big(\,\rho_{\rm an}(\tilde{\nabla})\,\big) \;=\; \tau_{b,\nabla}\big(\,\rho_{\rm an}(\nabla)\,\big)^2. \tag{4.11}$$

Hence, (1.3) *is equivalent to the equality*

$$\tau_{b,\tilde{\nabla}}\big(\,\rho_{\rm an}(\tilde{\nabla})\,\big) \;=\; e^{-\,4\pi i\,\big(\eta(\nabla)-{\rm rank}\,E\cdot\eta_{\rm trivial}\big)}. \tag{4.12}$$

4.7. **Deformation of the chirality operator.** We will prove (4.12) by a deformation argument. For $t\in[-\pi/2,\pi/2]$ introduce the rotation U_t on

$$\Omega^\bullet \;:=\; \Omega^\bullet(M,E)\,\oplus\,\Omega^\bullet(M,E),$$

given by

$$U_t \;=\; \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}.$$

Note that $U_t^{-1}=U_{-t}$. Denote by $\widetilde{\Gamma}(t)$ the deformation of the chirality operator, defined by

$$\widetilde{\Gamma}(t) \;=\; U_t\circ\begin{pmatrix}\Gamma & 0\\ 0 & -\Gamma\end{pmatrix}\circ U_t^{-1} \;=\; \Gamma\circ\begin{pmatrix}\cos 2t & \sin 2t\\ \sin 2t & -\cos 2t\end{pmatrix}. \tag{4.13}$$

Then

$$\widetilde{\Gamma}(0) \;=\; \begin{pmatrix}\Gamma & 0\\ 0 & -\Gamma\end{pmatrix},\qquad \widetilde{\Gamma}(\pi/4) \;=\; \begin{pmatrix}0 & \Gamma\\ \Gamma & 0\end{pmatrix}. \tag{4.14}$$

4.8. **Deformation of the odd signature operator.** Consider a one-parameter family of operators $\tilde{\mathcal{B}}(t):\Omega^\bullet\to\Omega^\bullet$ with $t\in[-\pi/2,\pi/2]$ defined by the formula

$$\tilde{\mathcal{B}}(t) \;:=\; \widetilde{\Gamma}(t)\,\tilde{\nabla} \;+\; \tilde{\nabla}\,\widetilde{\Gamma}(t). \tag{4.15}$$

Then

$$\tilde{\mathcal{B}}(0) \;=\; \begin{pmatrix}\mathcal{B} & 0\\ 0 & -\mathcal{B}'\end{pmatrix} \tag{4.16}$$

and

$$\tilde{\mathcal{B}}(\pi/4) \;=\; \begin{pmatrix}0 & \Gamma\nabla'+\nabla\Gamma\\ \Gamma\nabla+\nabla'\Gamma & 0\end{pmatrix}. \tag{4.17}$$

Hence, using (4.1), we obtain

$$\tilde{\mathcal{B}}(\pi/4)^2 \;=\; \begin{pmatrix}\Delta & 0\\ 0 & \Delta'\end{pmatrix} \;=\; \widetilde{\Delta}. \tag{4.18}$$

Set

$$\Omega^\bullet_+(t) \;:=\; \operatorname{Ker}\,\tilde{\nabla}\,\widetilde{\Gamma}(t);$$
$$\Omega^\bullet_- \;:=\; \operatorname{Ker}\tilde{\nabla} \;=\; \operatorname{Ker}\nabla\oplus\operatorname{Ker}\nabla'.$$

Note that $\Omega^\bullet_-$ is independent of t. Since the operators $\tilde{\nabla}$ and $\widetilde{\Gamma}(t)$ commute with $\tilde{\mathcal{B}}(t)$, the spaces $\Omega^\bullet_+(t)$ and $\Omega^\bullet_-$ are invariant for $\tilde{\mathcal{B}}(t)$.

Let $\mathcal{I}$ be an interval of the form $[0,\lambda]$ or (λ,∞). Denote

$$\Omega^\bullet_{\mathcal{I}}(t) \ := \ \Pi_{\tilde{\mathcal{B}}(t)^2,\mathcal{I}}\big(\,\Omega^\bullet(t)\,\big) \ \subset \ \Omega^\bullet(t),$$

where $\Pi_{\tilde{\mathcal{B}}(t)^2,\mathcal{I}}$ is the spectral projection of $\tilde{\mathcal{B}}(t)^2$ corresponding to $\mathcal{I}$. For $j=0,\dots,d$, set $\Omega^j_{\mathcal{I}}(t)=\Omega^\bullet_{\mathcal{I}}(t)\cap\Omega^j$ and

$$\Omega^j_{\pm,\mathcal{I}}(t) \ := \ \Omega^j_{\pm}(t)\cap\Omega^j_{\mathcal{I}}(t). \tag{4.19}$$

As $\Pi_{\tilde{\mathcal{B}}(t)^2,\mathcal{I}}$ and $\tilde{\mathcal{B}}(t)$ commute, one easily sees, cf. Subsection 11.9 of [5], that

$$\Omega^\bullet_{(\lambda,\infty)}(t) \ = \ \Omega^\bullet_{+,(\lambda,\infty)}(t)\,\oplus\,\Omega^\bullet_{-,(\lambda,\infty)}(t), \qquad t\in[-\pi/2,\pi/2]. \tag{4.20}$$

We define $\tilde{\mathcal{B}}^{\mathcal{I}}_j(t)$, $\tilde{\mathcal{B}}^{\mathcal{I}}_{\rm even}(t)$, $\tilde{\mathcal{B}}^{\mathcal{I}}_{\rm odd}(t)$, $\tilde{\mathcal{B}}^{\pm,\mathcal{I}}_j(t)$, $\tilde{\mathcal{B}}^{\pm,\mathcal{I}}_{\rm even}(t)$, $\tilde{\mathcal{B}}^{\pm,\mathcal{I}}_{\rm odd}(t)$, etc. in the same way as the corresponding maps were defined in Subsection 2.6.

4.9. **Deformation of the canonical element of the determinant line.** Since the operators $\tilde\nabla$ and $\tilde{\mathcal{B}}(t)^2$ commute, the space $\Omega^\bullet_{\mathcal{I}}(t)$ is invariant under $\tilde\nabla$, i.e., it is a subcomplex of $\Omega^\bullet$. The complex $\Omega^\bullet_{(\lambda,\infty)}(t)$ is acyclic and, hence, the cohomology of the finite dimensional complex $\Omega^\bullet_{[0,\lambda]}(t)$ is naturally isomorphic to

$$H^\bullet(M,E\oplus E') \ \simeq \ H^\bullet(M,E)\oplus H^\bullet(M,E').$$

Let $\widetilde\Gamma_{[0,\lambda]}(t)$ denote the restriction of $\widetilde\Gamma(t)$ to $\Omega^\bullet_{[0,\lambda]}(t)$. As $\widetilde\Gamma(t)$ and $\tilde{\mathcal{B}}(t)^2$ commute, it follows that $\widetilde\Gamma_{[0,\lambda]}(t)$ maps $\Omega^\bullet_{[0,\lambda]}(t)$ onto itself and, therefore, is a chirality operator for $\Omega^\bullet_{[0,\lambda]}(t)$. Let

$$\rho_{\widetilde\Gamma_{[0,\lambda]}(t)}(t) \ \in \ \operatorname{Det}\big(H^\bullet(M,E\oplus E')\big) \tag{4.21}$$

denote the refined torsion of the finite dimensional complex $\big(\Omega^\bullet_{[0,\lambda]}(t),\tilde\nabla\big)$ corresponding to the chirality operator $\widetilde\Gamma_{[0,\lambda]}(t)$, cf. Definition 2.3.

For each $t\in(-\pi/2,\pi/2)$ fix an Agmon angle $\theta=\theta(t)\in(-\pi/2,0)$ for $\tilde{\mathcal{B}}_{\rm even}(t)$ and define the element $\rho(t)\in\operatorname{Det}\big(\,H^\bullet(M,E\oplus E')\,\big)$ by the formula

$$\rho(t) \ := \ \operatorname{Det}_{{\rm gr},\theta}\big(\tilde{\mathcal{B}}^{(\lambda,\infty)}_{\rm even}(t)\big)\cdot\rho_{\widetilde\Gamma_{[0,\lambda]}(t)}(t), \tag{4.22}$$

where λ is any non-negative real number. It follows from Proposition 5.10 of [5] that $\rho(t)$ is independent of the choice of $\lambda\ge 0$.

For $t\in[-\pi/2,\pi/2]$, $\lambda\ge0$, set

$$T_\lambda(t) \ := \ \prod_{j=0}^d\Big(\ \operatorname{Det}_{2\theta}\big[\,\tilde{\mathcal{B}}^{(\lambda,\infty)}_{\rm even}(t)^2\big|_{\Omega^j_{(\lambda,\infty)}(t)}\,\big]\,\Big)^{(-1)^{j+1}j}, \tag{4.23}$$

Then, from (4.22) and (2.20) we conclude that

$$\tau_{b,\tilde\nabla}\big(\,\rho(t)\,\big) \ = \ \tau_{b,\tilde\nabla}\Big(\,\rho_{\widetilde\Gamma_{[0,\lambda]}(t)}(t)\,\Big)\cdot T_\lambda(t)\cdot e^{-2i\pi\,\eta\big(\tilde{\mathcal{B}}_{\rm even}(t)\big)}. \tag{4.24}$$

In particular,

$$\tau_{b,\tilde\nabla}\Big(\,\rho_{\widetilde\Gamma_{[0,\lambda]}(t)}(t)\,\Big)\cdot T_\lambda(t)$$

is independent of $\lambda \geq 0$.

4.10. **Computation for $t = 0$.** From (2.4) and definition (2.6) of the element ρ, we conclude that

$$\rho_{-\Gamma_{[0,\lambda]}}(\nabla', g^M) \;=\; \pm\,\rho_{\Gamma_{[0,\lambda]}}(\nabla', g^M).$$

Thus,

$$\tau_{b,\tilde{\nabla}}\big(\,\rho_{-\Gamma_{[0,\lambda]}}(\nabla', g^M)\,\big) \;=\; \tau_{b,\tilde{\nabla}}\big(\,\rho_{\Gamma_{[0,\lambda]}}(\nabla', g^M)\,\big).$$

Hence, from (4.8) and (4.14) we obtain

$$\tau_{b,\tilde{\nabla}}\big(\,\rho_{\widetilde{\Gamma}_{[0,\lambda]}(0)}(0)\,\big) \;=\; \tau_{b,\nabla}\big(\,\rho_{\Gamma_{[0,\lambda]}}(\nabla, g^M)\,\big)\cdot\tau_{b,\nabla'}\big(\,\rho_{\Gamma_{[0,\lambda]}}(\nabla', g^M)\,\big). \tag{4.25}$$

Using (4.16) and the definitions (2.19) and (4.23) of T_λ we get

$$T_\lambda(0) \;=\; T_\lambda(\nabla, g^M, \theta)\cdot T_\lambda(\nabla', g^M, \theta). \tag{4.26}$$

Combining the last two equalities with definitions (2.14), (4.22) of ρ and with (2.20), (4.2), and (4.7), we obtain

$$\tau_{b,\tilde{\nabla}}\big(\,\rho_{\widetilde{\Gamma}_{[0,\lambda]}(0)}(0)\,\big)\cdot T_\lambda(0) \;=\; \tau_{b,\nabla}\big(\,\rho_{\mathrm{an}}(\nabla)\,\big)^2\cdot e^{4\pi i\,\left(\,\eta(\nabla)-\operatorname{rank}E\cdot\eta_{\mathrm{trivial}}\right)}. \tag{4.27}$$

Comparing this equality with (4.11) we see that *in order to prove* (4.12) *and, hence,* (1.3) *it is enough to show that*

$$\tau_{b,\tilde{\nabla}}\big(\,\rho_{\widetilde{\Gamma}_{[0,\lambda]}(0)}(0)\,\big)\cdot T_\lambda(0) \;=\; 1. \tag{4.28}$$

4.11. **Computation for $t = \pi/4$.** From (4.18) and the definitions (3.4) and (4.23) of $\tau_{b,\tilde{\nabla},(\lambda,\infty)}$ and $T_\lambda(t)$, we conclude

$$T_\lambda(\pi/4) \;=\; 1/\tau_{b,\tilde{\nabla},(\lambda,\infty)}. \tag{4.29}$$

By (4.18) we have

$$\Omega^\bullet_{[0,\lambda]}(\pi/4) \;=\; \Omega^\bullet_{[0,\lambda]}(M,E)\oplus\Omega^\bullet_{[0,\lambda]}(M,E').$$

From (4.14) we see that the restriction of $\widetilde{\Gamma}(\pi/4)$ to $\Omega^\bullet_{[0,\lambda]}(\pi/4)$ preserves the bilinear form on $\Omega^\bullet_{[0,\lambda]}(\pi/4)$ induced by b. Hence we obtain from Lemma 3.2

$$\tau_{b,\tilde{\nabla},[0,\lambda]}\big(\,\rho_{\widetilde{\Gamma}_{[0,\lambda]}(\pi/4)}(\pi/4)\,\big) \;=\; 1.$$

Therefore, from (4.29) and the definitions (3.5) of τ, we get

$$\tau_{b,\tilde{\nabla}}\big(\,\rho_{\widetilde{\Gamma}_{[0,\lambda]}(\pi/4)}(\pi/4)\,\big)\cdot T_\lambda(\pi/4) \;=\; 1. \tag{4.30}$$

4.12. **Proof of Theorem 1.4.** Fix an Agmon angle $\theta \in (-\pi/2, 0)$ and set

$$\xi_{\lambda,\theta}(t) \ := \ -\frac{1}{2}\sum_{j=0}^{d}(-1)^{j+1}\, j\, \zeta_\theta'\big(0, \widetilde{B}_{\rm even}(t)^2|_{\Omega^j_{(\lambda,\infty)}(t)}\big),$$

where $\zeta_\theta'(0,A)$ denotes the derivative at zero of the ζ-function of the operator operator A. Then $T_\lambda(t) = e^{2\xi_{\lambda,\theta}(t)}$. Hence, from (4.30) we conclude that *in order to prove* (4.28) *(and, hence,* (4.12) *and* (1.3)*) it suffices to show that*

$$\tau_{b,\tilde{\nabla}}\big(\rho_{\widetilde{\Gamma}_{[0,\lambda]}(t)}(t)\big)\cdot e^{2\xi_{\lambda,\theta}(t)} \tag{4.31}$$

is independent of t.

Fix $t_0 \in [-\pi/2, \pi/2]$ and let $\lambda \geq 0$ be such that the operator $\tilde{\mathcal{B}}_{\rm even}(t_0)^2$ has no eigenvalues with absolute value λ. Choose an angle $\theta \in (-\pi/2, 0)$ such that both θ and $\theta+\pi$ are Agmon angles for $\tilde{\mathcal{B}}(t_0)$. Then there exists $\delta > 0$ such that for all $t \in (t_0-\delta, t_0+\delta)\cap[-\pi/2,\pi/2]$, the operator $\tilde{\mathcal{B}}_{\rm even}(t)^2$ has no eigenvalues with absolute value λ and both θ and $\theta+\pi$ are Agmon angles for $\tilde{\mathcal{B}}(t)$.

A verbatim repetition of the proof of Lemma 9.2 of [5] shows that

$$\frac{d}{dt}\,\rho_{\widetilde{\Gamma}_{[0,\lambda]}(t)}(t)\cdot e^{\xi_{\lambda,\theta}(t)} \ = \ 0. \tag{4.32}$$

Hence, (4.31) is independent of t. □

5. Properties of the Burghelea-Haller Quadratic Form

Combining Theorem 1.4 with results of our papers [4, 5, 3] we derive new properties and obtain new proofs of some known ones of the Burghelea-Haller quadratic form τ. In particular, we prove a weak version of the Burghelea-Haller conjecture, [9, Conjecture 5.1], which relates the quadratic form (1.2) with the Farber-Turaev torsion.

5.1. **Independence of τ of the Riemannian metric and the bilinear form.** The following theorem was established by Burghelea and Haller [9, Th. 4.2] without the assumption that M is oriented and odd-dimensional.

Theorem 5.2. [Burghelea-Haller] *Let M be an odd dimensional orientable closed manifold and let (E,∇) be a flat vector bundle over M. Assume that there exists a non-degenerate symmetric bilinear form b on E. Then the Burghelea-Haller quadratic form $\tau_{b,\nabla}$ is independent of the choice of the Riemannian metric g^M on M and is locally constant in b.*

Our Theorem 1.4 provides a new proof of this theorem and at the same time gives the following new result.

Theorem 5.3. *Under the assumptions of Theorem 5.2 suppose that b' is another non-degenerate symmetric bilinear form on E not necessarily homotopic to b in the space of non-degenerate symmetric bilinear forms. Then $\tau_{b',\nabla} = \pm\tau_{b,\nabla}$.*

Proof of Theorems 5.2 and 5.3. As the refined analytic torsion $\rho_{\mathrm{an}}(\nabla)$ does not depend on g^M and b, Theorem 1.4 implies that, modulo sign, $\tau_{b,\nabla}$ is independent of g^M and b. Since $\tau_{b,\nabla}$ is continuous in g^M and b it follows that it is locally constant in g^M and b. Since the space of Riemannian metrics is connected, $\tau_{b,\nabla}$ is independent of g^M. □

5.4. **Comparison with the Turaev torsion: proof of Theorem 1.10.** Let $\mathrm{PD}: H_1(M,\mathbb{Z}) \to H^{d-1}(M,\mathbb{Z})$ denote the Poincaré isomorphism. For $h \in H_1(M,\mathbb{Z})$ we denote by $\mathrm{PD}'(h)$ the image of $\mathrm{PD}(h)$ in $H^{d-1}(M,\mathbb{R})$. Turaev [23, §5.3] introduced a characteristic class $c(\varepsilon) \in H_1(M,\mathbb{Z})$ associated to an Euler structure ε. If ε is represented by a non-vanishing vector field X, $\varepsilon = [X]$, then

$$\mathrm{PD}'\big(c([X])\big) \;=\; -2\,[X^*\Psi(g^M)], \tag{5.1}$$

where $[X^*\Psi(g^M)] \in H^{d-1}(M,\mathbb{R})$ denotes the cohomology class of the closed form $X^*\Psi(g^M)$.

Following Farber [13], we denote by $\mathbf{Arg}_\nabla$ the unique cohomology class $\mathbf{Arg}_\nabla \in H^1(M,\mathbb{C}/\mathbb{Z})$ such that for every closed curve γ in M we have

$$\det\big(\,\mathrm{Mon}_\nabla(\gamma)\,\big) \;=\; \exp\big(2\pi i\langle \mathbf{Arg}_\nabla, [\gamma]\rangle\,\big), \tag{5.2}$$

where $\mathrm{Mon}_\nabla(\gamma)$ denotes the monodromy of the flat connection ∇ along the curve γ and $\langle\cdot,\cdot\rangle$ denotes the natural pairing $H^1(M,\mathbb{C}/\mathbb{Z}) \times H_1(M,\mathbb{Z}) \to \mathbb{C}/\mathbb{Z}$.

Let $L(p) = L_M(p)$ denote the Hirzebruch L-polynomial in the Pontrjagin forms of a Riemannian metric on M. We write $\widehat{L}(p) \in H_\bullet(M,\mathbb{Z})$ for the Poincaré dual of the cohomology class $[L(p)]$ and let $\widehat{L}_1 \in H_1(M,\mathbb{Z})$ denote the component of $\widehat{L}(p)$ in $H_1(M,\mathbb{Z})$.

Theorem 5.11 of [3] combined with formulae (5.4) and (5.6) of [3] implies that for each connected component $\mathcal{C} \subset \mathrm{Flat}(E)$ and each cohomological orientation $\mathfrak{o}$, there exists a constant $F^{\mathfrak{o}}_{\mathcal{C}}$ such that for every flat connection $\nabla \in \mathcal{C}$ and every Euler structure ε we have

$$|F^{\mathfrak{o}}_{\mathcal{C}}| \;=\; \big|\, e^{-2\pi i\langle \mathbf{Arg}_\nabla, \widehat{L}_1\rangle +\; 2\pi i\,\eta(\nabla)}\,\big|, \tag{5.3}$$

and

$$\left(\frac{\rho_{\varepsilon,\mathfrak{o}}(\nabla)}{\rho_{\mathrm{an}}(\nabla)}\right)^2 \;=\; F^{\mathfrak{o}}_{\mathcal{C}}\cdot e^{2\pi i\langle \mathbf{Arg}_\nabla, \widehat{L}_1 + c(\varepsilon)\rangle}. \tag{5.4}$$

Suppose now that ε is represented by a non-vanishing vector field X and set $\alpha_\varepsilon = X^*\Psi(g^M)$. Then it follows from (5.1) that

$$2\int_M \omega_{\nabla,b}\wedge\alpha_\varepsilon \;=\; -\,\langle[\omega_{\nabla,b}], c(\varepsilon)\rangle.$$

Hence, combining (5.4) with Theorem 1.4 and (1.2) we obtain

$$\tau^{\mathrm{BH}}_{b,\alpha_\varepsilon,\nabla}\big(\,\rho_{\varepsilon,\mathfrak{o}}(\nabla)\,\big) \;=\; \pm\,F^{\mathfrak{o}}_{\mathcal{C}}\cdot e^{2\pi i\langle \mathbf{Arg}_\nabla, \widehat{L}_1 + c(\varepsilon)\rangle -\; 2\pi i\,(\,\eta(\nabla) - \mathrm{rank}\,E\cdot\eta_{\mathrm{trivial}}\,) + \langle[\omega_{\nabla,b}],c(\varepsilon)\rangle}. \tag{5.5}$$

Assume now that ∇_t with $t \in [0,1]$ is a smooth path of flat connections. The derivative $\dot{\nabla}_t = \frac{d}{dt}\nabla_t$ is a smooth differential 1-form with values in the bundle of isomorphisms of E. We denote by $[\mathrm{Tr}\,\dot{\nabla}_t] \in H^1(M,\mathbb{C})$ the cohomology class of the closed 1-form $\mathrm{Tr}\,\dot{\nabla}_t$.

By Lemma 12.6 of [4], we have

$$2\pi i \frac{d}{dt}\,\mathbf{Arg}_{\nabla_t} \;=\; -\big[\operatorname{Tr}\dot{\nabla}_t\big] \;\in\; H^1(M,\mathbb{C}). \tag{5.6}$$

Let $\overline{\eta}(\nabla_t, g^M) \in \mathbb{C}/\mathbb{Z}$ denote the reduction of $\eta(\nabla_t, g^M)$ modulo $\mathbb{Z}$. Then $\overline{\eta}(\nabla_t, g^M)$ depends smoothly on t, cf. [16, §1]. From Theorem 12.3 of [4] we obtain[2]

$$\frac{d}{dt}\,\overline{\eta}(\nabla_t, g^M) \;=\; \frac{i}{2\pi}\int_M L(p)\wedge \operatorname{Tr}\dot{\nabla}_t \;=\; \frac{i}{2\pi}\,\big\langle \big[\operatorname{Tr}\dot{\nabla}_t\big], \widehat{L}_1\big\rangle. \tag{5.7}$$

By Lemma 2.2 of [9] we get

$$e^{-\langle[\omega_{\nabla,b}],c(\varepsilon)\rangle} \;=\; \pm\,\det \operatorname{Mon}_{\nabla}\big(c(\varepsilon)\big) \;=\; \pm\, e^{2\pi i\langle \mathbf{Arg}_{\nabla},c(\varepsilon)\rangle}. \tag{5.8}$$

(Note that $\operatorname{Mon}_{\nabla}(\gamma)$ is equal to the *inverse* of what is denoted by $\operatorname{hol}^E_x(\gamma)$ in [9]).

From (5.5)–(5.8) we obtain

$$\frac{d}{dt}\,\tau^{\mathrm{BH}}_{b,\alpha_\varepsilon,\nabla_t}\big(\,\rho_{\varepsilon,\mathfrak{o}}(\nabla_t)\,\big) \;=\; 0, \tag{5.9}$$

proving that the right hand side of (5.5) is independent of $\nabla \in \mathcal{C}$. Combining (5.3) with (5.8) and the fact that $\eta_{\mathrm{trivial}} \in \mathbb{R}$ we conclude that the absolute value of the right hand side of (5.5) is equal to 1. □

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