Prime geodisic theorem for the theta case II

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1 Introduction

Let $\Gamma_0(2)$ denote the group of matrices in $\Gamma = \mathrm{SL}(2,\mathbf{Z})$ with an even left–lower corner entry. Then the classical theta series

$$\vartheta_2(z) = \sum_{n\in\mathbf{Z}} e^{\pi i\left(n+\frac{1}{2}\right)^2 z} \quad , \quad \vartheta_3(z) = \sum_{n\in\mathbf{Z}} e^{\pi i n^2 z}, \quad \vartheta_4(z) = \vartheta_3(z+1)$$

are modular forms of weight $\frac{1}{2}$ for $\Gamma_0(2)$, $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \Gamma_0(2) \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \Gamma_0(2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ respectively. Put

$$\Theta(z) = (\vartheta_2(z),\, \vartheta_3(z),\, \vartheta_4(z))^t$$

then $\mathcal{V}$ is the multiplier system determined by

$$\Theta(Tz) = \mathcal{V}(T)(cz+d)^{\frac{1}{2}}\Theta(z)$$

for $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. In [8] we have proven the prime geodesic theorem

$$\pi_\Gamma(x, \mathcal{V}) = \mathrm{li}\left(x^{\frac{3}{4}}\right) + \mathcal{O}_\epsilon\left(x^{\frac{5}{8}+\epsilon}\right)$$

with

$$\pi_\Gamma(x,\mathcal{V}) = \sum_{\substack{\{P_0\},\\ \mathrm{N}P_0 \le x}} \mathrm{Tr}\mathcal{W}(P_0)\,.$$

where the sum is over all Γ–conjugacy classes of primitive hyperbolic elements P_0 with $\mathrm{Tr}P_0 > 4$ and $\mathrm{N}P_0 = \epsilon_D^2 < x$. Here D is the discriminant of the primitive binary quadratic form, for which the automorphism group is generated by $\pm P_0$, $\epsilon_D = \frac{t_0+u_0\sqrt{D}}{2}$ and $(t_0\,, u_0)$ is the fundamental solution of Pell's equation

$$t^2 - Du^2 = 4\,.$$

In the present paper we want to prove a better result, namely

Theorem 1.1

$$\pi_\Gamma(x,\mathcal{V}) = \mathrm{li}\left(x^{\frac{3}{4}}\right) + \mathcal{O}_\varepsilon\left(x^{\frac{59}{96}+\varepsilon}\right)$$

We should remark that this result formally corresponds to Iwaniec's result $\mathcal{O}(x^{35/48}+\varepsilon)$ in the error term for the prime geodesic theorem in the weight 0 case under the Shimura lift. In what follows we shall make improvements at three different places of [8]. Yet only the one which will be made in the 'Kloosterman' term will be responsible for the result stated in the theorem.

2 Rankin–Selberg Zeta function

We use the notation of [8]. $\rho_{k,j}$ are the Fourier coefficients of the normalized (with respect to the usual inner product) eigenfunction of the Laplace Beltrami operator for the eigenvalue $\lambda_k = 1/4 + t_k^2$, $\arg t_k \in \{0, -\pi/2\}$

$$\vec{u}_k(z) = \sum_{j=1}^{3} u_{k,j}(z)\vec{f_j}\,,$$

where

$$\begin{aligned} u_{k,j}(z) &= \sum_{\substack{n\equiv\alpha_j \bmod 1\\ n\neq 0}} \rho_{k,j}(n) W_{\frac{1}{4}sgn(n),it_k}(4\pi|n|y)e(nx) \\ &+ \delta_{j1}\begin{cases} \rho_{k,1}(0)y^{\frac{1}{2}-it_k} + \tilde{\rho}_{k,1}(0)y^{\frac{1}{2}+it_k} & ,\text{if } t_k\neq 0 \\ \rho_{k,1}(0)y^{\frac{1}{2}} + \tilde{\rho}_{k,1}(0)y^{\frac{1}{2}}\log y & ,\text{if } t_k = 0 \end{cases} \end{aligned} \quad (1)$$

with $\alpha_1 = 0\,, \alpha_2 = \frac{1}{2}\,, \alpha_3 = \frac{1}{8}$, then $\mathcal{V}(U)$ has eigenvalues $e(\alpha_1)\,, e(\alpha_2)\,, e(\alpha_3)$, the corresponding orthonormal set of eigenvectors is given by

$$\vec{f_1} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} \quad , \quad \vec{f_2} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix} \quad , \quad \vec{f_3} = \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

As in [1] introduce $\omega \in C^\infty(\mathbf{R})$ which has compact support in $[N, 2N]$ and satisfies

i) $||\omega^{(\nu)}||_{\sup} \ll \mathrm{N}^{-\nu} \quad , \quad \nu = 0, 1, 2, \ldots\,,$

ii) $\int_N^{2N} \omega(x)dx = \mathrm{N}\,.$

Our first result is an improvement of lemma 4.4 in [8]

Lemma 2.1 *For $T > 1$, $\gamma > -3/2$ and $j = 1, 2, 3$*

$$\sum_{j=1}^{3}\sum_{n>0} \frac{\omega(n+\alpha_j)|\rho_{k,j}(n)|^2(n+\alpha_j)}{\mathrm{ch}\pi t_k} = N\frac{b_{+,k}}{\mathrm{ch}\pi t_k} + r(N, t_k)$$

with

$$\sum_{0<t_k\leq T} t_k{}^{\gamma}|r(N,t_k)| = \mathcal{O}_\varepsilon\left(N^{\frac{1}{2}}T^{3/2+\gamma+\varepsilon}\right).$$

Proof. This follows from our result in [9]

$$\sum_{t_k<T} \frac{R_{+,k}(\frac{1}{2}+iv)}{\mathrm{ch}\pi t_k} \ll_\epsilon (vT)^{3/2+\epsilon} \tag{2}$$

which is the analogue to a conjecture of Iwaniec, which in the weight zero case was proven by Luo and Sarnak in [3]. Namely, put

$$\Omega(s) = \int_0^\infty \omega(t) t^{s-1} dt\,,$$

which is $\ll (1+\Im(s))^{-1994} N^{\Re(s)}$ (partial integration 1994 times) then the inversion theorem for Mellin transforms gives

$$\sum_{j=1}^{3}\sum_{n>0} \omega(n+\alpha_j)|\rho_{k,j,}(n)|^2(n+\alpha_j) = \frac{1}{2\pi i}\int_{(2)} \Omega(s) R_{+,k}(s) ds\,.$$

We shift the line of integration to $\Re(s)=\frac{1}{2}$. Then

$$\sum_{j=1}^{3}\sum_{n>0} \omega(n+\alpha_j)|\rho_{k,j}(n)|^2(n+\alpha_j) = \Omega(1) b_{+,k} + \mathcal{O}\left(N^{\frac{1}{2}}\int_{(\frac{1}{2})} \frac{|R_{+,k}(s)|}{|s|^{1994}} ds\right)$$

and the statement of the lemma follows from (2). □

3 Some Bessel transforms

We refer to section 4 of [8]. Recall we have chosen

$$\phi_j(w) = -\exp\left(-\frac{a_j}{w} + iw\mathrm{ch}b\right)\,,$$

where

$$b = \frac{1}{2}\log x + \frac{i}{2T} \quad , \quad a_j = 4\pi\frac{n+\alpha_j}{C}$$

with $1 \le \mathrm{N} \le n \le 2\mathrm{N}$ as test function in the Kuznecov–Bruggeman sum formula. We want to show

Lemma 3.1

$$\hat{\phi}_j\left(\frac{i}{4}\right) \ll \frac{\sqrt{C}}{\sqrt{N}} \exp\left(-\frac{\sqrt{N}}{\sqrt{2C}} x^{\frac{1}{4}}\right). \tag{3}$$

Recall that

$$J_{-\frac{1}{2}}(w) = \sqrt{\frac{2}{\pi}}\frac{\cos w}{\sqrt{w}}$$

cf.[4], p.73. So we get

$$\left|\hat{\phi}_j\left(\frac{i}{4}\right)\right| = \frac{\pi^2}{\sin(\frac{\pi}{4})\Gamma(\frac{1}{2})} \cdot \lim_{r\to\frac{1}{4}} \frac{(\frac{1}{4}-r)}{\frac{\pi}{2}(1-4r)} \cdot \left|\int_0^\infty J_{-\frac{1}{2}}(w)\phi_j(w)\frac{dw}{w}\right|$$

$$= \left| \int_0^\infty \cos(w)\, \phi_j(w) \frac{dw}{w^{\frac{3}{2}}} \right| .$$

From tables [10] we find for the Fourier cosinus transform

$$\int_0^\infty x^{-3/2} \exp\left(-\alpha x - \frac{\beta}{x}\right) \cos xy dx = \sqrt{\frac{\pi}{\beta}} e^{-u} \cos v$$

and the Fourier sine transform

$$\int_0^\infty x^{-3/2} \exp\left(-\alpha x - \frac{\beta}{x}\right) \sin xy dx = \sqrt{\frac{\pi}{\beta}} e^{-u} \sin v$$

with

$$u = \sqrt{2\beta}\left((\alpha^2 + y^2)^{1/2} + \alpha\right)^{1/2}$$

and

$$v = \sqrt{2\beta}\left((\alpha^2 + y^2)^{1/2} - \alpha\right)^{1/2}$$

and $\alpha,\, \beta > 0$.

Regarding this as a function in α we see that by analytic continuation the above formulas still hold for

$$\alpha = i\mathrm{ch} b = \frac{-E + iB}{2}.$$

We put $y = 1$ and observe that

$$B = \sqrt{x} + \mathcal{O}\left(\frac{\sqrt{x}}{T^2}\right) \quad , \quad E = \frac{\sqrt{x}}{2T} + \mathcal{O}\left(\frac{\sqrt{x}}{T^3}\right) .$$

Since

$$\left((\alpha^2 + 1)^{1/2} - \alpha\right)^{1/2} \ll |\alpha|^{-1/2}$$

we obtain (3). □

The same estimate then holds for $D_j(x, T; C; n)$, cf [8] equation (29). We therefore arrive at

Proposition 3.1 *Let $x > x_0$ for a sufficiently large x_0, $x^{\frac{1}{4}} < T < \sqrt{x}$ and $C,\, N > 1$ with $\frac{1}{\sqrt{x}} < 4\pi\frac{N}{C}$ and $\frac{C}{N} > x^{-\frac{1}{3}}$. Then we have for small $\varepsilon > 0$*

$$\sum_{k=1}^{\infty} \frac{x^{it_k}}{t_k} e^{-\frac{t_k}{T}} = \mathcal{O}_\varepsilon\left(T^\varepsilon\left(\frac{T}{N^{\frac{1}{2}}} + \frac{(NTx)^{\frac{1}{2}}}{C} + \frac{Nx^{\frac{1}{2}}}{C} + \left(\frac{C}{N}\right)^{\frac{1}{2}} \exp\left(-\frac{\sqrt{N}}{\sqrt{2C}} x^{\frac{1}{4}}\right)\right)\right)$$

$$+ \frac{1}{N} \sum_{j=1}^{3} \sum_{n>0} \omega(n + \alpha_j) S_j(x, T, C, n).$$

Compare this with proposition 5.1 in [8].

4 Proof of theorem

Now comes the crucial part of our paper. We put for $c > 0$

$$\rho^{\pm}(c, y) = \sum_{d \pm \bar{d} \equiv y \bmod c} 1,$$

and as in [8] we define for any even integer $c > 0$ the sum

$$\rho_e(c, y) = \sum_{\substack{d \bmod 8c, \\ d + \bar{d} \equiv y \bmod 8c}} \left(\frac{c}{d}\right) e\left(\frac{1-d}{8}\right).$$

and for odd $c > 0$

$$\rho_o(c, y) = \sum_{\substack{d \bmod 2c, \\ d + \bar{d} \equiv y \bmod 2c}} \left(\frac{d}{c}\right).$$

It is understood, that in the above sums d is coprime with c resp.$2c$ and $\bar{d}$ denotes the inverse of $d(\bmod c)$, resp. $d(\bmod 2c$ or $d(\bmod 8c))$.

Notice that ρ^+ is Iwaniec's ρ in [1]. We start with the formula, cf. [8] equation (85),

$$\sum_{j=1}^{3} \sum_{n>0} \sum_{c>0} \omega(n+\alpha_j) S_j^{(1)}(x, T, C, n) \ll \sum_{j=1}^{3} |\mathcal{D}_x^{e,j}(C, N, B, E)| + \sum_{k=1}^{2} |\mathcal{D}_x^{o,k}(C, N, B, E)| + \mathcal{O}(1), \tag{4}$$

where for arbitrary F, $G > 0$

$$\mathcal{D}_x^{e,j}(C, N, F, G) = \sum_{n>0} \sum_{\substack{\frac{NG}{\log^2 x} \le c \le C \log^2 x, \\ c \equiv 0 \bmod 2}} \frac{\omega(n+\alpha_j)}{c} e^{-\frac{c}{C}} \sum_{-\frac{c}{2} < y \le \frac{15}{2} c} e\left(\frac{8(n+\alpha_j)}{8c}(y + F + iG)\right) \rho_e(c, y),$$

$$\mathcal{D}_x^{o,k}(C, N, F, G) = \sum_{n>0} \sum_{\substack{\frac{NG}{\log^2 x} \le c \le C \log^2 x, \\ c \equiv 1 \bmod 2}} \frac{\omega(n+\alpha_k)}{c} e^{-\frac{c}{C}} e\left(-\frac{c}{8}\right) \sum_{-\frac{c}{2} < y \le \frac{3c}{2}} e\left(\frac{2(n+\alpha_k)}{2c}(y + F + iG)\right) \rho_o(c, y).$$

As distinguished from [8] we again split up the above expressions

$$\mathcal{D}_x^{e,j}(C, N, F, G) = \mathcal{D}_x^{e,j,*}(C, N, F, G) + \mathcal{D}_x^{e,j,**}(C, N, F, G)$$

where the star shall indicate that in the sum over y we exclude those summands with $y \pm 2$ a square, these being just contained in the expression with the two stars.

Our analysis in [8], section 6, shows that

Lemma 4.1 *Let $C,\ F,\ G > 0$, $N > (8F\log^2 x)/G > 1$ and $F > 2C\log^2 x/N^{1-a}$ for some small $a > 0$. Then for any $0 < \varepsilon \le a$ and $1 \le j \le 3$, $1 \le k \le 2$*

$$\mathcal{D}_x^{e,j,*}(C,N,F,G),\quad \mathcal{D}_x^{o,k,*}(C,N,F,G) \ll_\varepsilon (FCN)^\varepsilon C^{\frac{3}{4}} F^{\frac{3}{16}}.$$

The proof of the theorem now follows from the following

Proposition 4.1 *With the notations and assumptions as above*

$$\mathcal{D}_x^{e,j,**}(C,N,F,G),\quad \mathcal{D}_x^{o,k,**}(C,N,F,G) \ll_\varepsilon (FCN)^\varepsilon((N^{\frac{1}{2}}C^{\frac{1}{2}}F^{-\frac{5}{32}} + C^2N^{-1}F^{-\frac{3}{4}} + C^{\frac{1}{2}}).$$

Namely instead of propositon 6.2 in [8] we obtain from the above propositions together with lemma 4.1

$$\sum_{k=1}^{\infty} \frac{x^{it_k}}{t_k} e^{-\frac{t_k}{T}} \ll_\varepsilon$$
$$T^\varepsilon \left(\frac{T}{N^{\frac{1}{2}}} + \frac{(NTx)^{\frac{1}{2}}}{C} + \frac{Nx^{\frac{1}{2}}}{C} + \left(\frac{C}{N}\right)^{\frac{1}{2}} \exp\left(-\frac{\sqrt{N}}{\sqrt{2C}} x^{\frac{1}{4}}\right) \right.$$
$$\left. + \quad \frac{C^{\frac{1}{2}}}{N^{\frac{1}{2}} x^{\frac{5}{32}}} + \frac{C^2}{N^2 x^{\frac{3}{4}}} + \frac{C^{\frac{19}{16}} x^{\frac{3}{32}}}{N^{\frac{39}{32}}} \right).$$

Now choose $N = x^{15}$, $C = x^{15+37/96}$ to arrive at

$$\sum_{k=1}^{\infty} \frac{x^{it_k}}{t_k} e^{-\frac{t_k}{T}} \ll x^{\frac{11}{96}+\varepsilon}.$$

By standard calculations as in [8] and together with propostion 3.3 in [8] the theorem follows on choosing $T = x^{\frac{37}{96}}$.

5 Proof of proposition

For the proof of the proposition we shall need the following two lemmas.

Lemma 5.1 *Let $y \pm 2$ be a square, $y = z^2 \mp 2$, say. Then*

$$\rho_e(c,y) = \rho^+(8c,y),\quad \rho_o(c,y) = \rho^+(2c,y).$$

Further for any squarefree odd number k

$$\rho^+(k,y) = \rho^\pm(k,z).$$

Proof. First look at ρ_e. The condition $d + \bar{d} \equiv y \bmod 8c$ is equivalent to $d^2 \mp 2d + 1 \equiv dz^2 \bmod 8c$. From this follows that d must be a square $\bmod 8c$, hence $(c/d) = 1$ and $(2/d) = 1$. The latter implies $d \equiv 1 (\bmod 8)$ giving $e\left(\frac{1-d}{8}\right) = 1$, hence the statement for ρ_e. For ρ_o the proof is similar. For proving $\rho^+(.,y) = \rho^\pm(.,z)$ we use the product representation

$$\begin{aligned}\rho^+(k,y) &= \prod_{p|k}\left(1+\left(\frac{y^2-4}{p}\right)\right) = \prod_{p|k}\left(1+\left(\frac{y-2}{p}\right)\left(\frac{y+2}{p}\right)\right) \\ &= \prod_{p|k}\left(1+\left(\frac{y-2}{p}\right)\left(\frac{y+2}{p}\right)\right) \\ &= \prod_{p|k}\left(1+\left(\frac{z^2\mp 4}{p}\right)\right) = \rho^\pm(k,y).\end{aligned}$$

The last equality follows from the observation that the congruence $d^2-yd\pm1\equiv 0(\mathrm{mod}\, k)$ is equivalent to $(y^2\mp4)\equiv x^2(\mathrm{mod}\, k)$ in $x(\mathrm{mod}\, k)$. □

Put

$$F_h^\pm(A,B,C)=\sum_{c\le C}\sum_{|y+B|\le A}{}^{\pm,hc}\rho^+(c,y+hc),$$

where the upper index at the sum means that the sum is over those y for which $y\pm2+hc$ is a square. We have the following

Lemma 5.2 *i)* $F_0^\pm=\lambda A_0^\pm C+\mathcal{O}\left((A^{\frac12}B^{-\frac{5}{32}}C+AB^{-\frac12}C^{\frac12})(BC)^\epsilon\right)$.

ii) For $1\le h\le 7$ *we obtain*

$$F_h^\pm(A,B,C)=\mathcal{O}\left(AC^{\frac12}\right).$$

Proof. If we proceed as in the proof of theorem 3 in [1] we obtain by means of lemma 5.1

$$F_h^\pm(A,B,C)=\sum\sum\sum_{\substack{lrs\le C,\\ l\in\mathcal{L},(sr,2l)=1}}\mu^2(sr)\sum_{|y+B|\le A,y\pm2+hlrs=z^2}\rho^+(l,y)\left(\frac{z^2\mp4}{r}\right). \quad (5)$$

where $\mathcal{L}=\{n\in\mathbf{N}: n \text{ odd and } p|n => p^2|n \text{ for all primes } p\}$. Splitting this sum up into two sums $F_{h,0}^\pm(A,B,C)+F_{h,\infty}^\pm(A,B,C)$ restricted by $lr\le R$ and $lr>R$ respectively, with some parameter $R>0$, then the analysis of Iwaniec in [1] leads to

$$\begin{aligned}F_{h,0}^\pm(A,B,C) &= \sum_{l\in\mathcal{L},l\le C}\sum_{\alpha(\mathrm{mod}\, l)}\rho^+(l,\alpha)\sum_{\substack{lrs\le C,lr\le R\\(rs,2l)=1}}\mu^2(rs)\left(\frac{m^\pm(r)}{rl}A_{hlrs}^\pm\right. \\ &\quad +\mathcal{O}\left((\min\{\tau(r)\log(2r)r^{\frac12},A_{hlrs}^\pm\}\right)\Big)\end{aligned}$$

where $A_{hlrs}^\pm$ denotes the number of y with $|y+B|\le A$ for which $y\pm2+hlrs$ is a square. We have for $h\ne0$

$$A_{hlrs}^\pm\ll A\left((lrs+B)^{-\frac12}\right)$$

giving

$$F_{h,0}^\pm(A,B,C)=\mathcal{O}\left(AC^{\frac12}\right)$$

independently of R and in this case we choose $R = C$ i.e. $F_h^{\pm}(A,B,C) = F_{h,0}^{\pm}(A,B,C)$. this proves (ii).

If $h = 0$ then the analysis in *loc.cit.* gives

$$F_{h,0}^{\pm}(A,B,C) = \lambda A_0^{\pm} C + \mathcal{O}\left(A_0^{\pm} C^{\frac{1}{2}} + C R^{\frac{1}{2}} + A_0^{\pm} C R^{-\frac{1}{2}}\right)$$

for some $\lambda > 0$. Now completely analogous calculations as in [1] lead to

$$F_{0,\infty}^{\pm} \ll A_0^{\pm} B^{\frac{3}{16}} C R^{-\frac{1}{2}} (BC)^{\epsilon}.$$

Put $R = A_0^{\pm} B^{\frac{3}{16}}$ then we obtain

$$F_0^{\pm} = \lambda A_0^{\pm} C + \mathcal{O}\left((A^{\frac{1}{2}} B^{-\frac{5}{32}} C + A B^{-\frac{1}{2}} C^{\frac{1}{2}})(BC)^{\epsilon}\right)$$

since $A_0^{\pm} \ll A B^{-\frac{1}{2}}$. The proof of the lemma is thus complete. □

Consider now $\mathcal{D}_x^{\epsilon,1,--}(C,N,F,G)$. The sum over y equals

$$\begin{aligned}
&\sum_{\pm} e\left(\mp\frac{2}{c}\right) \sum_{-\frac{c}{2} < z^2 \mp 2 \le \frac{15c}{2}} e\left(\frac{n}{c}(z^2 + F + iG)\right) \rho^{+}(8c, z^2 \mp 2) \\
&= \sum_{\pm} e\left(\mp\frac{2}{c}\right) \sum_{-\frac{c}{2} < z^2 \mp 2 \le \frac{15c}{2}} e\left(\frac{n}{c}(z^2 - h(c,z)c + F + iG)\right) \rho^{+}(8c, z^2 \mp 2),
\end{aligned}$$

where $h(c,z) \in \mathbf{Z}$, $0 \le h(c,z) \le 7$ is such that $z^2 - h(c,z)c \in (-c/2, c/2]$. Next we proceed as in [1] and replace the sum over n by an integral via the Poisson summation formula. Put $M_{c,z} = z^2 - h(c,z)c + F + iG$, then

$$\begin{aligned}
\sum_{n\in\mathbf{Z}} \omega(n) e\left(\frac{n}{c} M_{c,z}\right) &= \sum_{n\in\mathbf{Z}} \int_{-\infty}^{+\infty} \omega(x) e\left(\frac{x}{c} M_{c,z} - xn\right) dx \\
&= \int_{-\infty}^{+\infty} \omega(x) e\left(\frac{x}{c} M_{c,z}\right) dx + \sum_{n\neq 0} \int_{-\infty}^{+\infty} \omega(x) e\left(x\left(\frac{M_{c,z}}{c} - n\right)\right) dx \\
&= \int_{-\infty}^{+\infty} \omega(x) e\left(\frac{x}{c} M_{c,z}\right) dx \\
&\quad + \sum_{n\neq 0} \left((n - \frac{M_{c,z}}{c}) 2\pi i\right)^{-f} \int_{-\infty}^{+\infty} \omega^{(f)}(x) e\left(x\left(\frac{M_{c,z}}{c} - n\right)\right) dx \\
&= \int_{-\infty}^{\infty} \omega(x) e\left(\frac{x}{c} M_{c,z}\right) dx + \mathcal{O}_f\left(N^{-f+1}\right)
\end{aligned}$$

for any integer $f \ge 2$, provided there is a positive real constant γ such that $\left|\frac{M_{c,z}}{c} - n\right| \ge \gamma > 0$ for all $n \in \mathbf{Z}$. The latter is assured by our assumption

$$N > \frac{8F}{G} \log^2 x\,.$$

For then $c \ge NG/\log^2 x > 8F$, therefore

$$\left|\frac{M_{c,z}}{c} - n\right|^2 = \frac{G^2}{c^2} + \left(\frac{z^2 - h(c,z)c}{c} + \frac{F}{c} - n\right)^2 \ge \left(n - \frac{5}{8}\right)^2 \ge \frac{9}{64}$$

for all $n \in \mathbf{Z}$. Further partial integration l–times gives

$$\int_{-\infty}^{+\infty} \omega(x) e\left(\frac{x}{c} M_{c,z}\right) dx \ll_l \left(\frac{c}{|M_{c,z}|}\right)^l N^{-l+1}, \tag{6}$$

which is $\ll_l N^{-l\epsilon+1}$ for $|M_{c,z}| > \frac{c}{N^{1-\epsilon}}$. Especially this is the case for all z with $|z^2 - h(c,z)c + F| > \frac{c}{N^{1-\epsilon}}$. Therefore for any $C, N, F, G > 0$ with $C > N > \log^2 x$

$$\mathcal{D}_x^{e,1,**}(C,N,F,G) =$$

$$\sum_{\pm} e\left(\mp\frac{2}{c}\right) \sum_{\substack{\frac{NG}{\log^2 x} \le c \le C \log^2 x, \\ c \equiv 0 \bmod 2}} \frac{\exp\left(-\frac{c}{C}\right)}{c} \sum_{|z^2 - h(c,z)c+F| \le \frac{c}{N^{1-\epsilon}}} \rho^+(8c, z^2 \mp 2)$$

$$\cdot \int_{-\infty}^{\infty} \omega(x) e\left(\frac{x}{c} M_{c,z}\right) dx + \mathcal{O}_\epsilon\left(\frac{C}{N^{1777}}\right).$$

The sum over c is split up into $\mathcal{O}(\log C)$ subsums, a typical sum of which is

$$\sum_{\substack{H \le c \le 2H \\ c \equiv 0(2)}} \exp\left(\frac{-c}{C}\right) \sum_{|z^2 - h(c,z)c+F| \le \frac{c}{N^{1-\epsilon}}} \rho^+(8c, z^2 \mp 2) \int_{\frac{N}{2H}}^{\frac{2N}{H}} \omega(c\xi) e(\xi M_{c,z}) d\xi$$

$$= \int_{\frac{N}{2H}}^{\frac{2N}{H}} \sum_{\substack{H \le c \le 2H \\ c \equiv 0(2)}} a^\pm(c,\xi,z) f(c,\xi) d\xi + \mathcal{O}_\epsilon\left(\frac{H}{N^{1777}}\right), \quad \frac{NG}{\log^2 x} \le H \le \frac{C}{2} \log^2 x, \tag{7}$$

with

$$a^\pm(c,\xi,z) = \sum_{|z^2 - h(c,z)c+F| \le 2H/N^{1-\epsilon}} e(\xi M_{c,z}) \rho^+(8c, z^2 \mp 2),$$

$$f(c,\xi) = \exp\left(\frac{-c}{C}\right) \omega(c\xi).$$

The $\mathcal{O}$–term in (7) arises, since we replaced $\sum_{|z^2 - h(c,z)c+F| \le 2c/N^{1-\epsilon}}$ by $\sum_{|z^2 - h(c,z)c+F| \le 2H/N^{1-\epsilon}}$, applying (6) with a sufficiently large l. Consider for $0 \le h < 7$, $C > 0$ and $B > A > 0$ the sums

$$F_{\xi,h}^\pm(A,B,C) = \sum_{c \le C} \sum_{|z^2 - hc+B| \le A} e\left(\xi(z^2 + B)\right) \rho^+(8c, z^2 \mp 2).$$

We apply lemma ?? to (5) and obtain for $h \ne 0$

$$F_{\xi,h}^\pm(A,B,C) \ll (AC^{\frac{1}{2}})$$

while

$$F_{\xi,0}^\pm(A,B,C) = \lambda B^{-\frac{1}{2}} \int_{\sqrt{B-A}}^{\sqrt{B+A}} e\left(\xi(\alpha^2 + B)\right) d\alpha \cdot$$
$$+\mathcal{O}\left((1+\xi A)(A^2 B^{-\frac{3}{2}} C + +A^{\frac{1}{2}} B^{-\frac{5}{32}} C + AB^{-\frac{1}{2}} C^{\frac{1}{2}})(BC)^\epsilon\right).$$

Again proceeding as in [1],p.157f gives

$$\mathcal{D}_x^{e,1,**}(C,N,F,G) \ll (FCN)^\epsilon (N^{\frac{1}{2}} C^{\frac{1}{2}} F^{-\frac{5}{32}} + C^2 N^{-1} F^{-\frac{3}{2}} + C^{\frac{1}{2}}).$$

This proves the proposition for the 'even' case. In the 'odd' case the proof runs similar.

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