KdV-invariant polynomial functionals

by

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Abstract

It is proved that the algebra of the KdV-invariant polynomial functionals on the space of C^{∞} functions on the one-dimensional torus is isomorphic to the polynomial algebra of the conserved quatities found by [MGK] .

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Introduction

It is now long since the Korteweg-de Vries equation

$$u_t = uu_x + u_{xxx}$$

is recognized as a completely integrable Hamiltonian system, for example, on the space of C^∞ functions on the one-dimensional torus T^1. A complete set of its first integrals or invariants is provided by the eigenvalues $\{\lambda_i[u]\}$ of the Hill's operator $-d^2/dx^2 + u(x)$ (cf. [MM]), which are however highly transcendental functionals of u.

An infinite set of invariants $\{\tilde{I}_i[u]\}$ which are "elementary" functionals of u can be constructed through the asymptotic expansion

$$\sqrt{4\pi t} \sum_i e^{-\lambda_i[u]t} \sim 1 + \sum_{i=1}^{\infty} \tilde{I}_i[u]t^j \quad (t \searrow 0) .$$

In fact $\tilde{I}_i[u]$ is the integral of a local conserved density $I_i[u]$. Although $I_i[u]$'s are known to exhaust the space of equivalence classes of local conserved densities (cf. [KMGZ] or Theorem (A.3.3.1)), it is obvious that these elementary invariants have less information than $\lambda_i[u]$'s and do not form a complete set of invariants of the KdV-flow.

In this paper we take up the problem whether or not there are other "elementary" invariants other than $\tilde{I}_i[u]$'s.

The functionals which we consider as elementary are such $K[u]$'s as are expressible as

$$K[u]=\sum_n \int_{T^n} K_n(x_1,\ldots,x_n)u(x_1)\ldots u(x_n)\ dx_1\ldots dx_n,$$

where K_n is a distribution on T^n and only a finite number of K_n's are non-zero. These will be called polynomial functionals. The space of the polynomial functionals is strictly larger than the space multiplicatively generated by those with local densities, since it includes those expressed as iterated integrals of local densiities.

Our main result asserts that the functionals expressed as polynomials of $\tilde{I}_i[u]$'s are the only invariants which are polynomial functionals.

Our proof of this is rather involved due mainly to the simple topological fact that for $k \geq 3$ the space $\{(x_1,\ldots,x_k) \in T^k;\ x_i \neq x_j\ (i \neq j)\}$ is not connected. This fact gives rise to the possibility of the existence of first integrals expressible as iterated integrals of local conserved densities, which we were able to eliminate only after a detailed analysis of the local conserved densities of the KdV equation.

It seems to be an interesting problem to find a simpler proof, which admits us to infer whether other soliton equations have the same property or not.

For the evolution equations of space dimension greater than one, it seems probable that the similar result can be rather easily established because the space $\{(x_1,\ldots,x_k) \in M^k;$

$x_i \neq x_j (i \neq j)\}$ is connected for each k when $\dim M \geq 2$.

In § 1 , we give basic definitions and state the main result (Theorem (1.4.1)). The rest of the paper is devoted to its proof. We start it first by describing the space of polynomial functionals by differential polynomials in § 2 applying the idea of Gelfand and Fuks ([GF]). In § 3 the derivation on the space of polynomial functionals corresponding to the KdV-flow is expressed in terms of a derivation on the algebra of differential polynomials. The outline of the proof of the main result is exposed in § 4. The sections 5 and 7 prove key lemmas used in § 4 and the section 6 proves the algebraic independency of diagonal functionals. In § 8 we give several remarks and raise a few related problems. We collected in the appendix certain facts and technical arguments in order to make it easier to see the main flow of the proof of the main result. In § A.1., we recall the structure theorem of distributions, with which we prove the propositions of the section 3. In § A.2., we give basic definitions about differential polynomials and recall some of the basic facts in the theory of formal calculus of variation ([GD]). The section A.3. recalls the result on the existence of infinite number of independent conserved densities ([MGK]) and derive from it various consequences, which play crucial roles in various parts of our proof of the main result.

The author is deeply indebted to T. Sunada who explained him his result ([S]) and suggested him the problem treated here.

Table of Symbols

A:	the algebra of differential polynomials (§ A.2.1).
$\tilde{A}, A^i, A_n, A(d)$ etc:	the subspaces of A defined in § A.2.1.
d:	the x-derivation of A (§ A.2.1).
d_t:	the derivation of A corresponding to the KdV-flow (§ A.3.1.).
$D'(X,Y)$:	the space of distributions on X with the supports in Y.
δ_u:	the variational operator $A \longrightarrow A$ (§ A.2.2.).
Δ:	the endomorphism of A, the kernel of which is isomorphic to the space of the equivalence classes of local conserved densities of the KdV-equation (§ A.3.1.).
Δ^M:	a twisted version of Δ (§ A.3.4).
$F(T)$:	the space of real valued C^∞ functions on T.
$F^kP(F(T))$:	the Gelfand-Fuks filtration on $PF(T)$ (§ 2.2.).
I_i:	the local conserved density of the KdV-equation determined by Theorem (A.3.3.1).
$\tilde{I}_i$:	the local conserved quantitiy of the KdV-equation, which is the integral of I_i.
$\{\Omega, d\}$:	the complex associated to the KdV-equation (§ A.3.1.).
$P(F(T))$:	the algebra of polynomial functionals on $F(T)$ (§ 1.1.).
R:	the field of real numbers.
S:	the symmetrization operator (§ 2.1.).
S_n:	the permutation group.
T:	the one-dimensional torus R/Z.
T_i:	$= \delta_u I_i$ (§ A.3.2).

T_{ij}: $= \delta_u(-I_iX_j + I_jX_i)$ (§ A.3.3.).

$T^n(k)$: $= \{(x_1,\dots,x_n) \in T^n ; \# \{x_1,\dots,x_n\} \le k\}$.

X_i: the flux for I_i determined by Theorem (A.3.3.1.)

X_K: the derivation of A corresponding to the evolutionary equation $u_t = K$ (§ A.2.3.).

χ_k: the map $D'(T^k) \otimes \tilde{A}^{\otimes k} \longrightarrow F^kP(F(T))$ defined in § 2.3.

Z: the ring of integers.

Z_+: the set of nonnegative integers.

All the vector spaces and all the tensor products are over R.
For a group G and a G-module V, the space of all the G-invariant elements is denoted by V^G .

§ 1. Statements of the main results.

1.1. Polynomial functionals.

Let $T = R/Z$ be the one-dimensional torus and $F(T)$ the space of all the real valued C^∞ functions on T, which we identify with the periodic real valued C^∞ functions on R with period 1.

We call a real valued functional $u \longmapsto K[u]$ on $F(T)$ a polynomial functional if for $u \in F(T)$

$$(1.1.1) \qquad K[u] = K_0 + \sum_{n=1}^{\infty} \langle K_n, u^{\otimes n} \rangle ,$$

where $K_0 \in R$, $T^n = R^n/Z^n$, $K_n \in D'(T^n) = \{$ distributions on $T^n\}$, $u^{\otimes n} = u \otimes \ldots \otimes u$ (n-times) is the C^∞ function on T^n defined by $u^{\otimes n}(x_1, \ldots, x_n) = u(x_1)\ldots u(x_n)$, and $K_n = 0$ except for finite n's.

Example (1.1.2). The following are some of the examples of polynomial functionals:

$$K_1 : u \longmapsto u(x_0) ,$$

$$K_2 : u \longmapsto \hat{u}(n) = \int_T e^{2\pi i n x} u(x)\,dx,$$

$$K_3 : u \longmapsto \int_T u(x) u'(x)^2 dx,$$

$$K_4 : u \longmapsto \int_T u(x) u(x + x_0)\,dx,$$

$$K_5 : u \longmapsto \int_{0 \le x_1 \le \ldots \le x_n \le 1} f_1[u](x_1) \ldots f_n[u](x_n)\,dx_1 \ldots dx_n ,$$

where $x_0 \in T$, $f_i = f_i(u_0, u_1, \ldots)$ are differential polynomials (cf. § A.2.1) and $f_i[u]$ denotes the function made by the substitutions: $u_i = d^i f/dx^i$.

The space of all the polynomial functionals is denoted by $P(F(T))$, which is a commutative algebra by the multiplication:

$$K_1K_2[u] := K_1[u]K_2[u] \ , \quad u \in F(T)$$

for $K_1, K_2 \in P(F(T))$.

A polynomial functional K is called diagonal or local if $\operatorname{supp} K_n$ is in the diagonal of T^n. For example, K_1, K_2, K_3 and K_5 with $n = 1$ are diagonal.

1.2. Spectral invariant functionals for the Hill operator.

For $u \in F(T)$, we denote the spectrum of the Hill operator $L_u := -d^2/dx^2 + u$ by

$$\operatorname{Spec}(u) = \{\lambda_0 < \lambda_1 \leq \lambda_2 < \ldots < \lambda_{2i-1} \leq \lambda_{2i} < \ldots\} \ .$$

A real valued functional K on $F(T)$ is called spectral invariant if $\operatorname{Spec}(u) = \operatorname{Spec}(v)$ implies $K[u] = K[v]$ for $u, v \in F(T)$. We denote by $P_{spec}(F(T))$ the subalgebra of $P(F(T))$ consisting of all the spectral invariant polynomial functionals.

Example ([MM]). For $u \in F(T)$ with $\operatorname{Spec}(u) = \{\lambda_i\}$, the following asymptotic expansion holds for $t \searrow 0$:

$$\sum_{i\geq 1} e^{-\lambda_i t} \sim \sqrt{4\pi t}^{\,-1}(1 + \sum_{i\geq 1} \tilde{I}_i[u]t^i) .$$

Moreover a universal differential polynomial I_i exists such that

$$\tilde{I}_i[u] = \int_0^1 I_i[u](x)dx.$$

Obviously $\tilde{I}_i$'s are spectral invariant functionals, which are also polynomial and diagonal. Note that the differential polynomials I_i's are not determined uniquely. We shall choose canonical ones by Theorem (A.3.3.1).

1.3. KdV-invariant functionals.

A functional K is called invariant under the KdV-flow, or KdV-invariant for short, if $K[u(\cdot,t)]$ is independent of t whenever $u(x,t)$ is a solution of the Korteweg-de Vries equation:

$$\partial u/\partial t = 3u\partial u/\partial x - (1/2)\partial^3 u/\partial x^3 . \tag{1.3.1}$$

We denote by $P_{KdV}(F(T))$ the subalgebra of $P(F(T))$ consisting of all the KdV-invariant polynomial functionals.

The Lax representation of (1.3.1):

$$\frac{d}{dt} L_u = \left[2\frac{d^3}{dx^3} - \frac{3}{2}\left(u\frac{d}{dx} + \frac{d}{dx}u\right), L_u\right] \tag{1.3.2}$$

implies

Proposition (1.3.3). The spectral invariant functionals are KdV-invariant: $P_{spec}(F(T)) \subset P_{KdV}(F(T))$.

1.4. Main theorem.

Theorem (1.4.1.) The algebra of the KdV-invariant polynomial functionals coincides with that of the spectral invariant ones and is isomorphic to the polynomial algebra generated by $\tilde{I}_i$'s:

$$P_{spec}(F(T)) = P_{KdV}(F(T)) \cong R[\tilde{I}_1, \tilde{I}_2, \ldots] .$$

This is an immediate consequence of Proposition (1.3.1) and the following

Theorem (1.4.2). The functionals $\tilde{I}_i$'s are algebraically independent and generates the algebra of the KdV-invariant polynomial functionals.

We remark that the algebraic independency of $\tilde{I}_i$'s has been already proved by Sunada (cf. [S]).

Remark. Our results imply that if a functional of iterated integral type such as K_5 in the Example (1.1.2) is spectral invariant, then there exists a unique polynomial $F(\tilde{I}_1, \ldots, \tilde{I}_N)$ with some N such that

$$K_5[u] = F(\tilde{I}_1[u], \ldots, \tilde{I}_N[u]), \quad u \in F(T).$$

For example Sunada ([S]) obtains such spectral invariants $A_i^n[u]$ as the coefficients in an asymptotic expansion:

$$F_n[u,t] := \int_0^1 dx \int_\Omega \exp\left(-t\int_0^1 u(x+n\tau+\sqrt{t}\omega(\tau))d\tau\right)d\mu(\omega)$$

$$\sim 1 + A_1^n[u]t + A_2^n[u]t^2 + \ldots \qquad (t \searrow 0),$$

where Ω is the space of all the continuous functions $\omega:[0,1] \longrightarrow R$ with $\omega(0) = \omega(1)$ and μ is the Wiener's measure on Ω. Our results implies that we can find polynomials $H_i^n \in R[\tilde{I}_1, \tilde{I}_2, \ldots]$ such that

$$A_i^n[u] = H_i^n(\tilde{I}_1[u], \tilde{I}_2[u], \ldots), \quad u \in F(T).$$

This is a weaker version of the Sunada's result, which gives much more precise information about the polynomials H_i^n.

§ 2. Description of polynomial functionals.

Using a filtration similar to the one introduced by Gelfand and Fuks ([GF]) in the computation of the continuous cohomology of the Lie algebra of vector fields, we describe the algebra of the polynomial functionals in terms of differential polynomials.

2.1. Identification of $P(F(T))$ with the symmetric algebra of $D'(T)$.

A polynomial function on a vector space can be identified with an element of the symmetric algebra of its dual space. Analogously a polynomial functional K given by (1.1.1.) can be identified with the sequence $(K_n)_{n=0,1,2,\dots}$, where K_n is a symmetric distribution on T^n, i.e., $K_n^s = K_n$ for all $s \in S_n$, where $\langle K_n^s, f\rangle := \langle K_n, {}^s f\rangle (f \in F(T^n))$ with

$$ {}^s f(x_1,\dots,x_n) = f(x_{s1},\dots,x_{sn}) \; . $$

Hereafter we make the following identification:

$$ P(F(T)) = \oplus_{n=0}^{\infty} D'(T^n)^{S_n} \quad , $$

where $D'(T^n)^{S_n}$ denotes the space of the symmetric distributions on T^n.

Note that for $K \in D'(T^n)$,

$$ \langle K, u^{\otimes n}\rangle = \langle S(K), u^{\otimes n}\rangle \quad (u \in F(T)) \quad , $$

where $S(K)$ denotes the symmetrisation of K:

$$S(K) = (1/n!) \sum_{s \in S_n} K^s .$$

2.2 Gelfand-Fuks filtration on $P(F(T))$.

For a subset X of T^n, denote by $D'(T^n,X)$ the subspace of $D'(T^n)$ consisting of all the distributions on T^n with supports in X. Define

$$T^n(k) := \{(x_1,\dots,x_n) \in T^n,\ ^{\#}\{x_1,\dots,x_n\} \le k\},$$

where $^{\#}A$ stands for the number of the elements of a set A. Then $D'(T^n,T^n(k))$ $(k = 1,2,\dots,n)$ is an S_n-invariant subspace of $D'(T^n)$. We define $F^0P(F(T)) = R$ and

$$F^kP(F(T)) = \oplus_{n \geq k} D'(T^n,T^n(k)), \text{ for } k \geq 1.$$

This is an increasing filtration: $F^0 \subset F^1 \subset F^2 \subset \dots$, which is multiplicative, i.e., $F^pF^q \subset F^{p+q}$. Note that $F^1P(F(T))$ is exactly the space of the diagonal polynomial functionals. Note also for example that $K_4 \in F_2 \setminus F_1$ and $K_5 \in F_n \setminus F_{n-1}$ (cf. Example (1.2.1)).

2.3. Description of F^k/F^{k-1}

Let A be the algebra of differential polynomials of u (cf. § A.2.1.), and denote by $\tilde{A}$ the subspace of A consisting of all the elements with the zero constant terms: $\widetilde{A} := \{f \in A,\ f(0) = 0\}$. Denote by $\widetilde{A}^{\otimes k}$ the tensor product of k copies of $\widetilde{A}$ over R.

Define

$$\widetilde{\chi}_k : D'(T^k) \otimes \widetilde{A}^{\otimes k} \longrightarrow P(F(T))$$

by

$$\widetilde{\chi}_k(w \otimes f_1 \otimes \ldots \otimes f_k)[u] = \langle w, f_1[u] \otimes \ldots \otimes f_k[u] \rangle,$$

where $w \in D'(T^k)$, $f_i \in \widetilde{A}$ and $u \in F(T)$. This induces

$$\chi_k : [D'(T^k) \otimes \widetilde{A}^{\otimes k}]^{S_k} \longrightarrow F_k P(F(T)) ,$$

where $s \in S_k$ acts on $D'(T^k) \otimes \widetilde{A}^{\otimes k}$ by

$$s(w \otimes f_1 \otimes \ldots \otimes f_k) = w^s \otimes f_{t1} \otimes \ldots \otimes f_{tk}$$

$(t = s^{-1})$.

Then we have

Proposition (2.3.1). χ_k is surjective.

This will be proved in § A.1 using the structure theorems of distributions.

Now we describe $\chi_k^{-1} F_{k-1}$.

Define endomorphisms d_i $(i = 1,\ldots,k)$ of $D'(T^k) \otimes \widetilde{A}^{\otimes k}$ by

$$d_i := \partial/\partial x_i \otimes 1 + 1 \otimes (1 \otimes \ldots \otimes 1 \otimes d \otimes 1 \otimes \ldots \otimes 1) ,$$

d being on the i-th factor. Then we have

Proposition (2.3.2). $\chi_k^{-1}(F_{k-1}P(F(T)))$ is spanned by

$$(\sum_{i=1}^{k} \operatorname{Im} d_i) \cap [D'(T^k) \otimes \tilde{A}^{\otimes k}]^{S_k}$$

and

$$[D'(T^k, T^k(k-1)) \otimes \tilde{A}^{\otimes k}]^{S_k} .$$

This will be also proved in § A.1.

§ 3. The KdV-derivation D_t on $P(F(T))$

We introduce a derivation D_t on $P(F(T))$, which is in fact the infinitesimal generator of the KdV-flow on $F(T)$ and $\operatorname{Ker} D_t = P_{KdV}(F(T))$ holds. In § 3.3, we find an operator which corresponds to D_t in the description of § 2.3.

From now on, we rescale (x,t,u) to $(-x,2t,-3u/2)$, so that the Korteweg-de Vries equation takes the simple form:

$$\partial u/\partial t = u\partial u/\partial x + \partial^3 u/\partial x^3. \tag{3.0.1}$$

Observe that the validity of the Theorem (1.4.2) does not change by this rescaling.

3.1. The infinitesimal generator of the KdV-flow.

To the KdV-flow on $F(T)$ corresponds the derivation D_t on $P(F(T))$ characterized by

$$(d/dt)K[u(\cdot,t)] = (D_t K)[u(\cdot,t)] \tag{3.1.1}$$

for $K \in P(F(T))$ and all the solutions $u(x,t)$ of (3.0.1).

It is easy to see that this derivation can be expressed for $K \in D'(T^n)^{S_n}$ as

$$D_t K = L_n + L_{n+1} , \tag{3.1.2}$$

where

$$L_n = -nS(\partial^3/\partial x_n^3 K_n(x_1,\dots,x_n)),$$

$$L_{n+1} = nS(K_n(x_1,\dots,x_n)\delta'(x_n - x_{n+1})).$$

Here $\delta(x-y) \in D'(T^2)$ denotes the delta functional defined by

$$<\delta(x-y),f> = \int_T f(x,x)\,dx, \qquad f \in F(T^2)$$

and

$$\delta'(x-y) = \partial\delta(x-y)/\partial x = -\partial\delta(x-y)/\partial y \quad .$$

3.2 Solvability of the Cauchy problem for the KdV-equation.

We recall the following

Theorem (3.2.1) ([MT]). For every $u_0 \in F(T)$, a unique solution $u \in F(T \times R)$ of the KdV equation exists and satisfy $u(x,0) = u_0(x)$.

This implies

Proposition (3.2.2). $P_{KdV}(F(T)) = \operatorname{Ker} D_t$.

Proof. Obviously $D_t K = 0$ implies that K is KdV-invariant by virtue of (3.1.1).

Conversely let K be a KdV-invariant polynomial functional.

For $u_0 \in F(T)$, let $u \in F(T \times R)$ be the solution of the KdV-equation with $u(x,0) = u_0(x)$. Then by (3.1.1),

$$D_t K[u_0] = (d/dt)K[u(\cdot,t)] = 0.$$

Hence $D_t K = 0$

Q.E.D.

We note that only the solvability of the KdV-equation in small time is necessary for the proof of this proposition.

We remark that we may as well define the notion of KdV-invariance of a polynomial functional K by $D_t K = 0$, which is a little technical condition but makes it unnecessary to rely on the above deep result.

3.3 Description of D_t in terms of differential polynomials.

Define a derivation d_t of A by $d_t = X_{uu_1+u_3}$ (cf. § A.2.3), i.e.,

$$d_t := \sum_{i=0}^{\infty} d^i(uu_1+u_3)\,\partial/\partial u_i \;,$$

and endomorphisms $d_{t,i}$ $(i = 1,\dots,k)$ of $D'(T^k) \otimes \tilde{A}^{\otimes k}$ by

$$d_{t,i} := 1 \otimes \dots \otimes d_t \otimes \dots \otimes 1,$$

d_t being on the i-th place, and put

$$\tilde{d}_t := d_{t,1} + \dots + d_{t,k} \,.$$

Then

<u>Proposition</u> (3.3.1). $\quad D_t \circ \chi_k = \chi_k \circ \tilde{d}_t$.

<u>Proof</u>. For $f \in A$ and a solution u of (3.0.1) we have obviously

$$\partial f[u]/\partial t = (d_t f)[u] \ .$$

Using this, we have for $K \otimes f_1 \otimes \ldots \otimes f_k \in D'(T^k) \otimes \tilde{A}^{\otimes k}$

$$\begin{aligned} d/dt\, \tilde{\chi}_k(K \otimes f_1 \otimes \ldots \otimes f_k) &= d/dt < K, f_1[u] \otimes \ldots \otimes f_k[u] > \\ &= \textstyle\sum_{i=1}^{k} < K, f_1[u] \otimes \ldots \otimes \partial f_i[u]/\partial t \otimes \ldots \otimes f_k[u] > \\ &= \tilde{\chi}_k(K \otimes \tilde{d}_t(f_1 \otimes \ldots \otimes f_k))\ [u] \ . \end{aligned}$$

Hence, by Theorem (3.2.1), $D_t \circ \tilde{\chi}_k = \tilde{\chi}_k \circ \tilde{d}_t$, from which the proposition follows immediately.

Q.E.D.

§ 4. Proof of the main Theorem (1.4.2).

The algebraic independency follows from a general Theorem (6.1). So we prove in this section, $P_{KdV}(F(T)) = P_0$, where P_0 denotes the subalgebra generated by $\tilde{I}_i$'s.

Let $K \in P_{KdV}(F(T))$. Let k be the integer satisfying

$$K \in F^k P(F(T)) \setminus F^{k-1} P(F(T)) .$$

We may suppose $k \geq 1$. We shall show that

$$K \in F^{k-1} P(F(T)) + P_0 .$$

Then by the induction on k it follows that $K \in P_0$.

First by Proposition (2.3.1), K can be expressed as

$$K = \chi_k(J)$$

for some $J \in [D'(T^k) \otimes \tilde{A}^{\otimes k}]^{S_k}$. Applying D_t to both sides, we obtain by Proposition (3.3.1)

$$\chi_k(\tilde{d}_t J) = 0 .$$

Then Proposition (2.3.2) implies

$$\tilde{d}_t J \in \sum_{i=1}^{k} \operatorname{Im} d_i + D'(T^k, T^k(k-1)) \otimes \tilde{A}^{\otimes k} . \tag{4.1}$$

For each positive integer i, fix I_i and $X_i \in A$ which satisfies the conditions of Theorem (A.3.3.1). Denote by C the subspace of A spanned by $\{I_i;\ i=1,2,3,\dots\}$.

Define a subset T_0 of T^k by

$$T_0 := \{([x_1],\dots,[x_k]);\ x_1<x_2<\dots<x_k<x_1+1\}\ ,$$

where $[x] \in R/Z$ denotes the class represented by $x \in R$. Then obviously T_0 is a connected component of $T^k \setminus T^k(k-1)$. Let H be its characteristic function, i.e., H is 1 on T_0 and 0 on $T^k \setminus T_0$, which we regard as a distribution on T^k .

For $L \in \tilde{A}^{\otimes k}$ define

$$J_L := (1/k!) \sum_{s \in S_k} s[H \otimes L] \in [D'(T^k) \otimes \tilde{A}^{\otimes k}]^{S_k} .$$

Denote by Z_k the cyclic subgroup of S_k generated by the cyclic permutation $(12\dots k)$.

<u>Lemma</u> (4.2). If an element J of $[D'(T^k) \otimes \tilde{A}^{\otimes k}]^{S_k}$ satisfies (4.1), then an $L \in [C^{\otimes k}]^{Z_k}$ exists such that

$$\chi_k(J - J_L) \in F^{k-1} P(F(T)) .$$

This is in fact one of the two key points in our proof of the main theorem and will be proved in § 5.

When $k \leq 2$, we have $J_L = 1 \otimes L$ because $T^k \setminus T^k(k-1)$ is connected. Hence we have modulo $F^{k-1}P(F(T))$

$$K \equiv \chi_k(1\otimes L) \in P_0 ,$$

which we wanted to show.

Suppose now $k \geq 3$.

We have proved that an $L \in [C^{\otimes k}]^{Z_k}$ exists which satisfies

$$K - \chi_k(J_L) \in F^{k-1}P(F(T)) .$$

Then by Proposition (2.3.1) we have an $N \in [D'(T^{k-1}) \otimes \tilde{A}^{\otimes(k-1)}]^{S_{k-1}}$ satisfying

$$K = \chi_k(J_L) + \chi_{k-1}(N) .$$

Applying D_t , we obtain

(4.3) $$D_t\chi_k(J_L) + \chi_{k-1}(\tilde{\partial}_t N) = 0 .$$

Now we calculate $D_t\chi_k(J_L)$. Define

$$\partial : C^{\otimes k} \longrightarrow \tilde{A}^{\otimes(k-1)}$$

by

$$\partial(I_{i_1} \otimes \ldots \otimes I_{i_k}) = S(i_k, i_1) \otimes I_{i_2} \otimes \ldots \otimes I_{i_k}$$

$$+ \sum_{j=1}^{k-1} I_{i_1} \otimes \ldots \otimes I_{i_{j-1}} \otimes S(i_j, i_{j+1}) \otimes I_{i_{j+2}} \otimes \ldots \otimes I_{i_k} .$$

Here $S(i,j) := -I_iX_j + I_jX_i$. Then we have

<u>Lemma</u> (4.4). For $L \in C^{\otimes k}$,

$$D_t \chi_k(J_L) = \chi_{k-1}(J_{\partial L}) \ .$$

<u>Proof</u>. For the sake of simplicity, we prove this when $k = 3$. Let $L = Z(I_a \otimes I_b \otimes I_c)$. Then, for $u \in F(T)$,

$$(\chi_3 J_L)[u] = \int_0^1 dx \int_x^{x+1} dy \int_y^{x+1} dz \ I_a[u](x)\, I_b[u](y)\, I_c[u](z) \ ,$$

Put $f_i = I_i[u]$ and $g_i = X_i[u]$ $(i = a,b,c)$ for brevity. Since $d_t I_i = dX_i$, we have

$$(d_t I_i)[u] = (dX_i)[u] = g_i' \ .$$

Hence we have

$$\begin{aligned} D_t \chi_3(J_L)[u] &= \int_0^1 g_a'(x)\,dx \int_x^{x+1} f_b(y)\,dy \int_y^{x+1} f_c(z)\,dz \\ &+ \int_0^1 f_a(x)\,dx \int_x^{x+1} g_b'(y)\,dy \int_y^{x+1} f_c(z)\,dz \\ &+ \int_0^1 f_a(x)\,dx \int_x^{x+1} f_b(y)\,dy \int_y^{x+1} g_y'(z)\,dz \ . \end{aligned}$$

The first term is

$$\int_0^1 dx \left\{ [g_a(x) \int_x^{x+1} f_b(y)\,dy \int_y^{x+1} f_c(z)\,dz]' + \right.$$

$$\left. + \ g_a(x) f_b(x) \int_x^{x+1} f_c(z)\,dz - g_a(x) \int_x^{x+1} f_b(y)\,dy f_c(x+1) \right\} =$$

$$= \int_0^1 g_a(x) f_b(x)\,dx \int_x^{x+1} f_c(y)\,dy - \int_0^1 g_a(x) f_c(x)\,dx \int_x^{x+1} f_b(y)\,dy \ .$$

The second is

$$\int_0^1 f_a(x)\,dx \int_x^{x+1} dy \left\{\frac{\partial}{\partial y}\,[g_b(y) \int_y^{x+1} f_c(z)\,dz] + g_b(y)\,f_c(y)\right\}$$

$$= -\int_0^1 f_a(x)\;g_b(x) \int_x^{x+1} f_c(y)\,dy + \int_0^1 f_a(x)\,dx \int_x^{x+1} g_b(y)\,f_c(y)\,dy \;.$$

The third is

$$\int_0^1 f_a(x)\,g_c(x) \int_x^{x+1} f_b(y)\,dy - \int_0^1 f_a(x)\,dx \int_x^{x+1} f_b(y)\,g_c(y)\,dy \;.$$

Hence we have

$$\begin{aligned} D_t\chi_3(J_L)\,[u] &= \int_0^1 \Big(-f_a(x)\,g_b(x) + g_a(x)\,f_b(x)\Big)dx \int_x^{x+1} f_c(y)\,dy \\ &+ \int_0^1 \Big(-f_c(x)\,g_a(x) + g_c(x)\,f_a(x)\Big)dx \int_x^{x+1} f_b(y)\,dy \\ &+ \int_0^1 f_a(x)\,dx \int_x^{x+1} \Big(-f_b(y)\,g_c(y) + g_b(y)\,f_c(y)\Big)dy \\ &= \chi_2(J_{\partial L})\,[u] \;. \end{aligned}$$

Q.E.D.

By this lemma, (4.3) implies

$$\chi_{k-1}(J_{\partial L} + \tilde{\partial}_t N) = 0 \;. \tag{4.5}$$

Lemma (4.6). If an $L \in [C^{\otimes k}]^{Z_k}$ satisfies (4.5) for an $N \in [D'(T^{k-1}) \otimes \tilde{A}^{\otimes(k-1)}]^{S_{k-1}}$, then L is S_k-invariant.

The proof of this is the most involved and will be given in § 7 .

This lemma implies that $L \in [C^{\otimes k}]^{S_k}$, whence $J_L = 1 \otimes L$. It follows then that modulo $F^{k-1}P(F(T))$

$$K \equiv \chi_k(1 \otimes L) \in P_0 \ .$$

This completes the proof of the main Theorem (1.4.2).

§ 5. Proof of Lemma (4.2).

We use the notations of § A.2.4 with $M = D'(T^k)$ and $\partial_j = \partial/\partial x_j$. Further we define $\Delta_j := D_{K,j} + X_{K,j}$ with $K = uu_1 + u_3$.

Applying δ to the both sides of (4.1), we obtain

$$(5.1) \qquad (\Delta_1 + \ldots + \Delta_k)\ \delta J \in D'(T^k, T^k(k-1)) \otimes \tilde{A}^{\otimes k}$$

because of Lemmas (A.2.4.1-2) and Corollary (A.2.4.3).

Denote by r the restriction map from $D'(T^k) \otimes \tilde{A}^{\otimes k}$ to $D'(T^k \setminus T^k(k-1)) \otimes A^{\otimes k}$. Then from (5.1) it follows

$$(5.2) \qquad (\Delta_1' + \ldots + \Delta_k')\ r(\delta J) = 0 ,$$

where Δ_i' $(i=1,\ldots,k)$ is the endomorphism of $D'(T^k \setminus T^k(k-1)) \otimes \tilde{A}^{\otimes k}$ denoted by Δ_i in § A.2.4 for $M = D'(T^k \setminus T^k(k-1))$ and $\partial_j = \partial/\partial x_j$.

Now we solve the equation (5.2).

Let T_i be the variational derivative of I_i : $T_i = \delta_u I_i$ (cf. § A.2.2 for the defintion of δ_u). Denote by δC the subspace of $\tilde{A}$ spanned by T_i's $(i=1,2,3\ldots)$, and by $LC(X)$, the space of locally constant real valued functions on a topological space X. Then

<u>Lemma</u> (5.3). $\mathrm{Ker}(\Delta_1' + \ldots + \Delta_k') = LC(T^k \setminus T^k(k-1)) \otimes (\delta C)^{\otimes k}$.

<u>Proof</u>. We apply Lemma (A.3.6.1) to $M = D'(T^k \setminus T^k(k-1)) \otimes \tilde{A}^{\otimes(k-1)} \otimes R$ with $\partial = \partial/\partial x_k$ and $G = \Delta_1' + \ldots + \Delta_{k-1}'$. Then we obtain

$$\mathrm{Ker}(\Delta_1' + \ldots + \Delta_k') = (\mathrm{Ker}(\partial/\partial x_k) \cap \mathrm{Ker}(\Delta_1' + \ldots + \Delta_{k-1}')) \otimes \delta C .$$

By induction, we obtain

$$\mathrm{Ker}(\Delta_1' + \ldots + \Delta_k') = (\mathrm{Ker}(\partial/\partial x_1) \cap \ldots \cap \mathrm{Ker}(\partial/\partial x_k)) \otimes (\delta C)^{\otimes k}$$

Q.E.D.

Hence we have

$$r(\delta J) = \textstyle\sum_c H_c \otimes \delta(L_c) ,$$

where c ranges over the set of connected components of $T^k \setminus T^k(k-1)$, H_c is the characteristic function of c and $L_c \in C$. Since J is S_k-symmetric, we have

$$\delta L_{sc} = \delta L_c$$

for all c and $s \in S_k$. Hence putting $L := L_{T_0}$, we have

$$r\delta(J - J_L) = 0 ,$$

which means

$$\delta(J - J_L) \in D'(T^k, T^k(k-1)) \otimes \tilde{A}^{\otimes k}$$

It follows then by Lemma (A.2.4.1) that

$$J - J_L \in \sum_{i=1}^{k} \mathrm{Im}\, d_i + D'(T^k, T^k(k-1)) \otimes \tilde{A}^{\otimes k} ,$$

whence by Proposition (2.3.2)

$$\chi_k(J-J_L) \in F_{k-1}P(F(T)) .$$

Finally we note that L is Z_k-invariant. In fact the Z_k-invariance of T_0 implies that, for $s \in Z_k$,

$$\delta(L - sL) = 0$$

whence $L = sL$ because of Lemma (A.2.4.4).

This completes the proof of Lemma (4.2).

§ 6. Algebraic independency.

We prove in this section the following

Theorem (6.1). Let $\{K_1,\dots,K_m\}$ be a linearly independent subset of $\bar{A} := \chi_1(R\otimes\tilde{A}) \subset F^1P(F(T))$. Then they are algebraically independent in the algebra $P(F(T))$.

Proof. It suffices to show the injectivity of the map

$$a : \oplus_{k=0}^{\infty}[\bar{A}^{\otimes k}]^{S_k} \longrightarrow P(F(T))$$

induced by the multiplication. Moreover we have only to show for each k the injectivity of the map

$$a_k : [\bar{A}^{\otimes k}]^{S_k} \longrightarrow F^kP(F(T))/F^{k-1}P(F(T))$$

induced from a , since

$$a([\bar{A}^{\otimes k}]^{S_k}) \subset F^kP(F(T)) \ .$$

Suppose a_k is not injective, i.e.,

$$a(\bar{g}) \in F^{k-1}P(F(T)) \ ,$$

for some $\bar{g} \in [\bar{A}^{\otimes k}]^{S_k}$. Choose $g \in [Q^{\otimes k}]^{S_k}$ such that $\bar{g} = \chi_k(1\otimes g)$ where Q is a complement of Imd in $\tilde{A}$. Then by Proposition (2.3.2)

$$1 \otimes g \in [D'(T^k,T^k(k-1)) \otimes \tilde{A}^{\otimes k}]^{S_k} + \sum_{i=1}^{k}\mathrm{Imd}_i \ .$$

Let T_0 be a connected component of $T^k \setminus T^k(k-1)$. Then on T_0 we have

$$1 \otimes g \in \sum_{i=1}^{k} \operatorname{Im} d_i$$

Now we use the results of § A.2.4 with $M = D'(T_0)$, $\partial_i = \partial/\partial x_i$. Then we have

$$\delta(1 \otimes g) = 0 .$$

But this implies $\delta' g = 0$, where δ' denotes the δ of § A.2.4 with $M = R$ and $\partial_i = 0$. By Lemma (A.2.4.4), we have $g = 0$, whence $\bar{g} = 0$, establishing the injectivity of a_k.

Q.E.D.

§ 7. Proof of Lemma (4.6)

By virtue of Proposition (2.3.2), it suffices to prove the following. Put $Z = 1/(k-1)!\sum_{s\in Z_{k-1}} s$.

Lemma (7.1). Let $k \geq 3$. Suppose an

$$L = \sum a_{i_1\ldots i_k} I_{i_1} \otimes \ldots \otimes I_{i_k} \in [C^{\otimes k}]^{Z_k}$$

satisfies

$$(1) \qquad Z(\partial L) = \sum_{i=1}^{k-1} d_i N_i + \tilde{d}_t N \quad \text{on} \quad T_0 ,$$

where T_0 is a component of $T^{k-1} \setminus T^{k-1}(k-2)$ and $N_i, N \in D'(T_0) \otimes \tilde{A}^{\otimes(k-1)}$. Then $L \in [C^{\otimes k}]^{S_k}$.

Proof. Since L is Z_k-invariant, we have

$$Z(\partial L) = \frac{k}{k-1} \sum_{j=1}^{k-1} a_{i_1\ldots i_k} I_{i_1} \otimes \ldots \otimes I_{i_{j-1}} \otimes S(i_j, i_{j+1}) \otimes I_{i_{j+2}} \otimes \ldots \otimes I_{i_k} .$$

We use the notations of § A.2.4 with $M = D'(T_0)$ and $\partial_j = \partial/\partial x_j$. Let $\Delta_j = D_{K,j} + X_{K,j}$ with $K = uu_1 + u_3$. Applying δ to (1), we obtain

$$(2) \qquad \sum_{j=1}^{k-1} a_{i_1\ldots i_k} T_{i_1} \otimes \ldots \otimes T_{i_{j-1}} \otimes T_{i_j i_{j+1}} \otimes \ldots \otimes T_{i_k} = (\Delta_1 + \ldots + \Delta_{k-1}) P ,$$

where $T_{ij} := \delta(S(i,j)) \in \tilde{A}$, $P = k/(k-1)\delta N$.

Sublemma (7.2). Let $k \geq 1$. The elements

$$\{T_{i_1}\otimes\ldots\otimes T_{i_{j-1}}\otimes T_{i_j i_{j+1}}\otimes\ldots\otimes T_{i_{k+1}}\ ;\ 1\le j\le k\ ,\ i_a\in Z_+,\ i_j<i_{j+1}\}$$

are linearly independent in $D'(T_0)\otimes\tilde{A}^{\otimes k}$ modulo $\mathrm{Im}(\Delta_1+\ldots+\Delta_k)$.

By this, (2) implies

$$a_{ij\,i_3\ldots i_k} = a_{ji\,i_3\ldots i_k}$$

for all $i,j,i_3,\ldots,i_k$. Since $a_{i_1\ldots i_k}$ is cyclic with respect to the suffixes, it follows that $a_{i_1\ldots i_k}$ is actually symmetric, whence $L\in[C^{\otimes k}]^{S_k}$.

Thus it remains to prove Sublemma (7.2).

Let $k=1$. Suppose

$$(3)\qquad \sum_{i<j} a_{ij}T_{ij} = \Delta P\ ,$$

for some $P\in D'(T)\otimes\tilde{A}$ with some $a_{ij}\neq 0$. We use the notations of § A.3.4 with $M=D'(T)$ and $\partial=\partial/\partial x$.
Let P_i be the A_M^i-component of P and put

$$m := \max[\{2(i+j);\ a_{ij}\neq 0\}\cup\{i+3;\ \Delta P_i\neq 0\}]\ .$$

Note that m must be even. In fact otherwise, we have $\Delta P_{m-3}=0$, contradicting to the definition of m.

Put $m=2s$. Then the A_M^{2s}-component of (3) is

$$\sum_{i+j} a_{ij} T_{ij} = \Delta P_{2s-3}$$

Applying Proposition (A.3.5.1), we obtain $a_{ij} = 0$ for $i+j = s$ and $P_{2s-3} = 0$ contradicting to the definition of $m = 2s$. Hence (3) implies $a_{ij} = 0$ for all i and j.

Let now $k \geq 2$ and

$$(4) \sum a^{\ell}_{i_1 \ldots i_{\ell-1} st i_{\ell+1} \ldots i_k} T_{i_1} \otimes \ldots \otimes T_{i_{\ell-1}} \otimes T_{st} \otimes T_{i_{\ell+1}} \otimes \ldots \otimes T_{i_k} = (\Delta_1 + \ldots + \Delta_k) P$$

with $P \in D'(T_0) \otimes \tilde{A}^{\otimes k}$. Suppose some of the a's is nonzero. We use the notations of § A.3.4 now with $M = D'(T_0) \otimes \tilde{A}^{\otimes (k-1)}$ and $\partial = \partial / \partial x_k$. Then (4) can be rewritten as

$$(5) \qquad a_i \otimes T_i + \sum_{i<j} a_{ij} \otimes T_{ij} = (G + \Delta^M) P ,$$

where

$$a_{ij} := \sum a^k_{i_1 \ldots i_{k-1} ij} \otimes T_{i_1} \otimes \ldots \otimes T_{i_{k-1}} ,$$

$$a_i := \sum_{\ell=1}^{k-1} a^{\ell}_{i_1 \ldots i_{\ell-1} st i_{\ell+1} \ldots i_{k-1} i} T_{i_1} \otimes \ldots \otimes T_{i_{\ell-1}} \otimes T_{st} \otimes T_{i_{\ell+1}} \otimes \ldots \otimes T_{i_{k-1}} ,$$

$$G := \Delta_1 + \ldots + \Delta_{k-1} .$$

Let P_i be the A^i_M-component of P. Let m be the maximum number in the union of $\{2(i+j);\ a_{ij} \neq 0\}$, $\{2i+2;\ a_i \neq 0\}$ and

$\{i \neq 3\ ;\ (G+\Delta^M)P_i \neq 0\}$.

Suppose m is odd and let $m = 2s+1$. Then the A_M^{2s+2} - and A_M^{2s} -components of (5) read

$$\Delta_3^M P_{2s-2} = 0$$

$$\Delta_2^M P_{2s-2} + \Delta_3^M P_{2s-3} = \sum_{\substack{i<j \\ i+j=s}} a_{ij} \otimes T_{ij} + a_{s+1} \otimes T_{s+1} .$$

Then by virtue of Proposition (A.3.5.1) ,

$$a_{s+1} + 3\, a_{1,s-1} = 0 \quad ,$$

which implies $a_{s+1} = a_{1,s-1} = 0$ by virtue of Corollary (A.3.3.3). Then by (ii) of Proposition (A.3.5.1) $a_{ij} = 0 (i+j=s)$ and $\Delta^M P_{2\ell-2} = 0$, which contradicts the definition of m .

Let $m = 2s$. Putting $P_{2s-2} = 0$, we can use the above arguments to show that $a_s = 0$, $a_{ij} = 0 (i+j=s)$ and $\Delta_3^M P_{2s-3} = 0$. But then $P_{2s-3} = 0$ by Lemma (A.3.4.2). We obtain again a contradiction.

This completes the proof of Sublemma (7.2) and hence that of Lemma (7.1).

Q.E.D.

§ 8. Remarks and Problems.

Some of the following have also been stated in [T2].

8.1. The validity of our result relies partly on the following algebraic fact: The map $\kappa : \Lambda^2 H^1 \longrightarrow H^2$ induced from the exterior product $\Omega^1 \times \Omega^1 \longrightarrow \Omega^2$ (cf. § A.3.1 for the notations) is injective, which is an easy consequence of Proposition (A.3.5.1).

Problem. Are the similar maps for other solition equations injective?

8.2. Conversely if some evolution equation $u_t = F$ $(F \in A)$ has the nonzero kernel of $\Lambda^2 H^1 \longrightarrow H^2$, then we can construct an invariant which cannot be expressed as a polynomial of local conserved quantities. For example, suppose there are three 1-forms $w_i = I_i dx + X_i dt \, (i=1,2,3)$ such that $D_F w_i = 0 \quad (i=1,2,3)$ and $w_1 \wedge w_2 = D_F w_3$, i.e.,

$$d_t I_i = dX_i \qquad (i=1,2,3),$$

$$I_1 X_2 + I_2 X_1 = dX_3 + d_t I_3 ,$$

where $d_t = X_F$ (cf. §A.2.3). Then it is easily shown that

$$K[u] := \int_{-\infty}^{\infty} I_1(x_1) dx_1 \int_{-\infty}^{x_1} I_2(x_2) dx_2 + \int_{-\infty}^{\infty} I_3(x) dx$$

($I_i(x) := I_i[u](x)$) is an invariant with respect to the flow on the Schwarz space $S(R)$ induced by the equation.

8.3. A similar result as Theorem (1.4.2) can be obtained when we consider the KdV-flow on the Schwarz space $S(R)$ on R .

8.4. When n is greater than one, there are spectral invariant polynomial functionals for the Laplace operator $\Delta + u$ on the n-dimensional torus which cannot be expressed as a polynomial of local spectral invariants (cf. [S]) .

Problem. Find all the spectral invariant polynomial functionals: $F(T^n) \longrightarrow R$ for $-\Delta+u(n\geq 2)$.

8.5. We can consider a sort of de Rham complex on $F(T)$ as follows: A map $w : F(T)\times F(T)^p \longrightarrow R$ is called a polynomial p-form if it can be written as

$$w(u,X_1,\ldots,X_p) = \sum_N \int_{T^N\times T^p} w_N(x_1,\ldots,x_N;y_1,\ldots,y_p)$$

$$u(x_1)\ldots u(x_N)X_1(y_1)\ldots X_p(y_p)dx_1\ldots dx_N dy_1\ldots dy_p$$

($u,X_i \in F(T)$) , where w_N is a distribution on T^{N+p} , symmetric in x_i's and antisymmetric in y_i's and only a finite number of w_N's are nonzero. Denote by $\Omega^p F(T)$ the space of all the polynomial p-forms. Define the exterior differentiation $d : \Omega^p F(T) \longrightarrow \Omega^{p+1} F(T)$ by

$$dw(u,X_1,\ldots,X_{p+1})$$
$$= \sum_{N\geq 1}\sum_{i=1}^{p+1}(-1)^i \int_{T^{N-1}\times T^{p+1}} Nw_N(x_1,\ldots,x_{N-1},y_i;y_1,\ldots,\hat{y}_i,\ldots,y_{p+1})$$
$$u(x_1)\ldots u(x_{N-1})X_1(y_1)\ldots X_{p+1}(y_{p+1})dx_1\ldots dx_{N-1}dy_1\ldots dy_{p+1}.$$

Then we obtain a complex $\{\Omega^*F(T),d\}$. This is easily seen to be acyclic.

Let D_t be the Lie derivation on $\Omega^*F(T)$ induced by the KdV-flow on $F(T)$. Since D_t commutes with d , we can define the subcomplex of invariant polynomial forms: $\Omega^*_{KdV}F(T) := \mathrm{Ker}D_t$. Note that our main result asserts that $\Omega^0_{KdV}F(T) \cong R[\tilde{I}_1,\tilde{I}_2,\ldots]$.

<u>Problem</u>. Determine the space $\Omega^*_{KdV}F(T)$ and then compute its cohomology.

The first step of the calculation goes just in the same way as in § 2-3. The second step corresponding to the determination of the space of local conserved densities is to calculate the $E_1^{1,p}$-terms($p\geq 1$) of the Vinogradov's spectral sequence (cf. [T1,V]) associated to the KdV-equation, which does not seem to be carried out yet.

§ A.1. Distributions.

In this section, we recall structure theorems of distributions and prove the propositions (2.3.1-2).

A.1.1. Structure theorem of distributions.

Let $R^{n+m} = R^n_x \times R^m_y$ be the Euclidean space with the standard linear coordinate system $(x_1,\dots,x_n,y_1,\dots,y_m)$ and X the submanifold defined by $y_1 = \dots = y_m = 0$, which we identify with R^n_x. Let K be a compact subset of X.

Define

$$Q : D'(R^n_x,K) \otimes R[\partial_x,\partial_y] \longrightarrow D'(R^{n+m},K)$$

by

$$\langle Q(w \otimes \partial^A_x \, \partial^B_y), f\rangle = \langle w, \partial^A_x \partial^B_y f \mid_{y=0}\rangle ,$$

for $w \in D'(R^n_x,K) := \{w \in D'(R^n_x) \; ; \; \mathrm{supp}(w) \subset K\}$, $A \in Z^m_+$, $B \in Z^n_+$, $f \in F_0(R^{n+m}) :=$ {smooth functions on R^{n+m} with compact supports}. Here $\partial_x = \partial/\partial x$, $R[\partial_x,\partial_y]$ is the polynomial algebra on ∂_x and ∂_y, and ∂^A_x stands for $(\partial_{x_1})^{A_1}\dots(\partial_{x_n})^{A_n}$ when $A = (A_1,\dots,A_n)$. The structure theorem of the distributions (cf. [Sch]) can be formulated as the following

<u>Theorem</u> (A.1.1.1). Q is an isomorphism on $D'(R^n_x,K) \otimes R[\partial_y]$.

We describe now the kernel of Q : Define endomorphisms $D_i (i=1,\dots,n)$ of $D'(R^n_x, K) \otimes R[\partial_x, \partial_y]$ by

$$D_i := \partial_{x_i} \otimes 1 + 1 \otimes \partial_{x_i} ,$$

where ∂_{x_i} stands also for the multiplication map:

$P(\partial_x, \partial_y) \longmapsto P(\partial_x, \partial_y)\partial_{x_i}$. Then obviously Q maps $\mathrm{Im}\, D_1 + \dots + \mathrm{Im}\, D_n$ to zero. In fact we can show

<u>Proposition</u> (A.1.1.2). (i) $\mathrm{Ker}\, Q = \sum_{i=1}^n \mathrm{Im}\, D_i$.

(ii) For a compact subset L of K ,

$$Q^{-1}(D'(R^{n+m}, L)) = D'(R^n_x, L) \otimes R[\partial_x, \partial_y] + \sum_{i=1}^n \mathrm{Im}\, D_i .$$

<u>Proof</u>. (ii) implies (i) if we put $L = \phi$.

Suppose

$$Q(\sum w_{AB} \otimes \partial_x^A \partial_y^B) \in D'(R^{n+m}, L) .$$

Since

$$w \otimes \partial_x^A \partial_y^B \equiv (-\partial_x)^A w \otimes \partial_y^B \ (\mathrm{mod.}\ \sum_{i=1}^n \mathrm{Im}\, D_i) ,$$

we have

$$Q(\sum (-\partial_x)^A w_{AB} \otimes \partial_y^B) \in D'(R^{n+m}, L) ,$$

whence, by the above theorem,

$$\sum(-\partial_x)^A w_{AB} \otimes \partial_y^B \in D'(R^n,L) \otimes R[\partial_y] \ .$$

Thus

$$\sum w_{AB} \otimes \partial_x^A \partial_y^B \in D'(R^n,L) \otimes R[\partial_y] + \sum \operatorname{Im} D_i \, .$$

Q.E.D.

A.1.2. <u>Decomposition of</u> $D'(T^n, T^n(k))$.

We denote by $P(n,k)$ $(1 \leq k \leq n)$ the set of all the partitions of $\{1,\ldots,n\}$ into k nonvoid subsets:

$$P(n,k) := \Big\{ p = \{p_1,\ldots,p_k\} \ ; \ p_1 \cup \ldots \cup p_k = \{1,\ldots,n\} \ , \\ p_i \neq \phi \ , \ p_i \cap p_j = \phi \, (i \neq j) \Big\}$$

For $p \in P$, put

$$T^n[p] := \{ (x_1,\ldots,x_n) \in T^n \ ; \ x_i = x_j \ \text{ if } \ i,j \in p_a \ \text{ for some } \ a \}.$$

Then

$$T^n(k) = \cup_{p \in P(n,k)} T^n[p]$$

Since the subset T^n is regular in the sense of [Sch] , we have

Lemma (A.1.2.1). The map

$$g : \oplus_{p\in P(n,k)} D'(T^n,T^n[p]) \longrightarrow D'(T^n,T^n(k))$$

induced from the inclusions is surjective.

Denote by $\bar{P}(n,k)$ the subset of $P(n,k)$ consisting of all the p's satisfying the following condition:
whenever $i<j$, $a\in p_i$, $b\in p_j$, one has $a<b$. Let S_n act on $P(n,k)$ by

$$sp := \{sp_1,\ldots sp_k\} \qquad (s\in S_n,\ p\in P(n,k))$$

and define for $p\in P(n,k)$ the subgroup

$$S(p) := \{s\in S_n\ ;\ sp=p\},$$

which leaves $D'(T^n,T^n(p))$ invariant.

Lemma (A.1.2.2). g induces a surjection:

$$g' := S\circ\bar{g} : \oplus_{p\in\bar{P}(n,k)} D'(T^n,T^n[p])^{S(p)} \longrightarrow D'(T^n,T^n(k))^{S_n},$$

where $\bar{g}$ is the restriction of g and S is the symmetrization.

Proof. Let $K\in D'(T^n,T^n(k))^{S_n}$. By Lemma (A.1.2.1),

$$K = \textstyle\sum_{p\in P(n,k)} K_p$$

for some $K_p\in D'(T^n,T^n[p])$. Then

$$
\begin{aligned}
K &= (1/n!) \sum_{s\in S_n} sK \\
&= (1/n!) \sum_{p\in P(n,k)} \sum_{s\in S_n} sK_p \\
&= (1/n!) \sum_{q\in P(n,k)} \sum_{s\in S_n} sK_{s^{-1}q} \quad .
\end{aligned}
$$

Put

$$\tilde{K}_q := (1/n!) \sum_{s\in S_n} sK_{s^{-1}q} \quad .$$

Since $\operatorname{supp}(sK_p) \subset T^n[sp]$, we have $\tilde{K}_q \in D'(T^n, T^n[q])$. Furthermore it is easy to see that $\tilde{K}_q$ is $S(q)$-invariant and that $s\tilde{K}_q = \tilde{K}_{sq}$. Hence noting that $S_n\bar{P}(n,k) = P(n,k)$, we obtain

$$
\begin{aligned}
K &= \sum_{q\in P(n,k)} \tilde{K}_q \\
&= \sum_{p\in\bar{P}(n,k)} (1/m(p)!) \sum_{s\in S_n} s\tilde{K}_p \\
&= S\Big(\sum_{p\in\bar{P}(n,k)} (n!/m(p)!)\tilde{K}_p\Big) \quad ,
\end{aligned}
$$

where $m(p) = \# S(p)$.

Q.E.D.

A.1.3. Description of $D'(T^n, T^n[p])^{S(p)}$.

Define for $p \in \bar{P}(n,k)$ the subspace $A\langle p\rangle$ of $A^{\otimes k}$ spanned by all such elements $f_1 \otimes \ldots \otimes f_k$ as f_i is homogeneous of degree $\# p_i$ $(i=1,\ldots,k)$. Denote the restriction of $\tilde{\chi}_k$ on $D'(T^k) \otimes A\langle p\rangle$ by $\tilde{\chi}_p$.

<u>Lemma</u> (A.1.3.1).

(i) $\mathrm{Im}\tilde{\chi}_p = g'(D'(T^n, T^n[p])^{S(p)})$,

(ii) $\mathrm{Ker}\tilde{\chi}_p = D'(T^k) \otimes A\langle p\rangle \cap \sum_{i=1}^k \mathrm{Im} d_i$,

(iii) $\chi_p^{-1}(D'(T^n, T^n(k-1))) = D'(T^k, T^k(k-1)) \otimes A\langle p\rangle + \sum_{i=1}^k \mathrm{Im}\, d_i$.

<u>Proof</u>. (i) Obviously the left hand side is in the right. Denote by $\bar{p}_i$ the number of the elements of p_i and denote the coordinates of R^n as $(x_{11}, \ldots, x_{1\bar{p}_1}, \ldots, x_{k\bar{p}_k})$. Then $T^n[p]$ is the submanifold defined by the equations:

$$x_{11} = \ldots = x_{1\bar{p}_1}, \ldots, \quad x_{k1} = \ldots = x_{k\bar{p}_k}.$$

Hence, by Theorem (A.1.1.1), for each $K \in D'(T^n, T^n[p])$ we can find such an element $\sum K_B \otimes \partial_x{}^B$ of $D'(T^k) \otimes R[\partial_x]$ as

$$(g'K)[u] = \sum \langle K_B, L_{B,1}u \otimes \ldots \otimes L_{B,k}u\rangle$$

for $u \in F(T)$, where

$$L_{B,i}\, u := \partial_{x_{i1}}^{B_{i1}} u \cdots \partial_{x_{i\bar{p}_i}}^{B_{i\bar{p}_i}} \qquad (1 \le i \le k)$$

$(B = (B_{11}, \ldots, B_{1\bar{p}_1}, \ldots, B_{k1}, \ldots, B_{k\bar{p}_k}))$. If we put

$$f_B := u_{B_{11}} \cdots u_{B_{1\bar{p}_1}} \otimes \ldots \otimes u_{B_{k1}} \cdots u_{B_{k\bar{p}_k}} \in A\langle p\rangle ,$$

then

$$(g'K)[u] = \bar{\chi}_k(\sum K_B \otimes f_B)[u] \ ,$$

which shows (i) . The other assertions follow immediately from Proposition (A.1.1.2).

Q.E.D.

A.1.4. Proof of Propositions (2.3.1-2).

Proposition (2.3.1) follows directly from Lemma (A.1.2.2) and (i) of Lemma (A.1.3.1).

Proof of Proposition (2.3.2). Let $K \in D'(T^k) \otimes \tilde{A}^{\otimes k}$.

Since

$$\tilde{A}^{\otimes k} = \oplus_{n \geq k} A(k,n)$$

with

$$A(k,n) := \oplus_{p \in \bar{P}(n,k)} A\langle p\rangle \ ,$$

we can write

$$K = \sum_{n \geq k} \sum_{p \in \bar{P}(n,k)} K_p$$

with $K_p \in D'(T^k) \otimes A\langle p\rangle$. Suppose $\tilde{\chi}_k(K) \in F^{k-1}P(F(T))$. Since $\tilde{\chi}_k(A(k,n)) \subset D'(T^n)^{S_n}$, we have for each n ,

$$\bar{\chi}_k(\sum_{p \in \bar{P}(n,k)} K_p)) \in D'(T^n, T^n(k-1))^{S_n} \ .$$

Hence for each $p \in \bar{P}(n,k)$ we have

$$\tilde{\chi}_k(K_p) = L - \sum_{q \in \bar{P}(n,k), q \neq p} \tilde{\chi}_k(K_q)$$

with $L \in D'(T^n, T^n(k-1))$. Since

$$\operatorname{supp} \tilde{\chi}_k(K_q) \subset \bigcup_{s \in S_n} T^n[sq] \quad ,$$

it follows

$$\operatorname{supp} \tilde{\chi}_p(K_p) \subset T^n(k-1) \cup \bigcup_{(q,s)} (T^n[p] \cap T^n[sq]) \quad ,$$

where $(q,s) \in \bar{P}(n,k) \times S_n$ satisfies the condition either that $q \neq p$ or that $q = p$ and $s \neq id$. Since

$$T^n[p'] \cap T^n[p''] \subset T^n(k-1)$$

for $p' \neq p''$, we obtain

$$\operatorname{supp} \tilde{\chi}_k(K_p) \subset T^n(k-1) \quad .$$

Then by (iii) of Lemma (A.1.3.1) ,

$$K_p \in D'(T^k, T^k(k-1)) \otimes A\langle p\rangle + \sum_{i=1}^{k} \operatorname{Imd}_i \quad .$$

Hence we have proved

$$\tilde{\chi}_k^{-1}(F^{k-1}P(F(T))) = D'(T^k, T^k(k-1)) \otimes \tilde{A}^{\otimes k} + \sum_{i=1}^{k} \operatorname{Imd}_i \quad .$$

Restricting this to $[D'(T^k) \otimes \tilde{A}^{\otimes k}]^{S_k}$, we obtain the proposition

Q.E.D.

§ A.2. Differential polynomials.

A.2.1. The differential algebra A of differential polynomials.

Let A denote the algebra of differential polynomials of u :

$$A := R[u_0, u_1, u_2, \ldots] = \bigcup_{k=1}^{\infty} R[u_0, u_1, \ldots, u_k]$$

endowed with the derivation d defined by $du_i = u_{i+1} (i=0,1,2,\ldots)$. We write often u_0 simply by u.

We define the weight and the degree of differential polynomials multiplicatively by $\mathrm{weight}(u_i) = i+2$ and $\mathrm{degree}(u_i) = 1$. We put

$$A^i := \{f \in A \; ; \quad \mathrm{weight}(f) = i\} ,$$

$$A_n := \begin{cases} R[u_0, u_1, \ldots, u_n] & \text{for} \quad n \geq 0 , \\ R & \text{for} \quad n = -1 , \end{cases}$$

$$A(d) := \{f \in A \; ; \quad \mathrm{degree}(f) = d\} ,$$

$$A^{(i)} := \textstyle\sum_{j \leq i} A^j ,$$

$$A[d] := \textstyle\sum_{c \geq d} A(c) ,$$

$$A^i(d) := A^i \cap A(d) ,$$

$$A^i_n := A^i \cap A_n ,$$

$$\tilde{A} := A[1]$$

$$\bar{A} := A/\mathrm{Im}\, D .$$

The following is well-known:

<u>Lemma</u> (A.2.1.1). $\quad \operatorname{Ker} d = R$.

<u>Proof</u>. Suppose $g \in A_n \setminus A_{n-1}$ satisfies $dg = 0$. Suppose $n \geq 0$. Then $\partial g/\partial u_n = \partial/\partial u_{n+1}(dg) = 0$, a contradiction. Hence we must have $n = -1$, i.e., $g \in R$.

Q.E.D.

A.2.2. <u>The variational operator</u> δ_u .

Define $\delta_u : A \longrightarrow A$ by

$$\delta_u := \sum_{i=0} (-d)^i \circ \partial/\partial u_i \ .$$

Then the following is well-known:

<u>Proposition</u> (A.2.2.1). $\quad \operatorname{Ker} \delta_u = R + \operatorname{Im} d$.

<u>Proof</u>. From $[\partial/\partial u_{i+1}, d] = \partial/\partial u_i$, it follows immediately $\delta_u \circ d = 0$.

Suppose $\delta_u g = 0$ for some $g \in A$. Then

$$\sum u_i \, \partial g/\partial u_i \equiv u \, \delta_u g = 0 \quad (\text{mod. } \operatorname{Im} d) \quad ,$$

whence $\sum i g_i \in \operatorname{Im} d$, g_i being the $A(i)$-component of g . By the induction on the integer $n = \max\{n \mid g_n \neq 0\}$ we can show $g \equiv g_0$ (mod. $\operatorname{Im} d$) .

Q.E.D.

A.2.3. Evolutional derivation X_K .

For $K \in A$, define a derivation of A by

$$X_K := \sum_{i=0}^{\infty} d^i K\, \partial/\partial u_i \ ,$$

which commutes with d. Define an endomorphism D_K of A by

$$D_K := \sum_{i=0}^{\infty} (-d)^i \circ \partial K/\partial u_i \ .$$

For example, if $K = uu_1 + u_3$, then

$$D_K := u_1 - d\circ u - d^3 = -d^3 - ud \ .$$

Lemma (A.2.3.1). $\delta_u \circ X_K = (D_K + X_K) \circ \delta_u$.

Proof. First we show

$$g\delta_u d_K f \equiv g(D_K + X_K)\delta_u f \qquad (\text{mod. Im}\, d)$$

for $f, g \in A$. In fact, modulo $\text{Im}\, d$,

$$\begin{aligned} g\delta_u X_K f &\equiv X_g X_K f \\ &= X_g \textstyle\sum_i d^i K\, \partial f/\partial u_i \\ &= \textstyle\sum_i d^i (\sum_j d^j g\, \partial K/\partial u_j)\, \partial f/\partial u_i \\ &\qquad + \textstyle\sum_{i,j} d^i K\, d^j g\, \partial^2 f/\partial u_i \partial u_j \\ &\equiv \textstyle\sum_j d^j g\, \partial K/\partial u_j \delta_u f + \sum_j d^j g\, X_K\, \partial f/\partial u_j \end{aligned}$$

$$= g(\sum(-d)^j(\partial K/\partial u_j\ \delta_u f) + \sum_j(-d)^j(X_K\partial f/\partial u_j))$$

$$= g(D_K\,\delta_u f + X_K\,\delta_u f) \quad .$$

Now it is not difficult to see that, if $f \in A$ satisfies $fg \in \mathrm{Im}\, d$ for all $g \in A$, then $f \in 0$ (cf. for example [K]). Thus we have the lemma.

Q.E.D.

A.2.4. The twisted multi-variational operators.

Let M be a vector space with k mutually commuting endomorphisms $\partial_1, \ldots \partial_k$. Put

$$A_{M,k} := M \otimes_R A^{\otimes k}$$

where $A^{\otimes k} = A \otimes_R \cdots \otimes_R A$ (k-times). Put

$$d_i = \partial_i \otimes 1 + 1 \otimes (1 \otimes \ldots \otimes 1 \otimes d \otimes 1 \otimes \ldots \otimes 1)$$

d being on the i-th factor. Define

$$\delta_i := \sum(-d_i)^n \circ \partial/\partial u_n \ ,$$

$$\delta := \delta_1 \circ \ldots \circ \delta_n \ .$$

Then we have

Lemma (A.2.4.1). Let P be a subspace of M invariant with respect to d_i's. Then

$$\delta^{-1}(A_{P,k}) = \sum_{i=1}^{k} \operatorname{Im} d_i + M \otimes R^{\otimes k} + A_{P,k} .$$

<u>Proof</u>. It is easy to see as before $\delta d_i = 0$ $(i=1,\ldots,k)$, whence δ maps the right hand side into $A_{P,k}$.

Suppose $g \in A_{M,k}$ satisfies $\delta g \in A_{P,k}$. Then, modulo $\sum \operatorname{Im} d_i$,

$$\sum (\partial_{1,i_1} \cdots \partial_{k,i_k} g) \cdot u_{i_1} \otimes \cdots \otimes u_{i_k} \equiv \delta g \cdot u \otimes \cdots \otimes u \in A_{P,k}$$

where $\partial_{a,m} = 1 \otimes (1 \otimes \cdots \otimes 1 \otimes \partial/\partial u_m \otimes 1 \otimes \cdots \otimes 1)$, $\partial/\partial u_m$ being on the a-th factor, and $A_{M,k}$ is considered as an $A^{\otimes k}$-algebra in the natural way. Now the argument in the proof of Lemma (A.2.2.1) shows $g \in M \otimes R^{\otimes k} + \sum \operatorname{Im} d_i + A_{P,k}$.

Q.E.D.

For $K \in A$, define endomorphisms $X_{K,i}, D_{K,i}$ $(i=1,\ldots,k)$ of $A_{M,k}$ by

$$X_{K,i} := 1 \otimes (1 \otimes \cdots \otimes 1 \otimes X_K \otimes 1 \otimes \cdots \otimes 1) ,$$
$$D_{K,i} := \sum_{n=0}^{\infty} (-d_i)^n \circ (1 \otimes \cdots \otimes 1 \otimes \partial K/\partial u_n \otimes 1 \otimes \cdots \otimes 1)$$

where X_K and $\partial K/\partial u_n$ are on the i-th factor. Then we have

<u>Lemma</u> (A.2.4.2).

(i) If $i \neq j$, then δ_i commutes with $X_{K,j}$ and $D_{K,j}$.

(ii) $\delta_i \circ X_{K,i} = (X_{K,i} + D_{K,i}) \circ \delta_i$.

Q.E.D.

Proof. (i) is obvious and (ii) can be proved just in the same way as Lemma (A.2.3.1).

Q.E.D.

Corollary (A.2.4.3). $\delta \circ X_{K,i} = (X_{K,i} + D_{K,i}) \circ \delta$.

Finally suppose $M = R$, $\delta_i = 0$. Then Lemma (2.4.1) implies the following

Lemma (A.2.4.4). Let Q be a complement of $R + \operatorname{Im} d$ in A . Then δ is injective on $Q^{\otimes k}$.

Proof. Obvious since $Q^{\otimes k}$ is a complement of $\sum \operatorname{Im} d_i$ in $A^{\otimes k}$.

Q.E.D.

§ A.3. Conservation laws of the KdV-equation.

A.3.1. A complex associated to the KdV equation.

Define a complex

$$0 \longrightarrow \Omega^0 \xrightarrow{D} \Omega^1 \xrightarrow{D} \Omega^2 \longrightarrow 0$$

as follows: Put

$$\Omega^i := A \otimes_R \Lambda^i [dx,dt] ,$$

where $\Lambda^*[dx,dt]$ stands for the exterior algebra on $R.dx \otimes R.dt$ and D is determined by

$$Df = dfdx + d_t fdt , \quad \text{for} \quad f \in A = \Omega^0 ,$$

$$D(g \wedge h) = Dg \wedge h + (-1)^a g \wedge Dh , \quad \text{for} \quad f \in \Omega^a, g \in \Omega^b ,$$

where $d_t = X_{uu_1+u_3}$ in the notation of § A.2.3. Since d and d_t commutes with each other, we obtain a complex $\{\Omega^*,D\}$. Denote by H^i the i-th cohomology space. We may call H^1 the space of the equivalence classes of conserved densities of the KdV-equation, since Propositions (2.3.1-2) and (3.3.1) for $k = 1$ imply that the map $fdx + gdt \longrightarrow \chi_1(1 \otimes f)$ induces a map

$$H^1 \longrightarrow \mathrm{Ker}D_t \cap F^1P(F(T)) ,$$

which in fact is an isomorphism by the argument before (4.1) in § 4.

Now we cite the result of [MGK] in the following form:

Theorem (A.3.1.1). For each positive integer i, an element $w_i = I_i dx + X_i dt$ of Ω^1 exists such that $Dw_i = 0$, weight$(w_i) = 2i$. Moreover the classes in H^1 represented by w_i's $(i=1,2,3,\dots)$ are non-zero.

Put $\Delta := D_K + X_K$ with $K = uu_1 + u_3$, i.e.,

$$\Delta := d_t - ud - d^3 .$$

Then by Proposition (A.2.2.1) and Lemma (A.2.3.1), we have the following realization of H^1:

Lemma (A.3.1.2). The map $I dx + X dt \longrightarrow \delta_u I$ induces an injection $H^1 \longrightarrow \operatorname{Ker} \Delta$.

A.3.2. Computation of Ker Δ

Put

$$\Delta_{3,0} = \sum_{i,j,k=0}^{\infty} u_{i+1} u_{j+1} u_{k+1} \partial^3/\partial u_i \partial u_j \partial u_k$$

$$+ 3 \sum_{i,j=0}^{\infty} u_{i+2} u_{j+1} \partial^2/\partial u_i \partial u_j ,$$

$$\Delta_{3,1} = -\sum_{i=0}^{\infty} \sum_{0<a\le i} \binom{i}{a} u_a u_{i-a+1} \partial/\partial u_i .$$

Then it is easy to show

<u>Lemma</u> (A.3.2.1).

(i) $\Delta = \Delta_{3,0} + \Delta_{3,1}$,

(ii) $\Delta_{3,i}(A^j(k)) \subset A^{j+3}(k+i)$.

(iii) $[\partial/\partial u , \Delta] = 0$.

Now we solve the equation $\Delta f = 0$.

<u>Lemma</u> (A.3.2.2). Suppose $f \in A_n \setminus A_{n-1}$ satisfies $\Delta f = 0$. Then n is even. Moreover, for $n \geq 4$,

$$f \equiv a(u_n + ((n+1)/3)uu_{n-2}) + bu_{n-1} + cu_{n-2} \pmod{A_{n-3}}$$

and

$$f = a(u_2 + u^2/2) + b \ , \quad \text{for} \quad n = 2$$

$$f = a \ , \quad \text{for} \quad n = 0$$

with $a,b,c \in R$.

<u>Proof</u>. From $\partial/\partial u_{n+2}(\Delta f) = 0$, we obtain

$$d(\partial f/\partial u_n) = 0 \ .$$

Hence by Lemma (A.2.1.1), f must be of the form

$$f \equiv au_n \pmod{A_{n-1}}.$$

Thus for $n = 0$ we obtain $f = au+b$. But then

$$\Delta f = au^2 \ ,$$

whence a must be zero.

Suppose now $n \geq 1$. Then

$$d(\partial f/\partial u_{n-1}) = \partial/\partial u_{n+1}\Delta f = 0 ,$$

whence

$$f = au_n + bu_{n-1} + f_{n-2}$$

with $a,b \in R$ and $f_{n-2} \in A_{n-2}$.

Suppose $n = 1$. Then $\Delta f = au_1^2 + bu^2$, whence we have $a = 0$, contradicting $f \notin A_0$.

Thus we have $n \geq 2$. Then from $\partial/\partial u_{n-2}\Delta f = 0$, we obtain

$$d(\partial f_{n-2}/\partial u_{n-2}) = (n+1)au_1/3 ,$$

whence

$$\partial f_{n-2}/\partial u_{n-2} = (n+1)au/3 + c$$

with $a,c \in R$.

Suppose now $n = 2$. Then

$$f_0 = u^2/2 + cu + e$$

with an $e \in R$, whence

$$f = a(u_2 + u^2/2) + bu_1 + cu + e .$$

But then $\Delta f = bu_1^2 + cu^2$, whence $b = c = 0$.

Suppose now $n \geqq 3$. Then

$$f_{n-2} \equiv (n+1)auu_{n-2}/3 + cu_{n-2} \quad (\text{mod. } A_{n-3}) .$$

Hence

$$f \equiv a(u_n + (n+1)uu_{n-2}/3) + bu_{n-1} + cu_{n-2} \quad (\text{mod. } A_{n-3}).$$

Thus it remains to show that n is even. By (iii) of Lemma (A.3.2.1), $\Delta \partial^k f/\partial u^k = 0$ for any k . Hence if n is odd, we obtain an element of the form $au_1 + g(u)$ in $\operatorname{Ker} \Delta$. Then by what we have shown above we must have $a = 0$, contradicting $f \notin A_{n-1}$.

Q.E.D.

<u>Corollary</u> (A.3.2.3). $\operatorname{Ker} \Delta \cap A_n^{n+3} = (0)$, for $n \geq 1$.

<u>Proof</u>. Suppose we have a nonzero element f in $\operatorname{Ker} \Delta \cap A_n^{n+3}$. Note that $A_n^{n+3} = A_{n-1}^{n+3}$, since $\text{weight}(u_n) = n+2$, and $\text{weight}(u_j) \geq 2$ for all j . Thus $f \in A_k \setminus A_{k-1}$ for some k with $0 \leq k \leq n-1$. By the above lemma, we have $f \equiv au_k$ (mod. A_{k-1}) . But $\text{weight}(u_k) = k+2 \leq n+3$, whence $a = 0$ contradicting $f \notin A_{k-1}$.

Q.E.D.

Put $T_i = \delta_u I_i$, where I_i's are those differential polynomials given in Theorem (A.3.1.1) . Then we have

Proposition (A.3.2.4).

(i) $$\operatorname{Ker}\Delta \cap A^n = \begin{cases} R.T_{i+1}\,, & \text{for } n = 2i\,, \\ (0)\,, & \text{for odd } n\,. \end{cases}$$

(ii) $$T_1 = c_1\,,$$
$$T_2 = c_2 u\,,$$
$$T_3 = c_3(u_2 + u^2/2)\,,$$
$$T_i = c_i(u_{2i-4} + (2i-3)uu_{2i-5}/3) \pmod{A_{2i-5}}$$

for $i \geq 4$ with nonzero c_i's .

Proof. Note first that $A^n = A^n_{n-2}$. Hence if n is odd, $\operatorname{Ker}\Delta \cap A^n = \operatorname{Ker}\Delta \cap A^n_{n-2} = \operatorname{Ker}\Delta \cap A^n_{n-3} = 0$ by Lemma (A.3.2.2) and Corollary (A.3.2.3).

Obviously we have $T_{i+1} \in A^{2i} = A^{2i}_{2i-2}$, which is nonzero Suppose now $T \in \operatorname{Ker}\Delta \cap A^{2i} = \operatorname{Ker}\Delta \cap A^{2i}_{2i-2}$. Then by Lemma (A.3.2.2)

$$T \equiv au_{2i-2} \pmod{A_{2i-3}}\,,$$

whence $T - bT_{i+1} \in A^{2i}_{2i-3} \cap \operatorname{Ker}\Delta = (0)$ for some $b \in R$. Hence $A^{2i} \cap \operatorname{Ker}\Delta = R.T_{i+1}$, which gives (i) .

Finally Lemma (A.3.2.2) gives (ii) .

Q.E.D.

§ A.3.3. Information on certain differential polynomials.

First we refine Theorem (A.3.1.1) using Lemma (A.3.1.2) and Proposition (A.3.2.4).

Theorem (A.3.3.1). For each positive integer i, an element $w_i = I_i dx + X_i dt$ of Ω^1 exists such that $Dw_i = 0$, weight$(I_i) = 2i$, weight$(X_i) = 2i+2$,

$$I_1 = u ,$$

$$I_i \equiv u_{i-2}^2 \quad (\text{mod. } A[3]) , \quad \text{for} \quad i \geq 2 ,$$

$$X_1 = u_2 + u^2/2 ,$$

$$X_i \equiv 2u_{i-2}u_i - u_{i-1}^2 \quad (\text{mod. } A[3]) , \quad \text{for} \quad i \geq 2 .$$

Moreover the classes in H^1 represented by w_i's$(i=1,2,\dots)$ constitute a basis of H^1.

Proof. Lemma (A.3.1.2) and (i) of Proposition (A.3.2.4) imply obviously the last assertion.

It is obvious that w_1 satisfies $Dw_1 = 0$ and $\delta_u I_1 = 1$ also spans $R.T_1$.

Let $i \geq 2$. Let I_i' be any element of A^{2i} such that $\delta_u I_i' = T_i$. As an element of A^{2i}, we can write

$$I_i' \equiv \sum a u_{2i-2} + \sum_{k=0}^{i-2} a_k u_k u_{2i-4-k} \quad (\text{mod. } A[3]) .$$

But modulo $\operatorname{Im} d$, $u_{2i-2} \equiv 0$ and $u_k u_{2i-4-k} \equiv (-1)^k u_{i-2}^2$.

Hence we may suppose $I_i' \equiv a_i u_{i-2}^2 \pmod{A[3]}$. But then $\delta_u I_i' \equiv 2a_i u_{2i-4} \pmod{A[2]}$. Hence we must have $a_i \neq 0$. Thus we can take $I_i = I_i'/a_i$.

Now modulo $A[3]$

$$\begin{aligned} d_t u_{i-2}^2 &\equiv 2u_{i-2}u_{i+1} \\ &= d[2u_{i-2}u_i - u_{i-1}^2] . \end{aligned}$$

Since $dA(m) \subset A(m)$ and $\operatorname{Ker} d = R$, the $X_i \in A^{2i+2}$ with $dX_i = d_t I_i$ must be of the form

$$X_i \equiv 2u_{i-2}u_i - u_{i-1}^2 \qquad \pmod{A[3]} .$$

Q.E.D.

Define $T_{ij} := \delta_u(S(i,j)) \in A^{2(i+j)}$, where we recall

$$S(i,j) = -I_i X_j + I_j X_i .$$

An easy calculation using the above Theorem shows the following

Lemma (A.3.3.2).

(i) For i,j with $j > i \geq 2$,

$$T_{ij} \equiv c_{ij}\, u_{i-2}^2\, u_{2j-2} \qquad \pmod{A[4]}$$

with $c_{ij} \neq 0$.

(ii) For $i \geq 2$,

$$T_{1i} \equiv c_i u u_{2i-2} \qquad \pmod{A[3]}$$

with $c_i \neq 0$.

Corollary (A.3.3.3). $T_k (k = 1,2,\ldots)$ and $T_{ij} (1 \leq i < j)$ are linearly independent.

Proof. It suffices to show that T_{i+1} and $T_{k,i-k} (1 \leq k < [i/2])$ are linearly independent, which is obvious by Proposition (A.3.2.4) and Lemma (A.3.3.2).

Q.E.D.

A.3.4. Twisting of Δ .

Let M be an R-vector space with an endomorphism $\partial : M \longrightarrow M$. Define $A_M := M \otimes_R A$ and

$$\mathrm{weight}(m \otimes f) = i ,$$

$$\mathrm{degree}(m \otimes f) = j ,$$

for $m \in M$ and $f \in A^i(j)$. We denote $M \otimes A^i$, $M \otimes A(j)$, $M \otimes \Lambda_n$, $M \otimes A^i(j)$, etc. respectively by A^i_M, $A_M(j)$, $A_{M,n}$, $A^i_M(j)$, etc..
Define

$$d^M := \partial \otimes 1 + 1 \otimes d ,$$

$$\Delta^M := 1 \otimes d_t - ud^M - (d^M)^3 : A_M \longrightarrow A_M .$$

Then we can decompose Δ^M :

$$\Delta^M = \sum^M_{i,j}$$

where $\Delta^M_{i,j}(A^k_M(n)) \subset A^{k+i}_M(n+j)$. Then it is easy to show

Lemma (A.3.4.1).

$$\Delta^M_{0,0} = \partial^3 \otimes 1 \quad ,$$

$$\Delta^M_{1,0} = 3\partial^2 \otimes d \quad ,$$

$$\Delta^M_{2,0} = 3\partial \otimes d^2 \quad ,$$

$$\Delta^M_{2,1} = \partial \otimes u \quad ,$$

$$\Delta^M_{3,0} = 1 \otimes \Delta_{3,0} \quad ,$$

$$\Delta^M_{3,1} = 1 \otimes \Delta_{3,1} \quad ,$$

and $\Delta^M_{i,j} = 0$ otherwise.

Put $\Delta^M_2 := \Delta^M_{2,0} + \Delta^M_{2,1}$, $\Delta^M_3 := \Delta^M_{3,0} + \Delta^M_{3,1} = 1 \otimes \Delta$.

By Proposition (A.3.2.4), we have

Lemma (A.3.4.2).

$$\operatorname{Ker} \Delta^M_3 \cap A^n_M = \begin{cases} M \otimes_R R.T_{i+1} , & \text{for } n = 2i , \\ (0) , & \text{for odd } n . \end{cases}$$

A.3.5. Independency of T_i and T_{jk} modulo $\operatorname{Im} \Delta$.

Let M and ∂ be as above.

Proposition (A.3.5.1). Suppose $f_j \in A^j_M$ ($j=2i-2$, $2i-3$) satisfies

(1) $$\Delta_3^M f_{2i-2} = 0 \quad ,$$

(2) $$\Delta_2^M f_{2i-2} + \Delta_3^M f_{2i-3} = a \otimes T_{i+1} + \sum_{1 \le k \le [(i-1)/2]} a_k \otimes T_{k,i-k} \; .$$

Then $a+3a_1 = 0$. Furthermore if a or a_1 is zero, then all the a_k's are zero and $\Delta^M f_{2i-2} = 0$.

We need the following

<u>Lemma</u> (A.3.5.2). Let $p_2 : A_M \longrightarrow A_M(2)$ be the projection. Then

$$P_2 \circ \Delta_3^M : A_M^{2i-3}(1) \oplus A_M^{2i-3}(2) \longrightarrow A_M^{2i}(2)$$

is injective and $M \otimes_R R.uu_{2i-4}$ is a complement of its image.

<u>Proof</u>. Since

$$p_2 \Delta_3^M (m \otimes u_j u_{2i-7-j}) = 3m \otimes (u_{j+2} u_{2i-6-j} + u_{j+1} u_{2i-5-j})$$

for $0 \le j \le i-4$, we have

$$m \otimes u_j u_{2i-4-j} \equiv (-1)^{i+j} m \otimes u_{i-2}^2 \;\; (\text{mod. } p_2 \Delta_3^M A_M^{2i-3}(2)) \quad ,$$

for $1 \le j \le i-3$. Further modulo $p_2 \Delta_3^M A_M^{2i-3}(2)$

$$\Delta_3^M(m\otimes u_{2i-5}) = -m\otimes \sum_{j=1}^{2i-5}\binom{2i-5}{j}u_j u_{2i-4-j}$$

$$\equiv -m\otimes \sum_{j=1}^{2i-5}\binom{2i-5}{j}(-1)^{i+j}u_{i-2}^2$$

$$= (-1)^i m\otimes u_{i-2}^2 .$$

Hence $p_2\circ\Delta_3(A_M^{2i-3}(1)\oplus A_M^{2i-3}(2)$ is spanned by $\{m\otimes u_j u_{2i-4-j};\ m\in M,\ 1\leq j\leq i-2\}$. The injectivity of $p_2\circ\Delta_3$ can be easily proved.

Q.E.D.

Proof of Proposition (A.3.5.1). By Lemma (3.4.2),

(1) implies $f_{2i-2}=m\otimes T_i$ with some $m\in M$. Then the $A_M^i(1)$-component of (2) gives $a=3\partial m$. On the other hand, by virtue of the above lemma, we obtain comparing the coefficients of uu_{2i-4} in (2)

$$(2i-2)\partial m = (2i-1)\ a/3 + a_1$$

whence $a+3a_1=0$.

Suppose now $a=a_1=0$. We may suppose $i\geqq 5$. Since $\partial m=0$, we have $\Delta^M f_{2i-2}=\Delta_3^M f_{2i-2}=0$, by virtue of Lemma (3.4.1). Thus we have

(3) $$\Delta_3^M f_{2i-3} = \sum_{2\le k\le[(i-1)/2]} a_k \otimes T_{k,i-k} .$$

Hence by Lemma (A.3.3.2),

$$\Delta_3^M f_{2i-3} \equiv 0 \qquad \text{(mod. } A[3]) .$$

By Lemma (A.3.5.2) above, we have

$$f_{2i-3} \in A[3].$$

Hence we can write

$$f_{2i-3} \equiv \sum_{j=0}^{2i-9} f_j u_j \qquad \text{(mod. } A_M[4]) ,$$

where $f_j \in A_M(2) \cap A_{M,j}$. If $j \ge i-4$, then

$$\text{weight}(u_a u_j^2) = 6+2j+a \ge 2i-2$$

whence f_j cannot have a nonzero term having u_j as a factor. Thus actually

(4) $$f_j \in A_{M,j-1} , \quad \text{for} \quad j \ge i-4 .$$

Thus (3) can be written as

(5) $$\Delta_3^M (f_{2i-9}u_{2i-9}+\dots+ f_{i-4}u_{i-4}+h_{i-5})$$

$$\equiv \sum_{2\le k\le[(i-1)/2]} a_k \otimes T_{k,i-k}$$

Modulo $A_M[4]$ with $f_j \in A_{M,j-1}$ and $h_{i-5} \in A_{M,i-5}$.

Consider now the following assertion:

$$(6)_k \qquad \begin{cases} a_j = 0\ , & \text{for} \quad j \le k\ , \\ f_j = 0\ , & \text{for} \quad j \ge 2i-2k-4 \end{cases}$$

What we must show is $(6)_{p-1}$ when $i = 2p$ and $(6)_p$ when $i = 2p+1$. In either case we have only to show $(6)_{[(i-1)/2]}$, which we shall prove by induction on k.

First comparing the coefficients of $u^2 u_{2i-6}$ of (5), we have $a_2 = 0$. Hence $(6)_2$ is valid if we consider f_j to be zero for $j \ge 2i-8$.

Suppose now that for some k such that $2 \le k \le [(i-1)/2]-1$ the assertion $(6)_k$ is true. Then (5) looks like

$$(7) \qquad \Delta_3^M(f_{2i-2k-5}u_{2i-2k-5} + f_{2i-2k-6}u_{2i-2k-6} + \ldots)$$
$$\equiv a_{k+1} \otimes (u_{k-1}^2 u_{2i-2k-4} + \ldots) + \ldots$$

modulo $A_M[4]$. Since $k \le [(i-1)/2 -1]$, we have $2i-2k-6 \ge i-4$, whence

$$f_s \in A_{M,s-1} \quad \text{for} \quad s = si-2k-5\ ,\ 2i-2k-6\ .$$

Comparing the coefficients of $u_{2i-2k-3}$ in (7), we obtain

$$(1 \otimes d) f_{2i-2k-5} = 0\ .$$

This implies by Lemma (A.2.1.1) $f_{2i-2k-5} = 0$. Comparing further the coefficients of $u_{2i-2k-4}$ in (7), we obtain

$$(1 \otimes d) f_{2i-2k-6} = a_{k+1} \otimes u_{k-1}^2 .$$

Applying $1 \otimes \delta_u$, we obtain $a_{k+1} \otimes u_{2k-2} = 0$, whence $a_{k+1} = 0$. Then we have $f_{2i-2k-6} = 0$, establishing $(6)_{k+1}$.

Q.E.D.

A.3.6. A refinement of Proposition (A.3.2.4).

Let M and ∂ be as in § A.3.4 and G an endomorphism of M.

Lemma (A.3.6.1). $\operatorname{Ker}(\Delta^M + G \otimes 1) = (\operatorname{Ker} G \cap \operatorname{Ker} \partial) \otimes \tilde{T}$,

where $\tilde{T}$ is the subspace of A spanned by T_i's .

Proof. Suppose $g \in A_M$ satisfies

(1) $$(\Delta^M + G \otimes 1)\, g = 0 .$$

Let g_k be the A_M^k-component of g and n the maximal number such that $g_n \neq 0$. The A_M^{n+3}-component of (1) gives $\Delta_3^M g_n = 0$, whence by Lemma (A.3.4.2), n is even: $n = 2i$ and $g_{2i} = a_1 \otimes T_{i+1}$ for some $a_1 \in M$. Then the A_M^{2i+2}-component of (1) gives

$$\partial a_1 \otimes (3d^2 T_{i+1} + u T_{i+1}) + \Delta_3^M g_{2i-1} = 0 .$$

Since $\mathrm{Im}\Delta_3^M \cap A_M(1) = 0$, we obtain $\partial a_1 = 0$ if we compare the coefficients of u_{2i} . Thus we have $\Delta^M(a_1 \otimes T_{i+1}) = 0$ and $\Delta_3^M g_{2i-1} = 0$, whence $g_{2i-1} = 0$.

Now the A_M^{2i+1}-component of (1) reads $\Delta_3^M g_{2i-2} = 0$, whence $g_{2i-2} = a_2 \otimes T_i$ for some $a_2 \in M$.

Finally the A_M^{2i}-component of (1) gives

$$(2) \qquad G(a_1) \otimes T_{i+1} + \partial a_2 \otimes (3d^2T_i) + uT_i) + \Delta_3^M g_{2i-3} = 0 .$$

By Lemma (A.3.5.2), the coefficients of u_{2i-2} and uu_{2i-4} in (2) give when $i \geq 3$

$$G(a_1) + 3\partial a_2 = 0$$

$$(3) \qquad (2i-1)G(a_1)/3 + (2i-2)\partial a_2 = 0 ,$$

whence $G(a_1) = 0$. When $i = 2$, (3) is replaced by

$$G(a_1) + \partial a_2 = 0 ,$$

whence $G(a_1) = 0$ again. Finally when $i = 1$, we have

$$g = a_1 \otimes u + a_2 \otimes 1 ,$$

$$(\Delta^M + G \otimes 1)g = G(a_1) \otimes u + G(a_2) \otimes 1 = 0 ,$$

whence $G(a_1) = 0$.

Thus we have

$$a_1 \otimes T_{i+1} \in (\mathrm{Ker}\, G \cap \mathrm{Ker}\, \partial) \otimes \tilde{T}$$

and

$$g - a_1 \otimes T_{i+1} \in A_M^{n-1}.$$

Hence by the induction on n, we obtain the Lemma.

Q.E.D.

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