NONCOMMUTATIVE AFFINE SCHEMES

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INTRODUCTION

The purpose of this work is to introduce the basics of noncommutative affine algebraic geometry. In other words, we consider here facts which are naturally expressed in the language of rings, ideals, and modules, without using categorical approach (as in [R6]).

Section 0 contains preliminaries about Gabriel localizations.

Section 1 presents the first notions and facts of the noncommutative local algebra: the left spectrum, localizations at points of the left spectrum, canonical topologies, supports of modules.

In Section 2, we prove the 'stability' of the left spectrum under localizations.

Section 3 is concerned with functorial properties of the left spectrum. If $f: A \longrightarrow B$ is a generic associative ring morphism, the preimage of an ideal from the left spectrum of B does not belong, in general, to the left spectrum of A.

A standart way to handle this situation is to single out the classes of ring morphisms which respect the left spectrum. One of the (practically) most important classes is the class of *left normal* morphisms which contains among others central extensions.

Another way which proved to be much more important for applications (cf. [R3], [R4], [R5]) is based on the observation that any ring morphism

$$f: A \longrightarrow B$$

induces a correspondence - a map from the left spectrum of the ring B into the set of subsets of the left spectrum of the ring A. In commutative case, this correspondence coincides with the preimage map.

The central fact of Section 4 is the description of the related to the left spectrum radical which is, by definition, the intersection of all ideals of the left spectrum. A surprising and important fact is that this radical equals to one of the classical objects of ring theory:

Theorem (4.10.2). *The intersection of all ideals of the left spectrum of a ring coincides with the biggest locally nilpotent ideal (Levitzki radical) of this* *ring.*

One of the consequences of this theorem is that the (introduced in Section 1) Zariski topology of the left spectrum of an arbitrary associative ring has a base of quasi-compact open sets. This fact is established, among others, in Section 5.

Section 6 is concerned with structure (pre)sheaves on the left spectrum. The central result is the *reconstruction theorem* (6.2) which, in commutative case, implies the equivalence of the category of modules over a ring and the category of quasi-coherent sheaves on the spectrum of the ring.

In Section 7, the noncommutative quasi-affine schemes and the projective spectrum are introduced.

Finally, we consider a couple of simplest examples. First, we describe the left spectrum of a left and right principal ideal domain (this happens to be useful for all examples). This description is applied then to produce the spectral picture of a generic quantum plane over an algebraically closed field. For 'real' applications, a reader is referred to [R3], [R4], and [R5].

0. PRELIMINARIES: LOCALIZATIONS AND RADICAL FILTERS.

0.1. Conventions and notations. Let R be an associative ring with unity, $I_l R$ the set of left ideals of R. For an arbitrary left ideal m of R and a subset w of elements of R, set $(m:w) := \{z \in R \mid zw \subset m\}$. It is easy to see that $(m:w)$ is a left ideal as well.

For any Z-module V, the symbol $P(V)$ will denote the set of all finitely generated Z-submodules of V.

Note that if $x, y \in P(R)$, then xy and $x + y$ also belong to $P(R)$.

The following relations are going to be used a lot:

$$(m:yx) = ((m:x):y) \quad \textit{and} \quad (m:x+y) = (m:x) \cap (m:y)$$

for any left ideal m *and Z-submodules* x, y *in* R.

0.2. Multiplication of filters. Denote by *fil-R* the set of filters (with respect to inclusion) of left ideals in the ring R. Define the *Gabriel multiplication*, $(F,G) \longmapsto F \odot G$, on *fil-R* as follows:

$$F \odot G = \bigcup_{m \in G} F \odot \{m\}$$

where

$$F \odot \{m\} := \{n \in I_l R \mid (n:w) \in F \text{ for any } w \in P(m)\}.$$

0.3. Radical filters. A filter F of left ideals of a ring R is called *a radical filter* if $F \odot \{R\} = F = F \odot F$. Other names: *a Gabriel filter*, and *an idempotent topologizing filter.*

0.3.1. Example: filters F_S. Let S be a subset of $P(R)$. Denote by F_S the set of all left ideals m in R such that $P((m:x))$ contains elements of S for any $x \in P(R)$.

Call a subset $S \subseteq P(R)$ a *multiplicative system* (or *set*) if $st \in S$ for any $s, t \in S$.

0.3.2. Lemma. *For any multiplicative system $S \subseteq P(R)$, the set F_S is a radical filter.*

Proof. a) If $m \in F_S$, then $(m:x) \in F_S$ for any $x \in P(R)$, since $((m:x):y)=(m:yx)$ for any $y \in P(R)$; and therefore $P(((m:x):y)) \cap S \neq \varnothing$.

b) Let $m \in F_S$ and $n \in F_S \odot \{m\}$; i.e. $(n:x) \in F_S$ for each $x \in P(m)$. Take an arbitrary $y \in P(R)$. Since $m \in F_S$, there exists $s \in S$ such that $sy \in P(m)$. Therefore $(n:sy) \in F_S$. In particular, there exists $t \in S$ such that $t(sy)=tsy \subset n$; or, equivalently, $ts \in P((n:y))$. Since $ts \in S$ and y has been chosen arbitrarily, this means that $n \in F_S$. ∎

0.4. Flat localizations and radical filters. *A flat localization* of an abelian category $\mathcal{A}$ is an exact functor, $Q: \mathcal{A} \longrightarrow \mathcal{B}$, which has a fully faithful right adjoint $J: \mathcal{B} \longrightarrow \mathcal{A}$. The category $\mathcal{B}$ here is called *the quotient category of* $\mathcal{A}$.

Localizations $Q: \mathcal{A} \longrightarrow \mathcal{B}$ and $Q': \mathcal{A} \longrightarrow \mathcal{B}'$ are called *equivalent* if there exists an equivalence $T: \mathcal{B} \longrightarrow \mathcal{B}'$ such that $T \circ Q = Q'$. It is possible to assign to any equivalence class of localizations its canonical (the largest in a certain sense) quotient category. This correspondence admits a particularly nice description in the case when $\mathcal{A}$ is the category $R\text{-}mod$ of left modules over an associative ring R.

Let F be a radical filter. Denote by $R\text{-}mod/F$ the full subcategory of the category $R\text{-}mod$ formed by all the left modules M such that the canonical map $M \longrightarrow Hom_R(m,M)$, which sends an element z of the module M into the morphism $r \longmapsto r \cdot z$, is a bijection for any ideal m from the filter F.

On the other hand, for any R-module M, set

$$H'_F(M) := colim\{Hom_R(m,M) \ : \ m \in F\}$$

(morphisms in F are inclusions). The Z-module $H'_F(M)$ possesses a natural structure, ζ, of R-module which is uniquely determined by the property:

the canonical map $\tau_{F,M} : M \longrightarrow H'_F(M)$ is an R-module morphism from M to $H_F(M):=(H'_F(M),\zeta)$. Moreover, the map $M \longmapsto H_F(M)$ is extended to a functor H_F: $R\text{-}mod \longrightarrow R\text{-}mod$ such that $\tau_F = \{\tau_{F,M}\}$ is a functor morphism from Id to H_F. Denote the square of the functor H_F by $\mathbb{G}_F$ (- *the Gabriel functor*), and set $j_F:=H_F\tau_F\circ\tau_F$.

0.4.1. Theorem. *(a) Let F be a radical filter. Then the functor $\mathbb{G}_F$ takes values in the subcategory R-mod/F. The corestriction Q_F of the functor $\mathbb{G}_F$ onto R-mod/F is a flat localization of the category R-mod, with the natural inclusion as the right adjoint functor.*

In particular, an R-module M belongs to the subcategory R-mod/F if and only if the canonical arrow $j_F(M): M \longrightarrow \mathbb{G}_FM$ is an isomorphism.

(b) If Q is a localization of the category R-mod, then the set FQ of all the left ideals m such that $Q(R/m) = 0$ is a radical filter.

(c) The map $F \longmapsto Q_F$ defines a bijection of the set of all radical filters of left ideals in the ring R onto the set of all equivalence classes of flat localizations of the category R-mod. More explicitly, $FQ_F = F$, and the localization Q_{FQ} is equivalent to Q for any radical filter F and any localization Q.

For any $M \in ObR\text{-}mod/F$ and an element z of M, the action

$$\cdot z: R \longrightarrow M, \quad r \longmapsto rz,$$

is extended uniquely to a morphism $\mathbb{G}_FR \longrightarrow M$ - the composition of

$$\mathbb{G}_F(\cdot z): \mathbb{G}_FR \longrightarrow \mathbb{G}_FM$$

and the isomorphism $\mathbb{G}_FM \longrightarrow M$ (cf. the assertion *(a)* of Theorem 0.4.1). These morphisms define a map

$$\mu(M) : \mathbb{G}_FR \otimes M \longrightarrow M \tag{1}$$

which depends functorially on M such that

$$\mu(\mathbb{G}_FR): \mathbb{G}_FR \otimes \mathbb{G}_FR \longrightarrow \mathbb{G}_FR \tag{2}$$

is an associative ring structure; and $\mu(M)$ is a left $\mathbb{G}_FR$-module structure for any module M from the subcategory R-mod/F.

Note that the ring structure (2) is uniquely defined by the requirement that $j_F(R): R \longrightarrow \mathbb{G}_FR$ is a ring morphism; and the $\mathbb{G}_FR$-module structure (1) is uniquely defined by the compatibility with the R-module morphism

$$j_F(M): M \longrightarrow \mathbb{G}_FM.$$

Thus, there is a functor $\mathfrak{G}_F$: $R\text{-}mod/F \longrightarrow \mathbb{G}_FR\text{-}mod$.

0.4.2. Proposition. *The functor* $\mathfrak{G}_F$ *is right adjoint to the localization at the radical filter* F' *of all left ideals* m *in the ring* $\mathbb{G}_F R$ *such that the preimage* $j_F^{-1}(m)$ *of* m *belongs to* F.

In particular, the category $R\text{-mod}/F$ *is naturally equivalent to the category* $\mathbb{G}_F R\text{-mod}/F'$.

Proofs of Theorem 0.4.1 and Proposition 0.4.2 can be found in [BD] or in [F,I], Chapter 16.

1. LEFT SPECTRUM.

1. A preorder on the set of left ideals. Define a relation $\le$ on the set $I_l R$ of left ideals in the ring R as follows:

$$m \le n \quad \text{if} \quad (m:x) \subseteq n \quad \text{for some} \quad x \in P(R).$$

1.1. Lemma. *The relation* $\le$ *is a preorder.*

Proof. Let $m' \le m$, and $m \le n$; i.e. $(m':x) \subseteq m$, and $(m:y) \subseteq n$ for some $x, y \in P(R)$. Then $((m':y):x) \subseteq (m:x) \subseteq n$. But $(m':xy) = ((m':y):x)$, and $xy \in P(R)$. ■

1.2. Remark. It is easy to see that if the ideal m is two-sided, then the relation $m \le n$ is equivalent to the inclusion $m \subseteq n$ (since in this case $m \subseteq (m:x)$ for any subset $x \in R$). In particular, if the ring R is commutative, then the preorder $\le$ coincides with the inclusion. ■

Call a set $\mathfrak{F}$ of left ideals in R *a uniform filter* if it is a filter with respect to $\le$; i.e. if $m \in \mathfrak{F}$ and $m \le n$, then $n \in \mathfrak{F}$.

1.3. Example: filters $[m]$. With a left ideal m, one can associate the set $[m] = \{n \in I_l R \mid m \le n\}$. Obviously, $[m]$ is the smallest among uniform filters containing m.

It is easy to see that the filter $[m]$ is topologizing.

In fact, if $(m:x) \subseteq n$ and $(m:y) \subseteq n'$ for some x and y from $P(R)$, then $(m:x+y) = (m:x) \cap (m:y) \subseteq n \cap n'$. ■

1.4. Example: filters $\langle m \rangle$. Given a left ideal m, denote by $\langle m \rangle$ the set $I_l R - \{n \in I_l R \mid n \le m\}$. It follows from Lemma 1.7.1 that $\langle m \rangle$ is a uniform filter. Clearly $\langle m \rangle$ *is the biggest uniform filter which does not contain the ideal* m.

1.4.1. Lemma. *For any two left ideals* m, n *in the ring* R,

$$m \leq n \quad \textit{if and only if} \quad \langle n \rangle \subseteq \langle m \rangle .$$

In particular, m *is equivalent to* n *with respect to* $\leq$ *if and only if* $\langle m \rangle = \langle n \rangle$.

Proof. If $m \leq n$, then $I_l R - \langle m \rangle \subseteq I_l R - \langle n \rangle$; or, equivalently, $\langle n \rangle$ is a subset of $\langle m \rangle$.

Conversely, the inclusion $I_l R - \langle m \rangle \subseteq I_l R - \langle n \rangle$ implies, evidently, that $m \leq n$. ■

In other words, the associate with $\leq$ order is isomorphic to the ordered set $(\{\langle m \rangle \mid m \in I_l R\}, \supseteq)$

Note that, for a generic left ideal m, the filter $\langle m \rangle$ needs not to be a cofilter. For example, if there exists a pair, α, β, of two-sided ideals such that $\alpha \cap \beta \subseteq m$, but $\alpha - m$ and $\beta - m$ are non-empty, then $\alpha, \beta \in \langle m \rangle$ and $\alpha \cap \beta \notin \langle m \rangle$.

1.5. Left spectrum. The *left spectrum,* $Spec_l R$, of the ring R consists of all left ideals p which have the following property:

(*) $(p{:}x) \leq p$ *for any* $x \in R - p$.

1.5.1. Note. Since, in commutative case, the relation $\leq$ is the inclusion (cf. Remark 1.2), the left spectrum of a commutative ring coincides with its prime spectrum. ■

The following lemma shows that the left spectrum is pretty ample.

1.5.2. Lemma. *The left spectrum,* $Spec_l R$, *contains the set* $Max_l R$ *of all maximal left ideals in the ring* R.

Proof. In fact, if n, m are left ideals in R such that the relation $(m{:}x) \leq m$ does not hold if $x \in n$, then it does not hold if $x \in n + m$. But if $m \in Max_l R$ and n is not contained in m, then $n + m = R$. In particular, $n + m$ contains the unity, 1, of R. Clearly $(m{:}1) = m \leq m$. Hence, if $m \in Max_l R$, then $n \subseteq m$. This means that $m \in Spec_l R$. ■

1.6. Completely prime left ideals. Call a left ideal p *completely prime* if the set $R - p$ is a multiplicative system. The set of completely prime left ideals (*completely prime spectrum*) will be denoted by $Spec_l^{\wedge} R$.

As a rule, completely prime spectrum is much poorer than the left spectrum, as one can see from the second assertion of the following lemma.

1.6.1. Lemma. *1)* $Spec_l^{\wedge}R \subseteq Spec_l R$.

2) A left maximal ideal m *in the ring* R *is completely prime if and only if* m *is two-sided.*

3) If every left ideal in the ring R *is two-sided (e.g.* R *is commutative), then* $Spec_l^{\wedge}R = Spec_l R$.

Proof. 1) Note that left ideal μ is *completely prime* if and only if $(\mu{:}x) \subseteq \mu$ for any $x \in R - \mu$. This implies immediately that $Spec_l^{\wedge}R \subseteq Spec_l R$.

2) If m is two-sided, then $m \subseteq (m{:}x)$ for each $x \in R$. Therefore, if m is a maximal left ideal, then $(m{:}x) = m$ for any $x \in R - m$; i.e. m belongs to $Spec_l^{\wedge}R$.

Conversely, suppose that a left maximal ideal μ is completely prime. The latter means that $(\mu{:}x) \subseteq \mu$ for all $x \in R - \mu$. Since $(\mu{:}x)$ is a maximal left ideal for any $x \in R$, the inclusion $(\mu{:}x) \subseteq \mu$ implies that $(\mu{:}x)$ coincides with μ. Therefore μ is a two-sided ideal.

3) For two-sided ideals, the preorder $\leq$ coincides with $\subseteq$. ∎

1.7. Remark. The difference between $Spec_l^{\wedge}R$ and $Spec_l R$ provides a number of examples of radical filters of the form F_S (cf. Example 0.3.1), where S is not a multiplicative system.

In fact, for any $p \in Spec_l R$, the radical filter $<p>$ coincides with F_S, where $S = R - p$, and with $F_{\mathfrak{S}}$, where $\mathfrak{S} = P(R) - P(p)$. ∎

1.8. The prime spectrum and the left spectrum. Recall that the *prime spectrum*, *SpecR*, of R consists of *prime* ideals. A two-sided ideal is called *prime* if, for any pair of two-sided ideals, m, n, the inclusion $mn \subseteq p$ implies that either $m \subseteq p$ or $n \subseteq p$.

1.8.1. Lemma. *For every* $p \in Spec_l R$, *the two-sided ideal* $(p{:}R)$ *is prime.*

Proof. Let m, n be two-sided ideals such that m is not contained in p, but $mn \subseteq p$. Since the ideal m is two-sided, $m \in <p>$, and, therefore, $(mn{:}x) \in <p>$ for any $x \in P(n)$. This implies that $n \subseteq p$. Since the ideal n is two-sided, and $(p{:}R)$ is the maximal among the two-sided ideals which are contained in p, the inclusion $n \subseteq p$ implies that $n \subseteq (p{:}R)$. ∎

1.8.2. Remark. We shall show later that if R is a left noetherian ring, then $SpecR \subseteq Spec_l R$.

1.9. The left spectrum and the filters $\langle m\rangle$**.** For each proper left ideal m in the ring R, denote by $m^\wedge$ the set

$$\{r \in R \mid (m:r) \in \langle m\rangle\}.$$

It is clear that

(a) $m \subseteq m^\wedge \neq R$;

(b) $xr \in m^\wedge$ for any $r \in m^\wedge$ and $x \in P(R)$, since $\langle m\rangle$ is a filter with respect to $\leq$, and $(m:r) \leq ((m:r):x) = (m:xr)$.

It follows from (b) that $m^\wedge$ is a left ideal if and only if it is a $\mathbb{Z}$-submodule of R. Nevertheless, it happens quite seldom that, for a given left ideal m, the set $m^\wedge$ turns out to be a left ideal.

Let $I^\wedge_l R$ denote the set of all proper left ideals m in the ring R for which the following condition holds:

($\wedge$) *if the elements* x, y *of the ring* R *are such that*

$$(m:x) \in \langle m\rangle \text{ and } (m:y) \in \langle m\rangle,$$

then $(m:\{x,y\}) \in \langle m\rangle$.

1.9.1. Proposition. *1) For any* $n \in I^\wedge_l R$, *the set* $n^\wedge$ *is an ideal from* $Spec_l R$.

2) The following conditions on a proper left ideal m *are equivalent:*

(a) $m^\wedge$ *is a left ideal, and* $m^\wedge \leq m$;

(b) the filter $\langle m\rangle$ *is radical.*

3) The following conditions on a proper left ideal m *are equivalent:*

(c) $m = m^\wedge$;

(d) $m \in Spec_l R$.

Proof. 1) Suppose that $n \in I^\wedge_l R$; i.e., for any pair x, y of elements of $n^\wedge$, the ideal $(n:x) \cap (n:y) = (n:\{x,y\})$ belongs to the filter $\langle n\rangle$. Since

$$(n:x) \cap (n:y) \subseteq (n:x+y),$$

the ideal $(n:x+y)$ also belongs to $\langle n\rangle$. This means that $n^\wedge$ is closed under the addition and is, therefore, a left ideal.

Let us show that $n^\wedge \in Spec_l R$. This fact is equivalent to the following condition:

$$\textit{If } z \in R \textit{ and } (n^\wedge:z) \in \langle n\rangle, \textit{ then } z \in n.$$

1.9.2. Lemma. *Let* $n \in I_l R$, *and let* w *be a finite subset in* R *such that* $(n^\wedge:w)$ *is not contained in* $n^\wedge$. *Then* $(n:w)$ *is not contained in* n.

Proof. Suppose that $(n:w) \subseteq n$. Then, for any $x \in (n:w)-n$, the following relations hold:

$$(n:xw) = ((n:w):x) \subseteq (n:x) \leq n.$$

On the other hand, since xw is a finite subset from $n^\wedge$, then $(n:xw) \in <n>$. Contradiction. ■

For an arbitrary $z \in R$, we have, by Lemma 1.9.2, the following implications: $[(n:z) \in <n^\wedge>] \Leftrightarrow [(n:xz)$ *is not contained in* $n^\wedge$ *for any finite subset* x *in* $R] \Rightarrow [(n:xz)$ *is not contained in* n *for any finite* $x \subset R] \Leftrightarrow [(n:z) \in <n>;$ *i.e.* $z \in n^\wedge]$.

This is exactly what we wanted to prove.

2) $(a) \Rightarrow (b)$. Given a left ideal n, we have:

$[(m:x) \in <m>$ *for every* $x \in n] \Rightarrow [n \subseteq m^\wedge$ *(by definition of* $m^\wedge$*)*$] \Leftrightarrow [n \leq m$ *(since* $m^\wedge \leq m$ *by condition*$]$.

$(b) \Rightarrow (a)$. It follows from the definition of $I^\wedge_l R$ that any left ideal m such that $<m>$ is a topologizing filter belongs to $I^\wedge_l R$. Therefore, according to the first assertion of this Proposition, $m^\wedge \in Spec_l R$. In particular, $p^\wedge \in Spec_l R$ if $<p>$ is a radical filter. Moreover, since $(p:x) \in <p>$ for any $x \in P(p^\wedge)$, the ideal $p^\wedge$ does not belong to $<p>$; i.e. $p^\wedge \leq p$.

3) The implication $(d) \Rightarrow (c)$ follows from the definition of $Spec_l R$. The converse implication is obvious. ■

1.9.3. Corollary. *The following properties of a left ideal* m *in the ring* R *are equivalent:*

(i) there exists an ideal p *from* $Spec_l R$ *and an* $x \in P(R)$ *such that* $(p:x) \subseteq m \subseteq p$ *(in particular* m *is equivalent to* p*);*

(ii) the filter $<m>$ *is radical.*

1.10. Topologies on the left spectrum. Define the specialization of a 'point' $p \in Spec_l R$ as the set of all $p' \in Spec_l R$ such that $p \leq p'$. We are interested only in the topologies compatible with specialization, i.e. topologies with the property:

$(\mathfrak{s})$ *the closure of a point* p *contains the set* $\mathfrak{s}(p)$ *of all the specializations of* p.

1.10.1. The topologies τ and τ_*. Denote by τ the strongest topology satisfying $(\mathfrak{s})$. The closure of a set $X \subseteq Spec_l R$ in τ is, evidently, the set

$$\bigcup_{p \in X} \mathfrak{s}(p).$$

It is easy to see that the set $Open(\tau)$ of open in the topology τ subsets of $Spec_l R$ is closed under arbitrary intersections.

We denote by τ_* the weakest topology with the property:

the closure of a point p coincides with the set $s(p)$ of its specializations.

The family of sets $\{s(p) \mid p \in Spec_l R\}$ is the base of the topology τ_*.

1.10.2. The topology $\tau^\wedge$. Given a two-sided ideal α, denote by $V_l(\alpha)$ the set $\{p \in Spec_l R \mid \alpha \subseteq p\}$ and by $U_l(\alpha)$ the set $Spec_l R - V_l(\alpha)$.

1.10.2.1. Lemma. *Let Ω be a subset of the set IR of two-sided ideals in the ring R; and let $\alpha, \beta \in IR$. Then*

$$U_l(sup\{\alpha' \mid \alpha' \in \Omega\}) = \bigcup_{\alpha' \in \Omega} U_l(\alpha'), \quad U_l(\alpha \cap \beta) = U_l(\alpha) \cap U_l(\beta)$$

Proof. 1) The first equality is equivalent to the equality

$$V_l(sup\{\alpha' \mid \alpha' \in \Omega\}) = \bigcap_{\alpha' \in \Omega} V_l(\alpha'),$$

which is obvious.

2) Clearly $U_l(\alpha \cap \beta) \subseteq U_l(\alpha) \cap U_l(\beta)$.

On the other hand, if $p \in Spec_l R$ and $\alpha \cap \beta \subseteq p$, but β is not contained in p, then, for any $x \in \beta\text{-}p$, we have:

$$\alpha \subseteq (\alpha \cap \beta{:}x) \subseteq (p{:}x) \leq p.$$

In particular, $\alpha \leq p$. Since α is a two-sided ideal, this means that $\alpha \subseteq p$.

Thus, we have checked that

$$V_l(\alpha \cap \beta) \subseteq V_l(\alpha) \cup V_l(\beta),$$

or, equivalently,

$$U_l(\alpha \cap \beta) \supseteq U_l(\alpha) \cap U_l(\beta). \ \blacksquare$$

Lemma 1.10.2.1 shows that the collection of the sets $U_l(\alpha)$, where α runs through the set IR of all the two-sided ideals in R, forms the set of open sets of a topology on $Spec_l R$ which we denote by $\tau^\wedge$.

The topology $\tau^\wedge$ is the less refined among reasonable topologies on the left spectrum. It is, obviously, trivial if the ring in question is simple (i.e. has no nonzero proper two-sided ideals). For example, it is trivial if R is the algebra of differential operators with polynomial coefficients. We shall see, however, that the topology $\tau^\wedge$ has the most desirable for algebraic geometry property: it has a base of quasi-compact open sets.

1.10.3. The topology τ^*. The topology τ^* is determined by its base of closed subsets which, by definition, consists of all sets of the form

$$V_l(m):=\{p \in Spec_l R \mid m \le p\},$$

where m runs through the set of all the proper left ideals in the ring R.

The topology τ^* is more refined, but, at the same time more capricious than $\tau^\wedge$. However, it behaves itself properly when the ring has a finite Krull dimension and in some other cases.

Clearly, if the ring R is commutative, then both $\tau^\wedge$ and τ^* coincide with the Zariski topology. If it is commutative and noetherian, then the topologies $\tau^\wedge$, τ^* and τ_* coincide.

1.11. The support of a module. The *support* of an R-module M is the set $Supp(M)$ of all $p \in Spec_l R$ such that $\mathbb{G}_{<p>}M \neq 0$.

Since the kernel of the canonical module morphism

$$j_{<p>}=j_{<p>,M}: M \longrightarrow \mathbb{G}_{<p>}M$$

coincides with the $<p>$*-torsion*, $<p>M=\{\xi \in M \mid Ann(\xi) \in <p>\}$, and the canonical map $\mathbb{G}_{<p>}M \longrightarrow \mathbb{G}_{<p>}(M/<p>M)$ is an isomorphism, the support of M can be described as the set

$$\{p \in Spec_l R \mid <p>M \neq M\} = \{p \in Spec_l R \mid Ann(\xi) \le p \text{ for some } \xi \in M\}.$$

Clearly the set $Supp(M)$ is closed in the topology τ for any module M (cf. 1.10.1).

If M is a finitely generated R-module, then $Supp(M)$ is closed in the topology τ^* (cf. 1.10.3).

1.11.1. Lemma. *$Supp(M) = \varnothing$ if and only if $M = 0$.*

Proof. 1) Clearly $Supp(M) = \varnothing$ if $M = 0$.

2) Let $M \neq 0$, and let ξ be a nonzero element of M. Then $Ann\xi = \{r \in R \mid r\xi = 0\}$, being a proper left ideal in R, is contained in some left maximal ideal, say μ. In particular, $Ann\xi \notin <\mu>$; i.e. $\xi \notin <\mu>M$. Since $\mu \in Spec_l R$ (cf. Proposition 1.4.1), this shows that $\mu \in Supp(M)$. ∎

1.11.2. Proposition. *1) If*

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

is an exact sequence of R-modules, then

$$Supp(M) = Supp(N) \cup Supp(L).$$

2) If M is the sum of a family $\{N_i \mid i \in J\}$ of its submodules, then

$$Supp(M) = \bigcup_{i \in J} Supp(M_i).$$

Proof. 1) Clearly $Supp(N) \subseteq Supp(M)$.

Let $\xi \in L$, and let ξ' belongs to the preimage of the element ξ in M. If $p \in Spec_l R$ is such that $Ann\xi \not\in \langle p\rangle$, then, obviously, $Ann\xi' \not\in \langle p\rangle$. Thus,

$$Supp(L) \subseteq Supp(M).$$

It remains to show that there is the inverse inclusion:

$$Supp(M) \subseteq Supp(N) \cup Supp(L).$$

In fact, since the functor $\mathbb{G}_{\langle p\rangle}$ is left exact, the sequence

$$0 \longrightarrow \mathbb{G}_{\langle p\rangle}N \longrightarrow \mathbb{G}_{\langle p\rangle}M \longrightarrow \mathbb{G}_{\langle p\rangle}L$$

is exact. Therefore, if $p \in Supp(M)$, i.e. $\mathbb{G}_{\langle p\rangle}M \neq 0$, then either $\mathbb{G}_{\langle p\rangle}N \neq 0$, or $\mathbb{G}_{\langle p\rangle}L \neq 0$. In the first case, $p \in Supp(N)$, in the second case p belongs to $Supp(L)$.

2) The inclusion $\bigcup_{i \in J} Supp(M_i) \subseteq Supp(M)$ is obvious, as well as the implication: $[\langle p\rangle M \neq M] \Rightarrow [\langle p\rangle M_i \neq M_i$ *for some* $i \in J]$. ∎

1.11.3. Corollary. *For any family* $\mathfrak{M}$ *of R-modules,*

$$Supp(\bigoplus_{M \in \mathfrak{M}} M) = \bigcup_{M \in \mathfrak{M}} Supp(M).$$

2. LOCALIZATIONS AND THE LEFT SPECTRUM.

2.0. Preorders $(I_lM,\leq)$ and localizations. Let R and B be associative rings. Fix an (R,B)-bimodule M. Denote by $(I_lM,\leq)$ the set I_lM of all the R-submodules of M with the preorder $\leq$ which is defined as follows:

$$N \leq N' \text{ if } (N:b) := \{z \in M \mid zb \subset N\} \subseteq N' \text{ for some } b \in P(B).$$

Clearly if M is the (R,R)-bimodule R, then the preorder $(I_lM,\leq)$ coincides with $I_lR = (I_lR,\leq)$.

Every functor $\mathbb{F}$: $R\text{-}mod \longrightarrow R'\text{-}mod$ defines uniquely the functor ${}^B\mathbb{F}$ from the category $(R,B)\text{-}bi$ of (R,B)-bimodules into the category $(R',B)\text{-}bi$ of (R',B)-bimodules. In particular, to the functor $\mathbb{G}_F$, there corresponds the functor ${}_B\mathbb{G}_F$.

2.1. Proposition. *Let F be a radical filter of left ideals in the ring R; and let M be an (R,B)-bimodule. Then, for any R-submodule N of M and any*

$b \in P(B)$, *we have:* $\mathbb{G}_F(N:b) = (\mathbb{G}_F N:b)$.

Proof. Let $b \in B$ be an arbitrary element in B, and $\cdot b$ the action of b on M. For any R-submodule N of M, the square

$$\begin{array}{ccc} N & \longrightarrow & M \\ \uparrow & & \uparrow \cdot b \\ (N:b) & \longrightarrow & M \end{array} \tag{1}$$

in which the horizontal arrows are the embeddings, is cartesian. Since the functor $\mathbb{G}_F$ is left exact, it sends the cartesian square (1) into the cartesian square

$$\begin{array}{ccc} \mathbb{G}_F N & \longrightarrow & \mathbb{G}_F M \\ \uparrow & & \uparrow \cdot b \\ \mathbb{G}_F(N:b) & \longrightarrow & \mathbb{G}_F M \end{array}$$

Therefore $\mathbb{G}_F(N:b)=(\mathbb{G}_F N:b)$.

Now, let $b' \in P(B)$ and $\{b_i \mid i \in I\}$ be a finite set of generators of b'. Thanks to the left exactness of $\mathbb{G}_F$, we have:

$$\mathbb{G}_F(N:b') = \mathbb{G}_F\Big(\bigcap_{i\in I}(N:b_i)\Big) = \bigcap_{i\in I}\mathbb{G}_F(N:b_i) = \bigcap_{i\in I}(\mathbb{G}_F N:b_i) = (\mathbb{G}_F N:b'). \ \blacksquare$$

2.2. Corollary. *The functor* ${}_B\mathbb{G}_F$ *determines a morphism*

$$(I_l M,\leq) \longrightarrow (I_l({}_B\mathbb{G}_F M),\leq).$$

of preordered sets.

2.3. Corollary. *(i) For every radical filter* F *of the left ideals in* R, *the map* $m \longmapsto \mathbb{G}_F m$, $m \in I_l\mathbb{G}_F R$, *is a morphism*

$$(I_l\mathbb{G}_F R,\leq) \longrightarrow (I_l\mathbb{G}_F R,\leq)$$

of preordered sets.

(ii) The map $n \longmapsto \mathbb{G}_F n$, $n \in I_l R$, *is a morphism*

$$(I_l R,\leq) \longrightarrow (I_l\mathbb{G}_F R,\leq).$$

of preordered sets.

2.4. Note. If N is an R-submodule of $\mathbb{G}_F M$, then $\mathbb{G}_F(j^{-1}_{F,M}(N))$ is canonically identified with $\mathbb{G}_F N$.

Indeed, by definition, the square

$$\begin{array}{ccc} N & \longrightarrow & \mathbb{G}_F M \\ {\scriptstyle i}\uparrow & & \uparrow {\scriptstyle j_{F,M}} \\ j_{F,M}^{-1}(N) & \longrightarrow & M \end{array} \qquad (1)$$

is cartesian; and $\mathbb{G}_F$ transforms cartesian squares into cartesian squares. In particular, the arrow $\mathbb{G}_F i: \mathbb{G}_F(j_{F,M}^{-1}(N)) \longrightarrow \mathbb{G}_F N$ is an isomorphism. ∎

2.5. Proposition. *Let* F *be a radical filter in* $I_l R$. *Then the map*

$$m \longmapsto \mathbb{G}_F m, \quad m \in I_l R,$$

sends the ideals from $Spec_l R - F$ *into the ideals from* $Spec_l \mathbb{G}_F R$.

Proof. Let $p \in Spec_l R - F$, $n \in I_l \mathbb{G}_F R$; and $(\mathbb{G}_F p{:}x)$ is not contained in $\mathbb{G}_F p$ for each $x \in P(n)$. In particular, $(\mathbb{G}_F p{:}j_{F,R}(x))$ is not contained in $\mathbb{G}_F p$ for any x from $P(j_{F,R}^{-1}(n))$. But, according to Proposition 2.1,

$$(\mathbb{G}_F p{:}j_{F,R}(x)) = \mathbb{G}_F(p{:}x)$$

for any $x \in P(R)$. Therefore the ideal $(p{:}x)$ is not contained in p for any $x \in P(j_{F,R}^{-1}(n))$. Since $p \in Spec_l R$, this implies that $j_{F,R}^{-1}(n) \subseteq p$. Thus, we have come to the inclusions (cf. Note 2.4) :

$$n \subseteq \mathbb{G}_F(j_{F,R}^{-1}(n)) = \mathbb{G}_F n \subseteq \mathbb{G}_F p.$$

Hence $\mathbb{G}_F p \in Spec_l \mathbb{G}_F R$. ∎

2.6. Proposition. *Let* $p' \in Spec_l \mathbb{G}_F R$, *and* $p := j_{F,R}^{-1}(p') \notin F$.
Then $p' = \mathbb{G}_F p$.

Proof. By Note 2.4, $p' \subseteq \mathbb{G}_F p$. Our task is to prove the inverse inclusion.

Suppose that $(p'{:}x) \subseteq p'$ for some $x \in P(\mathbb{G}_F p)$. Then there exists a left ideal $m \in F$ such that $j_F(m)x \subseteq j_F(p) \subset p'$. Since $p \notin F$, the set $j_F(m) - p'$ is non-empty. By condition, p' belongs to $Spec_l \mathbb{G}_F R$. Therefore $(p'{:}y) \simeq p'$ for any element y in $j_F(m) - p'$. In particular, $\mathbb{G}_F(p'{:}y) \neq \mathbb{G}_F R$.

Thanks to the inclusion $(p'{:}x) \subseteq p'$ and the equality $((p'{:}x){:}y) = (p'{:}yx)$, this implies that $\mathbb{G}_F(p'{:}yx) \neq \mathbb{G}_F R$. But $yx \subset p'$, and, therefore, $\mathbb{G}_F(p'{:}yx) = \mathbb{G}_F R$. Contradiction.

Thus, for any $x \in P(\mathbb{G}_F p)$, the ideal $(p'{:}x)$ is not contained in p'. Since $p' \in Spec_l \mathbb{G}_F R$, this implies that $\mathbb{G}_F p \subseteq p'$. ∎

2.7. Lemma. *Let* F *be a radical filter of left ideals in* R; *and let* p *be a* *left ideal in* R *such that* $\mathbb{G}_F p$ *is a completely prime left ideal of* $\mathbb{G}_F R$. *Then the ideal*

$$p_F := j^{-1}_{F,R}(\mathbb{G}_F p) = \{r \in R \mid (p{:}r) \in F\}$$

is completely prime.

Proof. If $y \in R - p_F$ and $x \in R$, then $[yx \in p_F] \Leftrightarrow [j_F(y) j_F(x) \in \mathbb{G}_F p] \Leftrightarrow [j_F(x) \in \mathbb{G}_F p$ *(since* $\mathbb{G}_F p$ *is completely prime and* $j_F(y) \notin \mathbb{G}_F p)] \Rightarrow [x \in p_F]$. ■

2.8. Corollary. *If the ring* R *is commutative (or, more generally, all the left ideals in* R *are two-sided), then, for every radical filter* F, *the map* $m \longmapsto \mathbb{G}_F m$ *determines a bijection of the set* $SpecR - F$ *onto the set of prime ideals* p' *of* $\mathbb{G}_F R$ *such that* $j^{-1}_{F,R}(p') \notin F$.

Proof. By Proposition 2.6, $p' = \mathbb{G}_F p$, where $p = j^{-1}_{F,R}(p')$. By Lemma 2.7, the ideal p is prime. The rest follows from Proposition 2.5. ■

2.9. Remark. In general situation, we cannot maintain that

a) the ideal $\mathbb{G}_F p$ is completely prime if p is completely prime;

b) the ideal $p_F = j^{-1}_{F,R}(\mathbb{G}_F p)$ belongs to $Spec_l R$ if $\mathbb{G}_F p$ belongs to the left spectrum of $\mathbb{G}_F R$.

However, the last assertion becomes true if we add the following condition:

(#) p *is a maximal with respect to* $\leq$ *element of the set*

$$\{(p{:}x) \mid x \in R,\ (p{:}x) \notin F\}.$$

In fact, by condition (#), $(p{:}x) \in F$ for any x such that $(p{:}x) \in \langle p\rangle$. This means that $p \in I^{\wedge}_l R$ (cf. 1.15); hence $p^{\wedge} := \{r \in R \mid (p{:}r) \in \langle p\rangle\}$ is a left ideal from $Spec_l R$.

On the other hand, $p^{\wedge} = \{r \in R \mid (p{:}r) \in F\}$; and the right-hand set coincides with $j^{-1}_{F,R}(\mathbb{G}_F p)$. ■

2.10. Localization of maximal left ideals. Here we have the following

2.10.1. Lemma. *Let* F *be a radical filter. If* μ *is a maximal left ideal in the ring* R, *and* $\mu \notin F$, *then* $\mathbb{G}_F \mu$ *is a maximal left ideal in the ring* $\mathbb{G}_F R$.

Proof. Let ν be a proper left ideal of $\mathbb{G}_F R$ which contains the ideal $\mathbb{G}_F \mu$. Then $j^{-1}_{F,R}(\nu)$ is a proper left ideal in R. Since $\mu \subseteq j^{-1}_{F,R}(\nu)$ and μ is maximal, $\mu = j^{-1}_{F,R}(\nu)$. Thus we have:

$$\mathbb{G}_F \mu \subseteq \nu \subseteq \mathbb{G}_F(j^{-1}_{F,R}(\nu)) = \mathbb{G}_F \mu;$$

i.e. $\nu = \mathbb{G}_F \mu$. ∎

3. Morphisms of Left Spectra.

The only one of the major 'commutative' properties of the spectrum which fails to have a straightforward noncommutative analogue is the functoriality with respect to arbitrary ring morphisms: for a generic associative ring morphism, $\phi: A \longrightarrow B$, ' and a generic $p \in Spec_l B$, the left ideal $\phi^{-1}(p)$ is not necessarily an element of $Spec_l A$.

The main goal of this section is to single out some important for applications classes (subcategories) of ring morphisms that preserve the left spectrum.

3.1. The category *LRings*. Let *LRings* denote the class of all ring morphisms, $\phi: R \longrightarrow R'$, such that

(L) If $p' \in Spec_l R'$, *and* m *is a left ideal in* R' *such that* $m \leq p'$, *then* $\phi^{-1}(m) \leq \phi^{-1}(p')$.

Or, equivalently,

If $p' \in Spec_l R'$, $m \in I_l R'$ *and* $\phi^{-1}(m) \in <\phi^{-1}(p')>$, *then the ideal* m *belongs to* $<p'>$.

3.1.1. Proposition. *Let* $\phi: R \longrightarrow R'$ *be a morphism from LRings. Then the map* $\phi: m \longrightarrow \phi^{-1}(m)$ *induces the map*

$$\phi_l: Spec_l R' \longrightarrow Spec_l R,$$

which is continuous with respect to the topologies τ, τ_* *and* $\tau^\wedge$ *(cf. 1.10).*

Proof. 1) If $p' \in Spec_l R'$ then, for any $x \in P(R)$, we have:

$$[(\phi^{-1}(p'):x) \in <p'>] \Rightarrow [(p':\phi(x)) \in <p'>] \Rightarrow$$

$$[p' \text{ contains } \phi(x)] \Leftrightarrow [\phi^{-1}(p') \text{ contains } x].$$

Therefore, since $x \in P(R)$ is arbitrary, $\phi^{-1}(p') \in Spec_l R$.

2) *The map* $\phi_l: Spec_l R' \longrightarrow Spec_l R$ *is continuous with respect to the topology* τ.

In fact, let W be a closed subset of $(Spec_l R, \tau)$; i.e $W = \{p' \in Spec_l R \mid p \leq p' \text{ for some } p \in W\}$. Suppose that $p' \in Spec_l R'$, and $'p \leq p'$ for some $'p$ from the preimage W' (with respect to ϕ_l) of the set W. Since ϕ is a morphism from *LRings*, $\phi^{-1}('p) \leq \phi^{-1}(p')$. Therefore, thanks to the closedness of the set W, the inclusion $\phi^{-1}('p) \in W$ implies that $\phi^{-1}(p')$ belongs to W; i.e. $p' \in W$.

3) Clearly the same argument shows that

The map ϕ_l*:* $Spec_l R' \longrightarrow Spec_l R'$ *is continuous with respect to the topology* τ_*.

4) It remains to show that

the map ϕ_l *is continuous with respect to the topology* $\tau^\wedge$.

More exactly, the preimage of the closed subset $V_l(\alpha)$ of the space $(Spec_l R, \tau^\wedge)$, $\alpha \in IR$, coincides with $V_l(\alpha\phi)$, where $\alpha\phi$ is the two-sided ideal in the ring R' generated by $\phi(\alpha)$.

In fact,

$$\phi_l^{-1}(V_l(\alpha)) = \{p \in Spec_l R' \mid \phi^{-1}(p') \text{ contains } \alpha\} = \{p' \in R' \mid p' \text{ contains } R'\phi(\alpha)\}.$$

Since ϕ is a morphism of the category *LRings*, the following implications hold:

[$\phi^{-1}(p)$ *contains* α, $p \le p'$, $p' \in Spec_l R'$*]* $\Rightarrow$ *[*$\alpha \le \phi^{-1}(p')$*]* $\Leftrightarrow$ *[*$\phi^{-1}(p')$ *contains* α*]* $\Leftrightarrow$ *[*p' *contains* $R'\phi(\alpha)$*]*.

In particular, since $p \in Spec_l R'$, we have:

[$\phi^{-1}(p) \supseteq \alpha$*]* $\Rightarrow$ *[*$(p:z) \supseteq R'\phi(\alpha)$ *for all* $z \in P(R)$*]* $\Rightarrow$ *[the two-sided ideal* $\alpha\phi$ *generated by* $\phi(\alpha)$ *is contained in* p*]*.

In other words, the preimage of $V_l(\alpha)$ is $V_l(\alpha\phi)$. ■

3.1.2. Corollary. *LRings is a subcategory of Rings.*

3.2. Left normal morphisms. Denote by L^-*Rings* the subcategory of the category *Rings* formed by all the ring morphisms $\phi: R \longrightarrow R'$ such that the map ϕ^{-1} is a morphism of preordered sets

$$(I_l R', \le) \longrightarrow (I_l R, \le).$$

Clearly L^-*Rings* is a subcategory of the category *LRings.* According to **Lemma** 3.1.1, all ring epimorphisms belong to the subcategory L^-*Rings*.

Now we are going to give much more subtle "estimate from below" of this category.

For an arbitrary ring morphism $\phi: R \longrightarrow R'$, set

$$N_l(\phi) := \{z \in R' : \phi(x)z \in R'\phi(x) \text{ for any } x \in R\}.$$

It is easy to see that $N_l(\phi)$ is a subring of the ring R'.

3.2.1. Definition. A ring morphism $\phi: R \longrightarrow R'$ will be called *left normal* if the subrings $\phi(R)$ and $N_l(\phi)$ generate R'.

3.2.2. Lemma. *The class* N_lRings *of left normal morphisms forms a subcategory of the category* $Rings$.

Proof is left to the reader. ■

3.2.3. Proposition. *The category* N_lRings *is a subcategory of* L^-Rings.

Proof. Let $\phi:R \longrightarrow R'$ be an arrow from N_lRings; and n, m be left ideals in the ring R' such that m contains $(n:y)$ for some $y \in P(R')$. We have to show that there exists a finite subset w of elements of the ring R such that $(\phi^{-1}(n):w)$ is a subset of the ideal $\phi^{-1}(m)$.

1) It follows from the definition of the subring $N_l(\phi)$ that, for any left ideal m' in the ring R' and for any element z from $N_l(\phi)$, the ideal $\phi^{-1}((m':z))$ contains $\phi^{-1}(m')$.

2) Suppose that a Z-submodule y is generated (over Z) by an element u of R', and consider different possibilities.

a) If $u \in N_l(\phi)$ then 1) implies that $\phi^{-1}(n) \subseteq \phi^{-1}(m)$.

b) If $u = \phi(x)z$ for some $z \in N_l(\phi)$ and $x \in R$, then

$$(n:u)=((n:z):\phi(x)).$$

Therefore

$$\phi^{-1}((n:u)) = \phi^{-1}(((n:z):\phi(x)))=(\phi^{-1}((n:z)):x) \supseteq (\phi^{-1}(n):x).$$

c) If $u = z\phi(x)$, where $z \in N_l(\phi)$, then

$$\phi^{-1}((n:u)) = \phi^{-1}(((n:\phi(x)):z)) \supseteq \phi^{-1}((n:\phi(x))) = (\phi^{-1}(n):x).$$

Thus, in both cases, *b)* and *c)*, $(\phi^{-1}(n):x) \subseteq \phi^{-1}(m)$.

d) Applying the standart induction to the situations *b)* and *c)*, one can easily check that if u is the product of several elements of the form $\phi(x_j)$, $0<j\leq k$, by elements z_i, $0<i<r$, from $N_l(\phi)$ (the factors are aranged in an arbitrary order), then the ideal $\phi^{-1}((n:u))$ contains the ideal $(\phi(n):x_{j_1}...x_{j_k})$, where $j_1,...,j_k$ are numbers of factors in the order of the appearence of $\phi(x_j)$ in the expression of u (from the left to the right).

e) Consider now the general case. Since ϕ is a morphism from N_lRings, every element $u \in R'$ is of the form

$$u_1 + u_2 + ... + u_s,$$

where each summand is the product of elements from $\phi(R)$ by elements from

$N_l(\phi)$. Therefore, for each summand u_i, there exists, according to d), an element $x_i \in R$ such that the ideal $\phi^{-1}((n:u_i))$ contains the ideal $(\phi^{-1}(n):x_i)$. Thus, we have:

$$\phi^{-1}((n:u)) \supseteq \bigcap_{1\le i\le s} \{\phi^{-1}((n:u_i))\} \supseteq \bigcap_{1\le i\le s} \{(\phi^{-1}(n):x_i)\} \supseteq (\phi^{-1}(n):\mathbf{x}),$$

where $\mathbf{x} := \{x_i: 1\le i\le s\}$.

3) This implies that, for any finite family u of generators of the Z-module y, there exists a finite subset x of elements from R such that $\phi^{-1}((n:u))$ contains $(\phi^{-1}(n):x)$. ■

3.3. Central extensions. A ring morphism $\phi: R \longrightarrow R'$ is called a *central extension* if its centralizer $Z(\phi):=\{z \in R': \phi(x)z=z\phi(x)$ for any $x \in R\}$ and its image, $\phi(R)$, generate the ring R'.

Clearly central extensions form a subcategory of the category $N_l Rings$. In particular, if ϕ is a central extension then the map

$$\phi^{-1}: m \longmapsto \phi^{-1}(m)$$

induces a continuous map

$$(Spec_l R',\tau') \longrightarrow (Spec_l R,\tau'),$$

where the topology τ' is either τ or $\tau^\wedge$ (cf. Proposition 3.1.1).

3.3.1. Lemma. *Let $\phi: R \longrightarrow R'$ be a central extension. Then the map ϕ^{-1} determines a continuous map $^a\phi: SpecR' \longrightarrow SpecR$.*

Proof. For any $p' \in SpecR$ and a pair α, β of two-sided ideals in the ring R, there are the following implications:
[$\phi^{-1}(p')$ contains $\alpha\beta$] $\Leftrightarrow$ [p' contains $\phi(\alpha)\phi(\beta)$] $\Rightarrow$ [p' contains $\phi(\alpha)\phi(\beta) + \phi(\alpha)\phi(R)Z(\phi)\phi(\beta) = (\phi(\alpha) + \phi(\alpha)R')\phi(\beta)$] $\Rightarrow$ [p' contains either $\phi(\alpha)$ or $\phi(\beta)$].

The verification of the identity $^a\phi^{-1}(V(\alpha)) = V(\alpha\phi)$, where $\alpha\phi$ is the two-sided ideal in the ring R' generated by $\phi(\alpha)$, is left to the reader. ■

3.4. A nonabelian functoriality. Most of ring morphisms are not compatible with the left spectrum in the way morphisms of *LRings* are. It is possible, however, to establish a weaker sort of functoriality for arbitrary ring morphisms.

First note that we are interested not in the left spectrum of a ring R, but in the quotient of $Spec_l R$ by the equivalence relation induced by the preorder $\le$. Denote this quotient ordered set by $\mathbf{Spec}_l R$. According to Lemma 1.4.1, the set $\mathbf{Spec}_l R$ can be canonically realized as the set $\{<p> \mid p \in Spec_l R\}$ with the order given by the inverse inclusion, $\supseteq$.

Let $\phi: A \longrightarrow B$ be a ring morphism. We can try to assign to ϕ a map from $\mathbf{Spec}_l B$ to $\mathbf{Spec}_l A$ as follows. Take $p \in Spec_l B$, set $p' := \phi^{-1} p$, and consider the set $\Omega_{p'} = \{(p':a) \mid a \in A - p'\}$ of left ideals in the ring A.

3.4.1. Lemma. *Suppose that a left ideal* ν *in the ring* R *is such that the set* $\Omega_\nu = \{(\nu:a) \mid a \in A - \nu\}$ *has a maximal element with respect to the preorder* $\leq$. *Then this maximal element belongs to* $Spec_l R$.

Proof. Let a be such an element of $A - \nu$ that $(\nu:a)$ is a maximal element of Ω_ν. This means that if $(\nu:a) \leq (\nu:a')$ for some $a' \in A - \nu$, then $(\nu:a) \approx (\nu:a')$.

Suppose that $x \in A - (\nu:a)$; or, equivalently, $xa \notin \nu$. Then $(\nu:xa) \leq (\nu:a)$. But, since $(\nu:xa) = ((\nu:a):x)$ and $(\nu:a) \leq ((\nu:a):x)$, this implies that $((\nu:a):x) \approx (\nu:a)$ for any $x \in A - (\nu:a)$; i.e. $(\nu:a) \in Spec_l A$. ■

Return now to our ideal $p' = \phi^{-1} p$. Suppose that the set

$$\Omega_{p'} = \{(p':a) : a \in A - p'\}$$

has a maximal with respect to $\leq$ element, say $(p':a)$. Since $(p':a) := (\phi^{-1} p:a) = \phi^{-1}(p:\phi(a))$, $p \in Spec_l B$ and $\phi(a) \notin p$, the left ideal $(p:\phi(a))$ is equivalent to the ideal p.

Denote by ${}^a\phi$ the map which assigns to any class $\langle p \rangle$ of elements of $Spec_l B$ the set

$$\{\langle \phi^{-1} p' \rangle \mid p' \approx p \text{ and } \phi^{-1} p' \in Spec_l A\}.$$

Clearly the map ${}^a\phi$ is well defined, and, if the preordered set $(I_l A, \leq)$ is noetherian (for instance, A is a commutative, noetherian ring), the set ${}^a\phi(\langle p \rangle)$ is nonempty for any $\langle p \rangle$.

If the morphism $\phi : A \longrightarrow B$ is from *LRings*, then ${}^a\phi(\langle p \rangle)$ equals $\{\langle \phi^{-1}(p) \rangle\}$; i.e. ${}^a\phi$ coincides in this case with the preimage map. Note that if the ring B is commutative, then any morphism to B is from *LRings*.

4. LEFT SPECTRUM AND LEVITZKI RADICAL.

4.0. Left radical. Fix an associative ring R. For any closed in the topology τ (cf. 1.10.1) subset W of $Spec_l R$, denote by $rad_l(W)$ the intersection of all ideals from W. Since $p \in W$ implies that the ideal $(p:x)$ is in W for every $x \in R - p$, we have:

$$rad_l(W) = \bigcap_{p \in W} \Big(\bigcap_{x \in R} (p:x) \Big) = \bigcap_{p \in W} (p:R).$$

In particular, $rad_l(W)$ is a two-sided ideal. We call the ideal $rad_l(W)$ *the radical of the set* W.

If $W = V_l(m) := \{p \in Spec_l R \mid m \le p\}$ for some left ideal m, we shall write $rad_l(R|m)$ instead of $rad_l(V_l(m))$ and call the ideal $rad_l(R|m)$ *the left radical of* m. Finally, we shall write $rad_l(R)$ instead of $rad_l(R|0) = rad_l(Spec_l R)$ and call this ideal *the left radical of the ring* R.

The goal of this section is to prove that $rad_l(R)$ coincides with the largest locally nilpotent ideal of the ring R.

4.1. ***l*-Systems.** A subset S of $P(R)$ will be called an *l-system* if, for any $t \in S$, there exists $a \in P(R)$ such that Sat is a subset of S; i.e. $t'at \in S$ for any $t' \in S$. Obviously, any *multiplicative system* S of $P(R)$ (i.e. $st \in S$ for any $s, t \in S$) is an l-system. Another series of examples of l-systems is provided with the following lemma.

4.2. Lemma. *A left ideal* p *of the ring* R *belongs to* $Spec_l R$ *if and only if the set* $S(p) := P(R) - P(p)$ *is an l-system.*

Proof. By definition, p belongs to $Spec_l R$ if and only if $(p:t) \le p$ for any $t \in S(p)$. This means exactly that $(p:at) = ((p:t):a)$ is a subset of the ideal p for some $a \in P(R)$. Clearly p contains $(p:at)$ if and only if $S(p)$ contains the set Sat. ∎

4.3. Proposition. *If* S *is an l-system of the ring* R *then the set of left ideals* $F_S := \{n \in I_l R : P((n:x)) \cap S$ *is non-empty for any* $x \in P(R)\}$ *is a radical filter.*

Proof. Let $m \in F_S$ and $n \in F_S \circledcirc \{m\}$; i.e. the intersection $P((n:y)) \cap S$ is non-empty for any $y \in P(m)$ and $P((m:x)) \cap S$ is non-empty for any $x \in P(R)$. We need to show that $P((n:x)) \cap S$ is non-empty for every $x \in P(R)$. Let t be an element of S such that $tx \in P(m)$, and let a be an element of $P(R)$ such that Sat is contained in S. Since $atx \in P(m)$, then there exists

$t' \in S$ such that $t'atx$ is a subset of the ideal n; i.e. $t'at \in P((n:x))$. But, since S is an l-system, $t'at \in S$.■

4.4. Levitzki radical. A ring R' is called *locally nilpotent* if every finite subset X of its elements generates a nilpotent subring. This means that there exists $N = N(X) \geq 1$ such that the product of any N elements of X is zero. The following facts are well known (cf. [J], Ch.8, Section 3):

4.4.1. Theorem. *1) A two-sided ideal generated by a left or right locally nilpotent ideal is locally nilpotent.*

2) The sum $L(R)$ of all the locally nilpotent ideals of R is a two-sided locally nilpotent ideal.

The (obviously) largest locally nilpotent ideal $L(R)$ is called *Levitzki radical* of the ring R.

4.5. Proposition. *The following properties of a left ideal m of the ring R are equivalent:*

(a) Any l-system S such that $S \cap P(m)$ is non-empty contains $\{0\}$.

(b) Any multiplicative subset S of $P(R)$ such that the intersection of S and $P(m)$ is non-empty contains $\{0\}$.

(c) the ideal m is locally nilpotent.

Proof. (a) $\Rightarrow$ (b), since any multiplicative subset of $P(R)$ is an l-system.

(b) $\Rightarrow$ *(c)*. Obviously, the ideal m is locally nilpotent if and only if for any $t \in P(R)$, there exists $N = N(t)$ such that $t^N=\{0\}$; i.e. $\{0\}$ belongs to the multiplicative system generated by t.

(c) $\Rightarrow$ *(a)*. Let S be an l-system, and let $t \in S \cap P(m)$. By definition, there exists $a \in P(R)$ such that Sat is a subset of S. In particular, tat, $(tat)at$, ..., $t(at)^k$ are elements of S for all $k \geq 1$. Since at belongs to $P(m)$, there exists (by hypothesis) $i \geq 1$ such that $(at)^i = \{0\}$. Therefore $\{0\} = t(at)^i$ belongs to S. ■

4.6. Corollary. *The intersection $rad_l(R)$ of all ideals of the left spectrum of an arbitrary associative ring R contains the Levitzki radical of this ring.*

Proof. Let m be a left ideal in R such that the set $m\text{-}rad_l(R)$ is not empty. This means that $m\text{-}p$ is non-empty for some $p \in Spec_l R$. If m were locally nilpotent then this would imply (by Proposition 10.5 and Lemma 10.2) that the set $S(p) = P(R) - P(p)$ contains $\{0\}$. But this is impossible. Hence we have the following implication:

If a left ideal m of the ring R is not contained in $rad_l(R)$, then m is not contained in the Levitzki radical $L(R)$; i.e. $L(R)$ is a subset of $rad_l(R)$. ∎

Thus, we have improved the estimate of the left radical from the low; i.e. from $B(R) \subseteq rad_l(R) \subseteq J(R)$ we have passed to

$$L(R) \subseteq rad_l(R) \subseteq J(R).$$

Our next step is to improve the estimate from the above.

4.7. The upper nil-radical. A ring is called a *nil-ring* if every its element is nilpotent. An ideal is called a *nil-ideal* if it is a nil-ring.

The following fact is well known ([J], Ch.8, 1):

4.7.1. Theorem. *The sum $K(R)$ of all two-sided nil-ideals of the ring R is a nil-ideal.*

Clearly the ideal $K(R)$ is the largest two-sided nil-ideal of the ring R. It is called the *upper nil-radical* or the *Kethe radical* of the ring R.

4.8. Proposition. *The left radical of an arbitrary associative ring is contained in its upper nil-radical.*

Proof. Obviously, it suffices to show that $rad_l(R|K(R)) = K(R)$ for any associative ring R (cf. 4.0). Let α be a two-sided ideal of R. There exists a natural isomorphism $rad_l(R/\alpha) \cong rad_l(R|\alpha)/\alpha$ which follows from the bijectivity of the map $V_l(\alpha) \longrightarrow Spec_l R/\alpha, \ p \longmapsto p/\alpha$.

Therefore $rad_l(R|\alpha) = \alpha$ if and only if $rad_l(R/\alpha) = 0$. In particular, $rad_l(R|K(R)) = K(R)$ if and only if $rad_l(R/K(R) = 0$.

Thus, we should show that the left radical of the ring $R' = R/K(R)$ is trivial. For this purpose, we shall use the following theorem of Amitsur ([16], 6.1.1):

4.9. Theorem. *If the ring R' has no non-zero two-sided nil-ideals, then the polynomial ring $R'[t]$ is semiprimitive (i.e. its Jacobson radical is zero).*

Since, for any ring A, we have $rad_l(A) \subseteq J(A)$, Theorem 4.9 implies that

$rad_l(R'[t]) = 0$. Here, as above, $R'=R/K(R)$.

Now notice that the natural embedding $R \longrightarrow R[t]$ is a central extension; hence, the map $\mu \longmapsto \mu \cap R$ is a preordered sets morphism

$$(I_l R[t], \le) \longrightarrow (I_l R, \le)$$

(cf. 3.3). In particular, the restriction map $\mu \longmapsto \mu \cap R$ sends $Spec_l R[t]$ into $Spec_l R$. Therefore $rad_l(R') \subset rad_l(R'[t]) \cap R' = 0$. ∎

4.10. Left radical and Levitzki radical. It remains to perform the last step: to pass from the estimate $L(R) \subseteq rad_l(R) \subseteq K(R)$ to the equality $rad_l(R) = L(R)$. In order to do it, consider the polynomial ring

$$R_\infty = R[t_1, t_2, ...]$$

in infinitely many non-commuting indeterminates.

4.10.1. Lemma. *If R has no non-zero locally nilpotent ideals then R_∞ has no non-zero nil-ideals; i.e. $K(R_\infty) = 0$.*

Proof. Denote by N_ω the set of all the finite ordered sets of positive integers. For every $\mathbf{i} = (i_1,...,i_n) \in N_\omega$, denote by $t^{\mathbf{i}}$ the product $t_{i_1} \cdot t_{i_2} \cdot ... \cdot t_{i_n}$.

Suppose that $K(R_\infty)$ is non-zero; and let $f(t) = \sum_{\mathbf{i} \in E} a_{\mathbf{i}} t^{\mathbf{i}}$ be a generic non-zero element of $K(R_\infty)$.

a) The subring of the ring R generated by the set of coefficients $\{a_{\mathbf{i}} : \mathbf{i} \in E\}$ is nilpotent.

In fact, by hypothesis, $xf(t)$ is a nilpotent element of R_∞ for each $x \in R_\infty$. Now, take $x = t_k$, where k is an index which is not encountered among the elements of $\mathbf{i}$. Since

$$\Big(\sum_{\mathbf{i} \in E} a_{\mathbf{i}} t_k t^{\mathbf{i}}\Big)^n = \sum_{\substack{\mathbf{i}_\nu \in E \\ 1 \le \nu \le n}} a_{\mathbf{i}_1} \cdot ... \cdot a_{\mathbf{i}_n} t_k t^{\mathbf{i}_1} t_k ... t_k t^{\mathbf{i}_n}$$

the condition *the n-th power of the element $t_k f(t)$ vanishes* means exactly that

$$a_{\mathbf{i}_1} \cdot ... \cdot a_{\mathbf{i}_n} = 0 \quad \text{for every} \quad (\mathbf{i}_1,...,\mathbf{i}_n) \in E \times ... \times E.$$

b) Now we shall show that the left ideal, generated by the set of coefficients $\{a_{\mathbf{i}} \mid \mathbf{i} \in E\}$, is locally nilpotent.

In other words, we should check that, for any finite set $\{b_{\mathbf{i},\nu} \mid (\mathbf{i},\nu) \in E \times \Omega\}$ of elements of the ring R, the subring generated by the set $\{a_{\mathbf{i}},$

$b_{\mathbf{i},\nu}a_{\mathbf{i}} \mid \nu \in \Omega,\ \mathbf{i} \in E\}$ is nilpotent.

Select positive integers k, $k_{\mathbf{i}}$, $\mathbf{i} \in E$, such that

1) $k_{\mathbf{i}} \neq k$ for every $\mathbf{i} \in E$, and $k_{\mathbf{i}} = k_{\mathbf{i}'}$ iff $\mathbf{i}$ coincides with $\mathbf{i}'$;

2) neither k nor any of $k_{\mathbf{i}}$ is encountered among the indeces of the sets $\mathbf{i}$, $\mathbf{i} \in E$.

Consider the linear form

$$g(t) = t_k + \sum_{(\mathbf{i},\nu) \in E \times \Omega} b_{\mathbf{i},\nu} t_{k_{\mathbf{i},\nu}}$$

By hypothesis, $g(t) \cdot f(t)$ is an element of the nil-radical $K(R_\infty)$. As it has been just shown, this implies that the set of coefficients of the polynomial

$$g(t) \cdot f(t) = \sum_{\mathbf{i} \in E} a_{\mathbf{i}} t_k t^{\mathbf{i}} + \sum_{(\mathbf{i},\nu) \in E \times \Omega} b_{\mathbf{i},\nu} a_{\mathbf{i}'} t_{k_{\mathbf{i},\nu}} t^{\mathbf{i}'}$$

generates a nilpotent subring of the ring R. Obviously, if a set of elements of R generates a nilpotent subring, then so does any of its subsets. In particular, $\{a_{\mathbf{i}},\ b_{\mathbf{i},\nu}a_{\mathbf{i}} \mid (\mathbf{i},\nu) \in E \times \Omega\}$ generates a nilpotent subring. ∎

4.10.2. Theorem. *The left radical of any associative ring R coincides with its Levitzki radical:* $rad_l(R) = L(R)$.

Proof. Since we have already established that $L(R) \subseteq rad_l(R)$, it remains to verify the inverse inclusion. Taking the quotient of R modulo $L(R)$, we reduce the desired assertion to the following one:

If R has no non-zero locally nilpotent ideals, then $rad_l(R) = \{0\}$.

Proof of this assertion follows the scenario of the proof of Proposition 4.8 with the ring $R[t]$ being replaced by the ring $R_\infty = R[t_1, t_2, \ldots]$.

The natural embedding $R \longrightarrow R_\infty$ is a central extension. Therefore the map $p \longmapsto p \cap R$ sends ideals from $Spec_l R$ into the ideals from $Spec_l R_\infty$. Hence the inclusion

$$rad_l(R) \subseteq R \cap rad_l(R_\infty)$$

holds. But, according to Proposition 4.8, $rad_l(R_\infty) \subseteq K(R_\infty)$; and, as Lemma 4.10.1 claims, $K(R_\infty) = \{0\}$ if $L(R) = \{0\}$. Therefore $rad_l(R_\infty) = \{0\}$, and $rad_l(R) = R \cap rad_l(R_\infty) = \{0\}$. ∎

5. The Levitzki Spectrum and the Left Spectrum.

5.1. Levitzki spectrum *LSpecR* of the ring R is the set of all the prime ideals p of the ring R such that the quotient ring R/p has no locally nilpotent ideals. The topology on *LSpecR* is induced by the Zariski topology on the prime spectrum $SpecR$.

5.2. Lemma. *(a) For any* $p \in Spec_l R$, *the two-sided ideal* $(p:R)$ *belongs to the Levitzki spectrum of* R.

(b) For any two-sided ideal α *in* R, *its left radical,* $rad_l(R|\alpha)$, *coincides with the preimage of the Levitzki radical of the quotient ring* R/α.

(c) In particular, a prime ideal $\mathbf{p}$ *belongs to* $LSpecR$ *if and only if* $\mathbf{p}$ *is equal to* $rad_l(R|\mathbf{p})$.

Proof. (a) Fix a $p \in Spec_l R$. By Lemma 1.8.1, the two-sided ideal $(p:R)$ is prime. Obviously, $(p:R)$ is the intersection of all the ideals $(p:x)$, where x runs through the set $R\text{-}p$. The left ideal $(p:x)/(p:R)$ belongs to $Spec_l R/(p:R)$. Therefore, by Theorem 4.10.2, *the ring* $R/(p:R)$ *has no locally nilpotent ideals.*

(b) By Theorem 4.10.2, $rad_l(R/\alpha)$ coincides with Levitzki radical of the ring R/α, and $rad_l(R/\alpha) = rad_l(R|\alpha)/\alpha$.

The assertion *(c)* is a special case of the assertion *(b)*. ■

Recall that a topological space X is called *sober* if every nonempty closed irreducible subset of X has a unique generic point.

5.3. Theorem. *(a) The map* $p \longmapsto (p:R)$ *is a quasi-homeomorphism*

$$(Spec_l R, \tau) \longrightarrow LSpecR.$$

(b) The space $LSpecR$ *sober.*

Proof. (a) 1) It follows from the assertion *a)* of Lemma 4.4 that the map

$$p \longmapsto (p:R)$$

sends the left spectrum of the ring R into its Levitzki spectrum.

2) *The map, which assigns to a subset* V *of* $Spec_l R$ *its radical - the intersection of all the ideals from* V *- induces a bijection of the set of closed subsets of the space* $(Spec_l R, \tau)$ *onto the set of all the two-sided ideals* α *such that* $\alpha = rad_l(\alpha)$.

3) Similarly, *the map, which assigns to a subset* V *of the Levitzki spectrum* $LSpecR$ *the intersection of all the ideals from* V *induces a bijection of the set of closed subsets of the space* $LSpecR$ *onto the set of all the two-sided ideals coinciding with their Levitzki radical.*

4) But, according to the assertion *b)* of Lemma 4.4, these two kinds of two-sided ideals coincide. Hence the map

$$q: Spec_l R \longrightarrow LSpecR, \quad p \longmapsto (p:R)$$

is a quasi-homeomorphism.

b) Note that

the closed subset X *of the space* *LSpecR* *is irreducible if and only if its radical* $r(X)$ *- the intersection of all ideals of* X *- is a prime ideal.*

In fact, let α and β be two-sided ideals such that $\alpha\beta \subseteq r(X)$. Suppose that α is not contained if $r(X)$; or, equivalently, $r(X)$ is not a subset of the closed subset $LV(\alpha):=\{p \in LSpecR \mid \alpha \subseteq p\}$. Therefore, since

$$X \subseteq LV(\alpha\beta) = LV(\alpha) \cup LV(\beta),$$

and, by assumption, the set X is irreducible, that $LV(\beta)$ contains X; or, equivalently, $\beta \subseteq r(X)$. ■

5.4. Remark. Now it is an appropriate moment to compare the left geometry with the right one. First of all, it follows from Theorem 4.5 that

$$rad_l(R|\alpha) = rad_r(R|\alpha) \text{ for any two-sided ideal } \alpha,$$

where the right radical $rad_r(R|\alpha)$ is the intersection of all the ideals of the right spectrum, $Spec_r R$, of the ring R.

The topological space $(Spec_l R, \tau^\wedge)$ is equivalent, from the sheaf-theoretic point of view, to its right analogue $(Spec_r R, \tau^\wedge)$, since there are canonical quasi-homeomorphisms

$$(Spec_l R, \tau^\wedge) \longrightarrow LSpecR \longleftarrow (Spec_r R, \tau^\wedge)$$

Of course, the categories of quasi-coherent (pre)sheaves on $(Spec_l R, \tau^\wedge)$ and $(Spec_r R, \tau^\wedge)$ (which are introduced in Section 6) can differ considerably. ■

5.5. Lemma. *Let* $\mathfrak{I}$ *be a directed (with respect to* $\subseteq$*) family of two-sided ideals which coincide with their Levitzki radical. Then the supremum of the family* $\mathfrak{I}$, $sup(\mathfrak{I})$, *also has this property.*

Proof. Let $x \in P(R)$ be such that x^n is a subobject of $sup(\mathfrak{I})$ for some $n > 0$. Since x^n is a finitely generated Z-module, and the family $\mathfrak{I}$ is directed, $x^n \subset \alpha$ for some ideal $\alpha \in \mathfrak{I}$. By hypothesis, α coincides with its Levitzki radical. Hence $x \subset \alpha$. ■

5.6. Theorem. *An open subset* U *of the space* $(Spec_l R, \tau^\wedge)$ *is quasi-compact if and only if* $U = U_l(\alpha)$ *for some finitely generated two-sided ideal* α.

Proof. 1) Let α be a finitely generated two-sided ideal; and let $\{ U_i \mid i \in I\}$ be an infinite cover of the open set $U_l(\alpha)$. Denote by **I** the directed (with respect to inclusion) set of finite subsets of I; and let $U^{\mathbf{i}}$ be the union of the family of sets $\{ U_i \mid i \in \mathbf{i} \}$ for each $\mathbf{i} \in \mathbf{I}$. We have to prove that $U = U^{\mathbf{i}}$ *for some* $\mathbf{i} \in \mathbf{I}$.

Denote by $\alpha_{\mathbf{i}}$ the *radical* of the complement to $U^{\mathbf{i}}$ closed subset; i.e. $\alpha_{\mathbf{i}}$ is the intersection of all the ideals from $Spec_l R - U^{\mathbf{i}}$. The ideals $\alpha_{\mathbf{i}}$ coincide with their left radicals : $\alpha_{\mathbf{i}} = rad_l(R \mid \alpha_{\mathbf{i}})$. By Theorem 4.10.2, this means exactly that each ideal $\alpha_{\mathbf{i}}$ coincides with its Levitzki radical. According to Lemma 4.7, the supremum β of the family $\{ \alpha_{\mathbf{i}} : \mathbf{i} \in \mathbf{I} \}$ also equals to its Levitzki (or, equivalently, left) radical: $\beta = rad_l(R \mid \beta)$. Therefore β is the largest two-sided ideal with the property $U_l(\beta) = U$. In particular, since $U = U_l(\alpha)$, the ideal α is contained in β.

By hypothesis, the ideal α is finitely generated (as a two-sided ideal). Therefore, since the family $\{ \alpha_{\mathbf{i}} : \mathbf{i} \in \mathbf{I} \}$ is directed, the inclusion $\alpha \subseteq \beta$ implies that $\alpha \subseteq \alpha_{\mathbf{i}}$ for some $\mathbf{i}$. Thus, $U = U_l(\alpha_{\mathbf{i}})$.

2) Let now $U_l(\alpha')$ be a quasi-compact open subset of $Spec_l R$. The ideal α' can be represented as a supremum of a directed family $\mathcal{F}$ of finitely generated two-sided ideals. The quasi-compactness of $U_l(\alpha')$ implies that $U_l(\alpha') = U_l(\alpha)$ for some ideal $\alpha \in \mathcal{F}$. ■

5.7. Corollary. *An open subset of the Levitzki spectrum of a ring* R *is quasi-compact if and only if it coincides with* $LU(\alpha)$ *for some finitely generated ideal* α.

Proof follows immediately from Theorem 4.5. ■

6. STRUCTURE PRESHEAVES. RECONSTRUCTION OF MODULES.

6.0. Structure presheaves. The definition of a radical filter (cf. 0.3) can be reformulated as follows:

a uniform (i.e. with respect to $\leq$) filter F of left ideals of a ring R is *radical* iff the following condition holds:

if $m \in F$ and a left ideal n is such that $(n{:}x) \in F$ for any $x \in P(R)$, then $n \in F$.

This reformulation makes clear that the intersection of an arbitrary family of radical filters is a radical filter. In particular, to any subset V of $Spec_l R$, we can assign a radical filter $\langle V \rangle := \bigcap_{p \in V} \langle p \rangle$.

Fix a topology $\mathfrak{T}$ on $Spec_l R$. Define a *presheaf of modules on* $\mathfrak{X} = (Spec_l R, \mathfrak{T})$ as any functor $F\colon Open\mathfrak{T}^{op} \longrightarrow R\text{-}mod$ such that $F(U)$ is an object of the subcategory $R\text{-}mod/<U>$ for every open set U. Denote by $\mathfrak{Ph}\mathfrak{X}$ (or by $\mathfrak{Ph}(Spec_l R,\mathfrak{T})$) the full subcategory of the category $Funct(Open\mathfrak{T}^{op},R\text{-}mod)$ generated by all presheaves of modules.

There is the *global sections* functor

$$\Gamma\colon \mathfrak{Ph}\mathfrak{X} \longrightarrow R\text{-}mod$$

which sends a presheaf of modules F it value at $Spec_l R$.

The structure presheaf of an R-module M is the presheaf of modules $M^{\sim} = M_{\mathfrak{T}}$ which assigns to every open subset U of the space $\mathfrak{X} = (Spec_l R,\mathfrak{T})$ the R-module $\mathfrak{G}_{<U>}M$.

Clearly the map $M \longmapsto M^{\sim}$ extends to a functor

$$\Delta\colon R\text{-}mod \longrightarrow \mathfrak{Ph}\mathfrak{X}.$$

6.0.1. Proposition. *The global sections functor is left adjoint to the functor* Δ. *The functor* Δ *is fully faithful.*

Proof. Set for convenience $X = Spec_l R$. Fix a presheaf of modules F. For any open subset U of X, the restriction map $F(X) \longrightarrow F(U)$ is uniquely decomposed (since $F(U) \in R\text{-}mod/<U>$) into the adjunction morphism

$$j_{<U>}\colon F(X) \longrightarrow F(X)^{\sim}(U) := \mathfrak{G}_{<U>}F(X)$$

and a morphism

$$\delta_F(U)\colon F(X)^{\sim}(U) \longrightarrow F(U).$$

The set $\delta_F := \{\delta_F(U) \mid U \in Open\mathfrak{T}\}$ is a functor morphism from $\Delta\circ\Gamma$ to $Id_{\mathfrak{Ph}\mathfrak{X}}$.

For every R-module M, we have an isomorphism (which can be chosen to be identical) $\varepsilon(M)\colon M \longrightarrow M^{\sim}(X)$. The set $\varepsilon := \{\varepsilon(M)\}$ is a functor isomorphism $Id_{R\text{-}mod} \longrightarrow \Gamma\circ\Delta$. One can see that

$$\Gamma\delta\circ\varepsilon\Gamma = id_{\Gamma}, \text{ and } \delta\Delta\circ\Delta\varepsilon = id_{\Delta}$$

which means that δ and ε are adjunction arrows. Since ε is an isomorphism, the functor Δ is fully faithful. ∎

Call a presheaf of modules *quasi-coherent* if it is isomorphic to a structure presheaf of some R-module. Denote the category of quasi-coherent presheaves on $\mathfrak{X}$ by $\mathfrak{Qh}\mathfrak{X}$.

The following assertion is a corollary of Proposition 6.0.1.

6.0.2. Proposition. *The functor* $\Delta\colon R\text{-}mod \longrightarrow \mathfrak{Ph}\mathfrak{X}$ *induces an equivalence of the category* $R\text{-}mod$ *and the category* $\mathfrak{Qh}\mathfrak{X}$ *of quasi-coherent presheaves on the* *topological space* $\mathfrak{X} = (Spec_l R, \mathfrak{T})$.

6.0.3. Associated sheaves? The next, standart, step is to go from presheaves to associated sheaves. There is no problem to produce a sheafification functor, $\mathcal{S}$, in our setting. Thus, we can assign to each R-module M its *structure sheaf* $M^a = \mathcal{S}(M^{\sim})$. We define a *quasi-coherent sheaf* as a (pre)sheaf which is isomorphic to $\mathcal{S}(M^{\sim})$ for some R-module M.

If the ring R is commutative and $\mathfrak{T}$ is the Zariski topology, the sheafification functor $\mathcal{S}$ induces an equivalence between the category of quasi-coherent presheaves and that of quasi-coherent sheaves. This fact, due to Serre, is one of the corner stones of (commutative) algebraic geometry.

If R is noncommutative, this is, usually, not true. Besides, in the noncommutative case, the Zariski topology might be not the best choice. For example, it is trivial if R is the algebra of differential operators with polynomial coefficients.

Note that the reason for using the sheafification functor is that sheaves, by their nature, could be reconstructed from local data - their values on coverings. Note also that the reconstruction is given by a procedure which works far beyond the limits of algebraic geometry.

There are two possibilities:

(a) Either to try to single out (classes of) modules which can be reconstructed from their structure sheaf.

b) Or, to look after a different reconstruction algorithm which recovers, hopefully, any module from its 'local data'. Up to isomorphism, of course.

The second way is, by many reasons, much more preferrable.

It occurs that there exists a very natural, specific for algebraic geometry (localization) setting procedure which allows to reconstruct quasi-coherent presheaves from local data for any topology.

Now we shall make this claim explicit and prove it.

6.1. The canonical diagram. Let Ω be a family of radical filters, and let F be the intersection of all the filters from Ω. Then the commutative diagrams of functors

$$\begin{array}{ccc} \mathbb{G}_{F'} & \xrightarrow{\mathbb{G}_{F'} j_{F''}} & \mathbb{G}_{F'} \circ \mathbb{G}_{F''} \\ {\scriptstyle j_{F,F'}} \uparrow & j_{F,F''} & \uparrow {\scriptstyle j_{F'} \mathbb{G}_{F''}} \\ \mathbb{G}_F & \longrightarrow & \mathbb{G}_{F''} \end{array} \tag{1}$$

where F', F'' run through Ω, define the diagram

$$\mathbb{G}_F \longrightarrow \prod_{F' \in \Omega} \mathbb{G}_{F'} \rightrightarrows \prod_{F',F'' \in \Omega} \mathbb{G}_{F'} \circ \mathbb{G}_{F''} \tag{2}$$

6.2. Theorem. *Let* Ω *be a family of radical filters, and let* $F = \bigcap_{F' \in \Omega} F'$. *Then*

1) The canonical morphism $\mathbb{G}_F \longrightarrow \prod_{F' \in \Omega} \mathbb{G}_{F'}$ *is a monomorphism.*

2) Suppose that there exists a finite subfamily Ω' *of* Ω *such that* $F = \bigcap_{F' \in \Omega'} F'$. *Then the diagram* (2) *is exact.*

Proof. 1) We have to prove that, for each R-module M, the canonical arrow

$$j: \mathbb{G}_F M \longrightarrow \prod_{F' \in \Omega} \mathbb{G}_{F'} M$$

is a monomorphism.

In fact, let ξ be an element of $Ker j$. This means that, for any $F' \in \Omega$, there exists a left ideal $m(F') \in F'$ such that $m(F')\xi = 0$; i.e. $Ann\xi$ belongs to the intersection, F, of all the filters from Ω. But $\mathbb{G}_F M$ is F-torsion free; hence $\xi = 0$.

2) It is pretty clear that the diagram (2) is exact if the diagram

$$\mathbb{G}_F \longrightarrow \prod_{F' \in \Omega'} \mathbb{G}_{F'} \rightrightarrows \prod_{F',F'' \in \Omega'} \mathbb{G}_{F'} \circ \mathbb{G}_{F''}$$

is exact for some subset Ω' of Ω. Hence we can (and will) assume that $\Omega = \Omega'$ is finite.

Let $\xi_{F'} \in \mathbb{G}_{F'} M$, $F' \in \Omega$, be elements such that , for any F', F'' from Ω, the images of $\xi_{F'}$ and $\xi_{F''}$ under the canonical morphisms

$$\mathbb{G}_{F'} j_{F''}: \mathbb{G}_{F'} M \longrightarrow \mathbb{G}_{F'} \circ \mathbb{G}_{F''} M \quad \text{and} \quad j_{F'} \mathbb{G}_{F''}: \mathbb{G}_{F''} M \longrightarrow \mathbb{G}_{F'} \circ \mathbb{G}_{F''} M$$

coincide.

Fix a filter F' from Ω, and let m' be an ideal from F' such that the morphism $\cdot \xi_{F'}|_{m'}: m' \longrightarrow \mathbb{G}_{F'} M$ of multiplying by $\xi_{F'}$ is a composition of a certain uniquely determined R-module morphism $u': m' \longrightarrow M/F'M$ and the canonical monomorphism

$$M/F'M \longrightarrow \mathbb{G}_{F'} M.$$

Choose, for any $x \in m'$, an element $\eta_x \in M$ such that $j_{F'}(\eta)=x\xi_{F'}$. Then, for any $F'' \in \Omega$, we have:

$$j_{F'}j_{F''}(\eta_x)=j_{F''}j_{F'}(\eta_x)=j_{F''}(x\xi_{F'})=j_{F'}(x\xi_{F''}).$$

The equality $j_{F'}j_{F''}(\eta_x)=j_{F'}(x\xi_{F''})$ means that, for every $F'' \in \Omega$, a left ideal $m_{F''} \in F'$ can be found such that

$$m_{F''}(j_{F''}(\eta_x) - x\xi_{F''})=0 \tag{3}$$

Since Ω is finite by hypothesis, then the left ideal

$$m_{F',x} := \bigcap_{F'' \in \Omega} m_{F''}$$

belongs also to the fllter F'. Therefore we can write (instead of (3)):

$$m_{F',x}(j_{F''}(\eta_x) - x\xi_{F''})=0 \tag{4}$$

Set $\xi := \{\xi_{F'} \mid F' \in \Omega\}$, and denote by $C(\xi)$ the set of all the left ideals n of R such that the morphism

$$\cdot\xi|_n : n \longrightarrow \prod_{F' \in \Omega} \mathbb{G}_{F'}M$$

of multiplication ξ by n factors through the canonical map

$$j: \mathbb{G}_F M \longrightarrow \prod_{F' \in \Omega} \mathbb{G}_{F'}M.$$

(a) It follows from the equalities (4) that the set $C(\xi)$ contains all the ideals of the form $m_{F',x}x$, where x runs through the set of all the elements of some ideal $m \in F'$.

In fact, consider the commutative diagram

$$\begin{array}{ccc}
m_{F',x} & \xrightarrow{\;g_x\;} & m_{F',x}x = n \\
h \big\downarrow & & \big\downarrow \cdot\xi|_n \\
\mathbb{G}_F M & \xrightarrow{\;j\;} & \prod_{F' \in \Omega} \mathbb{G}_{F'}M
\end{array}$$

Since j is a monomorphism and g_x is an epimorphism, there exists an arrow $\lambda: n = m_{F',x}x \longrightarrow M$ such that $j\circ\lambda = \cdot\xi|_n$.

(b) If the left ideals n, n' *belong to* $C(\xi)$, *then their sum,* $n + n'$, *also belongs to* $C(\xi)$.

Indeed, by hypothesis, the morphisms $\cdot\xi|_n$ and $\cdot\xi|_{n'}$ are of the form $j\circ\lambda$ and $j\circ\lambda'$ for some uniquely determined morphisms λ and λ' respectively. So, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{G}_F M & \longrightarrow & \prod_{F' \in \Omega} \mathbb{G}_{F'} M \\ {\scriptstyle (\lambda,\lambda')}\downarrow & & \downarrow {\scriptstyle \xi|_{n+n'}} \\ n \amalg n' & \xrightarrow{\varphi} & n+n' \end{array}$$

Since φ is an epimorphism and j is a monomorphism, there exists a unique R-module morphism $h: n+n' \longrightarrow M$ such that $\xi|_{n+n'} = j \circ h$.

(c) Finally, together with every ascending family W of ideals, the set $C(\xi)$ contains the sum of all the ideals from W.

(d) The assertions *(b)* and *(c)* allow to deduce (applying Zorn's Lemma) that the sum, $n(\xi)$, of all ideals from $C(\xi)$ belongs to $C(\xi)$.

Now, it follows from *(a)* that $n(\xi)$ belongs to $F' \circ F'$. Since F' is a radical filter, the ideal $n(\xi)$ belongs to F'.

(e) Through the whole argument above, F' was an arbitrary radical filter from Ω, the ideal $n(\xi)$ belongs to the intersection F of all the filters from Ω. ■

6.3. Quasi-coherent presheaves and ω-sheaves. Fix again a topology $\mathfrak{T}$ on $Spec_l R$, and denote, as in 6.0, the topological space $(Spec_l R, \mathfrak{T})$ by $\mathfrak{X}$.

Call a presheaf of modules F on $\mathfrak{X}$ *an ω-sheaf* if, for any finite cover $\mathfrak{U}$ of an open set U, the canonical diagram

$$F(U) \longrightarrow \prod_{U' \in \mathfrak{U}} F(U') \rightrightarrows \prod_{U', U'' \in \mathfrak{U}} F(U' \cap U'') \tag{1}$$

is exact.

Consider now the structure presheaf $M^{\sim}$ of an R-module M, $M^{\sim}(U) = \mathbb{G}_{<U>} M$ (cf. 6.0). We have the commutative diagram

$$\begin{array}{ccccc} \mathbb{G}_{<U>} M & \longrightarrow & \prod_{U' \in \mathfrak{U}} \mathbb{G}_{<U'>} M & \rightrightarrows & \prod_{U', U'' \in \mathfrak{U}} \mathbb{G}_{<U'>} \circ \mathbb{G}_{<U''>} M \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \gamma} \\ \mathbb{G}_{<U>} M & \longrightarrow & \prod_{U' \in \mathfrak{U}} \mathbb{G}_{<U'>} M & \rightrightarrows & \prod_{U', U'' \in \mathfrak{U}} \mathbb{G}_{<U' \cap U''>} M \end{array} \tag{2}$$

By Theorem 6.2, the upper row of the diagram (2) is exact. This implies that the lower row is exact if the canonical morphism γ is a monomorphism. In particular, if the canonical map

$$\gamma_{U',U''} : \mathbb{G}_{<U'>} \circ \mathbb{G}_{<U''>} M \longrightarrow \mathbb{G}_{<U' \cap U''>} M \tag{3}$$

is a monomorphism for every pair U', U'' of open subsets of $\mathbb{X}$, then the structure presheaf $M^{\sim}$ is an ω-sheaf.

6.3.1. Example. Suppose that the ring R is commutative.

(a) Let S' and S'' be multiplicative systems of elements in R, and F', F'' the corresponding radical filters: $F' = F_{S'}$, $F'' = F_{S''}$. Then

$$\mathbb{G}_{F'} \circ \mathbb{G}_{F''} \simeq \mathbb{G}_{F''} \circ \mathbb{G}_{F'} \simeq \mathbb{G}_{F_S},$$

where $S = S'S''$. Clearly F_S is the minimal among the radical filters which contain both F' and F''.

(b) If F is a radical filter such that either the functor $\mathbb{G}_F$ is exact, or F is of finite type (i.e. every ideal from F contains a finitely generated ideal from F), then $F = \bigcap_{p \notin F} <p>$.

(Note that filters of the form F_S satisfy the both conditions.)

Clearly $F \subseteq \bigcap_{p \notin F} <p>$. Let α be an arbitrary ideal of the ring R which does not belong to F.

1) Suppose that the functor $\mathbb{G}_F$ is exact. Since $\mathbb{G}_F\alpha$ is a proper ideal of the ring $\mathbb{G}_F R$, it is contained in some maximal ideal, μ. The exactness of the functor $\mathbb{G}_F$ implies that the natural functor

$$R\text{-}mod/F \longrightarrow \mathbb{G}_F R\text{-}mod$$

is an equivalence of categories. In particular, the preimage μ' of the ideal μ does not belong to F. Since R is commutative, μ' is prime. Thus, $\alpha \subseteq \mu'$, and $\mu' \in SpecR - F$. Therefore α is not contained in $\bigcap_{p \notin F} <p>$. This proves that $\bigcap_{p \notin F} <p> \subseteq F$.

2) Assume now that the filter F is of finite type. Then the supremum of any ascending chain of ideals from $IR - F$ does not belong to F. This implies that the ideal α is contained in a maximal ideal, μ, in $IR - F$. The set $\mu_F := \{x \in R \mid (\mu:x) \in F\}$ is a left ideal (since $(\mu:x+y) \supseteq (\mu:x) \cap (\mu:y)$, and $(\mu:rx) = ((\mu:x):r)$) which contains μ and is not contained in F (since the filter F is radical, the inclusion $\mu_F \in F$ would imply that $\mu \in F$). The maximality of μ implies that $\mu = \mu_F$.

If $(\mu:x) \neq \mu$, then, thanks to the same maximality of μ, $(\mu:x) \in F$ which implies that $x \in \mu_F = \mu$; i.e. μ is prime.

(c) It follows from *(a)* and *(b)* that if the functors $\mathbb{G}_{<U'>}$ and $\mathbb{G}_{<U''>}$ are exact or of finite type, then the canonical functor morphism

$$\gamma_{U',U''}: \mathbb{G}_{<U'>} \circ \mathbb{G}_{<U''>} \longrightarrow \mathbb{G}_{<U' \cap U''>} \tag{4}$$

is an isomorphism.

This implies, among others, a basic fact of algebraic geometry: for any R-module M the restriction of the structure presheaf $M^{\frown}$ to the topology of principal open sets is a sheaf.

If the ring R is noetherian, than the morphism (4) is an isomorphism for any sets U', U''. This means that the structure presheaf of any R-module is an ω-sheaf for any topology on $SpecR$ which is compatible with specializations of points. ■

6.3.2. Lemma. *Let now R be an arbitrary associative ring; and $\mathfrak{T}$ a topology on $Spec_l R$. If an R-module M is $<U>$-torsion free for every open set U, then the structure presheaf $M^{\frown}$ is an ω-sheaf.*

Proof. In fact, in this case the adjunction arrow

$$M \longrightarrow \mathbb{G}_{<U' \cap U''>}M \tag{5}$$

is a monomorphism. Since the functors $\mathbb{G}_{<U'>}$ and $\mathbb{G}_{<U''>}$ are left exact, and

$$\mathbb{G}_{<U'>} \circ \mathbb{G}_{<U' \cap U''>} \simeq \mathbb{G}_{<U' \cap U''>} \simeq \mathbb{G}_{<U''>} \circ \mathbb{G}_{<U' \cap U''>},$$

the monomorphness of (5) implies the monomorphness of

$$\gamma_{U',U''}: \mathbb{G}_{<U'>} \circ \mathbb{G}_{<U''>}M \longrightarrow \mathbb{G}_{<U' \cap U''>}M. \ ■$$

6.4. Structure presheaves of modules over semiprime Goldie rings. Recall that a ring R is called a *left Goldie* ring if

(a) any set of left annihilators in R (i.e. left ideals of the form $(0:x)$ for some $x \subseteq R$) has a maximal (with respect to the inclusion) element;

(b) R does not contain any infinite direct sum of nonzero left ideals.

Clearly any left noetherian ring is a left Goldie ring.

Recall that a ring R is *semiprime* if it has no nonzero nilpotent ideals or, equivalently, the intersection of all prime ideals in R is zero.

We need the following fact (Lemma 7.2.2 in [He]):

6.4.1. Lemma. *Any semiprime left Goldie ring satisfies the minimality condition for left annihilators.*

6.4.2. Lemma. *Let R be a semiprime left Goldie ring. Then, for any left annihilator m in R, there exists $x \in P(R)$ such that $(m:x) = (m:R)$. In particular, the ideal m is equivalent (with respect to $\leq$) to the two-sided ideal $(m:R)$.*

Proof. Clearly if m is a left annihilator, then $(m:y)$ is a left annihilator for any $y \subseteq R$. By Lemma 6.4.1, the set of left annihilators $\{(m:u) \mid u \in P(R)\}$ has a minimal (with respect to $\subseteq$) element $(m:x)$. Since, for any $u \in P(R)$,

$$(m:x+u) = (m:x) \cap (m:u) \subseteq (m:x),$$

and $(m:x)$ is minimal, $(m:x) \cap (m:u) = (m:x)$ for any $u \in P(R)$. This implies that

$$(m:R) = (m:x) \cap \Big(\bigcap_{u \in P(R)} (m:u)\Big) = (m:x).$$

The relations

$$(m:R) \subseteq m \leq (m:x) = (m:R)$$

show that m is equivalent $(m:R)$ ∎

6.4.3. Proposition. *(a) Let R be a prime left Goldie ring. Then any left annihilator in R is equivalent (with respect to $\leq$) to the zero ideal.*

(b) Any semiprime left Goldie ring satisfies the maximality and minimality conditions for left annihilators with respect to $\leq$.

Proof. (a) Recall that a ring R is called prime if the zero ideal in R is prime.

Let m be a left annihilator in R; i.e. $m = (0:x)$ for some subset $x \subseteq$

R. Since the zero ideal is prime,

$$(m:R) = ((0:x):R) = (0:Rx) = 0.$$

By Lemma 6.4.2, the left ideal m is equivalent to the ideal $(m:R) = 0$.

(b) Let X be an arbitrary subset of left annihilators of a semiprime left Goldie ring R. Consider the set $X' := \{(m:R) \mid m \in X\}$. Since X is also a set of left annihilators, it has a maximal element, $(\nu:R)$, $\nu \in X$. We claim that ν is a maximal element of X with respect to $\leq$.

In fact, let $m \in X$, and $\nu \leq m$; i.e. $(\nu:x) \subseteq m$ for some $x \in P(R)$. Then $(\nu:R) \subseteq (\nu:x) \subseteq m$, and, since the ideal $(\nu:R)$ is two-sided, and $(m:R)$ is the biggest two-sided ideal in m, the inclusion $(\nu:R) \subseteq m$ is equivalent to that $(\nu:R) \subseteq (m:R)$.

Since $(m:R) \in X'$ and $(\nu:R)$ is a maximal element in X', $(\nu:R) = (m:R)$. Now, by Lemma 6.4.2, $(m:R) = (m:u)$ for some $u \in P(R)$. So, we have the following relations:

$$m \leq (m:u) = (m:R) \subseteq (\nu:R) \subseteq \nu;$$

i.e. $m \leq \nu$. Since the ideal m in this argument is an arbitrary element of X, the maximality of ν is proved.

The similar argument shows the existence of a minimal element in $(X,\leq)$. ∎

6.4.4. Proposition. *Suppose that R is a prime left Goldie ring. And let M be a submodule of the product of an arbitrary family of projective R-modules. Then the corresponding to the module M structure presheaf in the Zariski topology is an ω-sheaf.*

Proof. Every open set in the Zariski topology is of the form

$$U_l(\alpha) = \{p \in Spec_l R \mid \alpha - p \neq \emptyset\},$$

where α is an arbitrary two-sided ideal in R. One can see that

$$\langle U_l(\alpha)\rangle = \{m \in I_l R \mid \text{if } m \leq p \text{ and } p \in Spec_l R, \text{ then } \alpha \subseteq p\}.$$

Thanks to Lemma 6.3.2, it suffices to show that the module M is $U_l(\alpha)$-torsion free for any two-sided ideal $\alpha \neq 0$.

Note that in the condition "M is a submodule of the product of a family of projective R-modules", can be replaced by "M is the product of a family of copies of R". Since any submodule of an F-torsion free module is F-torsion free (for any Gabriel filter F), it is sufficient to consider the case when M is the product of a family of copies of the ring R.

Clearly the annihilator of a nonzero element of the module M is a left annihilator in R. So, what we actually need to show is that the Gabriel filter

$U_l(\alpha)$ does not contain left annihilators of the ring R provided that the two-sided ideal α is nonzero.

Suppose that it is not the case; i.e. there exists an $\alpha \in IR$ such that the filter $<U_l(\alpha)>$ contains a left annihilator, m, of the ring R. By the assertion *(b)* of Proposition 6.4.3, the set

$$\{(m:x) \mid x \in P(R)\text{-}P(m)\},$$

being a set of left annihilators, contains a maximal with respect to $\leq$ element, say $(m:u)$. Clearly $(m:u) \in Spec_l R$, since, for any $x \in P(R)$, $(m:u) \leq ((m:u):x) = (m:xu)$ which, thanks to the maximality of $(m:u)$, implies that the left ideals $((m:u):x)$ and $(m:u)$ are equivalent.

It follows from the relation $m \leq (m:u)$, the the ideal $(m:u)$ is contained in the intersection of $Spec_l R$ and $F(\alpha)$ which means exactly that α is contained in $(m:u)$.

Since the ring R is prime (note that till this moment the primeness of R is not required), the ideal $(m:u)$, being a left annihilator, is equivalent to the zero ideal (cf. the assertion *(a)* of Proposition 6.4.3). Thus, $\alpha \leq 0$ which means, since α is two-sided, that α is the zero ideal. The latter contradicts to the initial assumption that α is nonzero. ■

6.4.5. Proposition. *Let R be a left semiprime Goldie ring. Then*

(a) For any left annihilator, m, in R, there is $u \in P(R)$ such that $(m:u) \in Spec_l R$.

(b) If a left ideal p from $Spec_l R$ is a left annihilator, then p is equivalent to the prime ideal $(p:R)$.

(a) Conversely, any prime ideal $\mathbf{p}$ *in R which is a left annihilator belongs to $Spec_l R$.*

Proof. The proof of the assertion *(a)* is contained in the proof of Proposition 6.4.4.

The assertion *(b)* is a special case of Lemma 6.4.2.

(c) Let a prime ideal $\mathbf{p}$ be a left annihilator in R. Then, by the assertion *(a)*, $(\mathbf{p}:u) \in Spec_l R$ for some $u \in P(R) - P(\mathbf{p})$. Since $(\mathbf{p}:u)$ is a left annihilator, it is equivalent, according to the assertion *(b)*, to the ideal $((\mathbf{p}:u):R) = (\mathbf{p}:Ru)$. Thanks to the primeness of $\mathbf{p}$, the ideal $(\mathbf{p}:Ru)$ coincides with $\mathbf{p}$. ■

6.4.6. Corollary. *For any left noetherian ring R, the set of all prime ideals, $SpecR$, is contained in the left spectrum of R.*

In particular, the prime spectrum of R coincides with its Levitzki spectrum.

Proof. In fact, for any prime ideal p in R, the quotient ring R/p is left noetherian (hence left Goldie) prime ring. This implies that the zero ideal in R/p belongs to $Spec_l R/p$ (cf. the assertion *(c)* in Proposition 6.4.5). Since ring epimorphisms respect the left spectrum, the ideal $p \in Spec_l R$. ∎

6.5. Structure sheaves of noetherian rings. Fix an associative ring R. For any left ideal m in R, let $[m]$ denote the intersection of all radical filters of left ideals which contain m. Clearly $m \subseteq m'$ implies that $[m'] \subseteq [m]$.

6.5.1. Lemma. *Let F be a radical filter of left ideals in R such that any two-sided ideal in F contains a finitely generated two-sided ideal. Then a two-sided ideal α belongs to F if and only if any prime ideal containing α belongs to F.*

Proof. Consider any increasing chain Ξ of two-sided ideals containing α. If the sum of all ideals of Ξ belongs to F, then one of them belongs to F. This implies (by Zorn's lemma) that there is a maximal two-sided ideal $\mathbf{p}$ which contains α, but does not belong to F. We claim that $\mathbf{p}$ is prime.

In fact, suppose that μ and ν are two-sided ideals which are not contained in $\mathbf{p}$, but $\mu\nu \subseteq \mathbf{p}$. Replacing μ by $\mu+\mathbf{p}$ and ν by $\nu+\mathbf{p}$, we can assume that both μ and ν contain $\mathbf{p}$ properly. This implies that they belong to F. Therefore the inclusion $\mu\nu \subseteq \mathbf{p}$ implies that $(\mathbf{p}:x) \in F$ for any $x \in \nu$ which means that $\mathbf{p} \in F$. Contradiction. ∎

6.5.2. Lemma. *Let R be a left noetherian ring and F a radical filter of left ideals in R. The following conditions on a two-sided ideal α are equivalent:*

(a) $\alpha \in F$;

(b) any prime ideal containing α belongs to F;

(c) $V_l(\alpha) := \{p \in Spec_l R \mid \alpha \subseteq p\} \subseteq F$.

Proof. Clearly *(a)* $\Rightarrow$ *(b)*. The implication *(b)* $\Rightarrow$ *(c)* follows from Lemma 1.8.1. The implication *(c)* $\Rightarrow$ *(b)* is a consequence of Corollary 6.4.3. Finally, *(b)* $\Rightarrow$ *(a)* according to Lemma 6.5.1. ∎

6.5.3. Corollary. *For any two-sided ideal α of a left noetherian ring R the filter $[\alpha]$ is the smallest among radical filters F having the property*

$$Spec_l R \cap F = V_l(\alpha) \qquad (1)$$

Note that $<U_l(\alpha)>$ is the biggest among radical filters F with the property (1).

For any R-module M, denote by $Ass(M)$ the set of those annihilators of elements of M which belong to $Spec_l R$. We call them *associated points of* M.

Clearly $Ass(M) \subseteq Supp(M)$.

6.5.4. Lemma. *Suppose that the ring* R *is such that, any nonzero R-module has an associated point. Then, for any radical filter* F, *the corresponding Serre subcategory* $\mathfrak{S}_F$ *is generated by all modules* M *such that* $Supp(M) \subseteq F$.

Proof. Let M be any nonzero R-module which does not belong to $\mathfrak{S}_F$; and let M' be the quotient of M by its F-torsion. By hypothesis, there exists an element x in M' such that $Ann(x) \in Spec_l R$. Since M' is F-torsion free, the ideal $Ann(x)$ does not belong to F. And, of course, $Ann(x)$ belongs to $Supp(M)$ (cf. Proposition 1.11.2).

This shows that if $Supp(M) \subseteq F$, then $M \in Ob\mathfrak{S}_F$. The inverse implication is evident. ∎

6.5.5. Corollary. *Let* R *be as in Lemma 6.5.4. Then, for any two-sided ideal* α, *the filter* $<U_l(\alpha)>$ *coinsides with the minimal radical filter* $[\alpha]$ *containing the ideal* α.

Call a ring R *left* $\leq$*-noetherian* if any subset of left ideals in R has a maximal element with respect to the preorder $\leq$.

6.5.6. Lemma. *If* R *is left* $\leq$*-noetherian, then any nonzero R-module has an associated point.*

Proof. It suffices to check the fact for quotient modules R/m. The set of annihilators of nonzero elements of R/m is $\Omega_m := \{(m:x) \mid x \in R-m\}$. By Lemma 3.4.1, the maximal with respect to $\leq$ element of Ω_m belongs to $Spec_l R$. ∎

6.6. Quasi-coherent presheaves on the structure space. Consider now the subspace of left maximal ideals, $(Max_l R, \mathfrak{Z}t)$, of the space $(Spec_l R, \tau^\wedge)$ (recall that $\tau^\wedge$ denotes the Zariski topology). The described in 4.0 procedure assigns to every

R-module M the *structure* presheaf $\mathfrak{Z}tM$ which sends an open set U of the space $(Max_l R, \mathfrak{Z}t)$ into the R-module: $\mathfrak{Z}tM(U) := \mathfrak{G}_{<U>}M$.

The map $\gamma : m \longmapsto (m:R)$ defines a quasi-homeomorphism

$$(Max_l R, \mathfrak{Z}t) \longrightarrow PrimR,$$

where $PrimR$ is the *Jacobson's structure space* of the ring R; i.e. the space of primitive ideals of the ring R. Thus, the categories of presheaves and sheaves on the spaces $(Max_l R, \mathfrak{Z}t)$ and $PrimR$ are equivalent. In particular, the direct image functor of the map γ transfers equivalently structure presheaves from $(Max_l R, \mathfrak{Z}t)$ onto $PrimR$.

7. Affine and quasi-affine schemes. Projective spectra.

The goal of this section is to make a couple of introductory steps towards a noncommutative scheme theory.

In Section 7.0, we are trying to argue what is a 'noncommutative space' and to single out minimal requirements on a space to be a scheme. The result of our reasoning is that the category of noncommutative schemes, whatever it is, should contain locally quasi-affine spaces and open imbeddings as morphisms.

In Section 7.1, we show that (non-affine) quasi-affine schemes are just affine schemes associated to rings without unity.

Section 7.2 is concerned with the projective spectrum. Thanks to the spectral theory, we are able to define the left projective spectrum associated to a graded ring approximately the same way as it is done in the commutative case. We show that an analog of the Serre's theorem [S] describing the category of quasi-coherent sheaves on noetherian projective scheme is true in the noncommutative setting. Only 'noetherian' is in the sense of the preorder $\leq$ which is quite natural. When R is an arbitrary (not necessarily noetherian) commutative ring, our definition of **Proj**(R) is equivalent to the classical one.

A serious study of projective spectra, or more general noncommutative schemes, is out of the scope of this work. And one of the reasons is that the language of rings and ideals is not quite adequate to the task. Especially as to applications. A (based on [R6]) sketch of noncommutative projective geometry shall appear in a forthcoming paper.

7.0. General remarks on noncommutative schemes. Geometrical objects of commutative algebraic geometry are locally ringed spaces; i.e. pairs $(X, \mathcal{O})$, where X

is a topological space and $\mathcal{O}$ is a sheaf of local rings on X satisfying some additional properties. What we really care about are certain subcategories of the category of $\mathcal{O}$-modules, such as categories of quasi-coherent or coherent sheaves. Luckily, these categories can be reconstructed from the pair $(X,\mathcal{O})$.

Moreover, the pair $(X,\mathcal{O})$ can be reconstructed by the category of quasi-coherent sheaves on X uniquely up to isomorphism. This fact is proved in [Gab] for noetherian schemes. The general case follows from [R6]. The reconstruction procedure is particularly straightforward when a scheme is known to be affine: it is the map $\mathcal{A} \longmapsto (SpecC(\mathcal{A}),\mathcal{O}_{C(\mathcal{A})})$, where $C(\mathcal{A})$ is the center of a category $\mathcal{A}$, i.e. the ring of endomorphisms of $Id_{\mathcal{A}}$.

In the noncommutative setting, the role of a ring, or a sheaf of rings, is less essential, and the choices are not canonical: Morita-equivalent rings have the same rights. Another, more important circumstance is that lots of natural objects of noncommutative geometry (to begin with open subspaces of spectra of rings) are not locally affine.

Thus, a right thing to do in the noncommutative setting, is to replace the sheaf of rings by the category of 'structure sheaves'. A straightforward formalization leads to the notion of a bundle of localizations.

7.0.1. A bundle of localizations is a triple $(X,\mathcal{E},\beta)$, where X is a topological space, $\mathcal{E}$ is a map which assigns to any open set $\mathcal{U} \subseteq X$ a category $\mathcal{E}_{\mathcal{U}}$, and to any inclusion $\iota: \mathcal{U} \subseteq \mathcal{U}'$ of open sets a flat localization

$$\mathcal{E}_\iota: \mathcal{E}_{\mathcal{U}'} \longrightarrow \mathcal{E}_{\mathcal{U}}.$$

Here 'flat' means 'exact and having a right adjoint'. Finally, β is a function which assigns to any pair $\iota: \mathcal{U} \subseteq \mathcal{U}'$, $\iota': \mathcal{U}' \subseteq \mathcal{U}''$ of inclusions a functor isomorphism

$$\beta_{\iota',\iota}: \mathcal{E}_\iota \circ \mathcal{E}_{\iota'} \longrightarrow \mathcal{E}_{\iota'\iota}$$

such that, for any three composable inclusions, ι, ι', ι'', the diagram

$$\begin{array}{ccc}
\mathcal{E}_\iota \circ \mathcal{E}_{\iota'} \circ \mathcal{E}_{\iota''} & \xrightarrow{\beta_{\iota',\iota}\mathcal{E}_{\iota''}} & \mathcal{E}_{\iota'\iota} \circ \mathcal{E}_{\iota''} \\
{\scriptstyle \mathcal{E}_\iota \beta_{\iota'',\iota'}} \downarrow & & \downarrow {\scriptstyle \beta_{\iota'',\iota'\iota}} \\
\mathcal{E}_\iota \circ \mathcal{E}_{\iota''\iota'} & \xrightarrow{\beta_{\iota''\iota',\iota}} & \mathcal{E}_{\iota''\iota'\iota}
\end{array} \tag{1}$$

is commutative and

$$\mathcal{E}_{id} = Id, \quad \beta_{\iota,id} = \beta_{id,\iota} = id. \tag{2}$$

Besides, we require that, for any covering $\mathcal{V}$ of an open set $\mathcal{U}$, the set of localizations $\{\mathcal{C}_\iota \mid \iota \in V\}$ reflects isomorphisms; i.e. if s is a morphism in $\mathcal{C}_{\mathcal{U}}$ such that $\mathcal{C}_\iota s$ is an isomorphism for all $\iota \in V$, then s itself is an isomorphism.

7.0.2. Note. Although β here is inavoidable by technical reasons, it is not really important. In fact, if we fix, for any open set $\mathcal{U}$, the canonical localization $\mathcal{C}_{\mathcal{U}\to X}$: $\mathcal{C}_X \longrightarrow \mathcal{C}_{\mathcal{U}}$ (cf. [GZ], I.1), then, we define restriction functors $\mathcal{C}_{\mathcal{U}\to\mathcal{U}'}$ by the requirement:

$$\mathcal{C}_{\mathcal{U}\to\mathcal{U}'} \circ \mathcal{C}_{\mathcal{U}\to X} = \mathcal{C}_{\mathcal{U}'\to X}. \qquad (3)$$

Thanks to the universal property of localizations, (3) defines the functor $\mathcal{C}_{\mathcal{U}\to\mathcal{U}'}$ uniquely. This implies that we can take $\beta = id$. ∎

7.0.3. Open imbeddings. Following the standart pattern, define a morphism of bundles of localizations $(X,\mathcal{C},\beta) \longrightarrow (X',\mathcal{C}',\beta')$ as a pair (ϕ,Φ), where ϕ is a morphism $X \longrightarrow X'$ of topological spaces, and Φ is a function which assigns to any open subset $\mathcal{U} \subseteq X'$ a functor

$$\Phi_{\mathcal{U}}: \mathcal{C}_{\phi^{-1}\mathcal{U}} \longrightarrow \mathcal{C}'_{\mathcal{U}}$$

and to any inclusion $\mathcal{U} \subseteq \mathcal{U}'$ of open sets a functor isomorphism

$$\Phi_{\mathcal{U}\to\mathcal{U}'}: \mathcal{C}'_{\mathcal{U}\to\mathcal{U}'} \circ \Phi_{\mathcal{U}} \longrightarrow \Phi_{\mathcal{U}'} \circ \mathcal{C}_{\phi^{-1}(\mathcal{U})\to\phi^{-1}(\mathcal{U}')}$$

satisfying natural compatibility requirements with respect to compositions which are expressed by a commutative diagram (this is exactly the place where β and β' get involved) left to a reader.

A morphism (ϕ,Φ) of bundles of localizations is called an equivalence (by abuse of language an isomorphism) if ϕ is a homeomorphism, and the functor $\Phi_{\mathcal{U}}$ is an equivalence of categories for all $\mathcal{U}$.

Let $\mathbf{X} = (X,\mathcal{C},\beta)$ be a bundle of localizations, and let $\phi: Y \longrightarrow X$ be an open map. Then we can induce a bundle of localizations $\phi^*\mathbf{X} = (Y,\phi^*\mathcal{C},\beta')$ on Y in the usual way:

$$\phi^*\mathcal{C}_{\mathcal{U}} := \mathcal{C}_{\phi(\mathcal{U})} \quad \text{and} \quad \phi^*\mathcal{C}_{\mathcal{U}\to\mathcal{U}'} := \mathcal{C}_{\phi(\mathcal{U})\to\phi(\mathcal{U}')}$$

for any two open subsets $\mathcal{U} \subseteq \mathcal{U}'$ of Y. In particular, we have a well defined restriction $\mathbf{X}|_U = (U,\mathcal{C}|_U,\beta')$ of a bundle $\mathbf{X}$ to any open subset U of X and a canonical morphism $\mathbf{X}|_U \longrightarrow \mathbf{X}$. We call a morphism $\mathbf{X}' \longrightarrow \mathbf{X}$ an open imbedding if it is a composition of $\mathbf{X}|_U \longrightarrow \mathbf{X}$ for some open subset U and of an equivalence $\mathbf{X}' \longrightarrow \mathbf{X}|_U$.

Open imbeddings are the only morphisms we need for what follows.

7.0.3. The local algebra setting. Suppose we are given a topological space X

and a flat localization $Q_x: \mathcal{A} \longrightarrow \mathcal{A}_x$ for all $x \in X$. For any open set $\mathcal{U}$ in X, take a localization $Q_{<\mathcal{U}>}$ at the Serre subcategory $<\mathcal{U}> := \bigcap_{x \in \mathcal{U}} KerQ_x$. This will define a bundle of localizations (cf. Note 7.0.2).

One of the attractions of this approach is that the topology plays the role of a parametre. Another 'global' parametre is the category $\mathcal{A} = \mathcal{A}(X)$ of structure presheaves on X:

Note that a local algebra setting on X induces in an obvious way a local algebra setting on any subspace Y of X. And the induced category of structure presheaves of Y is equivalent to the quotient category $\mathcal{A}(X)/<Y>$.

7.0.3.1. Basic examples. (a) Let $X = (Spec_l R, \mathcal{T})$, where $\mathcal{T}$ is one of the canonical topologies (cf. 1.10.2 and 1.10.3). To each point $p \in Spec_l R$, we assign a localization at $<p>$.

(b) If $\mathcal{T}$ is the Zariski topology, there is a quasi-isomorphism, ϕ, from $(Spec_l R, \mathcal{T})$ to the Levitzki spectrum $LSpecR$ of R (cf. Theorem 5.3). Thus, we assign to any associative ring R the bundle of localizations $(LSpecR, \mathcal{O})$, where $\mathcal{O}_{\mathcal{U}} = \mathcal{A}/<\mathcal{U}>$, $\mathcal{A} := R\text{-}mod$, $<\mathcal{U}> = \bigcap_{<p> \in \phi^{-1}(\mathcal{U})} <p>$.

We shall write $\mathcal{O}_R$ and $\mathcal{O}_{R,\mathcal{U}}$ when there is a need to mention the ring R (for instance in Section 7).

Note that this bundle of localizations can be obtained from a local algebra setting on $LSpecR$ which assigns to any point $\mathbf{p} \in LSpecR$ the localization at the Serre subcategory of $R\text{-}mod$ generated by all modules M such that $Ann(M)$ is not contained in $\mathbf{p}$. ■

7.0.4. Affine schemes. We define *a left affine (Zariski) scheme* as a bundle of localizations $(X, \mathcal{E}, \beta)$ which is isomorphic to the bundle $(LSpecR, \mathcal{O})$ of Example 7.0.3.1 (b).

We would like to underline that the affine scheme $(LSpecR, \mathcal{O})$ is not always the best choice of a geometrization, since it is trivial for all simple rings. If R is simple (e.g. a Weyl algebra), the bundle of Example 7.0.3.1 (a) with the topology $\mathcal{T}$ of 1.10.3 is in most cases an adequate geometrization. ■

7.0.4.1. Comparison with the construction by Van Oyestaeyen and Vershoren. Van Oyestaeyen and Vershoren [OV] assign to a left noetherian R its prime spectrum, and to any open subset $U(\alpha)$ the localization at the minimal radical filter $[\alpha]$ containing the two-sided ideal α.

Since the prime spectrum of any left noetherian ring coincides with its Levitzki spectrum, the underlying space is the same as that of a left affine scheme. But, it is not the same with structure presheaves. Because the radical filter $[\alpha]$ is, usually, a proper subset fo $<U_l(\alpha)>$ (cf. Section 6.5). The latter means that left affine schemes are locally simpler. They have greater similarity with locally ringed spaces (cf. [R6]).

7.0.5. The minimal requirements on the category of noncommutative schemes. Denote by $\mathcal{SCH}$ the class of bundles of localizations which should be regarded as schemes. We would like it to have the following properties:

(a) Affine spaces $(LSpecR,\mathcal{O})$ (cf. Example 7.0.3.1) should be schemes.

(b) If $(X,\mathcal{C})$ is a scheme, then, for any open subset $\mathcal{U}$ of X, the induced bundle $(\mathcal{U},\mathcal{C}|_{\mathcal{U}})$ is a scheme (an open subscheme of $(X,\mathcal{C})$).

(c) If $(X,\mathcal{C})$ is a bundle of localizations such that $(\mathcal{U},\mathcal{C}|_{\mathcal{U}})$ is a scheme for every $\mathcal{U}$ from some covering of X, then $(X,\mathcal{C})$ is a scheme.

Clearly $\mathcal{SCH}$ includes all commutative schemes. Moreover, in the commutative setting, (a) and (c) imply (b). But, if R is a generic associative ring, an open subscheme of the associated affine scheme is not usually locally affine.

Call a bundle of localizations *a left quasi-affine scheme* is it is equivalent to an open subscheme of a left affine scheme.

One can see that the minimal class of bundles of localizations which satisfies the conditions (a), (b), and (c) consists exactly of locally quasi-affine schemes. Thus quasi-affine schemes deserve a special attention.

7.1. Quasi-affine schemes and spectra of rings without unity. Let R be an arbitrary associative ring; i.e. not necessarily with unity. For any left ideal m of the ring R, denote by $<m>$ the set of all left ideals n of R such that m does not contain neither n nor any of the ideals $(n:y)$, where $y \in P(R)$. The rest of the notions and results of this paper can be transfered on rings without unity more or less straightforwardly (see [R1]). In paricular, for an arbitrary ring R, one can define its left spectrum $Spec_l R$ with canonical topologies and, given a topology, quasi-coherent ('structure') presheaves.

7.1.2. Theorem. *Let R be an arbitrary associative ring, and let α be a two-sided ideal of R. The map $p \longrightarrow p \cap \alpha$ determines an isomorphism of the open subscheme $(U_l(\alpha),\mathcal{O}_R|_{U_l(\alpha)})$, where $U_l(\alpha):=\{p \in Spec_l R \mid p$ does not contain $\alpha\}$, onto the bundle of localizations $(Spec_l\alpha,\mathcal{O}_\alpha)$.*

Proof. *(a)* Let $p \in U_l(\alpha)$, and let m be a left ideal of α such that $(p:x) \cap \alpha$ is not contained in $p \cap \alpha$ for any $x \in P(m)$. Then $(p:x)$ is not contained in p for any $x \in P(\alpha m)$. Since αm is a left ideal in R and $p \in Spec_l R$, $\alpha m \subseteq p$, and, therefore, $\alpha(R,m) \subseteq p$, where $(R,m):=Rm + m$. The claim is that $(R,m) \subseteq p$.

In fact, if it is not so, one can find $z \in (R,m) - p$. The ideal $(p:z)$, which contains α (cf. the argument above), is equivalent to p; i.e. $((p:z):w) \subseteq p$ for some $w \in P(R)$. Thus, we have:

$$\alpha \subseteq (\alpha:w) \subseteq ((p:z):w) \subseteq p$$

which contradicts to the inclusion $p \in U_l(\alpha)$.

(b) Now, let $\mu \in Spec_l \alpha$. Set $\mu_\alpha: = \{z \in R \mid \alpha z \subseteq \mu\}$. Clearly μ_α is a left ideal in R. Let m be a left ideal in R such that $(\mu_\alpha:x) - \mu_\alpha$ is not empty for any $x \in P(m)$. By definition, this means that for any $x \in P(m)$ there exists λ_x such that $\alpha\lambda_x x \subseteq \mu$ and $\alpha\lambda_x$ is not contained in μ. In particular, $(\mu:x) \cap \alpha$ is not contained in μ for any $x \in P(\alpha m)$. This implies, since $\mu \in Spec_l \alpha$, that $\alpha m \subseteq \mu$; or, equivalently, $m \in \mu_\alpha$. Therefore $\mu_\alpha \in Spec_l R$.

(c) If $\mu = p \cap \alpha$ for some $p \in U_l(\alpha)$ then

$$\mu_\alpha = \{z \in R \mid \alpha z \subseteq p\} = p.$$

On the other hand, $\mu_\alpha \cap \alpha = \{z \in \alpha \mid (\mu:z)_\alpha = \alpha\}$. Therefore, since $\mu \in Spec_l \alpha$, we have the equality $\mu_\alpha \cap \alpha = \mu$. I.e. the map $p \longmapsto p \cap \alpha$ determines a bijection u_α from $U_l(\alpha)$ onto $Spec_l \alpha$ with the inverse map

$$v_\alpha : \mu \longmapsto \mu_\alpha .$$

(d) The map u_α is continuous with respect to the topology $\tau^\wedge$, since, for any two-sided ideal β of the ring α, we have:

$$[\beta \subseteq u_\alpha(p): = p \cap \alpha] \Leftrightarrow [\alpha\beta \subseteq p]$$

(e) The map u_α is open.

In fact, it sends the open subset $U_l(\alpha') \cap U_l(\alpha)$ of $U_l(\alpha)$ into the open subset $U_l^{\alpha'}(\alpha' \cap \alpha)$ of $Spec_l \alpha$. Thus, *u_α is a homeomorphism.*

(f) It is easy to check that, for any localizing filter F of the left ideals of the ring R, which contains the ideal α, the set $F_\alpha: = \{m \cap \alpha \mid m \in F\}$ is a localizing filter of the left ideals of α. Since F_α is a cofinal subset of F, the F-torsion $FM: = \{z \in M \mid mz = \{0\}$ *for some* $m \in F\}$ of an arbitrary R-module M coincides with its F_α-torsion. Besides,

$$Hom_R(M',M'') = Hom_\alpha(M',M'')$$

provided the $\{\alpha\}$-torsion of the module M'' is zero.

Therefore we have (cf. 0.3):

$$\mathbb{G}_F(M) = colim\{Hom_R(m,M/FM) \mid m \in F\} = colim\{Hom_\alpha(m',M/F_\alpha M) \mid m' \in F_\alpha\} = \mathbb{G}_{F_\alpha}(M).$$

This implies that the homeomorphism u_α *induces an equivalence of the categories of quasi-coherent presheaves and, therefore, sheaves, on* $U_l(\alpha)$ *and* $Spec_l \alpha$ *respectively.* In other words, u_α induces an isomorphism of bundles of localizations $(U_l(\alpha), \mathcal{O}_R|_{U(\alpha)})$ onto $(Spec_l \alpha, \mathcal{O}_\alpha)$.■

7.1.3. Corollary. *The bundle of localizations* $\mathfrak{X}$ *is isomorphic to a left quasi-affine scheme if and only if it is isomorphic to* $(LSpecR, \mathcal{O}_R)$ *for some associative ring* R *(without identity element in general).*

Proof. In fact, any associative ring R is a two-sided ideal of the ring R^u generated by R and an element e (unity) which satisfies the relations:

$$e^2 = e, \quad ex = x \quad \text{for any} \quad x \in R.$$

Now the assertion follows from Theorems 7.1.2 and 5.3. ■

7.1.4. Local algebra setting for quasi-affine schemes. Let $(X,\mathcal{O})$ be a left affine (or any other) scheme given by a local algebra data

$$\big(X, Q_x: \mathcal{O}(X) \longrightarrow \mathcal{O}_x \mid x \in X\big)$$

(cf. Section 7.0.3). Then, any open subscheme of $(X,\mathcal{O})$ is given by

$$\big(\mathcal{U}, Q_x: \mathcal{O}(\mathcal{U}) \longrightarrow \mathcal{O}_x \mid x \in \mathcal{U}\big),$$

where $\mathcal{U}$ is an open subset of X and $\mathcal{O}(\mathcal{U})$ is equivalent to the quotient category $\mathcal{O}(X)/\langle\mathcal{U}\rangle$, $\langle\mathcal{U}\rangle := \bigcap_{x\in\mathcal{U}} KerQ_x$.

If $\mathcal{O}(X) \simeq R\text{-}mod$, then $\langle\mathcal{U}\rangle$ is the full subcategory of $R\text{-}mod$ generated by all modules with support contained in the closed subset $X - \mathcal{U}$.

7.2. Left projective spectrum. We begin with some generalities on the graded left spectrum.

7.2.1. Graded spectral theory. Let H be a commutative semigroup; and let R be an H-graded ring. Denote by $H\text{-}Spec_l R$ the subset of $Spec_l R$ formed by H-graded ideals. Similarly, we define the H-graded Levitzki spectrum of R, $H\text{-}LSpecR$.

Denote by $\mathcal{G}r_H I_l R$, or just $\mathcal{G}r I_l R$, when it is clear (or does not matter) what is H, the set of all homogenious left ideals in R. Clearly the imbedding $\mathcal{G}r_H I_l R \longrightarrow I_l R$ is a retract: the corresponding coretraction assigns to any left ideal m in R the left ideal $\hbar(m)$ generated by the set of all homogenious elements of m.

7.2.1.1. Lemma. *The map* $\hbar\colon I_l R \longrightarrow gr_H I_l R$ *induces a coretraction*

$$\mathcal{S}\hbar\colon Spec_l R \longrightarrow gr_H Spec_l R.$$

Proof. In fact, let $p \in Spec_l R$. And let $r \in R - \hbar(p)$. The latter means that one of the homogenious components of r, say r_ν, does not belong to p. Since p is in the left spectrum, $(p:r_\nu) \le p$ which means, by definition, that $((p:r_\nu):w) \subseteq p$ for some finite subset w of r_ν. Let W be the set of homogenious components of elements of w. We have:

$$(p:Wr_\nu) = ((p:r_\nu):W) \subseteq ((p:r_\nu):w) \subseteq p. \qquad (1)$$

Now, since Wr_ν is the set of homogenious elements,

$$\hbar((p:Wr_\nu)) = (\hbar(p):Wr_\nu). \qquad (2)$$

Finally,

$$(\hbar(p):r) \subseteq (\hbar(p):r_\nu). \qquad (3)$$

Combining (1), (2), and (3), we get the inclusion:

$$((\hbar(p):r):W) \subseteq ((\hbar(p):r_\nu):W) \subseteq \hbar(p)$$

which means that $(\hbar(p):r) \le \hbar(p)$. This proves that $\hbar(p) \in Spec_l R$. ■

One can see that the (induced) Zariski topology on $H\text{-}Spec_l R$ has the set of closed sets $H\text{-}V_l(\alpha) := H\text{-}Spec_l R \cap V_l(\alpha)$, where α runs through the set of H-graded two-sided ideals in R.

In fact, if V is a closed subset in $Spec_l R$, then

$$V' := H\text{-}Spec_l R \cap V = H\text{-}Spec_l R \cap V_l(\cap_{p \in V'} p). \qquad (4)$$

Theorem 5.6 implies the following

7.2.1.1. Proposition. *If* α *is a finitely generated homogenious two-sided ideal, then the open set* $H\text{-}U_l(\alpha)$ *is quasi-compact. In particular, for any* H*-graded ring* R *(not necessarily with unity) the space* $H\text{-}LSpecR$ *has a base of open quasi-compact subsets.*

7.2.1.2. Structure sheaves of graded modules. Denote by $\mathcal{F}$ the forgetting grading functor from the category $gr_H R\text{-}mod$ of H-graded left R-modules to the category of left R-modules. Since the functor $\mathcal{F}$ is exact and respects and reflects colimits, the preimage of any Serre subcategory $\mathbb{S}$ of $R\text{-}mod$ is a Serre subcategory of $gr_H R\text{-}mod.$, And we have a commutative diagram:

$$\begin{array}{ccc} gr_H R\text{-}mod & \xrightarrow{Q'} & gr_H R\text{-}mod/\mathcal{F}^{-1}\mathbb{S} \\ \mathcal{F}\downarrow & & \downarrow\mathcal{F}' \\ R\text{-}mod & \xrightarrow{Q} & R\text{-}mod/\mathbb{S} \end{array} \tag{1}$$

Since $\mathcal{F}^{-1}\mathbb{S}$ is a Serre subcategory of the Grothendieck category $gr_H R$-mod, the localization Q' has a right adjoint functor. And $gr_H R$-$mod/\mathcal{F}^{-1}\mathbb{S}$ is canonically identified with a full subcategory of the category $gr_H R$-mod. In particular, any object of $gr_H R$-$mod/\mathcal{F}^{-1}\mathbb{S}$ is an H-graded R-module.

Now we describe all this in the language of rings, left ideals, and Gabriel filters.

To the Serre subcategory $\mathbb{S}$, we assign a Gabriel filter $\mathcal{G} = \mathcal{G}_{\mathbb{S}}$ in the usual way: $\mathcal{G} := \{m \in I_l R \mid R/m \in Ob\mathbb{S}\}$. The analogous operation with respect to $\mathbb{S}' := \mathcal{F}^{-1}\mathbb{S}$ gives us the set

$$\mathcal{G}' := \{m \in H\text{-}I_l R \mid R/m \in Ob\mathbb{S}'\} = \mathcal{G} \cap H\text{-}I_l R,$$

where H-$I_l R$ denotes the set of H-graded left ideals in R.

Fix an H-graded R-module M. For any H-graded module L, we have a well defined "inner hom"

$$\mathcal{H}om_R(L,M) := \oplus_{h\in H}\mathcal{H}om_R(L,M)_h.$$

Here $\mathcal{H}om_R(L,M)_h$ consists of all R-module morphisms $f: L \longrightarrow M$ of degree h; i.e. $f(L_t) \subseteq M_{h+t}$ for all $t \in H$. Thus, we have an H-graded $\mathbb{Z}$-module

$$'\mathcal{H}_{\mathcal{G}'}(M) := colim\big(\mathcal{H}om_R(m,M) \mid m \in \mathcal{G}'\big). \tag{2}$$

And one can show that $'\mathcal{H}_{\mathcal{G}'}(M)$ has unique structure of H-graded R-module compatible with the structure of R-module on

$$H'_{\mathcal{G}}(M) = colim\big(Hom_R(m,M) \mid m \in \mathcal{G}\big) \tag{3}$$

(cf. 0.4). The compatibility means that the $\mathbb{Z}$-module morphism from $'\mathcal{H}_{\mathcal{G}'}(M)$ to $'H_{\mathcal{G}}(M)$ induced by the compositions of $\mathbb{Z}$-module morphisms

$$\mathcal{H}om_R(m,M) \longrightarrow Hom_R(m,M) \quad \text{and} \quad Hom_R(m,M) \longrightarrow {}'H_{\mathcal{G}}(M), \quad m \in \mathcal{G}',$$

is an R-module morphism. The map $'\mathcal{H}_{\mathcal{G}'}(M)$ is functorial in M. So, we have a well defined functor

$$\mathcal{H}_{\mathcal{G}'}: gr_H R\text{-}mod \longrightarrow gr_H R\text{-}mod. \tag{4}$$

The functor $\mathbb{G}_{\mathcal{G}'} := \mathcal{H}_{\mathcal{G}'} \circ \mathcal{H}_{\mathcal{G}'}$ is the Gabriel functor of the localization at the Serre subcategory $\mathbb{S}'$ (or, what is the same, at the filter $\mathcal{G}'$); i.e. $\mathbb{G}_{\mathcal{G}'}$ is isomorphic to the composition $Q'^{\wedge} \circ Q'$ of the localization Q' at $\mathbb{S}'$ with its right adjoint.

Moreover, for any H-graded module M, there is a canonical H-graded action

$$\mu_M: \mathfrak{G}_{\mathfrak{g}'}R \otimes \mathfrak{G}_{\mathfrak{g}'}M \longrightarrow \mathfrak{G}_{\mathfrak{g}'}M \tag{5}$$

such that μ_R is an H-graded associative ring structure and μ_M is a structure of an H-graded $\mathfrak{G}_{\mathfrak{g}'}R$-module. Of course, this structures are defined uniquely by the compatibility condition with the corresponding non-graded structures (cf. Section 0.4).

7.2.2. Left projective spectrum. Let now R be a $\mathbb{Z}_+$-graded ring; and let R_+ denote the (direct) sum of all components R_n, $n \geq 1$.

By analogy with the commutative case, denote by $Proj_l(R)$ the open set

$$grU_l(R_+) := U_l(R_+) \cap grI_lR = \{p \in Spec_lR \mid p \text{ does not contain } R_+\} \cap grI_lR$$

of the graded left spectrum of R.

Note that, thanks to Theorem 7.1.2, we can identify $U_l(R_+)$ with $Spec_lR_+$. But, we are not going to use this identification here.

Now, to any open subset $\mathfrak{U}$ of $U_l(R_+)$, we assign the localization

$$Q_{<\mathfrak{U}>}: gr_{\mathbb{Z}}R\text{-}mod \longrightarrow gr_{\mathbb{Z}}R\text{-}mod/<\mathfrak{U}>$$

at $<\mathfrak{U}> := \bigcap_{p \in \mathfrak{U}} <p>$ (in the sense of 7.1.1). Thus we have defined a bundle of localizations which we denote by $\mathbf{Proj}_l(R)$.

Clearly $\mathbf{Proj}_l(R)$ can be given by a local algebra data:

$$\mathbf{Proj}_l(R) \approx \big(X, Q_x: O(X) \longrightarrow O_x \mid x \in X\big),$$

where $X = Proj_l(R)$; the category $O(X)$ of structure presheaves is equivalent to $gr_{\mathbb{Z}}R\text{-}mod/<U_l(R_+)>$; Q_x, $x \in X$, are localizations at points of the (homogenious) left spectrum.

For any $\mathfrak{U} \subseteq grSpec_lR$, consider the composition $P_{\mathfrak{U}}$ of the fully faithful imbedding

$$gr_{\mathbb{Z}}R\text{-}mod/<\mathfrak{U}> \longrightarrow gr_{\mathbb{Z}}R\text{-}mod$$

and the functor

$$P_0: gr_{\mathbb{Z}}R\text{-}mod \longrightarrow R_0\text{-}mod \tag{1}$$

which assigns to any $\mathbb{Z}$-graded module (respectively graded module morphism) from $gr_{\mathbb{Z}}R\text{-}mod/<\mathfrak{U}>$ its zero component.

7.2.2.1. Lemma. *Let the ideal R_+ be generated by R_1. Then, for any subset $\mathfrak{U}$ of $U_l(R_+)$, the kernel of the functor*

$$P_{\mathfrak{U}}: gr_{\mathbb{Z}}R\text{-}mod/<\mathfrak{U}> \longrightarrow R_0\text{-}mod$$

consists of all modules $M = \bigoplus_{n \in \mathbb{Z}} M_n$ such that $M_n = 0$ for all $n \leq 0$.

Proof. If $\mathcal{U} \subseteq \mathcal{U}'$, then $P_{\mathcal{U}}$ is the composition of the fully faithful functor $gr_{\mathbb{Z}}R\text{-}mod/\langle\mathcal{U}\rangle \longrightarrow gr_{\mathbb{Z}}R\text{-}mod/\langle\mathcal{U}'\rangle$ (full faithfulness is due to the fact that this functor is right adjoint to a localization; cf. [GZ], Ch.I) and $P_{\mathcal{U}'}$. Therefore, it suffices to prove the assertion in the case $\mathcal{U} = U_l(R_+)$.

Since the functor $P_{\mathcal{U}}$ has a left adjoint, it is left exact. This implies that the faithfulness of $P_{\mathcal{U}}$ is equivalent to the property:

$$\text{for any object } M, \ P_{\mathcal{U}}(M) = 0 \ \text{ iff } \ M = 0.$$

Let M be a graded module from $gr_{\mathbb{Z}}R\text{-}mod/\langle\mathcal{U}\rangle$ such that $M_0 = 0$. Then, for any $n \geq 1$, $R_1{}^n M_{-n} = 0$. Since R_1 generates R_+, this implies that $\bigoplus_{n \leq 0} M_n$ is an R-submodule of M which is $[R_+]$-torsion. Since $\langle\mathcal{U}\rangle \supseteq [R_+]$, it is also a $\langle\mathcal{U}\rangle$-torsion. But, all objects of $gr_{\mathbb{Z}}R\text{-}mod/\langle\mathcal{U}\rangle$ are $\langle\mathcal{U}\rangle$-torsion free. Therefore $\bigoplus_{n \leq 0} M_n = 0$; i.e. $M = \bigoplus_{n \geq 1} M_n$. ■

Fix a set S of elements in R satisfying the left Ore conditions; i.e. for any $s \in S$ and $r \in R$, there exist $s' \in S$ and $r' \in R$ such that $r's = s'r$. Let U_S denote the subset of all $p \in Spec_l R$ such that $p \cap S = \varnothing$. Or, equivalently, $U_S = Spec_l R - F_S$, where $F_S = \{m \in I_l R \mid (m:x) \cap S = \varnothing\}$. Since, by Lemma 0.3.2, F_S is a radical filter, the set U_S is closed in the topology τ (cf. Section 1.10.1).

Suppose that S above consists of homogenious elements, and some of them are of positive degree. Then grU_S is, obviously, a subset of $U_l(R_+)$.

7.2.2.2. Lemma. *Suppose that R_+ is generated by R_1. And let S be a left Ore subset of R_+, $\mathcal{U} \subseteq grU_S$. Then the functor $P_{\mathcal{U}}: gr_{\mathbb{Z}}R\text{-}mod/\langle\mathcal{U}\rangle \longrightarrow R_0\text{-}mod$ is faithful.*

Proof. a) The functor $P_{\mathcal{U}}$ is (isomorphic to) the composition of a fully faithful functor $gr_{\mathbb{Z}}R\text{-}mod/\langle\mathcal{U}\rangle \longrightarrow gr_{\mathbb{Z}}R\text{-}mod/\langle U_S\rangle$ and the functor

$$P_{U_S}: gr_{\mathbb{Z}}R\text{-}mod/\langle U_S\rangle \longrightarrow R_0\text{-}mod.$$

So, it suffices to consider the case $\mathcal{U} = U_S$.

b) According to Lemma 7.1.2.1, the kernel of $P_{\mathcal{U}}$ consists of all modules M from $gr_{\mathbb{Z}}R\text{-}mod/\langle\mathcal{U}\rangle$ such that $M_n = 0$ for $n \leq 0$.

Since $M \in Ob\,gr_{\mathbb{Z}}R\text{-}mod/\langle\mathcal{U}\rangle$ the canonical (graded) morphism $M \longrightarrow \mathbb{G}_{gr\langle\mathcal{U}\rangle}M$ is an isomorphism. This implies that $M \longrightarrow \mathbb{G}_F M$ is an isomorphism for any radical filter F which is contained in $\langle\mathcal{U}\rangle$. Note now that $F_S \subseteq \langle\mathcal{U}\rangle$. And, due to the Ore conditions, $\mathbb{G}_{F_S} M \simeq S^{-1}R \otimes_R M$, where the tensor product is the graded tensor product. But, if $M \neq 0$, the graded module $S^{-1}R\otimes_R M$ has nonzero negative components which contradicts to the existence of a graded isomorphism from M to $S^{-1}R\otimes_R M$. ∎

7.2.2.3. A special case. If $\mathcal{U} = U_S$ in the conditions of Lemma 7.1.2.2, then the Gabriel functor $\mathbb{G}_{<\mathcal{U}>}$ is isomorphic to $M \longmapsto S^{-1}R\otimes_R M$, and the quotient category $gr_{\mathbb{Z}}R\text{-}mod/<\mathcal{U}>$ is equivalent to the category $gr_{\mathbb{Z}}S^{-1}R\text{-}mod$. One can see that the forgetting functor $grS^{-1}R\text{-}mod \longrightarrow R_0\text{-}mod$ is the composition of the functor

$$\mathcal{F}:\ grS^{-1}R\text{-}mod \longrightarrow (S^{-1}R)_0\text{-}mod \qquad (1)$$

(the zero component of a graded $S^{-1}R$-module is an $(S^{-1}R)_0$-module) and the 'pull-back' functor

$$(S^{-1}R)_0\text{-}mod \longrightarrow R_0\text{-}mod.$$

The functor $\mathcal{F}$ is right adjoint to the functor $\hat{}\mathcal{F} := S^{-1}R\otimes_{(S^{-1}R)_0}$ of graded tensor product; and one can see that the adjunction morphism

$$Id \longrightarrow \mathcal{F}\circ\hat{}\mathcal{F}$$

is an isomorphism which means that $\hat{}\mathcal{F}$ is a fully faithful functor. Therefore the functor $\mathcal{F}$ is a localization ([GZ], I.2). But, $\mathcal{F}$ is faithful. And the only faithful localizations are equivalences of categories.

It remains to notice that U_S is naturally homeomorphic to the left spectrum of the ring $(S^{-1}R\text{-}mod)_0$.

All together shows that the structure of the projective spectrum of a general associative ring over the open sets U_S is the same as in the commutative case.

Suppose there is a family Ω of finitely generated left Ore subsets of homogenious elements in R_+ such that $Proj_l R = \bigcup_{S\in\Omega} U_S$. The sets U_S are open in the topology (induced by) τ^* (cf. 1.10.3). And the restriction of the bundle of localizations to each U_S is equivalent to the bundle of localizations of $((Spec_l(S^{-1}R)_0,\tau^*),O)$ (cf. Example 7.0.3.1(a)).

For **Proj** to be locally affine, we need a stronger requirement. Suppose that there is a set Ξ of elements in R_+ such that Ru is a two-sided ideal for any $u \in \Xi$. This implies that, for any $u \in \Xi$, the multiplicative system $(u) = \{u^n \mid n \geq 0\}$ is left a Ore set. Clearly $U_l(Ru) \subseteq U_l(R_+)$ for any $u \in \Xi$.

Suppose that the (Zariski open) sets $grU_l(Ru)$, $u \in \Xi$, cover $grU_l(R_+)$. Then $\mathbf{Proj}_l(R)$ is locally affine. Explicitly, the restriction of $\mathbf{Proj}_l(R)$ to $grU_l(Ru)$ is isomorphic to the Zariski bundle $(Spec_l((u)^{-1}R)_0,\tau_{Zar}),O)$.

If we consider Zariski topology, it is better to replace $\mathbf{Proj}_l(R)$ by the corresponding Levitzki spectrum $\mathbf{LProj}_l(R) = (X;Q_x:\mathcal{O}(X) \longrightarrow \mathcal{O}_x| \ x \in X)$, where, this time, $X = LProj(R):= grLU(R_+)$; $\mathcal{O}(X) \simeq gr_{\mathbb{Z}}R\text{-}mod/<U_l(R_+)>$; and Q_x is a localization of $gr_{\mathbb{Z}}R\text{-}mod$ at the Serre subcategory generated by all modules M such that the annihilators of nonzero elements of M do not contain x.

7.2.2.4. Example. Let R be a 'quantum space', i.e. an algebra over a field k generated by the indeterminates x_i, $0 \le i \le n$, subject to the relations:

$$x_i x_j = q_{ij} x_j x_i \tag{1}$$

where $0 \le i,j \le n$, and $q_{ij} \in k^*$ for all i, j. Taking the standart grading, we define the $\mathbf{LProj}_l(R)$. Clearly the ideal Rx_i is two-sided for all i. Therefore $\mathbf{LProj}_l(R)$ is locally affine. And the restriction of $\mathbf{LProj}_l(R)$ to $LU(Rx_i)$ is isomorphic to the affine scheme of the ring generated by the indeterminates $z_j = x_j/x_i$, $0 \le j \le n$, $j \ne i$, satisfying the (following from (1)) relations:

$$z_j z_m = (q_{mi} q_{jm} q_{ji}^{-1}) z_m z_j \tag{2}$$

$0 \le j,m \le n$, $j \ne i \ne m$. ■

7.2.3. Serre's theorem. Recall the description due to Serre of the category of quasi-coherent sheaves on $\mathbf{X} = \mathbf{Proj}(R)$, where R is a $\mathbb{Z}_+$-graded commutative noetherian ring generated as an R_0-algebra by R_1:

The category $Qcoh(\mathbf{X})$ of quasi-coherent sheaves on $\mathbf{X}$ is equivalent to the quotient category of $gr_{\mathbb{Z}}R\text{-}mod$ by the Serre subcategory $\mathbb{S}_+$ spanned on modules having finite number of nonzero component.

We shall see in a moment that there is a natural generalization of Serre's theorem to the noncommutative setting.

Note that the category $\mathbb{S}_+$ in the Serre's theorem is exactly the minimal Serre subcategory containing all graded modules annihilated by the ideal R_+. In other words, $\mathbb{S}_+$ is the Serre subcategory corresponding to the minimal radical filter $[R_+]$ containing R_+. The Serre's theorem holds if and only if $[R_+] = <U_l(R_+)>$. Recall that the category $<U_l(R_+)>$ is generated by all $\mathbb{Z}$-graded R-modules M such that $Supp(M) \subseteq V_l(R_+):= \{p \in Spec_l R \mid R_+ \subseteq p\}$, or, what is equivalent, by those graded modules M for which $grSupp(M) \subseteq grV_l(R_+)$.

We have the following analog of Lemma 6.5.4:

7.2.3.1. Lemma. *Suppose that the ring R is such that any nonzero graded R-module has an associated point. Then, for any radical filter F in grI_lR, the Serre subcategory $\mathbb{S}_F$ is generated by all graded modules M such that $Supp(M)$*

is contained in F.

In particular, the filter $[R_+]$ *coincides with* $<U_l(R_+)>$.

Proof repeats word by word the proof of Lemma 6.5.4. ■

A sufficient condition for nonzero modules to have associated points is the following version of noetherian property (cf. Lemma 6.5.6):

(Max≤) Any nonempty set of graded left ideals in R has a maximal element with respect to ≤.

If R is commutative, the property (Max≤) says exactly that R is noetherian which gives immediately the Serre's theorem.

But, in the noncommutative case, the class of ≤-noetherian rings was not properly studied yet, and there are no known methods of checking weather a ring is ≤-noetherian or not.

The situation with the using Lemma 7.2.3.1 directly is a little bit better. For instance, one can show that nonzero modules over a quantum space (cf. Example 8.2.4) do have associated points.

7.2.3.2. Note. The Serre's theorem (or equivalent to it description of the category of coherent sheaves on **X** as the quotient of the category of noetherian graded modules by the subcategory of noetherian modules with finite number of nonzero components [S]) is taken usually as a definition of the noncommutative projective spectrum (see [A], [M], [SmT], [Sm]). As Lemma 7.2.3.1 and the following discussion shows, the adopting such a definition means even more than just restricting to the noetherian case.

One of the advantages of our approach is that it does not require noetherian hypothesis of any kind. ■

APPENDIX: PRINCIPAL IDEAL DOMAINS, THE QUANTUM PLANE.

We consider here two simplest examples of computing the left spectrum. The first of them (principal ideal domains) proved to be very useful, the second one (the quantum plane) is ruther an illustration. The reader who is interested in 'real life' applications of the developed here (and in [R6]) spectral theory to the study of representations of algebras of mathematical physics is invited to look into [R3], [R4], and [R5].

A.1. The left spectrum of a principal domain. Recall that a ring R is called *a left and right principal ideal domain* if R is a ring without zero-divisors such that each left and right ideal of R is generated by one element.

A.1.1. Proposition. *Let R be a left and right principal domain. Then every nonzero ideal from $Spec_l R$ is equivalent to a left maximal ideal. Every left maximal ideal of the ring R is of the form $R\mathfrak{f}$, where $\mathfrak{f}$ is an irreducible element of the ring R.*

Proof. Let $p \in Spec_l R$. Since R is a left principal ideal domain, $p = R\mathfrak{f}$ for some element $\mathfrak{f} \in R$. It is easy to see that the absence of zero-divisors garantees that the right ideal $\mathfrak{f}R$ is proper.

In fact, if $\mathfrak{f}\mathfrak{q} = 1$ then $(1-\mathfrak{q}\mathfrak{f})\mathfrak{q} = \mathfrak{q}(1-\mathfrak{f}\mathfrak{q}) = 0$; therefore, $\mathfrak{q}\mathfrak{f}$ is also equal to 1; i.e. $p = R\mathfrak{f} = R$.

Being a proper ideal, $\mathfrak{f}R$ is contained in a right maximal ideal μ. Since R is a right principal ideal domain, $\mu = \mathfrak{q}R$ for some irreducible element $\mathfrak{q}$ of the ring R. The inclusion $\mathfrak{f}R \subseteq \mathfrak{q}R$ means that $\mathfrak{f} = \mathfrak{q}h$ for some h. Note that $h \notin p$.

Indeed,

$$[h \in p\ ] \Leftrightarrow [h = h'\mathfrak{f} \ \ for\ some\ \ \mathfrak{f} \in R] \Leftrightarrow [\mathfrak{q}h' = 1] \Leftrightarrow [\mu = \mathfrak{q}R = R]$$

Since $p \in Spec_l R$ and $h \notin p$, the left ideal $(p:h)$ is equivalent to p. Clearly $R\mathfrak{q} \subseteq (p:h)$. But $R\mathfrak{q}$ is a maximal left ideal (thanks to the irreducibility of $\mathfrak{q}$); hence $R\mathfrak{q} = (p:h)$.■

A.1.2. Lemma. *Let R be a left principal domain. Then every radical filter of left ideals is of the form F_S for some Ore multiplicative subset S.*

Proof. Let F be a localizing filter of left ideals. Denote by S the set of all the elements $t \in R$ such that $Rt \in F$. Since R is a left ideal principal ring, $(Rt:x) = Rt'$; i.e. for any $t \in S$ and any $x \in R$ there exist $y \in R$ and $t' \in S$ such that $t'x = yt$. The second Ore condition - *if $sa = 0$ for some $s \in S$ then there exists $s' \in S$ such that $as' = 0$* - holds automatically.

It remains to show that

$$if\ \ s, t \in S \ \ then\ \ st \in S;\ \ i.e.\ \ Rst \in F.$$

In fact, for any element $x = at$ of the left ideal Rt, we have:

$$(Rst:x) = ((Rst:t):a) \supseteq (Rs:a).$$

Since $(Rs:a) \in F$ for any $a \in R$, then $(Rst:x) \in F$ for any $x \in Rt$.

Therefore $Rst \in F$. ■

A.1.3. Topology τ_*. If a ring R is such that $Spec_l R/\approx$ coincides with $Max_l R/\approx$, then, obviously, a subset of $Spec_l R/\approx$ is closed in the topology τ_* (cf. 1.10.1) if and only if it is finite. In particular, this gives the description of τ_* when R is a left and right principal ideal domain.

A.1.4. Proposition. *Let R be a left and right principal ideal domain such that every nonzero element of R is the product of a finite number of irreducible elements. Then the topologies τ and τ_* coincide.*

Proof. Clearly, for an arbitrary ring R, every closed in the topology τ_* subset of $Spec_l R$ is also closed in the topology τ. To prove the inverse inclusion, it is sufficient to show that, under the conditions of the proposition A.1.4, each set of the form $V_l(m)$, for an arbitrary left ideal m of R, consists of a finite number of equivalence classes.

Since R is a principal left ideal domain, $m = Rq$ for some element q of the ring R. Consider the decompositon of q into irreducible factors: $q = q_1 \cdot \ldots \cdot q_n$. To this decomposition, there corresponds the sequence of monoarrows

$$R/Rq \rightarrowtail R/Rq^{(2)} \rightarrowtail \ldots \rightarrowtail R/Rq^{(n-1)} \rightarrowtail R/m \qquad (1)$$

where $q^{(\nu)} := q_1 \cdot \ldots \cdot q_\nu$.

Clearly

$$(R/Rq^{(i)})/(R/Rq^{(i-1)}) \cong R/Rq_i$$

is an irreducible module; i.e. (1) is a Jordan-Helder decomposition.

Now, note that the closed subset $V_l(m)$ coincides with $Supp(R/m)$; and

$$Supp(R/m) = \bigcup_{1 \le i \le n} Supp(R/Rq_i)$$

But $Supp(R/Rq_i)$ is the equivalence class of the left maximal ideal Rq_i. Therefore $V_l(m)$ consists of no more than n equivalence classes.

A.1.5. Localizations at points. Let $p \in Max_l R$; i.e. $p = Rq$ for some irreducible element q. The localization at $<p> = <Rq>$ can be obtained by inverting all the irreducible elements of R, which are not equivalent to q.

A.1.6. Localizations at open subsets. Now, let U be an arbitrary open subset of $Spec_l R$. By Proposition A.1.4, $U = Spec_l R - V$, where $V = \bigcup_{q \in X} V_l(Rq)$ for some finite set X of irreducible elements of the ring R. The localization at the open subset U is the inverting of all the elements q, $q \in X$.

A.2. Left normal morphisms and quantum plane. Let k be a field. The *quantum plane* is the k-algebra

$$k[x,y]_q = k<x,y>/(xy - qyx),$$

where $k<x,y>$ is the associative k-algebra freely generated by x, y.

A.2.1. Left spectrum of $k[x,y]_q$. Fix a left ideal p in $Spec_l k[x,y]_q$. There are the following possibilities:

(a) *p contains a nonzero polynomial in x;*

(a′) *p contains a nonzero polynomial in y;*

(b) $p \cap k[x] = \{0\}$;

(b′) $p \cap k[y] = \{0\}$.

Consider each of these alternatives.

(b′) Consider the localization $k[x,y]'_q$ of the algebra $k[x,y]_q$ at the set $k[y] - \{0\}$ of all the nonzero polynomials in y. This localization sends the ideal p into the ideal p' from $Spec_l k[x,y]'_q$. Now, note that the ring $k[x,y]'_q$ is an euclidean domain. In particular, $k[x,y]'_q$ is a ring of left and right principal ideals. This means that $Spec_l k[x,y]'_q$ consists of the principal ideals generated by irreducible elements of $k[x,y]'_q$.

In particular, $p' = k[x,y]'_q \cdot h$ for some irreducible element h of the algebra $k[x,y]'_q$.

(a) *and* (b′). The natural embedding $k[x] \longrightarrow k[x,y]_q$ is a left (and right) normal morphism (cf. **3.**). In particular, the intersection $p_x := p \cap k[x]$ is a prime ideal of the ring $k[x]$; i.e. $p_x = k[x]\mathfrak{f}(x)$, where $\mathfrak{f}(x)$ is an irreducible polynomial. Clearly $\mathfrak{f}(x)$ is an irreducible element of the ring $k[x,y]'_q$. Therefore the irreducible element h (cf. (a)) is equivalent to $\mathfrak{f}$; i.e. $p' = k[x,y]'_q \cdot \mathfrak{f}$.

(a) *and* (a′). Then the ideal p contains irreducible polynomials $\mathfrak{f}(x)$ and $\mathfrak{q}(y)$. In the commutative case, when $q=1$, every pair of irreducible polynomials, $<\mathfrak{f}(x),\mathfrak{q}(y)>$, defines a maximal ideal of the ring $k[x,y]$. It is not so in the noncommutative situation.

Suppose that the field k is algebraicly closed. Then $\mathfrak{f}(x) = x - c$ and $\mathfrak{q}(y) = y - d$ for some elements c, d of k . The ideal p contains, together with $\mathfrak{f}$ and $\mathfrak{q}$, the elements

$$y(x - c) = qxy - cy \quad \text{and} \quad xy - dx.$$

This implies that the element $qdx - cy$ belongs to p. And, therefore, the element

$$qdx - cy - qd(x - c) + c(y - d) = (q - 1)cd$$

belongs to p. Since the ideal p is proper, $(q - 1)cd = 0$; i.e. either $c = 0$ or $d = 0$ or both of them.

(b) and (b'). Then $p = p_h = k[x,y]'_q \cdot h \cap k[x,y]_q$, where h is an irreducible element of the ring $k[x,y]'_q$, which is not equivalent to any polynomial in x or in y.

A.2.2. The topology $\tau^\wedge$ and τ on $Spec_l k[x,y]_q$. Every proper closed subset of $(Spec_l k[x,y]_q, \tau^\wedge)$ is a finite subset of the cross $l_x \cup l_y$, where

$$l_x := \{(c,0) \mid c \in k\}, \quad l_y = \{(o,d) \mid d \in k\}.$$

The topologies τ and τ_* coincide. Moreover, every closed in the topology τ subset of $Spec_l k[x,y]_q$ is a finite union of the specializations of the points; i.e. the sets $\mathfrak{s}(p)$, $p \in Spec_l k[x,y]_q$.

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