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CYCLIC POLYGONS AS CRITICAL POINTS

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Abstract. We describe a nonconventional setting for studying cyclic polygons which provides a new point of view at the conjectures formulated in 90-s by D.Robbins and studied by R.Connelly, I.Pak, I.Sabitov and V.Varfolomeev. The exposition is centered around our interpretation of cyclic polygons as the critical points of oriented area considered as a function on the planar configuration space of the corresponding polygonal linkage. We present four general conjectures about critical points of area on configuration space and describe the state-of-the-art of the topic. In particular, we explain how one can count cyclic polygons with the fixed lengths of the sides using the so-called signature formulae for the mapping degree and Euler characteristic developed in our earlier papers. Connections of our results with formulae of Brahmagupta-Robbins type for areas of cyclic polygons are clarified and placed in a more general algebraic context. We also describe a general paradigm enabling one to prove nondegeneracy of critical points of area and briefly discuss the problem of calculating the Morse index of area at a cyclic configuration. In conclusion we mention a few extensions and generalizations for which our conjectures serve as a paradigm.

Key words: cyclic polygon, polygonal linkage, configuration space, oriented area, critical point, Euler characteristic, mapping degree, Brahmagupta formula, generalized Heron polynomial, Coulomb potential, tensegrity

Introduction

Cyclic polygons (i.e., polygons which can be inscribed in a circle) gained considerable attention in the last decade, partially due to the results and conjectures of D.Robbins [43] (see, e.g., [10], [19], [44], [51]). Recently, a number of new results on the geometry of cyclic polygons have been obtained in the framework of an approach suggested by the present author in [33], [34] which was based on the concept of *configuration space of polygonal linkage* (see, e.g., [11], [29]). In particular, it was shown that cyclic polygons can be interpreted as critical points of the oriented area considered as a function on the planar configuration space of the corresponding polygonal linkage [33], [35] and can be effectively counted or estimated using the so-called *signature formulae for topological invariants* [30], [17], [31]. Further results along these lines can be found in [18], [36], [22], [5], [41].

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The aim of the present paper is to describe the state-of-the-art of these topics and present a few (apparently) new observations and results. We begin by describing the setting and recalling necessary results from [34], [35]. Throughout the paper we freely use basic results about configuration spaces of polygonal linkages, which can be found in [11], [29], and a few standard paradigms of differential topology and singularity theory for which we refer to [26], [2]. We also need several results from [31] which are reproduced below for the sake of reader's convenience.

An important ingredient of our approach is the representation of configuration spaces considered as the fibres of proper quadratic mappings, which enables one to calculate their topological invariants and count critical points of regular functions using signature formulae from [17], [30], [31]. Specifically, we consider the oriented area as a function A on the planar configuration space $C_2(L)$ of a polygonal linkage L and embark on studying the critical points of A. One of the main outputs of our study is a general method of counting critical points of A which we describe in some detail with a view toward studying critical points of other natural functions on configuration spaces. For this reason we begin with recalling the aforementioned signature formulae and some of their applications in the context of configuration spaces of polygonal linkages. We also explain how these formulae can be applied to the study of so-called *cabled linkages* [37] and *tensegrity frameworks* [42] which do not seem to have been discussed in the literature from this point of view.

As was revealed in [33] and proven in full generality in [35], for a generic polygonal linkage L with non-singular configuration space $C_2(L)$, the critical points of A on $C_2(L)$ are given by the cyclic configurations of linkage L. This fact is central for our exposition so we present its generalization applicable to arbitrary polygonal linkages (Theorem 2.3) and a version for *open polygonal chains* (or *planar multiple penduli*) (Theorem 2.2) obtained in [36].

Motivated by these results and conjectures of D.Robbins, in Section 3 we present a few remarks on the geometry of cyclic configurations. Specifically, we show that coefficients of the generalized Heron polynomial introduced by D.Robbins can be algorithmically computed using the multidimensional logarithmic residue [50] (a version of the so-called Grothendieck residue symbol [31]). This enables us to formulate a general paradigm (Paradigm 1) which, to our mind, clarifies a number of results of I.Sabitov, I.Pak and V.Varfolomeev concerned with conjectures of D.Robbins on computation of areas of cyclic polygons [43].

In [33], [34] some examples and evidence were presented which encouraged us to conjecture that generically A is a Morse function on $C_2(L)$. This conjecture was further discussed in [36], where the parametric transversality theorem (see, e.g., [2]) was used to prove it for certain open polygonal chains (robot arms). In Section 4 we briefly describe the approach used in [36] and formulate another general paradigm (Paradigm 2) which, in particular, can be applied to a bunch of natural functions (of Coulomb electrostatic energy type) on configuration spaces of generic polygonal linkages. In the same section we present brief comments on the problem of calculating the Morse indices of A. In the last section we outline some of the possible generalizations and extensions of our results.

Most of the new results presented in these paper have been obtained during the author's visit to Max-Planck-Institut für Mathematik in Bonn in September-October of 2010. It's my pleasure to acknowledge the excellent working conditions and warm hospitality of the whole staff of MPIM.

1. Signature formulae and configuration spaces

The algebraic formulae for the mapping degree [17], [30] and Euler characteristic [30], [8] were developed in late 70-s and have found many applications to concrete problems of geometry, topology, singularity theory and nonlinear analysis (cf. [33]). We call them signature formulae [33] because they express the topological invariants in question in terms of signatures of effectively constructible quadratic forms. As was outlined in [30] and further developed in [8], [32], those formulae, in particular, provided an effective method for calculating the Euler characteristic of an explicitly given compact algebraic or semi-algebraic subset. It should be added that thanks to the existing computer programs for calculating the mapping degree (see, e.g., [38]) such calculations can nowadays be done quite effectively. We use these formulae to obtain information on the topological structure of configuration spaces of planar polygonal linkages (or *planar polygonal chains*) and count critical points of various differentiable functions on such spaces. To make exposition self-contained, at least formally, we recall now the main concepts and results related to signature formulae for the mapping degree [17], [30] and Euler characteristic.

It is convenient to begin with considering a real polynomial mapping

$$F : \mathbb{R}^s \to \mathbb{R}^t$$

defined by collection of t polynomials $F_1, \dots, F_t$ in s variables with real coefficients. We will only deal with the cases when $s \geq t$. If $s = t$, then F is called a *polynomial endomorphism* (or a *polynomial vector field* as in [2]). Let us for completeness fix some notation and conventions.

For any $y \in \mathbb{R}^t$, the set $F^{-1}(y)$ is called a fiber of F over the point y. A point $x \in \mathbb{R}^s$ is called a regular point of F if the rank of Jacobi matrix $J(F)(x)$ is maximal, i.e., equal to t (for $s = t$ this is obviously equivalent to $\det JF(x) \neq 0$). In the opposite case (i.e., if this rank is less than t) point x is called a singular point of F. A fiber $F^{-1}(y)$ is called regular (or smooth) if it does not contain singular points of F. In this case the point y is called regular value of F. The set of regular values of F is denoted $\operatorname{Reg} F$.

As is well known, each regular fiber is a smooth manifold of dimension $s-t$ [6]. A mapping F is called *proper* if preimage $F^{-1}(X)$ of any compact set $X \subset \mathbb{R}^t$ is a compact set in $\mathbb{R}^s$ (here and below we only consider the topologies induced by Euclidean metric). For convenience and brevity, a (real) *proper polynomial mapping* F as above will be referred to as a *propomap.* If $s = t$ then we'll speak of a *propofield.*

Thus the fibers of a propomap are compact algebraic varieties. Correspondingly, regular fibers of a propomap F are compact smooth manifolds. By Sard's lemma [2], the set of singular values of F has measure zero so a "generic" fiber of F is a smooth compact manifold of dimension $s-t$.

In particular, if $s = t$ then each fiber $F^{-1}(y)$ consists of a finite amount of points and it appears reasonable to consider the algebraic number of preimages of y, which leads to the concept of the mapping degree. More precisely, one can define the *mapping degree* $\deg F$ by the well-known formula

$$\deg F \;=\; \sum_{x\in F^{-1}(y)} \operatorname{sign}\det JF(x), \tag{1.1}$$

where sign denotes the sign of a real number, and $y \in \operatorname{Reg} F$ is an arbitrary regular value of F.

An important role is also played by a local version of this notion, the *local (mapping) degree* of an endomorphism which is often called the *index of an isolated zero* of a vector field [2]. The local degree $\deg_p F$ of a given polynomial endomorphism F is defined at any point p which is isolated in $F^{-1}(F(p))$. What is especially important, from the results of [17] and [31] it follows that $\deg_p F$ can be computed in a purely algebraic way as the signature of a certain non-degenerate quadratic form which is explicitly constructible using the coefficients of components F_i of F. The same refers to the (global) topological degree $\deg F$. This method of computing topological degree can be implemented as a computer algorithm based on the computation of a Gröbner basis of the ideal (F) generated by the components of F in the algebra of real polynomials in s indeterminates (see, e.g., [38]).

In the sequel we need to refer to a general result stating that the Euler characteristic of any fiber of a given propomap can be expressed through the local degree as follows. For our purposes it is sufficient to deal only with the case when all components of the map are polynomials of the same (algebraic) degree.

Let $F : \mathbb{R}^s \to \mathbb{R}^t$ be a propomap with the components F_i of degree d. Let $y \in \mathbb{R}^t$, $X_y = F^{-1}(y)$, and $f_i = F_i - y_i, i = 1, \dots, t$. Notice that X_y is a compact real algebraic variety so its Euler characteristic is well-defined [6]. Set

$$h_i(x_0, \dots, x_s) = x_0^{d+1} f_i(\frac{x_1}{x_0}, \dots, \frac{x_s}{x_0}), H_y = \sum_{j=1}^{t} h_i^2 - \sum_{k=0}^{s} x_k^{2d+4}.$$

Denote by $\mathrm{grad}\, H_y$ the gradient of H_y considered as a polynomial endomorphism of $\mathbb{R}^{s+1}$.

Theorem 1.1. *([30], [8]) With assumptions and notation as above, polynomial H_y has an isolated critical point at the origin and the Euler characteristic of X_y is given by the formula*

$$2\chi(X_y) = (-1)^s - deg_{\mathbb{O}}\, grad\, H_y, \tag{1.2}$$

where $deg_{\mathbb{O}}\, grad\, H_y$ denotes the local degree of $grad\, H_y$ at the origin of $\mathbb{R}^{s+1}$.

Since the local degree is equal to the signature of the so-called Gorenstein quadratic form on the local algebra of mapping [17], [31], this theorem shows that the Euler characteristic is expressible through the signature of an explicitly constructible quadratic form. For this reason, we refer to formula (1.1) as the signature formula for the Euler characteristic. We will also need a similar result for a compact *semi-algebraic* subset which can be proven using formula (1.1) and a standard process of eliminating inequalities by considering two-fold coverings branched along a subvariety [6], [31].

Theorem 1.2. *([31], [16]) The Euler characteristic of an explicitly given compact semi-algebraic subset can be calculated in algorithmic way in terms of signatures of explicitly constructible quadratic forms.*

We now introduce the setting of configuration spaces of planar polygonal linkages which we are going to deal with. Mechanical linkages, in particular planar mechanical linkages, and their configuration (moduli) spaces were studied in many papers. Linkages may be thought of as mechanisms build up from rigid bars (sticks) joined at flexible links (pin-joints). Some links may be fixed in the ambient space and the rest are supposed to be movable. In many problems it is important to know the totality of possible positions of the links in the ambient space which led to a (nowadays classical) mathematical definition of configuration space of a mechanical linkage discussed in big detail in [11], [28]. In some problems related to modelling *conformations of molecules* it appears natural to use a slightly more general concept defined in terms of weighted graphs. As will be mentioned in the sequel, some of our constructions and results make sense in this general setting so we recall the corresponding concept for reader's convenience.

Recall that a *weighted graph* is defined as a triple $\Gamma = (V, E, d)$ consisting of a set of vertices V, a set of edges $E = \{(V_{i_k}, V_{j_k})\}$, and a weight function $d : E \to \mathbb{R}_+$ which assigns to every edge (V_{i_k}, V_{j_k}) certain non-negative number (length) $d(V_{i_k}, V_{j_k}) \in \mathbb{R}_+$. We always assume that Γ is connected, i.e., each pair of its vertices can be connected by a sequence of elements of E.

A connected weighted graph is called *N-realizable* (or realizable in $\mathbb{R}^N$) if there exists a mapping $f : V \to \mathbb{R}^N$ such that the Euclidean distance

$|f(V_i) - f(V_j)|$ is equal to $d(V_i, V_j)$ for all $(V_i, V_j) \in E$ and each such mapping f is called N-realization of Γ. Often by N-realization is simply understood the corresponding collection $(f(V_1), \dots, f(V_n)) \in \mathbb{R}^{nN}$ such that $|p_i - p_j| = d(V_i, V_j)$ if $(V_i, V_j) \in E$ but we prefer to use the term *configuration* of Γ.

A connected weighted graph Γ is called a *planar mechanical linkage* if it is realizable in $\mathbb{R}^2$. For example, a regular tetrahedron is (by definition) 3-realizable but not 2-realizable.

The N-th configuration space of Γ can be defined as

$$C_N(\Gamma) = \{\text{N-realizations of } \Gamma\}/Iso_+(\mathbb{R}^N),$$

where $Iso_+(\mathbb{R}^N)$ denotes the group of all orientation preserving isometries of $\mathbb{R}^N$ and the factor is taken with respect to its obvious diagonal action on N-realizations. For $N = 2$ and $N = 3$ we speak of planar and spatial configuration space, respectively.

In other words, we factor out the motions of a realization as a rigid whole. There are versions of this definition where factoring is over the group of all (not necessarily orientation preserving) isometries or over the extension of $Iso_+(\mathbb{R}^N)$ by homotheties. These differences are inessential for our tasks so we use the simplest definition presented above. Notice that, using the second interpretation of N-realizations, the same space can be defined as

$$C_N(\Gamma) = \{x \in \mathbb{R}^{nN} : |x_i - x_j| = d(V_i, V_j) \text{ if } (V_i, V_j) \in E\}/Iso_+(\mathbb{R}^N),$$

where $Iso_+(\mathbb{R}^N)$ is the group of orientation preserving isometries of $\mathbb{R}^N$ acting diagonally on $\mathbb{R}^{nN}$.

Configuration spaces are sometimes called the *moduli spaces* of mechanical linkage [29] but we do not use this term in order to avoid misunderstanding because it has quite a number of different meanings. All configuration spaces are considered with a natural topology induced by the Euclidean distance. If the linkage is just a polygon with the fixed sidelengths one obtains configuration spaces of polygon studied in [49], [29].

From a topological point of view, planar configuration spaces are especially interesting because it turned out that for any smooth *closed* (i.e., compact and without boundary) manifold M there exists a planar mechanical linkage Γ such that one of the components of configuration space $C_2(\Gamma)$ is diffeomorphic to M [49], [29]. In many cases Γ can be chosen so that $C_2(\Gamma)$ is connected and itself diffeomorphic to M.

Given a graph with a fixed combinatorial structure, one obtains a mechanical linkage by choosing a N-realizable weight function d as above. We say that some property of a mechanical linkage is *generic* if it holds for almost all choices of the weight function. Using Sard's lemma it is easy to see that, for a generic mechanical linkage Γ, the N-th configuration space $C_N(\Gamma)$ is a compact smooth manifold of the dimension $Nn - \operatorname{card} E - \frac{1}{2}N(N+1)$, where n is the number of vertices of Γ and card E is the total number of edges. Configuration spaces of planar polygons, which gained a lot of attention in last two decades [49], [29], appear as particular case of this general definition. Recall that Euler characteristics of configuration spaces of polygons were calculated in many cases (see, e.g., [27]). Our first observation is that, in principle, all such results could be obtained using Theorem 1.1.

Corollary 1.1. *The Euler characteristic of N-th configuration space of any mechanical linkage $\Gamma = (V, E, d)$ as above can be calculated in an algorithmic way from its combinatorial data (V, E) and weight function d.*

This easily follows from the fact that configuration space is naturally represented as the fiber of a certain explicitly given quadratic mapping Q_Γ combined with Theorem 1.1. The components of the mapping Q_Γ are obtained by writing down (in canonical coordinates on $\mathbb{R}^{nN}$) the conditions that the squared distance from $f(V_i)$ to $f(V_j)$ should be equal to $[d(V_i, V_j)]^2$ for each edge $(V_i, V_j) \in E$. Obviously one has exactly $t(\Gamma) = \operatorname{card} E$ such conditions. The action of $Iso_+(\mathbb{R}^N)$ can be used to place one of the vertices at the origin of $\mathbb{R}^N$ and direct one of the edges starting from this vertex along the first coordinate axis. Then it is obvious that $C_N(\Gamma)$ can be identified with the set of all N-realizations of Γ satisfying the above normalization. Thus Q_Γ naturally emerges as a quadratic mapping from $\mathbb{R}^{N(n-2)}$ to $\mathbb{R}^{t(\Gamma)-1}$. For evident geometric reasons, Q_Γ is proper and so one can directly apply formula 1.2 by taking $F = Q_\Gamma$. The result follows by noticing that in virtue of [17], [31] the local topological degree in the formula (1.2) can be calculated in algorithmic way. It may be added that the Euler characteristic of any fiber of Q_Γ can also be calculated using results of [1].

This result has several concrete applications. As was shown in [34], it enables one to list all possible topological types of configuration spaces of planar pentagons. Using the scheme of [34] one can also describe all possible topological types of configuration spaces for planar and spatial quadrilaterals and for mechanical linkages with two-dimensional configuration spaces [33], [21].

Remark 1.1. Since configuration spaces of linkages can be represented as the fibres of quadratic mappings it is worthy of mentioning that, for quadratic mappings, another way of calculating the Euler characteristics of fibers was proposed in [1]. However we are not aware of computer implementations of this method so in concrete situations one has to use Theorems 1.1 and 1.2.

Recently, we realized that similar results can be obtained for the so-called *cabled linkages* [37] and *tensegrity linkages* [42], which give a natural mathematical framework for investigating certain mechanical constructions. Informally, tensegrity linkages are rigid (non-deformable) spatial (3d) constructions consisting of pin-jointed bars, struts and cables (see, e.g., [42]). Real life prototypes of this concept are often called *truss structures* [42]. A rigorous general definition of tensegrity linkage is somewhat involved and not quite relevant to our main topics so we'll stick to the framework of cabled linkages.

It is possible to say that cabled linkages are obtained from spatial linkages by substituting some of the edges by cables instead of rigid bars where it is assumed that cables are flexible but nonexpandable. Realizations of a cabled linkage are defined using an obvious modification of the above definition, the only difference being that, for each cable element, one imposes the condition that the distance between corresponding vertices does not exceed the length of this cable. Subclass of tensegrity linkages can be (informally) characterized by existence of rigid configurations. Instead of explicating the sense in which rigidity should be understood we adopt a convention that a cabled linkage will be called a tensegrity linkage if it has a non-empty finite set of 3-realizations (cf. [42]).

Examples of real life prototypes of tensegrity linkages were constructed by Kenneth Snelson in 1948, which has eventually led to appearance of the above general concept, particularly due to the activity of Buckminster Fuller (who also stimulated investigation of molecular conformations nowadays called fullerenes). Main applications of tensegrity linkages are in architecture, engineering and chemistry while the mathematical aspects of the topic seem to be less developed [42]. As is easy to understand, our approach enables one to find the number of 3-realizations of a tensegrity linkage with prescribed combinatorial structure and fixed lengths of bars and cables.

To this end, notice first that the combinatorial structure of a cabled linkage T can be described by a weighted graph as above. Next, the fact that cabled linkages are spatial structures corresponds to considering the 3-realizations of a given graph in $\mathbb{R}^3$. Thus it is appropriate to deal with the 3-rd configuration space, i.e., take $N = 3$. Finally, we can also formalize the fact that there are edges of two kinds. Namely, divide E in two subsets E_b (bars) and E_c (cables) and modify the above definition of configuration space as follows. The conditions corresponding to edges from E_b remain unchanged, while for each pair $(V_i, V_j) \in E_c$ one imposes a quadratic inequality $|x_i - x_j|^2 \leq [d(V_i, V_j)]^2$ instead of the corresponding equality. In this way we obtain a semi-algebraic set called the configuration space $C(T) = C_3(T)$ of a cabled linkage T. The above definition just means that, for a tensegrity linkage, $C(T)$ is a finite semi-algebraic set. Hence the number of 3-realizations (or stable configurations) of T is equal to the Euler characteristic of $C(T)$. These remarks combined with Theorem 1.2 immediately yield an analog of Corollary 1.1 for tensegrity linkages.

Corollary 1.2. *The number of distinct* 3*-realizations of a tensegrity linkage with given combinatorial type and lengths of elements, can be algorithmically computed in terms of signatures of explicitly constructible quadratic forms.*

Since it is recognized that the problem of counting all possible configurations of tensegrity linkage is far from trivial [42], this result may appear useful for investigating concrete tensegrity linkages. In fact, one may speak of a certain analogy between 3-realizations of a tensegrity linkage and cyclic configurations of polygonal linkages because all of them can be interpreted as critical points of certain functions on the corresponding configuration space. However we will not develop this analogy here.

Concluding this section we wish to add that similar results can be obtained for configuration spaces of various other types. For instance, one may consider flexible polyhedra or spatial polygonal chains where each two adjacent faces can be rotated around their common side. One can also consider linkages such that certain vertices can slide along prescribed subspaces or submanifolds of the ambient space (in other words, possible positions of those vertices should belong to prescribed submanifolds). Such situations arise in circle packing problems [46] (the author owes this indication to E.Wegert) and in a well-known topic of numerical integration concerned with the concept of n-design [14]. In all those cases one can find the Euler characteristic of configuration space using the formulae presented above but discussion of these extensions is irrelevant to our aims.

2. Oriented area as a function on configuration space

We proceed by investigating the oriented area of planar polygon (see, e.g., [15]) as a function on configuration space of a planar linkage. Recall that given an ordered set of n points $p_1 = (x_1, y_1), \ldots, p_n = (x_n, y_n)$ in the plane, the oriented area A of corresponding n-gon is defined by the formula

$$2A = (x_1 y_2 - y_1 x_2) + \ldots + (x_n y_1 - y_n x_1). \tag{2.1}$$

In this section we use symbol $C_2(n)$ to denote the configuration space of planar n-gon with unspecified but fixed sidelength vector l. In other words, $C_2(n)$ is the collection of all possible planar configurations of n-gon linkage. In this context "n-gon linkage" means essentially the same as "n-gon with fixed (prescribed) sidelengths".

We wish to deal first with *generic sidelengths*, which as usual means that vector of sidelengths l is taken from an (unspecified) open dense subset U_n of the parameter space of planar n-gons. In the case of planar polygonal linkages as above, the parameter space is an open subset P_n of the positive n-orthant $\mathbb{R}^n_+$ and one can explicitly indicate a collection of hyperplanes B_n, called the bifurcation diagram of configuration space $C_2(n)$, such that the aforementioned open subset U_n of generic sidelengths is equal to the complement of B_n in P_n. Thus most of our considerations make sense for planar polygons with the sidelength vector belonging to U_n. The connected components of U_n, called chambers, play an essential role in the topological study of configuration spaces [29].

By what was said above, for generic sidelengths (in the above sense), the configuration space $C_2(n)$ is a compact orientable smooth manifold of dimension n–3. Moreover, it is known that the topological type (homeomorphism class) of $C_2(n)$ is the same for all sidelengths belonging to the same chamber [29]. As was explained, $C_2(n)$ can be defined as a fibre of a proper quadratic mapping $Q_l : \mathbb{R}^{2n-4} \to \mathbb{R}^{n-1}$ and the bifurcation diagram B_n can be identified with the set of singular values of Q_l [29].

To make this quite precise, notice that the action of $Iso_+(\mathbb{R}^2)$ obviously does not change the totality of configurations of a planar linkage. Thus we can change its position in the plane using the action of this group. In this way, we obviously can place the first vertice at the origin and the first side along the Ox-axis. It is also easy to realize that a homothety transformation of polygon does not change the topological type of its configuration space. Thus we may assume that the first side is of length 1, i.e. $l_1 = 1$, without loss of generality. It follows that we may assume that the first two vertices are $v_1 = (0,0), v_2 = (1,0)$ and we keep this assumption from now on. This done, it becomes obvious that the configuration space can be defined as a fibre of a quadratic mapping as above.

In order to complete the description of our setting, notice that the oriented area of polygon naturally defines a (infinitely) differentiable function $A : C_2(n) \to \mathbb{R}$ on each configuration space. Our aim is to study the critical points of A for a generic sidelength vector belonging to U_n. For $n = 4$ (planar quadrilaterals) and $n = 5$ (planar pentagons), it is easy to obtain considerable information about singularities of A using the signature formulae. To describe our approach in a consistent way, let us present relevant results about the topological structure of configuration spaces $C_2(4)$. In doing so we freely use standard topological concepts and methods of singularity theory [2]. The topological structure of configuration spaces of generic quadrilaterals can be described as follows.

Proposition 2.1. *The bifurcation diagram B_4 consists of eight hyperplanes $\{c_1a_1 + c_2a_2 + c_3a_3 = 1, c_i = \pm 1\}$ and its complement in P_4 has eight connected components.*

Proposition 2.2. *Each generic configuration space $C_2(4)$ is homeomorphic either to circle S^1 or to the disjoint union of two circles.*

Actually, one can also describe all possible topological types of $C_2(4)$ for non-generic sidelengths as well and the signature formulae appear helpful to this end. For example, using the method of Section 2 one easily finds that the Euler characteristic of the planar configuration space of square is equal to -3. With some additional work it can be shown that this space is homeomorphic to the union of three circles each pair of which has a common point, which of course gives the same value of the Euler characteristic. In fact, performing similar calculations for each component of U_4 one can show that the Euler characteristic of moduli spaces of non-degenerate planar quadrilaterals can only take four values: $0, -1, -2, -3$ (formally, one could also consider degenerate quadrilaterals where the length of the longest side is equal to the sum of the other three sidelengths in which case the configuration space is obviously one point). Moreover, one can describe the local geometric structure of possible singularities of $C_2(4)$ so that the topology of $C_2(4)$ is known in great detail, which gives a sufficient background for determining the global behaviour of area function on such configuration spaces.

Let us now count the critical points of A. Assume as above that $l_1 = 1, v_1 = (0, 0), v_2 = (1, 0)$. Then $C_2(4)$ is defined by three obvious quadratic equations with four unknown coordinates of movable vertices v_3, v_4. The area function in this case is given by $2A = x_3y_4 – x_4y_3 + y_3$. One can now introduce Lagrange multipliers c_1, c_2, c_3 and obtain a (7×7)-system of quadratic equations, (first four coordinates of) solutions to which give the critical points of A. It is quite easy to compute the jacobian of these equations and show that the equations are algebraically independent and the set of solutions is finite. Hence the number of critical points of A is also finite and equal to the number of real solutions to this system. Since the Euler characteristic of solution set can be effectively computed using the signature formulae, we conclude that the number of critical points of A can be found for each concrete sidelength vector $l \in U_4$. Using standard topological arguments it is easy to show that the qualitative behaviour of A remains the same for each component of U_4. Thus to achieve a complete investigation of area function it is sufficient to choose a vector of sidelengths in each component of U_n and find the number of critical points for those concrete sidelengths using Theorem 1.1.

Realization of this program yields the following results. If a generic configuration space has one component (homeomorphic to circle), then the number of critical points is equal to two (one maximum and one minimum). If a generic configuration space has two components, then the number of critical points is equal to four (one maximum and one minimum on each component). It can be verified that all these critical points are non-degenerate and so A is a Morse function on $C_2(4)$.

One can further explicate this conclusion as follows. According to a classical result of J.Steiner, each polygonal linkage L has a convex cyclic configuration and A attains its maximum, say M, precisely at this configuration (see, e.g., [19]). Due to the skew-symmetry of oriented area, the minimum value of A on $C_2(L)$ is $-M$ and it is attained at the configuration obtained from the preceding one by reflection in the first side of linkage L, which in our setting obviously reduces to changing the signs of ordinates of all vertices (since the first side is placed on the Ox-axis). Moreover, one can express M in terms of sidelengths by Brahmagupta formula and get:

$$M^2 = \frac{1}{16}(2a^2b^2 + \ldots + 2c^2d^2 - a^4 - b^4 - c^4 - d^4 + 8abcd), \tag{2.2}$$

where a, b, c, d are the sidelengths and p is the half-perimeter of L [43].

Next, using Lagrange multipliers it is fairly easy to show that each critical point is given by a cyclic configuration (cf. [34] for details). If the configuration space has two components, then A is positive on one of them and negative on another one. In this case there also exists a self-intersecting cyclic configuration of L which corresponds to the minimum of A on the positive component. It is known (see, e.g., D.Robbins [43]), that the area of a self-intersecting cyclic configuration is given by the following analog of Brahmagupta formula:

$$m^2 = \frac{1}{16}(2a^2b^2 + \ldots + 2c^2d^2 - a^4 - b^4 - c^4 - d^4 - 8abcd). \tag{2.3}$$

Thus the minimum of A on the positive component can also be computed in terms of sidelengths. Taking into account that m should be real, the latter formula can only be applied when the right hand side is positive. It's easy to verify that the r.h.s. is positive if and only if the configuration space has two components. Elementary as it is, this observation seems remarkable since it establishes a precise relation between the topology of configuration space and cyclic configurations.

Next, with a little more work one can verify that, for generic sidelengths, the critical points of A are non-degenerate, i.e., A is a Morse function on $C_2(4)$, and their indices (in the sense of Morse theory) can be read off their geometry (shape). These observations were in fact quite instructive because their formulations were independent of the number of sides of linkage and they fitted nicely to some results and conjectures of D.Robbins. So it became tempting to conjecture that similar statements remain true for linkages with arbitrary number of sides [33]. Namely, the following four conjectures have been formulated in [33].

Conjectures CA. *For a generic n-gon linkage L with smooth planar configuration space $C_2(L)$, the following four statements hold true:*

(CA1) the critical points of area on $C_2(L)$ are given by cyclic configurations of L;

(CA2) the critical values of area can be calculated as the roots of a certain explicitly constructible polynomial;

(CA3) the critical points are non-degenerate;

(CA4) the Morse indices of cyclic configurations can be read off their shape.

Let us at once add that these conjectures obviously make sense for a *planar open polygonal chain* (or *planar multiple pendulum*), in which setting they can be (conveniently) denoted as (CA1*, ... , CA4*). Notice that in the case of an open n-gonal chain the configuration space is (obviously) diffeomorphic to $(n-1)$-torus T^{n-1}. It should also be noted that there is no direct reduction of one setting to another and in fact there are essential differences between these two settings. In this paper we basically confine ourselves to the case of polygonal linkage and often speak of a *closed polygonal chain* in order to make more visual the parallels and differences between two settings. The results on open chains will be mentioned only occasionally.

All these conjectures appeared to be true (some under additional assumptions) in considerable generality and our main aim here is to describe these results and present a number of new ones related to conjectures (CA1-CA4). In fact, we believe that they may serve as a (sort of) paradigm for studying the critical points of other geometrically or physically meaningful functions on configuration spaces, like oriented volume, Coulomb energy of unit charges placed at vertices, sum of (or sum of pairwise products of) diagonals or normalized determinant considered by M.Atiyah [3] (more detailed comments on such perspectives are given in the last section). Having in mind these and some other generalizations, in this paper we avoid going into details specific for polygonal linkages and concentrate on more general aspects of the topic.

It should be noted that (CA2) is in fact equivalent to a (weakened form of) conjecture formulated by D.Robbins in [43] as a statement concerned with calculation of the areas of cyclic polygons in terms of the lengths of their sides. However D.Robbins did not use the concepts of linkage and configuration space so the aforementioned relation between the two conjectures can only be stated after having proven (CA1).

We now briefly describe the state-of-the-art regarding the four conjectures above and then present more extended comments and new developments. (CA1) has been proven [35] for arbitrary n. The Robbins conjecture has been thoroughly studied and proven in full generality in [44], [10], [19]. Thus our proof of (CA1) automatically implies validity of (CA2) for generic polygonal linkages with arbitrary number of sides. (CA1*) has been proven for generic n-arms with n arbitrary in [36] (even in a bit more precise form). Thus (CA2*) also holds true in full generality. Thus the first two conjectures hold true for generic closed and open planar polygonal chains with arbitrary number of sides.

The third pair of conjectures appeared more hard, although there is practically no doubt that they are also correct. (CA3*) has been proven in [36] for triple planar penduli using the *parametric transversality* theorem (or paradigm) described, e.g., in [2]. The argument used in [36] is obviously applicable for arbitrary n as well as for closed chains but the calculations become too involved already for quadruple penduli and we failed to see the pattern for general argument. So maybe it is not reasonable to apply this method in straightforward way in general case and one should look for a more ingenious approach. However one can prove (CA3) for generic quadrilateral linkages by a straightforward calculation of Hessian of area.

Situation with (CA4) and (CA4*) was obviously even more complicated since one should first invent a way of calculating indices of cyclic configurations from their geometry and this was by no means obvious. For $n = 4$ and $n = 5$ the indices have been calculated in [33], [18], but the form of those results did not suggest a reasonable general algorithm. Such an algorithm has been recently announced in [41] with outline of proof and the result looks quite plausible but one should still wait for a detailed proof, which is promised to follow soon. Granted this it should be already much easier to work out a modification of the algorithm and prove its validity for arbitrary planar multiple penduli. Thus there is good evidence that all these conjectures are correct.

Quite complete and satisfactory as they are, these results by no means exhaust the suggested paradigm but rather suggest further perspectives as we are going to show in the sequel. Before passing to generalizations we present a few more comments. Recall that the following two basic results were obtained in [35] and [36], respectively.

Theorem 2.1. *(*[35]*) For a generic n-gon linkage L with nonsingular planar configuration space, all critical points of A on $C_2(L)$ are given by the cyclic configurations of L.*

In order to formulate an analogous result for planar (robot) n-arms (or multiple planar penduli [36]) we need an ad hoc definition. For each configuration $v_1, \dots, v_n$ of a planar n-arm define the *connecting side* as the segment $v_n v_0$. A cyclic configuration of a planar n-arm is called diacyclic if the center of its circumscribed circle lies on the connecting side (thus $v_n v_0$ is a diameter of the circumscribed circle). Obviously, the planar configuration space of n-arm is diffeomorphic to torus T^{n-1}.

Theorem 2.2. *(*[36]*) For a generic planar n-arm R, all critical points of A on $C_2(R)$ are given by the diacyclic configurations of L.*

Both these theorems were proved by geometric methods but since all objects are of algebraic nature we had in mind finding purely algebraic proof, which appeared possible and suggested an algebraic reformulation of the basic result. The main idea of the algebraic proof is to compare the polynomial system defining the cyclic configurations with the one obtained from the Lagrange multipliers method.

It is easy to see that the critical points of A in a generic configuration space $C_2(n)$ can be counted as the real solutions to a certain $(2n-4)\times(2n-4)$-system S_l of polynomial equations depending on parameters l_i. Indeed, according to Lagrange rule the gradient ∇A at a critical point should be linearly dependent with the gradients of defining quadratic equations $g_i = l_i^2, i = 1, \dots, n-1$. In other words, the rank of Jacobi matrix $(\nabla g_1, \ldots, \nabla g_{n-1}, \nabla A)^T$ should be equal to $n-1$, which is equivalent to vanishing of all of its $(n \times n)$-minors. Since the number of variables is $2n-4$, generically this can be expressed by vanishing of any collection of $n-3$ minors. Joining the arising $n-3$ polynomial equations to the defining equations $g_i = l_i^2$ we obtain a system S_l mentioned above. Notice that the left-hand-sides of this system do not depend on the sidelengths l_i.

Analogously, using the well-known determinantal criterion of concyclicity for four points (see, e.g., [13]) it is also easy to see that the cyclic configurations correspond to the roots of another $(2n-4) \times (2n-4)$-system T_l of polynomial equations in the same $2n-4$ unknowns. By Theorem 2.1, for all generic values of parameters l_i, the projections of sets of real solutions to these two systems on the ambient space of linkage coincide: $P(Z_R(S_l)) = P(Z_R(T_l))$. In view of Nullstellensatz this fact indicates that there exists some kind of strong relation between the two systems of equations and it is natural to have a closer look at the ideals generated by their left-hand-sides.

The essence of matters can be clearly seen in the case of quadrilateral. The Lagrange condition in this case is expressed by vanishing of the determinant of Jacobi matrix $(\nabla g_1, \nabla g_2, \nabla g_3, \nabla A)^T$. Thus the first system S_l in this case is obtained by adding just one polynomial equation $\{P_1 = 0\}$ of algebraic degree four to the defining equations of configuration space. At the same time, the aforementioned determinantal concyclicity criterion for four points is also expressed by a polynomial equation $\{P_2 = 0\}$ of (algebraic) degree four (all equations are written in terms of Cartesian coordinates of movable vertices). A direct computation shows that P_1 is a scalar multiple of P_2, which means that all critical points satisfy the concyclicity condition and vice versa. Notice that this conclusion now holds for all sidelengths and not only for generic ones. Moreover, we see that the ideals $I(S_l)$ and $I(T_l)$ generated by the left-hand-sides of both systems in the polynomial ring $\mathbb{R}_4$ coincide, which gives us a pattern for the general case.

Proposition 2.3. *For each quadrilateral linkage, one has the equality of ideals* $I(S_l) = I(T_l)$ *in* $\mathbb{R}_4$.

It is now natural to extend our discussion of critical points of A to the case of a singular configuration space. Taking into account that the underlying space is a real algebraic variety with isolated singularities it becomes possible to use the setting and results of the stratified Morse theory [23]. The general definition of critical point from [23] in our situation reduces to the zero-set of the ideal S_l. Finally, remembering that the concyclicity condition is also fulfilled if the four points lie on the same straight line, we arrive to the following result.

Proposition 2.4. *For each quadrilateral linkage Q, all critical points of A on the planar configuration space $C_2(Q)$ are given by either cyclic or aligned configurations of Q.*

To show that this explication is essential, it is sufficient to look at the set of critical points of area on the configuration space of a rhomboid (all four sides of equal length). As is easy to see, it consists of three aligned configurations (one of which is cyclic as well), four open arcs of cyclic configurations and two isolated cyclic configurations.

To generalize the above discussion to n-gonal linkages with arbitrary n, let us define the two ideals $I(S_l)$ and $I(T_l)$ in $\mathbb{R}_{2n-4}$ as the corresponding Fitting ideals of the two matrices introduced above. Now, using Proposition 2.3, specific sparse structure of the two matrices and induction one easily arrives to the desired generalization of Theorem 2.1.

Theorem 2.3. *For arbitrary n, the ideals $I(S_l)$ and $I(T_l)$ coincide.*

One can now derive various conclusions in the spirit of Proposition 2.4 which we will not dwell upon. Instead we notice that the same reasoning enables one to obtain a similar extension of Theorem 2.2. However there arise some nuances related to the condition of diacyclicity which we do not wish to discuss here so we omit the corresponding statement for open linkages.

Returning to the case of quadrilateral notice that, taking into account the squares of sidelengths in the r.h.s. of the above equations, each of the two systems is a so-called *free term deformation* [2] of the same proper polynomial endomorphism F_4 of $\mathbb{R}^4$. In view of discussion in Section 1, its mapping degree $\operatorname{Deg} F_4$ is well-defined. For quadrilateral linkage this does not give anything interesting, since one can use our formula (1.2) to show that $\operatorname{Deg} F_4 = 0$. However, notice that similar considerations yield a proper polynomial endomorphism F_n for arbitrary n. It would be interesting to find out if its degree is sometimes non-zero and if so, what useful information one can derive from the value of its degree.

Remark 2.1. The above comments show that polygonal linkage itself can be considered as a *deformation* of a certain nonhomogeneous real quadratic mapping $Q : \mathbb{R}^{2n-4} \to \mathbb{R}^{n-1}$. One can complexify it and it's easy to check that it has an isolated singular point at the origin. Thus its Milnor number is well-defined and can be computed by the well-known formula due to Lê [39]. It is now interesting to investigate how this can be used to obtain topological information on configuration spaces of n-linkages (say, to obtain an estimate for the sum of Betti numbers of configuration space).

3. Areas of cyclic polygons

We now intend to describe some applications of the above constructions and results to the study of cyclic polygons by their own in the spirit of conjectures of D.Robbins [43]. Recall that the problem of counting and constructing cyclic polygons was considered in [43], [51]. The geometric results contained in those papers agree with the ones obtained by our methods.

As we have seen, the critical values of area function on configuration space of a generic polygon can be explicitly calculated in terms of sidelengths. The results of [43], [51] suggested a stronger version of this conjecture, namely, that the set of critical values coincides with the set of real roots of a certain explicitly computable polynomial (the generalized Heron polynomial of D.Robbins [43]). This is indeed true and we'll now outline how this can be derived from the properties of multidimensional logarithmic residue (MLR) [50] (one of the guises of Grothendieck residue symbol [31]). The next statement is formulated as a *paradigm* because we make no attempt to explicate the condition of *genericity* and provide a rigorous proof. We believe that this makes sense since this formulation gives a sort of "raison d'être" for the results of [43], [51], [40] and may serve as a guide for further research.

Paradigm 1. *Let $f, g_1, \ldots, g_k \in \mathbb{R}_n, k \leq n-1$ be a generic set of real polynomials in n variables. Suppose that the level set $X = \{g_1 = 0, \ldots, g_k = 0\}$ is smooth and compact. Then the critical values of restriction $f|X$ are the real roots of a real polynomial in one variable whose coefficients can be algebraically expressed through coefficients of $f, g_1, \ldots, g_k$.*

This can be derived from elimination theory by an argument similar to the ones used by D.Robbins [43] and [10] but such an argument does not provide a way of computing the corresponding polynomial. We describe a different approach based on the properties of multidimensional logarithmic residue [50] which, in principle, enables one to effectively calculate the coefficients of sought polynomial since logarithmic residues can be computed using results of [50].

We start by some general facts concerned with MLR. Recall that the (global) MLR is defined for a polynomial $f \in \mathbb{C}_N$ with respect to a generic system $G = (g_1, \ldots, g_N)$ of N complex polynomials by a well-known formula of Cauchy integral type for which we refer to [50]. It is a complex number which will be denoted by $Res_G\, f$. In fact, it is required that polynomials g_i form a system of parameters, hence their zero set is finite [50]. If all polynomials in question are real then $Res_G\, f \in \mathbb{R}$. It is also known that, if all roots $z_i \in \mathbb{C}_n$ of polynomial system $\{G = 0\}$ are simple, then $Res_G\, f$ is equal to the sum of fractions $f(z_i)/J_G(z_i)$ over all roots of $\{G = 0\}$, where $J_G = J(G)$ is the jacobian of G. Thus $Res_G\, f^k J_G$ is equal to the sum of k-th powers of values of f at the roots of $\{G = 0\}$. In other words, if one introduces a (generalized Heron) polynomial $H(f, G)$ whose roots are the numbers $f(z_i)$ then $Res_G\, f^k J_G$ are the Newton sums of its roots. Hence the coefficients of $H(f, G)$ can be algebraically expressed through the first d Newton sums, where d is the degree of $H(f, G)$ (i.e., the number of roots of $\{G = 0\}$). At the same time the numbers $Res_G\, f^k J_G$ can be effectively computed using the formulae and algorithms from [50]. Hence the coefficients of $H(f, G)$ are also computable in an algorithmic way. Thus the values of f at the roots of $\{G = 0\}$ can be computed as the roots of $H(f, G)$. Moreover, if all polynomials in question are real then the values of f at the real roots of $\{G = 0\}$ can be computed as the real roots of equation $H(f, G) = 0$.

It is now obvious how to apply this scheme in the situation of our paradigm. Namely, writing down the Lagrange equations for the critical points of $f|X$ in the "multiplier-free" form used above, one obtains a system S of n real polynomial equations in n variables first k of which are the given polynomials g_i. Notice that the critical values of $f|X$ are by definition the values of f at the real roots of $\{S = 0\}$. Hence they can be computed as the real roots of the polynomial $H(f, G)$ introduced above. If the system of polynomials g_i is generic then the number of roots of $\{S = 0\}$ is equal to the product of degrees of equations (Bezout number), which provides the number d used in the above considerations. Now we can calculate the residues $Res_S\, f^k J_S$ for $k = 1, \dots, d$ and find the coefficients of $H(f, S)$, which gives the desired statement.

We do not make attempts to make the above argument rigorous because for us this paradigm is basically of methodological importance. Indeed, combining these considerations with our approach to configuration spaces and Theorem 2.1, one concludes that the results of [43], [10], [51] concerned with the existence of generalized Heron polynomials for areas of cyclic polygons appear to be very special cases of Paradigm P1. What seems even more important, they can be applied to many other functions on configuration spaces of linkages, in particular, to the energy functions of the type considered in the last section.

4. Area as a Morse function

In this section we present short comments on conjectures (CA3) and (CA4). As was already mentioned, (CA3) for quadrilaterals can be proved by a direct verification. For 3-arms, (CA3*) has been proven in [36] using the parametric transversality theorem. A thorough examination of the proof shows that the same reasoning can be applied for n-arm with arbitrary n. We do not formulate and prove those results here because we have recently realized that they are special cases of the following general statement which we again formulate in the form of a paradigm since we do not possess a rigorous proof in full generality. However there is little doubt that it can be proved using the parametric transversality theorem along the lines of [36].

Paradigm 2. *Let $f, g_1, \dots, g_k \in \mathbb{R}_n, k \leq n-1$ be algebraically independent real polynomials in n variables such that $g_1, \dots, g_k$ define a propomap $G : \mathbb{R}^n \to \mathbb{R}^k$ which is generically of maximal rank k. Then, for generic $l = (l_1, \dots, l_k) \in \mathbb{R}^k$, the level surface $X_l = \{g_1 = l_1, \dots, g_k = l_k\}$ is smooth and all critical points of restriction $f|X_l$ are nondegenerate (in the sense of Morse theory).*

Let us now describe the scheme of the proof. Smoothness of generic level surface follows (of course) from Sard's lemma. Consider now a generic level X_l such that Jacobi matrix DG has maximal rank k at each point $x \in X_l$. Without loss of generality we can assume that $l = \mathbf{0_k}$ and the minor formed by the first k columns is nowhere vanishing on $X_0 = X_{\mathbf{0}_k}$, which implies that condition $\{G = l\}$ is (locally) equivalent to $u = R(v, l)$, where u denotes the vector of first k coordinates of x, v denotes the vector of remaining $n - k$ coordinates and R is a differentiable function provided by implicit function theorem. Then, for l close to $\mathbf{0_k}$, the set $S(\hat{f}_l)$ critical points of $\hat{f}_l = f|X_l$ is defined by the system of equations $\nabla_u \hat{f}_l = \mathbf{0}$ which can be written down explicitly in terms of partial derivatives of f, g_j by the same implicit function theorem.

We now wish to show that, for generic l, the hessian $h(\hat{f}_l) = \det H(\hat{f}_l)$ of $\hat{f}_l$ is nonvanishing at all critical points of $\hat{f}_l$. Having in mind to apply the parametric transversality theorem, let us consider the "full" gradient $\nabla_{vl} f_l$ and "full" Hessian matrix $\tilde{H}(f_l) = D(\nabla_{vl}\hat{f}_l)$ of $\hat{f}_l$ taken with respect to all n coordinates v, l appearing in $\hat{f}_l$. To apply the parametric transversality theorem we need to show that $\nabla_{vl}\hat{f}_l$ is a surjection over the origin of $\mathbb{R}^k$. To this end we calculate the second partial derivatives of $\hat{f}$ by implicit function theorem and examine the "rightmost" $k \times k$ minor of $\tilde{H}(f_l)$ on the critical set $S(\hat{f}_l)$. For $k = 1$, it is straightforward to write down the entries of $\tilde{H}(f_l)$ and simplify them using equations $\nabla_u \hat{f}_l = \mathbf{0}$ to see that nonvanishing of $h(f_l)$ is equivalent to functional independence of f and $G = g_1$. Thus, for $k = 1$, one can indeed use the parametric transversality theorem to obtain the desired conclusion. It seems that it should be possible to establish the paradigm in full generality using induction but we are not yet able to overcome the arising technical difficulties.

Anyway, we were able to proof the nondegeneracy of critical points for many functions on configuration spaces so it is quite reasonable to consider the problem of calculating the Morse index of a critical point. A general method for calculating the Morse indices in the setting of Paradigm 2 is provided by the results of [24]. Namely, one just needs to calculate the so-called bordered Hessian of the Lagrange function $L = f + \sum \lambda_i g_i$ and then the formulae from [24] can be used to calculate the Morse index at each nondegenerate critical point of $f|X_l$. Notice that this setting is meaningful even for $k = 1$, where it enables one to establish a relation between Morse indices of $f|\{g_1 = b\}$ and $g_1|\{f = a\}$ by looking at the behaviour of their gradients near a point $p \in \mathbb{R}_n$, where the level surfaces $\{f = a\}$ and $\{g_1 = b\}$ have a first order tangency. One may now wish to extend the above paradigm in such a way that it becomes applicable to singular level surfaces as well. It seems that a natural framework for such an extension is provided by the stratified Morse theory of M.Goresky and R.McPherson [23].

Regarding the signed area function on planar configuration space of a polygonal linkage, it is especially interesting to find ways of computing the Morse indices in terms of geometry of a cyclic (critical) configuration. As was already mentioned, an algorithm for computation of Morse index of A in terms of geometry of a cyclic configuration of polygonal linkage was recently suggested in [41]. There is little doubt that the method and the main result of [41] may be modified to solve the problem for robot n-arms as well. It would also be interesting to investigate if some information about the Morse indices of cyclic configurations can be obtained using the generalized Heron polynomials provided by [43] and our Paradigm 1.

5. Concluding remarks

We are going to describe a number of similar settings for which our conjectures and results may serve as a paradigm. First of all, it is quite interesting to investigate the critical points of area in singular configuration spaces. Quite a lot is known about the structure of singular points of such spaces (see, e.g., [28]) and many examples have been investigated in big detail (see, e.g., [28], [33]). As we have already seen, unlike to the generic case, critical points can be non-isolated and they may also be given by aligned configurations. Both these phenomena are observed for rhomboid linkage. This can be seen directly and also follows from the algebraic considerations in Section 2. A natural way to study singular cases is to examine the zero locus of the ideal $I(S_l)$ introduced in Section 2 using Gröbner bases and other tools of computer algebra.

Taking into account our Paradigm P1, it seems plausible that (CA2) is also valid in this wider context. As to (CA3) and (CA4), they can be investigated in the framework of stratified Morse theory of M.Goresky and R.McPherson [23]. Namely, one can await that the area is a Morse function in the sense of [23]. In singular case, instead of calculating Morse indices one can try to describe the normal and tangential Morse data introduced in [23] and this should be possible in terms of the shape of a critical configuration.

Next, one may try to treat in a similar way certain other natural functions on the configuration space of polygonal linkage L which are sometimes called the *energy functions* due to their interpretations and applications [9], [20]. For each positive r and configuration $V \in C_2(L)$, function E_r can be defined as the sum of lengths of diagonals of V taken to power r. The most important examples of such functions are given by the sum of diagonals $D = E_1$ and sum of inverses of lengths of diagonals $E = E_{-1}$ (Coulomb potential) but some other functions on $C_2(L)$ have also been considered in the literature [3], [9], [12]. For an integer r, critical points of E_r can, in principle, be treated using our approach, in particular, their critical values should be algebraically computable in terms of sidelengths in the spirit of generalized Heron polynomials of D.Robbins [43]. For brevity, we will only present a few remarks concerned with D and E.

Notice that the absolute minimum of E defines the equilibrium of the system of equal charges placed at the vertices of linkage and is thus relevant to some problems emerging in electrostatics (see, e.g., a recent paper [20]). In particular, it is known that, for a regular linkage, the absolute minimum of E is given by its convex cyclic configuration (having the shape of regular polygon) [20] (the same is true for D). For regular quadrilaterals and pentagons, one can directly verify that the absolute minimum of E is nondegenerate (Hessian matrix is nondegenerate). Granted this, a routine application of implicit function theorem proves that the same is true for linkages sufficiently close to the regular one and the global minima of E for such linkages are close to the regular polygon. In particular, the minimal positive root of the generalized Heron polynomial for E would give the effective electrostatic energy in equilibrium. In both these cases using Lagrange method and some elementary algebraic manipulations it's easy to write down a system of polynomial equations describing the critical points and estimate their quantity using our signature formulae. However, it appears difficult to derive any conclusions about the shape of critical configurations directly from the corresponding system of polynomial equations.

So a more elaborate strategy is needed to achieve some understanding of critical points of E and our conjectures (CA1)-(CA4) suggest a possible line of thinking. However, it was a priori clear that there was no hope for results similar to Theorem 2.1 since it was known that already the number of local minima of E is growing quite fast with n (see, e.g., [20], [4]). So there are typically many more critical points of E on $C_2(L)$ than cyclic configurations of L. In many concrete cases this can be shown by merely computing the number of critical points using our signature formulae and comparing it with the maximal number of cyclic configurations given by Robbins formula [43].

For linkages with a small number of sides, the interior angles of cyclic configurations can be explicitly expressed through sidelengths. Taking the angles as local coordinates on $C_2(L)$ and substituting them into equations obtained by Lagrange method, one can verify if cyclic configurations are indeed critical. For quadrilaterals, the cosines of interior angles of cyclic configurations are easily computable in general case and one can check that cyclic configurations are no longer critical points of E even in the case of a kite with sidelength vector (a, a, b, b) with $a \neq b$. As a certain substitute for (CA1) one can compare the critical points of D with those of E. There are no reasons for them to coincide always but, as we have seen, sometimes this happens for the global minimum of E. One can formulate several natural problems in this topic but there are yet no results worthy of mentioning.

Next, it should be possible to prove the existence of a Heron-Robbins polynomial in both these cases in the framework of our general paradigm $P1$.

However, it remains obscure how to write down these polynomials in an explicit form. In principle this can be done algorithmically by computing a finite number of Grothendieck residues, which can be done quite effectively (see, e.g., [50]). The nondegeneracy of the critical points of D and E should follow in the framework of our Paradigm $P2$. This can be rigorously done for $n = 4$ but the general case remains unsolved. And of course there is also the problem of calculating Morse indices for typical critical configurations. Results and constructions from [41] may appear helpful for working out an algorithm for Morse indices.

Furthermore, notice that functions D_r can also be defined for arbitrary planar graphs. It's unclear what could be a reasonable extension of our first conjecture to this setting but the three others still make sense and may be conveniently denoted as $(C\Gamma 2 - C\Gamma 4)$. Moreover, our paradigms $P1$ and $P2$ provide good evidence for validity of $C\Gamma 2$ and $C\Gamma 3$ in reasonable generality. If Γ is just an open polygonal chain then these statements are especially plausible (cf. Theorem 2.2). Thus our approach may apparently yield a number of further developments concerned with critical points of functions on configuration spaces of planar linkages.

In conclusion, we outline similar settings for higher-dimensional configuration spaces. For example, the oriented volume V is defined for a polygonal linkage in an ambient space of arbitrary dimension, while energies D_r are defined even for arbitrary graphs. In three dimensions one can in addition consider the *normalized determinant* introduced by M.Atiyah [3]. Our paradigms suggest that in all these cases one generically obtains a Morse function on configuration space $C_k(L)$ and its critical values can be calculated as the roots of a certain explicitly computable polynomial in one variable. In the case of normalized determinant this may help to prove the Atiyah conjecture on the positivity of normalized determinant for configuration spaces of polygonal linkages.

Finally, part of our discussion makes sense for linkages in the spaces of constant curvature, in particular, for spherical linkages. Some results for spherical linkages have been obtained in [22] but this issue remains largely unexplored.

References

[1] A.Agrachev, R. Gamkrelidze, Quadratic mapping and smooth vector-functions: Euler characteristic of the level set, (Russian) Itogi Nauki i Techniki. Sovremennye problemy matematiki. Noveishie dostizhenia 35(1989), 179–239.

[2] V.Arnold, A.Varchenko, S.Gusein-Zade, Singularities of differentiable mappings, (Russian) Nauka, Moscow, 2005.

[3] M.Atiyah, Configurations of points, Roy. Soc. Lond. Philos. Trans. Ser. A, Math. Phys. Eng. Sci. 359(2001), 1375–1387.

[4] M.Atiyah, R.Beliawski, Nahm's equations, configuration spaces and flag manifolds, Bull. Braz. Math. Soc. (New Series) 33(2002), No.2, 157–176.

[5] G.Bibileishvili, G.Khimshiashvili, On circular configurations of polygonal linkages, Bull. Georgian Nat. Acad. Sci. 4(2010), No.3, 13–16.

[6] J.Bochnak, M.Coste, M.-F.Roy, Real algebraic geometry, Springer, Berlin-Heidelberg-New York, 1998.

[7] R.Bucholz, J.MacDougall, Cyclic polygons with rational sides and area, J. Number Theory 128(2008), 17–48.

[8] J.Bruce, Euler characteristics of real varieties, Bull. London Math. Soc. 22(1990), 213–219.

[9] J.Cantarella, E.Demaine, H.Iben, J.O'Brien, An energy-driven approach to linkage unfolding, Proc. 20th ACM Symp. Comp. Geom., ACM Press, New York, 2004, 134–143.

[10] R.Connelly, Comments on generalized Heron polynomials and Robbins' conjectures, Discr. Math. 309(2009), 4192–4196.

[11] R.Connelly, E.Demaine, Geometry and topology of plygonal linkages, Handbook of discrete and computational geometry, 2nd ed., CRC Press, Boca Raton, 2004, 197–218.

[12] R.Connelly, E.Demaine, G.Rote, Straightening polygonal arcs and convexifying polygonal cycles, Discr. Comp. Geom. 30(2003), 205–239.

[13] H.Coxeter, Introduction to geometry, 2nd ed. John Wiley, New York, 1969.

[14] P.Delsartes, J.M.Goethals, J.J.Seidel, Spherical codes and designs, Geometria Dedicata 6(1977), 363–388.

[15] B.Dubrovin, S.Novikov, A.Fomenko, Modern geometry, (Russian) Nauka, Moscow, 1988.

[16] N.Dutertre, On the Euler characteristic of semi-analytic and semi-algebraic sets, Math. Proc. Camb. Phil. Soc. 135(2003), 527–538.

[17] D.Eisenbud, H.Levine, An algebraic formula for the degree of a C^∞ map germ, Ann. Math. 106(1977), 19–44.

[18] E.Elerdashvili, M.Jibladze, G.Khimshiashvili, Cyclic configurations of pentagon linkages, Bull. Georgian Nat. Acad. Sci. 2(2008), No.4, 29–32.

[19] M.Fedorchuk, I.Pak, Rigidity and polynomial invariants of convex polytopes, Duke Math. J. 129(2005), No.2, 371–404.

[20] A.Gabrielov, D.Novikov, B.Shapiro, Mystery of point charges. Proc. Lond. Math. Soc. 95(2007), No.2, 443–472.

[21] G.Giorgadze, Quadratic mappings and configuration spaces, Banach Center Publ. 62(2004), 159–172.

[22] G.Giorgadze, G.Khimshiashvili, Remarks on spherical linkages, Bull. Georgian Natl. Acad. Sci. 4(2010), No.2, 8–12.

[23] M.Goresky, R.McPherson, Stratified Morse theory, Springer, 1990.

[24] C.Hassell, E.Rees, The index of a constrained critical point, Amer. Math. Monthly 100(1993), 772–778.

[25] J.-C.Hausmann, Sur la topologie de bras articules, Lect. Notes Math. 1474(1991), 146–159.

[26] M.Hirsch, Differential topology, Springer, Berlin-Heidelberg-New York, 1976.

[27] Y.Kamiyama, The Euler characteristic of the moduli space of polygons in higher-dimensional Euclidean space, Kyushu J. Math. 54(2000), 333–369.

[28] M.Kapovich, J.Millson, Moduli spaces of linkages and arrangements, Progr. Math. 172(1999), 730–770.

[29] M.Kapovich, J.Millson, Universality theorems for configuration spaces of planar linkages, Topology 41(2002), 1051–1107.

[30] G.Khimshiashvili, On the local degree of smooth mapping, Bull. Georgian Acad. Sci. 85(1977), No.2, 31–34.

[31] G.Khimshiashvili, Signature formulae for topological invariants, A.Razmadze Math. Inst. Proc. 125(2001), 1–121.

[32] G.Khimshiashvili, On configuration spaces of planar pentagons, Zap. Nauch. Sem. S.-Peterb. Otdel. Mat. Inst. RAN 292(2002), 120–129.
[33] G.Khimshiashvili, Cyclic polygons as critical points, Proc. I.Vekua Inst. Appl. Math. 58(2008), 74–83.
[34] G.Khimshiashvili, Configuration spaces and signature formulae, J. Math. Sci. 160(2009), No.6, 727–736.
[35] G.Khimshiashvili, G.Panina, Cyclic polygons are critical points of area, J. Math. Sci. 158(2009), No.6, 899–903.
[36] G.Khimshiashvili, D.Siersma, Preprint ICTP, IC/2009/047. 11 p.
[37] R.King, Planar linkages and algebraic sets, Turkish J. Math. 23(1999), 33–56.
[38] A.Lecki, Z.Szafraniec, An algebraic method for calculating the topological degree, Banach Center Publ. 35(1996), 73–83.
[39] Lê Dung Trang, Calculation of the Milnor number for an isolated singularity of a complete intersection, (Russian) Funkts. Anal. Prilozhen. 8(1974), 45–52.
[40] I.Pak, The area of cyclic polygons: recent progress on Robbins' conjectures, Adv. Appl. Math. 34(2005), 690–696.
[41] G.Panina, A.Zhukova, Morse index of a cyclic polygon, Central Eur. J. Math. 2011. (accepted) (ArXiv: 1007.2740v2 [math.MG] 7 p.)
[42] A.Pugh, An introduction to tensegrity, Univ. Calif. Press, Berkeley, 1976.
[43] D.Robbins, Areas of polygons inscribed in a circle, Discrete Comput. Geom. 12(1994), 223–236.
[44] I.Sabitov, The volume as a metric invariant of polyhedra, Discr. Comp. Geom. 20(1998), 405–425.
[45] D.Shimamoto, C.Vanderwaart, Spaces of polygons in the plane and Morse theory, Amer. Math. Monthly 112(2005), 289–310.
[46] K.Stephenson, Introduction to circle packing, Camb. Univ. Press, New York, 2005.
[47] D.Svrtan, I.Urbiha, On cyclic characterization of regular pentagons and heptagons, Mathem. communications 7(2002), 71–89.
[48] Z.Szafraniec, A formula for the Euler characteristic of a real algebraic manifold, Manusr. Math. 85(1994), 345–360.
[49] W.Thurston, Shapes of polyhedra and triangulations of the sphere, Geom. Topol. Monogr. 1(1998), 511–549. (electronic)
[50] A.Tsikh. Multidimensional residues and their applications, Amer. Math. Soc., Providence, RI, 1992.
[51] V.Varfolomeev, Inscribed polygons and Heron polynomials, (Russian) Math. Sb. 194(2003), 3–24.

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