LOWER ESTIMATES FOR THE SUPREMUM OF SOME RANDOM PROCESSES

B. Kashin* and L.Tzafriri**

*
Steklov Mathematical Institute
Moscow

**
Hebrew University of Jerusalem
Israel

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI / 94-111

LOWER ESTIMATES FOR THE SUPREMUM OF SOME RANDOM PROCESSES

B. KASHIN AND L. TZAFRIRI*

In this paper we consider random processes of the type

$$\sum_{j=1}^{n} \xi_j(t)\varphi_j(x),$$

where $\{\xi_j\}_{j=1}^n$ is a system of independent random variables on a probability space $(T, \mathcal{T}, \tau)$ and $\Phi = \{\varphi_j\}_{j=1}^n$ is a system of functions in an $L_2(X, \Sigma, \mu)$ -space with (X, Σ, μ) being another probability space.

Many problems in functional analysis and probability theory lead to the investigation of the expectation (relative to τ)

$$\mathbb{E} \parallel \sum_{j=1}^{n} \xi_j(t)\varphi_j(x) \parallel_{L_\infty(\mu)} . \tag{1}$$

A well-known result of this type having many applications is the estimate

$$\mathbb{E} \parallel \sum_{j=1}^{n} r_j(t)e^{2\pi ijx} \parallel_{L_\infty} \leq C(n \log n)^{1/2}, \tag{2}$$

(with $\{r_j(t)\}_{j=1}^\infty$ being the usual Rademacher variables), first formulated in an explicit manner by R. Salem and A. Zygmund [7]. In the same paper [7], it was shown that the estimate (2) is exact in the sense that

$$\mathbb{E} \parallel \sum_{j=1}^{n} r_j(t)e^{2\pi ijx} \parallel_{L_\infty} \geq c(n \log n)^{1/2},$$

for all n and some constant $c > 0$.

A similar question was considered for stationary Gaussian processes while trying to find necessary and sufficient conditions for the continuity a.s. of the trajectory of processes of the type

$$\sum_{j=1}^{\infty} a_j\gamma_j(t)e^{2\pi ijx},$$

*This work was performed while both authors visited the Max-Planck-Institute in Bonn, Germany.

with $a_j \in \mathbb{R}$ for all j, and $\{\gamma_j\}_{j=1}^{\infty}$ being a system of independent Gaussian variables satisfying $\mathbb{E}\gamma_j = 0$ and $\mathbb{E}|\gamma_j|^2 = 1$, for all j (cf. X. Fernique [2]).

The behaviour of the expression (1) in the case when $\{\varphi_j\}$ is a system of characters of a locally compact abelian group G restricted to a compact symmetric neighbourhood V of the identity element $O \in G$ was considered in papers of Marcus, Pisier, Talagrand and others. A detailed presentation of this topic can be found in the books [5] and [4]. The approach to obtain lower estimates used in [5] and [4] is based on the comparison of the expectation (1) with the quantity

$$\mathbb{E} \parallel \sum_{j=1}^{n} \gamma_j(t)\varphi_j(x) \parallel_{L_\infty(\mu)}, \tag{3}$$

which in turn is estimated by using variants of Slepian's lemma. This lemma enables us to estimate from below expressions of the form

$$\mathbb{E} \max_{1\leq k\leq m} \Big| \sum_{j=1}^{n} \gamma_j(t)\varphi_j(x_k)\Big|,$$

and thus also the expression (3), provided that the points $\{x_k\}_{k=1}^{m}$ are chosen in V in such a manner so that the vectors

$$W_{x_k} = (\varphi_1(x_k), \varphi_2(x_k), \dots, \varphi_n(x_k)) \in \mathbb{C}^n,$$

$k = 1, 2, \dots, m$, are far away from each other in the euclidean metric in $\mathbb{C}^n$. In this way, the lower estimate of (3) is controlled by the ϵ -entropy in the ℓ_2^n -metric of the set $\Gamma_\Phi = \{W_x\}_{x\in V}$. The realization of the above program is not trivial even in relatively simple cases.

The method used in the present paper relies instead on a sharper version (with precise estimates of the error term) of the central limit theorem for sequences of independent vectors in $\mathbb{R}^2$. Our approach is, in some sense, a return to the original method of R. Salem and A. Zygmund [7] though their method cannot be applied directly in this case. The argument described below is not limited in use for the theorem proved in the sequel but can be applied also to more general situations (see also the remarks at the end of the paper). The lower estimate of (1) is connected with a special selection of points $\{x_k\}_{k=1}^{m}$ in X which maximizes some energy type functions. In the case considered in the paper, the main role of the selection is to ensure that the average

$$\frac{1}{n^2} \sum_{k,\ell=1}^{m} | < W_{x_k}, W_{x_\ell} > |^2$$

is relatively small, which is much easier than guaranteeing the condition appearing in the entropy estimate of Γ_Φ that all the scalar products $\langle W_{x_k}, W_{x_\ell}\rangle$, $k \neq \ell$, are small in absolute value.

Theorem. *For every $M < \infty$ there exists a constant $c = c(M) > 0$ such that, whenever $\{\varphi_i\}_{i=1}^n$ is a system of functions in an $L_2(\mu)$ -space satisfying:*

1° $\| \varphi_i \|_{L_2(\mu)} = 1$ *and* $\| \varphi_i \|_{L_3(\mu)} \leq M$, *for all* i,

2° $\| \sum_{i=1}^n a_i\varphi_i \|_{L_2(\mu)} \leq M(\sum_{i=1}^n |a_i|^2)^{1/2}$, *for all* $\{a_i\}_{i=1}^n$,

and $\{\xi_i\}_{i=1}^n$ *are independent random variables over a probability space* $(T, \mathcal{T}, \tau)$ *with*

3° $\mathbb{E}(\xi_i) = 0$, $\mathbb{E}|\xi_i|^2 = 1$ *and* $(\mathbb{E}|\xi_i|^3)^{1/3} \leq M$, *for all* i,

then

$$\mathbb{E} \| \sum_{i=1}^n \xi_i\varphi_i \|_{L_\infty(\mu)} \geq c(n \log n)^{1/2}$$

The proof consists of several steps.

<u>*Step I*</u>. For $\epsilon > 0$ consider the set

$$E_1 = \{x; \frac{1}{n}\sum_{i=1}^n |\varphi_i(x)|^3 < M^3/\epsilon\}$$

and notice that, by our assumption,

$$\mu(E_1^c)\frac{M^3 n}{\epsilon} \leq \int \sum_{i=1}^n |\varphi_i(x)|^3 d\mu \leq M^3 n$$

i.e.

$$\mu(E_1^c) \leq \epsilon$$

so that

$$\mu(E_1) \geq 1 - \epsilon.$$

Next notice that if

$$\varphi(x) = \frac{1}{n}\sum_{i=1}^n |\varphi_i(x)|^2$$

then $\| \varphi \|_{L_1(\mu)} = 1$ and

$$\| \varphi \|_{L_{3/2}(\mu)} = \frac{1}{n}(\int (\sum_{i=1}^n |\varphi_i|^2)^{3/2} d\mu)^{2/3} \leq \frac{1}{n}\sum_{i=1}^n \| \varphi_i \|^2_{L_3(\mu)} \leq M^2.$$

Consider the set

$$E_2 = \{x : \varphi(x) > \frac{1}{4}\}$$

and observe that

$$\int_{E_2^c} \varphi(x) d\mu \leq \frac{1}{4}$$

so

$$\frac{3}{4} \leq \int_{E_2} \varphi(x) d\mu \leq \| \varphi \|_{L_{3/2}(\mu)} \mu(E_2)^{1/3} \leq M^2 \mu(E_2)^{1/3}$$

i.e.

$$\mu(E_2) \geq (3/4\, M^2)^3.$$

Now put

$$E_3 = \{x \in E_2 : \varphi(x) \leq 2(\frac{4M^2}{6})^3\}$$

and notice that

$$2(\frac{4M^2}{3})^3(\mu(E_2) - \mu(E_3)) < \int_{E_2 \sim E_3} \varphi(x) d\mu < 1$$

from which it follows that

$$\mu(E_3) \geq \frac{1}{2}(3/4\, M^2)^3.$$

The final conclusion is that if $\epsilon = \frac{1}{4}(3/4\, M^2)^3$ then the corresponding set $E = E_1 \cap E_3$ has the following properties:

(i) $\mu(E) \geq \alpha(M) = \frac{1}{4}(\frac{3}{4M^2})^3 > 0$

(ii) $x \in E \Rightarrow \frac{1}{n} \sum_{i=1}^n |\varphi_i(x)|^3 < \frac{256}{27} M^9$

(iii) $x \in E \Rightarrow \varphi(x) = \frac{1}{n} \sum_{i=1}^n |\varphi_i(x)|^2$ satisfies

$$\frac{1}{4} < \varphi(x) < \gamma(M) = 2(\frac{4M^2}{3})^3$$

Step II. Change of density. Consider the measure ν defined by:

$$d\nu = \begin{cases} \chi_{E^c}(x) d\mu \\ \chi_E(x)(\varphi(x)\mu(E) / \int_E \varphi(u) d\mu) d\mu \end{cases}$$

and notice that ν is a probability measure on the same measure space as μ so that if

$$\psi_h(x) = \begin{cases} \varphi_h(x) \quad ; \quad x \in E^c \\ \varphi_h(x)(\int\limits_E \varphi(u)d\mu/\varphi(x)\mu(E))^{1/2} \quad ; \quad x \in E \end{cases} \quad , \quad 1 \le h \le n,$$

then

(i) $\| \psi_i \|_{L_2(\nu)} = 1$, for all i,

(ii) $\| \sum_{i=1}^n a_i\psi_i \|_{L_2(\nu)} = \| \sum_{i=1}^n a_i\varphi_i \|_{L_2(\mu)} \le M(\sum_{i=1}^n |a_i|^2)^{1/2}$, for all $\{a_i\}_{i=1}^n$,

(iii) $\psi(x) = \frac{1}{n}\sum_{i=1}^n |\psi_i(x)|^2 = \int\limits_E \varphi(u)d\mu/\mu(E) = K^2$, $x \in E$,

where

$$1/4 \le K^2 \le \gamma(M),$$

(iv) $\frac{1}{n}\sum_{i=1}^n |\psi_i(x)|^3 \le \beta(M) = 10^5M^{18}$, $x \in E$.

Finally, notice that for $x \in E$ and $t \in T$,

$$|\sum_{i=1}^n \xi_i(t)\psi_i(x)| \le 5M^3 |\sum_{i=1}^n \xi_i(t)\varphi_i(x)|$$

so that

$$\mathbb{E} \| \sum_{i=1}^n \xi_i(t)\psi_i(x)\chi_E \|_{L_\infty(\nu)} \le 5M^3\mathbb{E} \| \sum_{i=1}^n \xi_i(t)\varphi_i(x) \|_{L_\infty(\mu)} .$$

Hence, it suffices to prove our assertion for the system $\{\psi_i\}_{i=1}^n$ on the set E, introduced above.

Step III. There exist points $\{x_j\}_{j=1}^n$ in E so that

$$\frac{1}{n^2}\sum_{j,k=1}^n |\sum_{i=1}^n \psi_i(x_j)\psi_i(x_k)|^2 \le \frac{M^2n}{\alpha(M)^2}.$$

Moreover, one can assume without loss of generality that $\{x_j\}_{j=1}^n$ are points of approximative continuity for all the functions $\{\psi_i\}_{i=1}^n$. Indeed, notice that

$$\frac{1}{\nu(E)^n}\int\limits_E \cdots \int\limits_E \frac{1}{n^2}\sum_{j,k=1}^{n} |\sum_{i=1}^{n} \psi_i(x_j)\psi_i(x_k)|^2 d\nu(x_1)\dots d\nu(x_n) \le$$

$$\le \frac{1}{(\nu(E)n)^2}\sum_{j,k=1}^{n}\iint |\sum_{i=1}^{n} \psi_i(x_j)\psi_i(x_k)|^2 d\nu(x_j)d\nu(x_k) \le$$

$$\le \frac{M^2}{(\nu(E)n)^2}\sum_{j,k=1}^{n}\int \sum_{i=1}^{n} |\psi_i(x_k)|^2 d\nu(x_k) \le \frac{M^2 n}{\nu(E)^2} = \frac{M^2 n}{\mu(E)^2} \le \frac{M^2 n}{\alpha(M)^2},$$

which of course completes the argument.

Step IV. For $x \in E$ and $\rho > 0$ fixed, put

$$E_\rho(x) = \{t \in T, \sum_{i=1}^{n} \xi_i(t)\psi_i(x) > \rho\sqrt{n \log n}\}.$$

Our aim is to show that, for some $\rho = \rho(M)$, $n \ge n_0(M)$ and $d = d(M) > 0$, we have that

$$(*) \qquad \tau(\bigcup_{j=1}^{n} E_\rho(x_j)) \ge d.$$

Then, for $t \in \bigcup_{j=1}^{n} E_\rho(x_j)$ and $n \ge n_0$,

$$\| \sum_{i=1}^{n} \xi_i(t)\psi_i(x)\chi_E(x) \|_{L_\infty(\nu)} \ge \rho\sqrt{n \log n}$$

so that

$$\mathbb{E} \| \sum_{i=1}^{n} \xi_i(t)\psi_i(x)\chi_E(x) \|_{L_\infty(\nu)} \ge \rho\, d\sqrt{n \log n},$$

thus proving the assertion for $n \ge n_0$, by using the observation made at the end of Step II. The extension to all n is immediate.

Step V. Put $f(t) = \sum_{j=1}^{n} \chi_{E_\rho(x_j)}(t)$; $t \in T$, and observe that if $\tau(\bigcup_{j=1}^{n} E_\rho(x_j)) < d$, for some $d > 0$, then by the Cauchy-Schwartz inequality,

$$\mathbb{E}|f| \le (\mathbb{E}|f|^2)^{1/2} \cdot \tau(\bigcup_{j=1}^{n} E_\rho(x_j))^{1/2} < d^{1/2}(\mathbb{E}|f|^2)^{1/2},$$

which means that the assertion

$$(**) \qquad \mathbb{E}|f| \ge d^{1/2}(\mathbb{E}|f|^2)^{1/2}$$

implies (*). So now we shall find a $d = d(M) > 0$ for which (**) holds.

Step VI. In order to estimate $\mathbb{E}|f|$ and $(\mathbb{E}|f|^2)^{1/2}$, we shall use a sharper version of the one and two-dimensional central limit theorem containing an estimate of the error term. Among many results of this type exposed in the book of R.N. Bhattacharya and R. Ranga Rao [1], we shall use the following one due to V. Rotar' [6] (see Corollary 17.2 in [1]).

Proposition 1. *There exists a constant $C_1 = C_1(m) < \infty$ so that, whenever $\{X_i\}_{i=1}^h$ are independent random vectors in $\mathbb{R}^m$ for which $\mathbb{E}(X_i) = 0$; $1 \le i \le h$, then*

$$\sup_{A \in C} |Q_h(A) - \Phi_{0,V}(A)| \le C_1 h^{-1/2} \rho_3 \lambda^{-3/2},$$

where C denotes the class of all Borel measurable convex sets in $\mathbb{R}^m$,

$$\rho_r = h^{-1} \sum_{i=1}^{h} \mathbb{E} \parallel X_i \parallel^r ; \; r \ge 1,$$

λ is the smallest eigenvalue of the matrix $V = h^{-1} \sum_{i=1}^h cov(X_i)$ (recall that if $X_i = (X_{i,1}, \ldots, X_{i,m})$; $i = 1, 2, \ldots, h$, then $cov(X_i) = (\mathbb{E}(X_{i,\ell} \cdot X_{i,j}))_{\ell,j=1}^m$), $Q_h(A)$ is the probability of the event that $(X_1 + \ldots + X_h)/h^{1/2}$ belongs to to convex set A and, finally, $\Phi_{0,V}$ denotes the normal distribution whose density function is

$$\varphi_{0,V}(Y) = (2\pi)^{-m/2} (\det V)^{-1/2} \exp\{-\frac{1}{2}\langle Y, V^{-1} Y\rangle\},\ Y \in R^m.$$

The first application of the proposition above is done in the one-dimensional case, when, for fixed $1 \le j \le n$, we put

$$X_i(t) = \xi_i(t)\psi_i(x_j);\ 1 \le i \le n,\ t \in T.$$

Then, it is easily verified that

$$\rho_3 = n^{-1} \sum_{i=1}^{n} \mathbb{E}|\xi_i|^3 |\psi_i(x_j)|^3 \le M^3 \beta(M),$$
$$cov(X_i) = \mathbb{E}|\xi_i(t)|^2 |\psi_i(x_j)|^2 = |\psi_i(x_j)|^2$$

and

$$V = n^{-1} \sum_{i=1}^{n} |\psi_i(x_j)|^2 = K^2.$$

It follows that

$$|\tau(E_\rho(x_j)) - (2\pi)^{-1/2} K^{-1} \int_{\rho\sqrt{\log n}}^{\infty} exp\{-y^2/2K^2\}\, dy| \le$$
$$\le C_1 M^3 \beta(M) K^{-3} n^{-1/2} < \beta'(M) n^{-1/2},$$

for some new constant $\beta'(M) < \infty$.

For functions g_1 and g_2 of a parameter ζ, we shall use the expression $g_1 \asymp g_2$, whenever there exists a universal constant $0 < C < \infty$ such that

$$C^{-1}\, g_1(\zeta) \leq g_2(\zeta) \leq C\, g_1(\zeta),$$

for all values of ζ in the domain under consideration, and $g_2 \ll g_1$ if only the right hand side inequality above is true. With this notation, it is well known that

$$\int_{\zeta}^{\infty} exp\,(-y^2/2)\, dy \asymp \zeta^{-1}\, exp\,(-\zeta^2/2);\ \zeta > 1.$$

Hence,

$$\int_{\rho\sqrt{\log\, n}}^{\infty} exp\,(-y^2/2K^2)\, dy = K \int_{K^{-1}\rho\sqrt{\log\, n}}^{\infty} exp\,(-w^2/2) dw \asymp$$
$$\asymp \frac{K^2}{\rho\sqrt{\log n}}\, exp\,\{-\rho^2 \log\, n/2K^2\} \asymp \frac{K^2}{\rho\sqrt{\log\, n}} n^{-\rho^2/2K^2}.$$

Therefore, if we choose $\rho < \frac{1}{2}$ then

$$\frac{\rho^2}{2K^2} < \frac{1}{2}.$$

Hence, the error term, which is $\leq \beta'(M) n^{-1/2}$, can be neglected relative to the term

$$(2\pi)^{-1/2} K^{-1} \int_{\rho\sqrt{\log\, n}}^{\infty} exp\,\{-y^2/2K^2\}\, dy.$$

The outcome of these considerations is that, for all $1 \leq j \leq n$,

$$\tau(E_\rho(x_j)) \asymp \int_{\rho\sqrt{\log\, n}}^{\infty} exp\,\{-y^2/2K^2\}\, dy \asymp \frac{K^2 n^{-\rho^2/2K^2}}{\rho\sqrt{\log\, n}}$$

which implies that the function f introduced in Step V satisfies

$$\mathbb{E}|f| = \sum_{j=1}^{n} \tau(E_\rho(x_j)) \asymp \frac{K^2 n^{1-\rho^2/2K^2}}{\rho\sqrt{\log\, n}}.$$

Also

$$\mathbb{E}|f|^2 = \sum_{j=1}^{n} \tau(E_\rho(x_j)) + \sum_{\substack{j,\,k=1 \\ j\neq k}}^{n} \tau(E_\rho(x_j) \cap E_\rho(x_k))$$
$$= \mathbb{E}|f| + \sum_{\substack{j,\,k=1 \\ j\neq k}}^{n} \tau(E_\rho(x_j) \cap E_\rho(x_k))$$

Therefore, if we will prove the existence of a constant $D = D(M) < \infty$ so that (for some $\rho = \rho(M) < 1/2$),

$$(***) \qquad \sum_{\substack{j,\,k=1 \\ j\neq k}}^{n} \tau(E_\rho(x_j) \cap E_\rho(x_k)) \le D(\mathbb{E}|f|)^2$$

then, since $\mathbb{E}|f| \le (\mathbb{E}|f|)^2$ (as a consequence of the fact that $\mathbb{E}|f|$ is large), it would follow that

$$\mathbb{E}|f|^2 \le (1+D)(\mathbb{E}|f|)^2,$$

thus proving condition (**) and the theorem.

Step VII. In order to prove (***), we divide the set of all pairs (j,k) with $1 \le j \ne k \le n$ into two sets:

$$\sigma_1 = \{(j,k);\ 1 \le j \ne k \le n,\ |\frac{1}{n}\sum_{i=1}^{n} \psi_i(x_j)\psi_i(x_k)| < 1/8\}$$

and $\sigma_2 = \sigma_1^c$. By Step III, we conclude that

$$|\sigma_2| \le 64\, M^2 n/\alpha(M)^2$$

so that

$$\sum_{(j,k)\in\sigma_2} \tau(E_\rho(x_j) \cap E_\rho(x_k)) \le |\sigma_2| \max_{1\le j\le n} \tau(E_\rho(x_j)) \ll$$
$$\ll M^2 \frac{n^{1-\rho^2/2K^2}}{\alpha(M)^2\rho\sqrt{\log n}} \ll \frac{M^2\mathbb{E}|f|}{K^2\alpha(M)^2} \le \frac{M^2(\mathbb{E}|f|)^2}{K^2\alpha(M)^2}.$$

Next we consider the pairs $(j,k) \in \sigma_1$. To this end, we fix such a pair $s = (j,k) \in \sigma_1$, and consider the random vectors in $\mathbb{R}^2$ defined by

$$X_i^s(t) = (\xi_i(t)\psi_i(x_j),\ \xi_i(t)\psi_i(x_k));\ 1 \le i \le n,\ t \in T.$$

In the case, it is readily verified that

$$\rho_s^3 = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}|\xi_i(t)|^3(|\psi_i(x_j)|^2 + |\psi_i(x_k)|^2)^{3/2} \le$$
$$\le \frac{M^3}{n}\sum_{i=1}^{n}(|\psi_i(x_j)|^2 + |\psi_i(x_k)|^2)^{3/2} \le 8M^3\beta(M).$$

Since $\mathbb{E}|\xi_i|^2 = 1;\ 1 \le i \le n$, it follows that

$$cov\,(X_i^s) = \begin{pmatrix} |\psi_i(x_j)|^2 & \psi_i(x_j)\psi_i(x_k) \\ \psi_i(x_j)\psi_i(x_k) & |\psi_i(x_k)|^2 \end{pmatrix}$$

and

$$V^s = \begin{pmatrix} K^2 & \frac{1}{n}\sum_{i=1}^n \psi_i(x_j)\psi_i(x_k) \\ \frac{1}{n}\sum_{i=1}^n \psi_i(x_j)\psi_i(x_k) & K^2. \end{pmatrix}$$

Hence,

$$\det V^s = K^4 - \frac{1}{n^2}|\sum_{i=1}^n \psi_i(x_j)\psi_i(x_k)|^2 > \frac{1}{22}$$

and

$$\operatorname{trace} V^s = 2K^2 \leq 2\gamma(M)$$

Denote the eigenvalues of V^s by λ_1 and λ_2 and suppose that $0 \leq \lambda_1 \leq \lambda_2$. Then, on one hand, $\lambda_1 + \lambda_2 \leq \operatorname{trace} V^s \leq 2\gamma(M)$ which yields that $\lambda_2 \leq 2\gamma(M)$. Therefore, since $\det\, V^s = \lambda_1\,\lambda_2$, we get that

$$\frac{1}{22} \leq \lambda_1 \cdot \lambda_2 \leq 2\gamma(M)\lambda_1.$$

By using the estimate described in Step VI, it follows that

$$|\tau(E_\rho(x_j) \cap E_\rho(x_k)) - (2\pi)^{-1}(\det\, V^s)^{-1} \int_{\rho\sqrt{\log\, n}}^\infty \int_{\rho\sqrt{\log\, n}}^\infty \exp\{-\frac{1}{2}\langle Y, (V^s)^{-1}Y\rangle\}dY| \leq$$
$$\leq C_1 2^{12} M^3 \beta(M)\gamma(M)^{-3/2} n^{-1/2} = C_2 n^{-1/2},$$

for some constant $C_2 = C_2(M) < \infty$. Hence,

$$\tau(E_\rho(x_j) \cap E_\rho(x_k)) \leq C_2 n^{-1/2} + \frac{11}{\pi} \int_{\rho\sqrt{\log\, n}}^\infty \int_{\rho\sqrt{\log\, n}}^\infty exp\,\{-\frac{1}{2}\langle Y, (V^s)^{-1}Y\rangle\}dY,$$

which yields that, for some new constant $C_3 = C_3(M) < \infty$,

$$\tau(E_\rho(x_j) \cap E_\rho(x_k)) \leq C_3 \left(n^{-1/2} + \int_{\rho\sqrt{\log\, n}}^\infty \int_{\rho\sqrt{\log\, n}}^\infty exp\,\{-\frac{1}{2}\langle Y, (V^s)^{-1}Y\rangle\}dY \right).$$

Thus,

$$\sum_{s=(j,k)\in\sigma_1} \tau(E_\rho(x_j) \cap E_\rho(x_k)) \leq C_3 \left(n^{3/2} + \int_{\rho\sqrt{\log\, n}}^\infty \int_{\rho\sqrt{\log\, n}}^\infty (\sum_{s\in\sigma_1} exp\,\{-\frac{1}{2}\langle Y, (V^s)^{-1}Y\rangle\}dY \right).$$

Since

$$(\mathbb{E}|f|)^2 = \sum_{j,k=1}^{n} \tau(E_\rho(x_j)) \cdot \tau(E_\rho(x_k))$$

it is only natural to compare the expression $\sum_{s\in\sigma_1} \tau(E_\rho(x_j) \cap E_\rho(x_k))$ with $\sum_{s\in\sigma_1} \tau(E_\rho(x_j)) \cdot \tau(E_\rho(x_k))$. Notice that

$$\sum_{s\in\sigma_1} \tau(E_\rho(x_j)) \cdot \tau(E_\rho(x_k)) = \frac{1}{2\pi K^2} \int_{\rho\sqrt{\log n}}^{\infty} \int_{\rho\sqrt{\log n}}^{\infty} \sum_{s\in\sigma_1} exp\{-\frac{1}{2K^2}(y_1^2+y_2^2)\} dy_1 dy_2 + \delta,$$

where $|\delta| \leq C_4 n^{-1/2}$, for some constant $C_4 = C_4(M) \leq \infty$. In order to compare these two expressions, observe first that

$$(V^s)^{-1} = \frac{1}{\det V^s} \begin{pmatrix} K^2 & -\frac{1}{n}\sum_{i=1}^n \psi_i(x_j)\psi_i(x_k) \\ -\frac{1}{n}\sum_{i=1}^n \psi_i(x_j)\psi_i(x_k) & K^2 \end{pmatrix}$$

so if we introduce the notation

$$W_s = \frac{1}{n}\sum_{i=1}^{n} \psi_i(x_j)\psi_i(x_k)$$

then

$$(V^s)^{-1} = \begin{pmatrix} \frac{1}{K^2 - \frac{W_s^2}{K^2}} & -\frac{W_s}{K^4 - W_s^2} \\ -\frac{W_s}{K^4 - W_s^2} & \frac{1}{K^2 - \frac{W_s^2}{K^2}} \end{pmatrix}$$

which yields that

$$\sum_{s\in\sigma_1} \tau(E_\rho(x_j) \cap E_\rho(x_k)) \leq C_3(n^{3/2} +$$
$$+ \int_{\rho\sqrt{\log n}}^{\infty} \int_{\rho\sqrt{\log n}}^{\infty} \sum_{s\in\sigma_1} exp\{-\frac{1}{2(K^2 - \frac{W_s^2}{K^2})}(y_1^2 + y_2^2) + \frac{y_1 y_2 W_s}{K^4 - W_s^2}\} dy_1 dy_2.$$

Fix $s = (j,k) \in \sigma_1$, put $a_s = \frac{1}{K^2 - \frac{W_s^2}{K^2}}$ and $b_s = \frac{W_s}{K^4 - W_s^2}$ and notice that, by the definition of σ_1,

$$22 \geq a_s^2 - b_s^2 = \frac{1}{K^4 - W_s^2} \geq 1/\gamma(M)^2.$$

It follows that, with $L = L(M) > 16\gamma(M)$,

$$\int_{L\rho\sqrt{\log n}}^{\infty}\int_{\rho\sqrt{\log n}}^{\infty} exp\{-\frac{a_s}{2}(y_1^2+y_2^2)+b_s\, y_1y_2\}dy_1dy_2 \leq$$

$$\leq \int_{L\rho\sqrt{\log n}}^{\infty}\int_{\rho\sqrt{\log n}}^{\infty} exp\{-\frac{(a_s-b_s)}{2}(y_1^2+y_2^2)\}dy_1dy_2 \asymp$$

$$\asymp \frac{1}{L\rho^2\log n} exp\{-\frac{(a_s-b_s)}{2}(L^2+1)\rho^2\log n\} = \frac{n^{-\frac{(a_s-b_s)}{2}(L^2+1)\rho^2}}{L\rho^2\log n}.$$

However, the choice of L ensures that, for large n,

$$2n^{-\frac{(a_s-b_s)}{2}(L^2+1)\rho^2} \leq n^{-\rho^2/K^2}$$

and thus that

$$\sum_{s\in\sigma_1}\tau(E_\rho(x_j)\cap E_\rho(x_k)) \leq$$

$$\leq C_5(n^{3/2}+(\mathbb{E}|f|)^2+\int_{\rho\sqrt{\log n}}^{L\rho\sqrt{\log n}}\int_{\rho\sqrt{\log n}}^{L\rho\sqrt{\log n}}\sum_{s\in\sigma_1} exp\{-\frac{a_s}{2}(y_1^2+y_2^2)+b_s\, y_1y_2\}dy_1dy_2),$$

for some constant $C_5 = C_5(M) < \infty$.

Step VIII. In order to estimate the double integral above and to complete the proof of the theorem, it is enough to show that, for some constant $C = C(M)$ and for any pair (y_1, y_2) with $\rho\sqrt{\log n} \leq y_1, y_2 \leq L\rho\sqrt{\log n}$, the following (pointwise) estimate holds:

$$\sum = \sum_{s\in\sigma_1} exp\{\frac{-a_s}{2}(y_1^2+y_2^2)+b_s\, y_1y_2\} \leq$$
$$\leq C\sum_{s\in\sigma_1} exp\{\frac{-(y_1^2+y_2^2)}{2K^2}\} = C|\sigma_1|exp\{-(\frac{y_1^2+y_2^2}{2K^2})\}.$$

For this purpose set

$$\delta_r = \{s\in\sigma_1 : 2^{-r} \leq |W_s| < 2^{-r+1}\}, 4 \leq r \leq \infty$$

and observe that

$$\sigma_1 = \bigcup_{r=4}^{\infty}\delta_r.$$

Moreover, the selection of the points done in Step III implies that

$$2^{-2r}|\delta_r| \leq \sum_{s\in\delta_r} |W_s|^2 \leq \frac{M^2 n}{\alpha(M)^2},$$

i.e.

$$|\delta_r| \leq min\left(n^2,\, 2^{2r}\frac{nM^2}{\alpha(M)^2}\right),$$

for $r = 4, 5, \cdots$. Also notice that, for any $s \in \delta_r$ and $y_1, y_2 \geq 0$,

$$|a_s(y_1^2 + y_2^2) - 2b_s\, y_1 y_2 - K^{-2}(y_1^2 + y_2^2)| =$$

$$= \left|\frac{(W_s/K)^2(y_1^2 + y_2^2) - 2W_s\, y_1 y_2}{K^4 - W_s^2}\right| \leq$$

$$\leq 22[2^{-2r+4}(y_1^2 + y_2^2) + 2^{-r+2}y_1 y_2] \leq C_6 2^{-r}(y_1^2 + y_2^2),$$

for some constant $C_6 < \infty$. Hence,

$$exp\{-\frac{a_s}{2}(y_1^2 + y_2^2) + b_s y_1 y_2\}/exp\{-\frac{1}{2K^2}(y_1^2 + y_2^2)\} \leq$$

$$\leq exp\{C_6 2^{-r}(y_1^2 + y_2^2)\} \leq n^{C_6 2^{-r+1}L^2\rho^2},$$

provided $|y_1|$, $|y_2| \leq L\rho\sqrt{\log n}$. From the last inequality, taking into account that $n^{C_6 2^{-r+1}L^2\rho^2} \leq C_7 = C_7(M)$ if $r \geq \log_2 \log_2 n$, we get that

$$\sum \leq C_7\left(\sum_{r=4}^{\log_2\log_2 n} n^{C_6\, 2^{-r+1}L^2\rho^2}|\delta_r| + |\sigma_1|\right) \times$$
$$\times\, exp\{-\frac{1}{2K^2}(y_1^2 + y_2^2)\} \leq$$
$$\leq C_7 exp\{-\frac{1}{2K^2}(y_1^2 + y_2^2)\}(|\sigma_1| +$$
$$+ \sum_{r=4}^{\log_2\log_2 n} n^{C_6\, 2^{-r+1}L^2\rho^2} \cdot 2^{2r}\frac{nM^2}{\alpha^2(M)})$$

The last thing that we have to check is that for $\rho = \rho(M) > 0$ small enough,

$$\sum_{r=4}^{\log_2\log_2 n} n^{C_6\, 2^{-r+1}L^2\rho^2} \cdot 2^{2r}\frac{nM^2}{\alpha^2(M)} \leq C_8\,|\sigma_1|,$$

for some constant $C_8 = C_8(M)$. This completes the proof of the theorem. □

Remarks. 1. The comparison of the L_1 and L_2-norms, considered in the Step V of the proof above, has already been used in [7] and recently also in [3], in order to estimate the minimum on the unit circle of the absolute value of random polynomials with coefficients equal to ± 1.

2. As a generalization of the result mentioned as a remark in [7] p. 282, we can obtain the following proposition, by using essentially the same method as above.

Proposition 2. *With $\{\xi_i\}_{i=1}^n$ and $\{\varphi_i\}_{i=1}^n$ as in the statement of the Theorem, $0 < \sigma < 1/2$, Z as arbitrary subset of X of measure $\mu(Z) \geq n^{-\sigma}$ and*

$$\frac{1}{n}\sum_{i=1}^{n} |\varphi_i(x)|^2 \geq K_1 > 0, \ \frac{1}{n}\sum_{i=1}^{n} |\varphi_i(x)|^3 \leq K_2, \ x \in Z,$$

then

$$\mathbb{E} \parallel \sum_{i=1}^{n} \xi_i \varphi_i \parallel_{L_\infty(Z,\mu)} \geq c'(n \log n)^{1/2},$$

where $c' = c'(\sigma, M, K_1, K_2) > 0$.

Let us add that condition 2 in the statement of the Theorem above can be also weakened by replacing M with n^σ, $0 < \sigma < 1/2$.

3. The condition of boundedness in the L_3 -norm for both $\{\xi_i\}_{i=1}^n$ and $\{\varphi_i\}_{i=1}^n$ in the statement of the Theorem can be replaced by the weaker condition of uniform boundedness in some L_p -space, $p > 2$. The proof remains the same except that instead of Proposition 1 one must use Corollary 18.3 from [1].

References

1. R.N. Bhattacharya and R. Ranga Roo, *Normal approximation and Asymptotic Expansions*, J. Wiley (1976).
2. X. Fernique, *Regularité des trajectoires des fonctions aleatoires gaussiennes*, Lect. Notes in Math. 480 (1975), 1-96.
3. S. Konyagin, *On the minimum of the modulus of random trigonometric polynomials*, Math. Zametki **56** (1994), N3, 80-101.
4. M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Springer Verlag (1991).
5. M. Marcus and G. Pisier, *Random Fourier series with applications to harmonic analysis*, Princeton Univ. Press (1981).
6. V. Rotar.
7. R. Salem and A. Zygmund, *Some properties of trigonometric series whose terms have random signs*, Acta Math. **91** (1954), 245-301.

Steklov Mathematical Institute, Moscow, Hebrew University of Jerusalem.