The base of the formal versal deformation of a singularity and Lech inequalities

Jörg Jahnel

Friedrich-Schiller-Universität Jena
Mathematische Fakultät
Kollegiengasse
O-6900 Jena

Germany

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

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Abstract

We consider the problem of Lech, whether for a local deformation $(A, m) \longrightarrow (B, n)$ of a local singularity $B_0 = B/mB$ the inequality $e_0(A) \leq e_0(B)$ between the multiplicities is true, and give a positive answer in the case B_0 corresponds to a point of the Hilbert scheme (with respect to some formal embedding), having regular reduction and being Cohen-Macaulay itself.

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Introduction

In 1959 C. Lech [Le 59] stated the problem whether the multiplicities of local rings (A, m) and (B, n) being base, respectively total space, of a deformation $(A, m) \longrightarrow (B, n)$ of a local ring $B_0 = B/mB$ satisfy the inequality

$$e_0(A) \leq e_0(B). \tag{1}$$

Note that the only condition on such a homomorphism to be a deformation is its flatness.

A generalization of this is the analogous inequality

$$H_A^{d+i} \leq H_B^i \tag{2}$$

between sum transforms of the Hilbert series (d denotes the dimension of the fiber B_0). Here a sum transform is defined by

$$H_A^j := (1 - T)^{-j} \cdot H_A^0,$$

where H_A^0 is the usual Hilbert series

$$H_A^0 := \sum_{l=0}^{\infty} \dim_{A/m} m^l/m^{l+1} \cdot T^l.$$

The inequality between two formal power series $H = \sum_{l=0}^{\infty} H(l) \cdot T^l$ and $H' = \sum_{l=0}^{\infty} H'(l) \cdot T^l$ is always to be understood in its total sense, i.e.

$$H(l) \leq H'(l)$$

for all l.

In 1970 H. Hironaka [Hi 70] asked whether inequality (2) is always true with $i = 1$, since that would simplify his proof of the existence of a resolution of singularities in characteristic zero [Hi 64].

Unfortunately, this paper does not deal with that problem, but only with the inequality (1) between the multiplicities.

But also this inequality is established in very few cases, only. The most interesting result in that direction is due to Lech himself. It says that

$$H_A^1 \leq H_B^1$$

in the case, that the special fiber B_0 is a zero dimensional complete intersection [Le 64]. B. Herzog generalized this to the situation that B_0 corresponds to a regular point of the Hilbert scheme [He 90]. This includes all complete intersections (well known) and all singularities with an embedding dimension less than 3 (That is a result due to Hartshorne, but may be found in [Fo], see also [Gra].).

Further, Larfeldt and Lech [LL] (see also [Le 64] for one direction) showed that the general problem (1) of Lech is equivalent to the following statement:

For every local ring A and every coheight one prime P in A the inequality

$$e_0(A_P) \leq e_0(A) \tag{3}$$

is true.

This one, its immediate corollaries and the analogous inequalities for Hilbert series are usually referred as Bennett's inequality. Note that these problems can not be easy, since they generalize, at least in the Hilbert series version

$$H_{A_P}^1 \leq H_A^0,$$

Serre's result [Se], that the localization of a regular local ring by a prime ideal is again regular. They are solved in the case A is excellent (cf. Lemma 1.7 below), but that does not imply anything for Lech's problem, even for excellent rings!

We note, that there is also a completely different approach to the Lech- Hironaka problem. One can consider singularities with tangentially flat deformations only as in [He 91]. A generalization of that may be found in the doctoral thesis of the author [J].

In this paper we will follow the philosophy of [He 90] proving Lech's inequality, when B_0 corresponds to a mild singularity of the Hilbert scheme. Concretely, we require, that the base of the formal versal embedded deformation of B_0 [Schl] has a regular reduction and is Cohen-Macaulay itself. To say the truth, we also give in Theorem 1.6 a more general condition, but that seems to be very difficult to handle.

We shall use the conventions and notations of commutative algebra as in [Ma]. Further all local rings are assumed to be Noetherian. An A-algebra is a homomorphism of the ring A into some ring, a homomorphism of A-algebras is a commutative triangle. k will always denote a fixed ground field. Note that we use "local k-algebra" for algebras $k \longrightarrow (A, m)$, where (A, m) is local and $k \longrightarrow A/m$ is an isomorphism. In particular, "complete local k-algebras" form just the category $\hat{C}$ of [Schl]. By a local deformation of a local k-algebra B_0 we mean a flat local homomorphism of local k-algebras with special fiber B_0.

At some point we use the technical concept of tangential flatness. A local homomorphism $f : (A, m) \longrightarrow (B, n)$ of local rings such that the induced homomorphism $gr(A) \longrightarrow gr(B)$ of the associated graded rings

$$gr(A) := \bigoplus_{l=0}^{\infty} m^l/m^{l+1}$$

makes $gr(B)$ into a flat $gr(A)$ -module is called *tangentially flat.* The fundamental facts about tangential flatness may be found in [He 91].

We shall use the language of Schlessinger's paper [Schl]. Note, that we call the "pro- representable hull" of the deformation functor, he constructs, "formal versal embedded deformation". It is well known, and can easily be derived from the universal property of the Hilbert scheme, that the completed local ring of the Hilbert scheme at $[B_0]$ is nothing but the base of that formal versal embedded deformation of the singularity B_0.

At the end of the introduction the following principal remark: We consider only local deformations $f : (A, m) \longrightarrow (B, n)$ of local k-algebras, where A and B are equicharacteristic and f is residually rational. Using Cohen's structure theory one could really generalize that, at least one can replace "residually rational" by "residually separable". We will omit the proof for that, since it does not seem to make sense to consider the abstract situation, when almost nothing is known in the "geometric case".

1 A condition on the base of the versal deformation implying Lech's inequality

1.1 In this paper we are particularly interested in the class of local rings described in the Definition below.
Definition. *We will call a local ring* R permissible, *when it satisfies the following condition.*

There exists a system of parameters $\{x_1, \dots, x_s\}$ of R such that

$$\ell(R/(x_1, \dots, x_s)) \leq i(R).$$

Here the invariant $i(R)$ *is defined to be*

$$i(R) := \min_{\mathfrak{p} \in \mathrm{Specmin}(R)} \ell(R_{\mathfrak{p}}).$$

1.2 Remark. A word of interpretation for the invariant i. It measures how far the scheme Spec R is from having a component, in the generic point of which it is reduced. When we consider the special case, that R is Cohen-Macaulay and has only one minimal prime nil R, then i could be called "integrality defect". Indeed, in this case R has no embedded primes ([Ma], Theorem 17.3), i.e. $\mathrm{Ass}\,(R) = \{\mathrm{nil}\, R\}$ such that the complement of nil R contains regular elements only. Therefore, $\mathrm{nil}\, R = 0$ if and only if $\mathrm{nil}\, R \cdot R_{\mathrm{nil}\, R} = 0$, hence R is integral, if and only if $R_{\mathrm{nil}\, R}$ is a field, the latter being equivalent to $i(R) = 1$.

1.3 Now we come to our fundamental Proposition, implying ewerything what follows. Note that it is a direct generalization of [He 90], Theorem 6.
Proposition. *Let the commutative diagram*

$$\begin{array}{ccc} (R, M) & \longrightarrow & (S, N) \\ \downarrow & & \downarrow \\ (A, m) & \longrightarrow & (B, n) \end{array}$$

of local rings and local homomorphisms be cartesian, i.e. $B \cong A \otimes_R S$, *and assume the following conditions to be fulfilled.*

1. $R \longrightarrow A$ *is a homomorphism of local k-algebras.*

2. The special fiber of $R \longrightarrow S$ *has minimal dimension, i.e.*

$$\dim S = \dim R + \dim S/MS$$

(e.g. $R \longrightarrow S$ *is flat).*

3. R *is permissible.*

Then there exists a constant c (depending on the diagram) such that

$$H_A^{d+1}(n) \leq H_B^1(n + c)$$

for all n $(d := \dim B/mB)$.

In particular,

$$e_0(A) \leq e_0(B).$$

Proof. We are even in the position to specify the constant c, for which we will prove the assertion above:

c is the minimal natural number such that

$$(pS'_p)^{c+1} = 0$$

in S'_p for all minimal prime ideals p in S'.

Here we put $S' := S^\wedge[[T_1, \dots, T_t]]$, where $S^\wedge$ denotes the completion of S, $t\epsilon\mu_A(m)$ is the minimal number of generators of the maximal ideal m in A and $T_1, \dots, T_t$ are indeterminates.

Note that this definition makes sense, since there are only finitely many minimal prime ideals in S' and all the localizations by them are Artin local rings.

We start with several straightforward steps.

First step. *We may assume A to be an Artin local ring.*

For proving

$$H_A^{d+1}(n) \leq H_B^1(n+c)$$

for arbitrary given n the local rings A and B can be replaced by A/m^{n+c+1} and $B/m^{n+c+1}B$, respectively.

Second step. *We may assume, that B, R and S are complete local algebras (S and B possibly over an other ground field.)*

Replace the local rings of the diagram above by their completions. Since $A \otimes N$ is an n-primary ideal in $B = A \otimes_R S$, the canonical topology of B is that as a finite S-module. Therefore

$$B^\wedge = (A \otimes_R S)^\wedge = A \otimes_{R^\wedge} S^\wedge.$$

Further, Cohen's structure theory ([Ma], Theorem 28.3 or [EGA IV_0], §19) gives algebra structures on S and B over their coefficient fields.

Third step. *In the cartesian diagram above we may replace R and S in such a way that $R \longrightarrow A$ becomes surjective.*

Adjoin indeterminates T_i to R and S, which are mapped to a minimal system of generators of the maximal ideal m in A and consider the resulting commutative diagram.

$$\begin{array}{ccc} R[[T]] & \longrightarrow & S[[T]] \\ \downarrow & & \downarrow \\ A & \longrightarrow & B, \end{array}$$

where T denotes $\{T_1, \dots, T_t\}$.

Since A is Artin, $R[[T]] \longrightarrow A$ factors through $R_l := R[[T]]/(T)^l$ for some l. Therefore we see, using that $S[[T]]/(T)^l$ is a free S-module,

$$B \cong A \otimes_R S \cong A \otimes_{R_l} R_l \otimes_R S \cong A \otimes_{R_l} S[[T]]/(T)^l \cong$$

$$\cong A \otimes_{R_l} R_l \otimes_{R[[T]]} S[[T]] \cong A \otimes_{R[[T]]} S[[T]],$$

meaning that the new commutative square is cartesian, too.

We replace R and S by $R[[T]]$ and $S[[T]]$, respectively. Then a system of generators of m may be lifted to R. Since all rings are complete and $R \longrightarrow A$ is residually rational, this implies it is surjective.

Note that S', occuring in the definition of our constant c, is nothing but our new S. Further the permissibility of R is not affected by the replacement (Remark 1.10) and the dimension of the special fiber of $R \longrightarrow S$ is still the minimal one.

Fourth step. *In the cartesian diagram above we may replace S and B such that S/MS becomes an Artin local ring.*

Choose some prime ideal P in S satisfying $MS \subseteq P$ and

$$\dim S/P = \dim S/MS \quad (= d).$$

Then the special fiber of the induced homomorphism $R \longrightarrow S_P$ becomes zero dimensional and the commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & S_P \\ \downarrow & & \downarrow \\ A & \longrightarrow & B_P \end{array}$$

is again cartesian: $B_P \cong B \otimes_S S_P \cong A \otimes_R S \otimes_S S_P \cong A \otimes_R S_P$. Further, B_P is a local ring as a factor of the local ring S_P and all the homomorphisms in the diagram above are local. That is trivial, except for $A \longrightarrow B_P$, and there it follows from the simple reason that m consists of nilpotent elements only, which cannot be mapped to units.

Since B is complete, one has Bennett's inequality (Lemma 1.7)

$$H^{d+1}_{B_P} \leq H^1_B.$$

So we have to prove $H^{d+1}_A(n) \leq H^{d+1}_{B_P}(n+c)$, for which $H^1_A(n) \leq H^1_{B_P}(n+c)$ would be sufficient, obviously.

Further it turns out that the dimension of the fiber of $R \longrightarrow S_P$ is minimal. Here that means simply $\dim S_P = \dim R$. But this is clear by $\dim S = \dim R + d$ and $\dim S/P = d$, when one notes that S is complete and, therefore, catenary.

We replace S and B by S_P and B_P, respectively. Note that c conserves the property, that $(pS_p)^{c+1} = 0$ in S_p for all minimal primes p in S.

Fifth step. *This is the key step. We will prove that $B = A \otimes_R S$ is a factor of $B' := A \otimes_k S$ in a very specific way.*

Note that B' is Noetherian as a finitely generated S-algebra. Let n' be a maximal ideal in B'. Then $(m) := m \otimes S \subsetneq n'$, since m is nilpotent, and $n'/(m)$ is maximal in $B'/(m) = A \otimes_k S/m \otimes S \cong S$. Therefore,

$$n' = m \otimes S + A \otimes N,$$

which shows B' to be local. Further we see

$$\begin{aligned} B &\cong A \otimes_R S \\ &\cong A \otimes_k R/(x_1, \dots, x_s) \otimes_{R/(x_1,\dots,x_s)} R/M \otimes_R S \\ &\cong A \otimes_k R/(x_1, \dots, x_s) \otimes_R R/M \otimes_R S, \end{aligned}$$

where $\{x_1, \dots, x_s\}$ is the special system of parameters for R, required for permissibility. Using the commutativity of the tensor product, one obtains, denoting $R/(x_1, \dots, x_s)$ by $\overline{R}$

$$\begin{aligned} B &\cong A \otimes_k S \otimes_R \overline{R} \otimes_R \overline{R}/\overline{M} \\ &\cong A \otimes_k S \otimes_R \overline{R} \otimes_{\overline{R}} \overline{R}/\overline{M} \\ &\cong A \otimes_k S/(x_1, \dots, x_s) \otimes_{\overline{R}} \overline{R}/\overline{M} \\ &\cong B'/(x_1, \dots, x_s) \otimes_{\overline{R}} \overline{R}/\overline{M}. \end{aligned}$$

Here we remark, that $\overline{R}$ is Artin, hence $\overline{M}$ is nilpotent, which implies

$$\dim B'/(x_1, \dots, x_s) = \dim B'/(x_1, \dots, x_s) \otimes_{\overline{R}} \overline{R}/\overline{M} = \dim B = 0$$

by our reduction steps before. On the other hand $\dim S = \dim R$, since the dimension of the fiber of $R \longrightarrow S$ is the minimal one and we reduced that fiber to be Artin, and, furthermore,

$$\dim B' = \dim A \otimes_k S = \dim A/m \otimes_k S = \dim S = \dim R,$$

when we use m is nilpotent. Altogether that means, that $\{x_1, \dots, x_s\}$ is a system of parameters not only for R but also for B'.

Sixth step. *Now we are in the position to complete the proof, applying a method very similar to that used by B. Herzog in the case R is regular.*

Let

$$C_i := B'/(x_1, \dots, x_i).$$

By the fact that $\{x_1, \dots, x_s\}$ is a system of parameters for B' we know

$$\dim C_i = s - i.$$

Therefore we may take a chain of prime ideals

$$P_0 \subsetneq \dots \subsetneq P_s$$

in B' such that

i) $\dim B'/P_i = \dim C_i = s - i$,

ii) $P_{i+1} \supseteq (P_i, x_{i+1})$.

Then by Lemma 1.7

$$H^1_{(C_i)_{P_i}} \leq H^0_{(C_i)_{P_{i+1}}} \leq H^1_{(C_{i+1})_{P_{i+1}}},$$

hence

$$H^1_{B'_{P_0}} \leq H^1_{(B'/(x_1,\ldots,x_s))_{P_s}} \leq H^1_{B'/(x_1,\ldots,x_s)}.$$

Note that on the right hand side we simply use Bennett's inequality. Further, Lemma 1.8 gives

$$H^1_{B'/(x_1,\ldots,x_s)} \leq \ell(\overline{R}) \cdot H^1_{B'/(x_1,\ldots,x_s)\otimes_{\overline{R}}\overline{R}/\overline{M}} = \ell(\overline{R}) \cdot H^1_B$$

by the previous step. Altogether we found

$$H^1_{B'_{P_0}} \leq \ell(\overline{R}) \cdot H^1_B,$$

such that it would be sufficient to show

$$\ell(\overline{R}) \cdot H^1_A(n) \leq H^1_{B'_{P_0}}(n + c) \tag{11}$$

for all n.

For that we identify S with its canonical image in $B' := A \otimes_k S$ and put $P := P_0 \cap S$. Then the canonical homomorphism

$$S_P \longrightarrow B'_{P_0}$$

is well defined, local and factors through $A \otimes_k S_P$, a ring being local with maximal ideal

$$M_P := m \otimes_k S_P + A \otimes PS_P$$

(use that m is nilpotent). The induced homomorphism

$$A \otimes_k S_P \longrightarrow (A \otimes_k S)_{P_0}$$

turns out to be local, when we note once more that m is nilpotent. But, on the other hand, the ring on the right is obtained from the ring on the left by a further localization. So this homomorphism is even an isomorphism. In particular,

$$H^1_{B'_{P_0}} = H^1_{A\otimes_k S_P}.$$

By the completeness of the local k-algebra A we may assume that $A = k[[X]]/J$ for some finite set $X = \{X_1, \ldots, X_l\}$ of indeterminates and some ideal J in $k[[X]]$. The canonical imbedding $k \longrightarrow S_P$ is trivially tangentially flat, hence so is

$$k[[X]] \longrightarrow S_P[[X]], \qquad X_i \longmapsto X_i.$$

Therefore, by [He 91], Remark (1.4.i) or [J], Folgerung (1.12.i), the induced homomorphism

$$A = k[[X]]/J \longrightarrow S_P[[X]]/J \cdot S_P[[X]] \cong (k[[X]]/J) \otimes_k S_P = A \otimes_k S_P$$

is tangentially flat, too. Note that $S_P[[X]]/J \cdot S_P[[X]]$ may be written as the tensor product above, since the ring A is Artin. Now the reformulation of tangential flatness in terms of Hilbert series (see [He 91], Theorem (1.2.ii.e)) implies

$$H^1_{B'_{P_0}} = H^1_{A \otimes_k S_P} = H^1_A \cdot H^0_{S_P}.$$

Writing down that explicitly one sees

$$\begin{aligned} H^1_{B'_{P_0}}(n+c) &= \sum_{j=0}^{n+c} H^1_A(n+c-j) \cdot H^0_{S_P}(j) \\ &\geq \sum_{j=0}^{c} H^1_A(n) \cdot H^0_{S_P}(j) \\ &= H^1_A(n) \cdot \ell(S_P/(PS_P)^{c+1}) \\ &= H^1_A(n) \cdot \ell(S_P), \end{aligned}$$

using that H^1_A is monotonically increasing and that $(PS_P)^{c+1} = 0$ in S_P by construction of the constant c. Comparing that with (11) it turns out to be sufficient to prove

$$\ell(S_P) \geq \ell(\overline{R}) \quad (= \ell(R/(x_1, \ldots, x_s))). \tag{12}$$

For doing that we put $p := P \cap R$. Then the canonical homomorphism

$$R_p \longrightarrow S_P$$

is well defined, flat and local. Lemma 1.9 implies

$$\ell(S_P) \geq \ell(R_p).$$

Furthermore, we have

$$\ell(R_p) \geq i(R)$$

by the definition of $i(R)$. Note here, that, in the case that p is not a minimal prime, $\ell(R_p)$ is even infinite, which would also satisfy the required inequality, but easily turns out to be impossible. Finally we can use the permissibility of R

$$i(R) \geq \ell(R/(x_1, \ldots, x_s)).$$

All these three inequalities together give

$$\ell(S_P) \geq \ell(R/(x_1, \ldots, x_s)),$$

being just the required inequality (12).

□

1.4 Remark. We note, that, in the assumption of our Proposition, we did not need $R \longrightarrow S$ to be flat. We only required the minimality of the dimension of its fiber. The author does not know whether this fact is useful for the Lech-Hironaka problem. One should "lift" a formal versal deformation in such a way that the base becomes permissible (e.g. regular). Of course such a lift will not be flat, but it is reqired to have a fiber of minimal dimension. This "lift"-problem does not seem to be easy.

1.5 Remark. The following Theorem is, in some sense, the main result of this paper. We assume the base of the formal versal deformation of a singularity to be permissible. That condition seems to be difficult to handle. In the next section we will analyse that problem and construct a class of local rings being permissible.

1.6 Theorem. *Let (B_0, n_0) be a local k-algebra. Then consider its completion as an embedded singularity.*

$$B_0^\wedge = k[[X_1, \ldots, X_r]]/I_0$$

Suppose, that the base of the formal versal embedded deformation of $B_0^\wedge$ is permissible.

Then for every local deformation

$$(A, m) \longrightarrow (B, n)$$

of the local k-algebra B_0 the Lech inequality

$$e_0(A) \leq e_0(B)$$

is true.

Proof. We will prove the following better assertion.

There exists a natural number c (depending on $A \longrightarrow B$) such that

$$H_A^{d+1}(n) \leq H_B^1(n + c)$$

for all n. Here d denotes the dimension of B_0.

First step. *We may assume the local k-algebras B_0, A and B to be complete.*

Replace A and B by their completions. Then the induced homomorphism $A^\wedge \longrightarrow B^\wedge$ is again flat and its fiber is $B^\wedge/m^\wedge B^\wedge = B_0^\wedge$. Of course, there is no effect on the Hilbert series.

Second step. *$A \longrightarrow B$ is a base change of the formal versal deformation of B_0.*

Let the homomorphism

$$\xi : (R, M) \longrightarrow (S, N) \quad (= R[[X_1, \dots, X_r]]/I)$$

of local k-algebras be the formal versal embedded deformation of $B_0 = k[[X_1, \dots, X_r]]/I_0$. (The fact that ξ is of that special form can easily be deduced from [Ar], Remark 1.1.)

Using the language of Schlessinger [Schl], the couple (R, ξ) induces a morphism

$$h_R \longrightarrow D_{B_0/k},$$

where $h_R, D_{B_0/k}$: {Artin local k-algebras (with residue field k)} $\longrightarrow$ {Sets} are the *Hom*-functor of R and the deformation functor of B_0, respectively. This morphism is a pro- representable hull for the functor $D_{B_0/k}$ ([Schl], (3.10) and (2.7)), therefore it is smooth, which implies that the induced morphism

$$h_R^\wedge = \mathrm{Hom}_{\mathrm{local}\, k\text{-alg}}(R, \, .\,) \longrightarrow D_{B_0/k}^\wedge$$

between the canonical prolongations to

{complete (Noetherian) local k-algebras (with residue field k)}

is objectwise surjective ([Schl], (2.2) and (2.4)).

Down the earth this means nothing but the existence of a cartesian diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ A & \longrightarrow & B. \end{array}$$

So the claim comes from the Proposition above.

□

1.7 Lemma. *Let (A, m) be a local ring and $x \in m$ be an element. Then*

$$H_A^0 \leq H_{A/xA}^1.$$

If, moreover, A is excellent (e.g. complete), then for any prime ideal $P \in \mathrm{Spec}(A)$

$$H^d_{A_P} \leq H^0_A$$

$(d := \dim A/P)$.

Proof. The first statement is easily proved by the reader. Alternatively, see [Si], Theorem 1. The second part of the Lemma is just Bennett's inequality ([Be], Theorem (2)) in the improved version due to Singh (see [Si], p.202). For a comment on Singh's proof see [He 90], Proof of Lemma 2.

□

1.8 Lemma. *Let* $f : (A, m) \longrightarrow (B, n)$ *be a local homomorphism of local rings and assume A to be Artin. Then*

$$H^1_B \leq \ell(A) \cdot H^1_{B \otimes_A A/m}.$$

Proof. Note that this is a weakification of Theorem (1.2.i) of [He 91]. To give a direct proof, it will be sufficient to show $\ell(M) \leq \ell(A) \cdot \ell(M \otimes_A A/m)$ for every A-module M. But this is clear, since $\ell(M \otimes_A A/m) = \mu_A(M)$ is the minimal number of generators of M.

□

1.9 Lemma. *Let* $f : (A, m) \longrightarrow (B, n)$ *be a flat local homomorphism of local rings. Then*

$$\ell(A) \leq \ell(B) \qquad \textit{and} \qquad \mathrm{i}(A) \leq \mathrm{i}(B),$$

where ℓ *denotes the length and* i *is the invariant* i *from* Definition 1.1.

Proof. Of course, the first statement is interesting only when A is Artin. Then taking a composition series of A as an A-module and tensoring with B we obtain

$$\ell(B) = \ell(A) \cdot \ell(B/mB) \geq \ell(A).$$

For the second assertion let $P \in \mathrm{Spec}(B)$ be such a prime that $\ell(B_P)$ becomes minimal. When one puts $p := P \cap A$, then $A_p \longrightarrow B_P$ is flat and local, which implies $\ell(A_p) \leq \ell(B_P)$.

□

1.10 Remark. Let the local ring R be permissible. Then replacing R by

a) its completion $R^\wedge$

or

b) the formal power series ring $R[[T_1, \ldots, T_t]]$

does not affect the permissibility.

Actually in both cases Lemma 1.9 above implies that $i(R)$ can increase only. On the other hand, in case a) we can use for $R^\wedge$ the system of parameters coming from R, while in case b) we may extend it by $\{T_1, \dots, T_t\}$, to obtain the same length on the left hand side and, therefore, to conserve the permissibility inequality from Definition 1.1.

2 The case that the base of the versal deformation has regular reduction and is Cohen-Macaulay itself

2.1 Here we will apply Theorem 1. For that we have to construct examples of local rings being permissible in the sense of section 1. We will show that those, announced in the title, admit this property.

2.2 Theorem. *Let (B_0, n_0) be a local k-algebra. Then consider its completion $B_0^\wedge$ as an embedded singularity.*

$$B_0^\wedge = k[[X_1, \dots, X_r]]/I_0$$

Suppose that the reduction $R/\operatorname{nil} R$ of the base R of the formal versal embedded deformation of $B_0^\wedge$ is regular and that R itself is Cohen-Macaulay.

Then for every local deformation

$$(A, m) \longrightarrow (B, n)$$

of the k-algebra B_0 the Lech inequality

$$e_0(A) \leq e_0(B)$$

is true.

Proof. This is a direct consequence of the Theorem above and the Fact below.

□

2.3 Remark. Let B_0 be a local singularity corresponding to a point $[B_0]$ of the Hilbert scheme (with respect to some formal embedding), which has regular reduction and is Cohen-Macaulay itself.

This implies Lech's inequality for every local deformation of B_0.

In fact, the base of the formal versal deformation of B_0 is the completion of the Hilbert scheme at $[B_0]$ and completing preserves the properties having regular reduction and being Cohen-Macaulay.

2.4 Fact. *Let R be a complete local k-algebra such that the reduction $R/\operatorname{nil} R$ is regular and R itself is Cohen-Macaulay.*

Then R is permissible.

We need several Lemmata.

2.5 Lemma. *Let R be a complete local k-algebra such that its reduction $R/\operatorname{nil} R$ is regular.*

Then the canonical surjection $p : R \longrightarrow R/\operatorname{nil} R$ admits a section.

Proof. By completeness we may identify the regular local k-algebra $R/\operatorname{nil} R$ with $k[[T_1, \dots, T_l]]$, where $T_1, \dots, T_l$ are indeterminates. Then choose elements $T'_1, \dots, T'_l$ from the maximal ideal M of R, which are mapped to $T_1, \dots, T_l$, respectively. They induce a homomorphism on the free k-algebra.

$$k[T_1, \dots, T_l] \longrightarrow R \qquad , T_i \mapsto T'_i$$

Here $(T_1, \dots, T_l)$ is mapped into M and any polynomial, not contained in that ideal, has a constant term, therefore it is not mapped to zero in R/M, hence it is mapped to a unit in R. We get a local homomorphism $k[T_1, \dots, T_l]_{(T_1, \dots, T_l)} \longrightarrow R$. By the completeness of R this one may be continued to

$$i : k[[T_1, \dots, T_l]] \longrightarrow R.$$

It remains to prove $pi : k[[T_1, \dots, T_l]] \longrightarrow k[[T_1, \dots, T_l]]$ is the identity. But by construction we have $T_i \mapsto T_i$, implying that at least on $k[T_1, \dots, T_l]_{(T_1, \dots, T_l)} \subset k[[T_1, \dots, T_l]]$. The continuity of pi with respect to the natural topologies completes the proof.

□

2.6 Lemma. *Let R be a complete local k-algebra such that $p : R \longrightarrow R/\operatorname{nil} R$ has a section*

$$i : R/\operatorname{nil} R \hookrightarrow R.$$

Then R is, via i, a finite module over $R/\operatorname{nil} R$.

Proof. Let $\{x_1, \dots, x_v\}$ be a system of generators of the maximal ideal M in R. Then, obviously,

$$\{x_1 - ip(x_1), \dots, x_v - ip(x_v), ip(x_1), \dots, ip(x_v)\} \tag{21}$$

is such a system, too. Consider the following homomorphism of $R/\operatorname{nil} R$-algebras.

$$q : R/\operatorname{nil} R[[Y_1, \dots, Y_v]] \longrightarrow R \qquad , Y_i \mapsto x_i - ip(x_i)$$

This one is well defined, local and also a homomorphism of k-algebras. The system (21) of generators of M may be lifted to $\{Y_1, \dots, Y_v, p(x_1), \dots, p(x_v)\}$.

Since both rings are complete and q is residually rational, this implies q is surjective.

Further, we observe that

$$p(x_i - ip(x_i)) = 0,$$

maening $x_i - ip(x_i)$ is nilpotent for all i. Choose a_i such that $(x_i - ip(x_i))^{a_i} = 0$. Then q induces a surjection

$$R/\operatorname{nil} R[[Y_1, \ldots, Y_v]]/(Y_1^{a_1}, \ldots, Y_v^{a_v}) \longrightarrow R,$$

proving that R is finite as a module over $R/\operatorname{nil} R$.

□

2.7 Lemma. *Let $i : R' \hookrightarrow R$ be an injective local homomorphism of local rings, where R' is regular. Suppose that R is, via i, a finite R'-module.*

Then, if R is Cohen-Macaulay, it is even free as an R'-module.

Proof. This is just one direction of [Na], Theorem (25.16).

□

2.8 Proof of the Fact. The canonical surjection $p : R \longrightarrow R/\operatorname{nil} R$ has a section $i : R/\operatorname{nil} R \hookrightarrow R$ by Lemma 2.5. Further, the $R/\operatorname{nil} R$-module R (via i) is finite by Lemma 2.6 and, by Lemma 2.7, it is even free. Say r is its rank.

Identify the regular local k-algebra $R/\operatorname{nil} R$ with $k[[T_1, \ldots, T_s]]$, where $T_1, \ldots, T_s$ are indeterminates, and put

$$x_i := i(T_i).$$

Then $R/(x_1, \ldots, x_s)$ is free of rank r over $k[[T_1, \ldots, T_s]]/(T_1, \ldots, T_s) = k$, i.e. $\ell(R/(x_1, \ldots, x_s)) = r$. On the other hand, $\operatorname{nil} R$ is nilpotent, therefore

$$\dim R = \dim R/\operatorname{nil} R = s.$$

So we found a system of parameters $\{x_1, \ldots, x_s\}$ of R such that

$$\ell(R/(x_1, \ldots, x_s)) = r. \tag{22}$$

Furthermore, one can use localization instead of factorization. Let S be the multiplicative system

$$S := i(R/\operatorname{nil} R \setminus \{0\}).$$

in R. Then R_S is free of rank r over the quotient field $Q(R/\operatorname{nil} R)$, so

$$\ell(R_S) = r.$$

Obviously, $S \subsetneq R \backslash \operatorname{nil} R$. Therefore $R_{\operatorname{nil} R}$ is a further localization of R_S. We will prove, that, actually, $R_{\operatorname{nil} R} = R_S$.

For that we have to show, that every $x \in R \backslash \operatorname{nil} R$ is mapped into a unit in R_S. But

$$x = ip(x) + (x - ip(x)),$$

where the first summand is in S and the second one is nilpotent, since $p(x - ip(x)) = 0$. So the image of x in R_S is a sum of a unit and a nilpotent element and, therefore, in fact a unit ([Ma], §1, very first remarks).

We have shown

$$\ell(R_{\operatorname{nil} R}) = r.$$

But $R/\operatorname{nil} R$ is regular, hence integral. Therefore $\operatorname{nil} R$ is the only minimal prime ideal in R. Our definition gives

$$i(R) = r.$$

Combined with (22) this is the claim.

□

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