COMBINATIONS OF RATIONAL DOUBLE POINTS ON THE DEFORMATION OF QUADRILATERAL SINGULARITIES. I

by

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§0. Introduction

We would like to study hypersurface quadrilateral singularities in this article. Because the study of them can be reduced to the study of elliptic K3 surfaces with a section and with a singular fiber of type I_0^*, we study such K3 surfaces, too. We show that the possible combinations of rational double points on fibers in the semi-universal deformations of such singularities can be described by a certain law from the view-point of Dynkin graphs. This is equivalent to saying that the possible combinations of singular fibers in elliptic K3 surfaces which have a section and a singular fiber of type I_0^* can be described by a certain law using Dynkin graphs.

We always work over the complex number field $\mathbb{C}$ in this article.

Now, there are 6 kinds of hypersurface quadrilateral singularities (Arnold [1], [2], Looijenga [8]). Each of them has the following normal form of the defining function and the Milnor number μ.

$$J_{3,0} : x^3 + ax^2y^3 + y^9 + bxy^7 + z^2, \quad (4a^3 + 27 \neq 0),$$
$$\mu = 16.$$

$$Z_{1,0} : x^3y + ax^2y^3 + bxy^6 + y^7 + z^2, \quad (4a^3 + 27 \neq 0),$$
$$\mu = 15.$$

$$Q_{2,0} : x^3 + yz^2 + ax^2y^2 + bx^2y^3 + xy^4, \ (a^2 \neq 4),$$
$$\mu = 14.$$

$$W_{1,0} : x^4 + ax^2y^3 + bx^2y^4 + y^6 + z^2, \ (a^2 \neq 4),$$
$$\mu = 15.$$

$$S_{1,0} : x^2z + yz^2 + y^5 + ay^3z + by^4z, \ (a^2 \neq 4),$$
$$y = 14.$$

$$U_{1,0} : x^3 + xz^2 + xy^3 + ay^3z + by^4z, \ (a(a^2 + 1) \neq 0),$$
$$\mu = 14.$$

All of them have modules number 2.

We deal mainly with the four cases $J_{3,0}$, $Z_{1,0}$, $Q_{2,0}$ and $W_{1,0}$ in this article. The remaining two cases $S_{1,0}$, $U_{1,0}$ are treated only in the beginning part and will be studied further in a forthcoming article.

To state theorems we need two definitions (Urabe [12], [13], [14]). As for the precise definition of connected Dynkin graphs, see section 3.

<u>Definition 0.1.</u> (An elementary transformation)
A disjoint finite union of connected Dynkin graphs is called a Dynkin graph. The following procedure is called an <u>elementary transformation</u> of such a Dynkin graph:

(1) Replace each connected component by the corresponding extended Dynkin graph.

(2) Choose in an aribitrary manner at least one vertex from each component (of the extended Dynkin graph) and then remove these vertices together with the edges issuing from them.

<u>Definition 0.2.</u> (A tie transformation)

Assume that applying the following procedure to a Dynkin graph G, we have obtained the Dynkin graph $\overline{G}$. Then we call the following procedure a <u>tie transformation</u> of Dynkin graphs:

(1) Attach an integer to each vertex of G by the following rule: Now, let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be a root basis associated with a connected component G_0 of G. Let $\sum_{i=1}^{k} n_i\alpha_i$ be the associated maximal root. Then the attached integer to the vertex corresponding to α_i is n_i.

(2) Add one vertex and a few edges to each component of G and make it into the extended Dynkin graph of the corresponding type. Attach moreover the integer 1 to each new vertex.

(3) Choose in an arbitrary manner subsets A, B of the set of vertices of the extended graph $\tilde{G}$ satisfying the following conditions:

<a> $A \cap B = \phi$

<b> Choose arbitrarily a component $\tilde{G}_1$ of the extended graph $\tilde{G}$ and let V be the set of vertices in $\tilde{G}_1$. Let N be the sum of the numbers attached to elements in $B \cap V$. (If $B \cap V = \phi$, $N = 0$.) Let ℓ be the number of elements in $A \cap V$, and $m_1, m_2, \ldots, m_\ell$ be the attached integers to $A \cap V$. Then, the greatest common divisor of $\ell + 1$ numbers $N, m_1, m_2, \ldots, m_\ell$ is necessarily 1.

(4) Erase out all attached integers.

(5) Remove vertices belonging to A together with the edges issuing from them.

(6) Draw another new vertex called θ which corresponds to a long root. Connect θ and each vertex in B by an edge.

Remark. Often the resulting graph $\bar{G}$ after the above procedure (1) - (6) is not a Dynkin graph. We consider only the cases where the resulting graph $\bar{G}$ is a Dynkin graph and then we call the above procedure a tie transformation.

The number $\#(B)$ of elements in the set B satisfies $0 \leq \#(B) \leq 3$. $\ell = \#(A \cap V) \geq 1$.

Note that any connected Dynkin graph of type A, D or E corresponds to a singularity on a surface (Durfee [6]).

When the Dynkin graph G contains a_k of connected components of type A_k, b_ℓ of components of type D_ℓ, c_m of components of type E_m, d_n of components of type B_n, ..., we identify the formal sum

$G = \sum a_k A_k + \sum b_\ell D_\ell + \sum c_m E_m + \sum d_n B_n + \ldots$ with the graph G.

Let L be a lattice (i.e. a free $\mathbb{Z}$-module of finite rank equipped with an integral symmetric bilinear form). By p we denote a prime number. The Hasse symbol of the inner product space $L \otimes \mathbb{Q}$ over the rational number field $\mathbb{Q}$ is denoted by $\epsilon_p(L) = \pm 1$. The symbol $(\, , \,)_p$ is the Hilbert norm residue symbol. By $d(L)$ we denote the discriminant of L. $\mathbb{Q}_p$ is the field of p-adic numbers. $\mathbb{Q}_p^{*2} = \{a^2 | a \in \mathbb{Q}_p, \ a \neq 0\}$ (Cassels [4], Serre [11]).

Let X denote one of the 6 kinds of quadrilateral singularities. Let $PC(X)$ be the set of Dynkin graphs G with components of type A, D or E only such that there exists a fiber Y in the semi-universal deformation family of the singularity X satisfying the following two conditions depending on G.

(1) The fiber Y has only rational double points as singularities.

(2) The combination of rational double points on Y just corresponds to the graph G.

<u>Theorem 0.3.</u> Set $X_1 = J_{3,0}$, $X_2 = Z_{1,0}$, and $X_3 = Q_{2,0}$. According as $m = 1, 2$ or 3, we deal with the hypersurface quadrilateral singularity X_m. We denote the number of vertices in the Dynkin graph G by r.

[I] The following two conditions (a) and (b) are equivalent.

(a) $G \in PC(X_m)$ and one of the following conditions <1>, <2>,

<3> and <4> holds for the root lattice $Q = Q(G)$ of type G.

<1> $r = 13 - m$, $md(Q)$ is a square number, and for every prime number p $\epsilon_p(Q) = 1$.

<2> $r = 12 - m$, and for every prime number p $\epsilon_p(Q) = (-m, d(Q))_p$.

<3> $r = 11 - m$, and for every prime number p $-md(Q) \notin Q_p^{*2}$ or $\epsilon_p(Q) = (-m, -1)_p$.

<4> $r \leq 10 - m$.

(b) G contains no vertex corresponding to a short root and it can be obtained from one of the following basic Dynkin graphs by elementary transformations repeated twice.

The basic Dynkin graphs:

The case of $m = 1$, $J_{3,0}$: $E_8 + F_4$, B_{12}.

The case of $m = 2$, $Z_{1,0}$: $E_7 + F_4$, $E_8 + CB_3$, $B_{10} + CB_1$.

The case of $m = 3$, $Q_{2,0}$: $E_6 + F_4$, $E_8 + F_2$, B_9.

[II] The following two conditions (A) and (B) are also equivalent.

(A) $G \in PC(X_m)$ and one of the following conditions <1>, <2>, <3> holds for the root lattice $Q = Q(G)$ of type G.

<1> $r = 14 - m$, and for every prime number p $\epsilon_p(Q) = (m, -d(Q))_p$.

<2> $r = 13 - m$, and for every prime number p $md(Q) \notin Q_p^{*2}$ or $\epsilon_p(Q) = 1$.

<3> $r \leq 12 - m$.

(B) G contains no vertex corresponding to a short root and it can be obtained from one of the following essential basic Dynkin graphs by one of the following 3 kinds of procedures.

The procedures:

<1> elementary transformations repeated twice

<2> an elementary transformation following after a tie transformation

<3> a tie transformation following after an elementary transformation.

The essential basic Dynkin graphs:

The case of $m = 1$, $J_{3,0}$: $E_8 + F_4$

The case of $m = 2$, $Z_{1,0}$: $E_7 + F_4$, $E_8 + CB_3$

The case of $m = 3$, $Q_{2,0}$: $E_6 + F_4$, $E_8 + F_2$

[III] Let G be a Dynkin graph with components of type A, D or E only. Assume that we can obtain G from one of the basic Dynkin graph by tie transformations repeated twice. Then, $G \in PC(X_m)$.

<u>Remarks.</u> (1) (The Hilbert norm residue symbol) Let a, b and c be non-zero rational numbers. $(a,b)_p = \pm 1$, $(a,b)_p = (b,a)_p$, $(a,bc)_p = (a,b)_p(a,c)_p$, $(a,b^2)_p = 1$, $(a,-a)_p = 1$, and $(a,1-a)_p = 1$ for $a \neq 0, 1$.

Consider the case where a and b are integers with $a = p^{\alpha}u$, $b = p^{\beta}v$, where u and v are integers not divisible by the prime number p. If p is odd,

$$(a,b)_p = (-1)^{\alpha\beta(p-1)/2}\left[\frac{u}{p}\right]^{\beta}\left[\frac{v}{p}\right]^{\alpha} ,$$

where $\left[\frac{-}{p}\right]$ is the Legendre quadratic residue symbol. For $p = 2$,

$$(a,b)_2 = (-1)^{(u-1)(v-1)/4+\alpha(v^2-1)/8+\beta(u^2-1)/8}$$

(2) $r = \text{rank } Q(G)$.

(3) If $G = G' + G''$ for Dynkin graphs G, G', G'',

$Q(G) = Q(G') \oplus Q(G'')$ (orthogonal direct sum)

(4) If $L = L' \oplus L''$ (orthongal direct sum) for lattices L, L' and L'',

$d(L) = d(L')d(L'')$, $\epsilon_p(L) = \epsilon_p(L')\epsilon_p(L'')(d(L'),d(L''))_p$.

(5) $d(Q(A_k)) = k+1$, $d(Q(D_\ell)) = 4$, $d(Q(E_6)) = 3$,

$d(Q(E_7)) = 2$, and $d(Q(E_8)) = 1$.

(6) $\epsilon_p(Q(A_k)) = (-1,k+1)_p$, and $\epsilon_p(Q(D_\ell)) = \epsilon_p(Q(E_m)) = 1$.

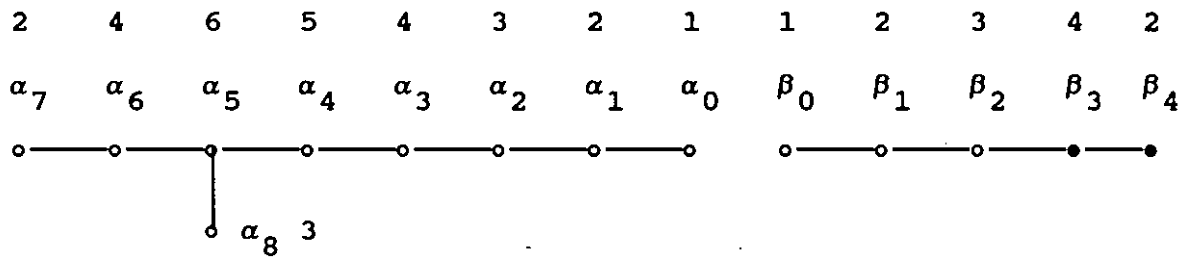
(7) As for the Dynkin graph of type CB_k $(k = 1, 2, 3)$ and the Dynkin graph of type B_1, F_2, F_3 $(= C_3)$, see section 3.

(8) It is easy to see that the condition [I](a) implies the condition [II](A).

(9) The maximal number of vertices of $G \in PC(X_m)$ is $15-m$. For example, E_8+E_6, $2E_7 \in PC(J_{3,0})$.

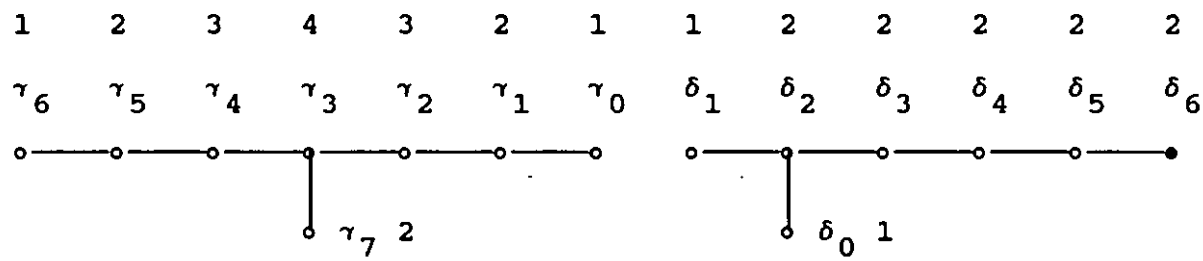
<u>Example.</u> Let us show that $2E_7$ and A_7+D_6 are members of $PC(J_{3,0})$.

First consider the Dynkin graph E_8+F_4. This is the essential basic Dynkin graph for $J_{3,0}$. We apply a tie transformation to this graph. At the second step we have the following graph.



Set $A = \langle \alpha_1, \beta_4 \rangle$ and $B = \langle \alpha_0, \beta_0 \rangle$. We can check that the condition on G.C.D. is satisfied for each component. Under this choice we get the graph E_7+B_6 as the result of the tie transformation.

Now, we can apply a transformation to E_7+B_6 once more. At the start we have the following graph.



If an elementary transformation is applied and if we erase out the vertices γ_7 and δ_6, we get the graph A_7+D_6. If a tie transformation is applied and if we choose $A = \langle \gamma_0, \delta_6 \rangle$ and $B = \langle \delta_1 \rangle$, we get the graph $2E_7$.

By the above theorem [II], [III], one knows that E_7+D_6, $2E_7 \in PC(J_{3,0})$.

Here we would like to explain the relation between the above theorem and elliptic K3 surfaces for those who are interested in elliptic surfaces (Kodaira [7]). Let $\Phi : Z \longrightarrow C(\cong \mathbb{P}^1)$ be an elliptic K3 surface. It has no multiple fibers. By Kodaira's result we have an elliptic K3 surface $\Phi' : Z' \longrightarrow C'$ with a section $C' \longrightarrow Z'$ whose combination of singular fibers is same as that of Φ. Therefore we can assume from the beginning that Φ itself has a section. Then we can associate each singular fiber with a connected Dynkin graph of type A, D or E in a natural manner.

$$
\begin{array}{llllll}
I_b & \longrightarrow & A_{b-1}, & I_b^* & \longrightarrow & D_{b+4} \\
II & \longrightarrow & \phi, & II^* & \longrightarrow & E_8 \\
III & \longrightarrow & A_1, & III^* & \longrightarrow & E_7 \\
IV & \longrightarrow & A_2, & IV^* & \longrightarrow & E_6
\end{array}
$$

Let $\hat{G}$ denote the formal sum of all connected Dynkin graphs associated with the singular fibers of Φ. Let PC be the set of all Dynkin graphs $\hat{G}$ obtained from elliptic K3 surfaces $\Phi : Z \longrightarrow C$. Note that $\hat{G}$ has a component of type D_4 if and only if Φ has a fiber of type I_0^*. Now, by Looijenga [8] it is known that $G+D_4$ belongs to PC if and only if G belongs to $PC(J_{3,0})$. (See section 1.) Therefore one knows by the above theorem that possible combinations of singular fibers in elliptic K3 surfaces with a singular fiber of type I_0^* are subject to the law described above. The sets $PC(Z_{1,0})$ and $PC(Q_{2,0})$ describe possible combinations of singular fibers in elliptic K3 surfaces with additional conditions. (See section 1.)

Now, we guess here that readers would like to know whether the converse of the above [II] and [III] holds or does not. Indeed, in the case $m = 2$, $Z_{1,0}$, the converse statement containing only the essential basic Dynkin graphs is true.

A Dynkin graph G with components of type A, D or E only belongs to $PC(Z_{1,0})$ if and only if we can make G from one of the essential basic Dynkin graphs E_7+F_4, E_8+CB_3 by elementary or tie transformations repeated 2 times. (We can apply 2 different kinds of transformations once for each, or can apply 2 transformations of the same kind.)

In the case $m = 1$, $J_{3,0}$, we have a unique exception $G = 3A_3+2A_2$. The Dynkin graph $3A_3+2A_2$ belongs to $PC(J_{3,0})$. However, we cannot make $3A_3+2A_2$ from either the basic Dynkin graph E_8+F_4 or B_{12} by a combination of 2 of elementary transformations and tie transformations.

If G contains components of type A, D, E only and if $G \neq 3A_3+2A_2$, G belongs to $PC(J_{3,0})$ if and only if we can make G from the essential basic Dynkin graph E_8+F_4 by 2 kinds of transformations repeated twice.

In the case $m = 3$, $Q_{2,0}$, $G = 3A_3+A_2$ is the unique exception.

We do not discuss the converse of [II] and [III] further in this article. We will deal with it in the forthcoming article.

The list of all maximal graphs in $PC(J_{3,0})$ with respect to the inclusion relation has been first given by F.-J. Bilitewski. He has used the calculation based on Nikulin's criterion about lattice embeddings. Here we express deep thanks to Professor Bilitewski for showing me his list.

At the same time Bilitewski has given the following description for $PC(Z_{1,0})$ and $PC(Q_{2,0})$. First we consider $PC(Z_{1,0})$. Set

$$\mathcal{M}_1 = \{E_6,\ E_7,\ E_8,\ A_1\} \cup \{D_\ell \mid \ell = 4,\ 5,\ \ldots\}$$
$$\mathcal{G}_1 = \{(G,\ G_0) \mid G \in PC(J_{3,0}),\ G_0 \in \mathcal{M}_1,\ G_0 \text{ is a component of } G.\}.$$

Consider an element $(G,\ G_0) \in \mathcal{G}_1$. We can write $G = G_1 + G_0$. We associate G_0' with G_0 in the following manner, depending on the type of G_0. Then, we set $G' = G_1+G_0'$.

$G_0 \longrightarrow G_0'$	
$E_8 \longrightarrow E_7$,	$D_4 \longrightarrow 3A_1$,
$E_7 \longrightarrow D_6$,	$D_5 \longrightarrow A_3+A_1$,
$E_6 \longrightarrow A_5$,	$D_\ell \longrightarrow D_{\ell-2}+A_1$ $(\ell \geq 6)$,
$A_1 \longrightarrow \phi$.	

Let $\mathcal{G}_1'$ be the set of all G' obtained from elements $(G, G_0) \in \mathcal{G}_1$. Then, $\mathcal{G}_1' = PC(Z_{1,0})$.

For $PC(Q_{2,0})$ the description is like the following. Set

$\mathcal{A}_2 = \{E_6, E_7, E_8, A_2\}$

$\mathcal{G}_2 = \{(G, G_0) | G \in PC(J_{3,0}), G_0 \in \mathcal{A}_2, G_0$ is a component of $G.\}$.

For $(G, G_0) \in \mathcal{G}_2$, we can write $G = G_1 + G_0$. Associating G_0' with G_0 in the following manner, we set $G' = G_1 + G_0'$.

$G_0 \longrightarrow G_0'$	
$E_8 \longrightarrow E_6$,	$E_6 \longrightarrow 2A_2$,
$E_7 \longrightarrow A_5$,	$A_2 \longrightarrow \phi$.

Let $\mathcal{G}_2'$ be the set of all G' obtained from elements in $\mathcal{G}_2$. Then $\mathcal{G}_2' = PC(Q_{2,0})$.

Bilitewski's replacement depends on the theory of singular fibers in elliptic surfaces. It is clear and easy to understand if the set $PC(J_{3,0})$ is known.

In order to state the theorem for $W_{1,0}$, we need introduce another new concept "obstruction components". Some of the components of the Dynkin graph are distinguished from the others as obstruction components and they follow special rules.

<u>Definition 0.4.</u> When a component G_1 of the Dynkin graph G is an <u>obstruction component</u>, G_1 follows the rules below.

[The rule under an elementary transformation].

Assume that making the corresponding extended Dynkin graph $\tilde{G}$ from G, and erasing out several vertices and edges issuing from them, we have obtained the Dynkin graph G'.

(1) Let $\tilde{G}_1$ be the component of $\tilde{G}$ corresponding to G_1. If the vertex erased from $\tilde{G}_1$ is unique, we can make any component G'_1 of G' derived from $\tilde{G}_1$ an obstruction component of G'. (We can also make G'_1 a non-obstruction component of G', if we want to.)

(2) When two or more vertices are erased from $\tilde{G}_1$, any component of G' derived from $\tilde{G}_1$ is <u>not</u> an obstruction component.

(3) Obstruction components of G' are only those obtained from obstruction components of G following the above rules (1) and (2).

[The rule under a tie transformation].

Assume that making the extended Dynkin graph $\tilde{G}$ from G and choosing subsets A and B of the set of vertices in $\tilde{G}$ satisfying the condition, we have made the new Dynkin graph G' depending A and B.

(1) Assume that the sets A and B satisfy the following condition #.

$\begin{cases} \text{Let } V_1 \text{ be the set of vertices in the extended Dynkin} \\ \text{graph } \tilde{G}_1 \text{ corresponding to } G_1.\ V_1 \cap B = \phi \text{ and} \\ V_1 \cap A \text{ consists of a unique element.} \end{cases}$

Then, $V_1 - A$ is the set of vertices in a component G_1' of G'. (G_1' has the same type as G_1.) This G_1' is necessarily an obstruction component of G'.

(2) When the sets A and B do not satisfy the condition #, any component of G' containing a vertex belonging to $V_1 - A$ is not an obstruction component.

(3) Obstruction components of G' are only those obtained from obstruction components of G following the above rules (1) and (2).

Remark. Usually we assume further that an obstruction component is of type A_k with $k \geq 4$. (See Definition 3.7 (2) and Theorem 3.9.)

Theorem 0.5. Let r denote the number of vertices in a Dynkin graph G.

[I] The following conditions (a) and (b) are equivalent.

(a) $G \in PC(W_{1,0})$ and one of the following conditions <1>, <2>, <3> and <4> holds for the root lattice $Q = Q(G)$ of type G.

<1> $r = 11$, $3d(Q)$ is a square number, and for every prime number p $\epsilon_p(Q) = (-1,3)_p$.

<2> $r = 10$, and for every prime number p
$\epsilon_p(Q) = (-3,d(Q))_p(-1,3)_p$.

<3> $r = 9$, an for every prime number p
$-3d(Q) \notin \mathbb{Q}_p^{*2}$ or $\epsilon_p(Q) = (-1,-1)_p$.

<4> $r \leq 8$.

(b) G contains no vertex corresponding to a short root and it can be obtained from one of the following basic Dynkin graphs by elementary transformations repeated twice.

The basic Dynkin graphs:

$E_8+B_1+G_2$, $E_7+B_3+G_1$, B_9+G_2, A_{11}.

[II] The following two conditions (A) and (B) are also equivalent.

(A) $G \in PC(W_{1,0})$ and one of the following conditions <1>, <2> and <3> holds for the root lattice $Q = Q(G)$ of type G.

<1> $r = 12$, and for every prime number p $\epsilon_p(Q) = (3,d(Q))_p$.

<2> $r = 11$, and for every prime number p
$3d(Q) \notin \mathbb{Q}_p^{*2}$ or $\epsilon_p(Q) = (-1,3)_p$.

<3> $r \leq 10$.

(B) G contains no vertex corresponding to a short root and it can be obtained from one of the above basic Dynkin graphs by one of the 3 kinds of procedures in Theorem 0.3 [II] (B).

[III] Among the connected Dynkin graphs appearing as a component of the above basic Dynkin graphs, we define that one of type A_{11} is an obstruction component and any one of other type is not an obstruction one.

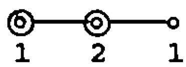
Let G be a Dynkin graph with components of type A, D or E only. Assume moreover that G contains no obstruction component. If we can make G from one of the basic Dynkin graph by 2 of elementary or tie transformations, then $G \in PC(W_{1,0})$.

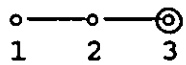
Remark. (1) As for the Dynkin graphs of type B_1 and G_1, see section 3.
(2) The maximal number of vertices of $G \in PC(W_{1,0})$ is 13. For example A_{13}, D_{13}, E_8+D_5, $E_7+D_6 \in PC(W_{1,0})$.

As for $W_{1,0}$, the converse statement of [III] is also true.

Under the definition of obstruction components as in the above [III], if a Dynkin graph G has components of type A, D, or E only and if G has no obstruction components, G belongs to $PC(W_{1,0})$ if and only if we can make G from one of the above 4 basic Dynkin graphs by 2 kinds of transformations repeated 2 times.

However, we do not discuss the converse in this article. We will show it in the forthcoming article.

Now, 2 kinds of quadrilateral singularities $S_{1,0}$ and $U_{1,0}$ are remaining. We can easily formulate the corresponding theorems to them. The basic Dynkin graphs for $S_{1,0}$ seem to be A_9+CB_1, E_8+CB_1, B_8+A_1, E_7+CB_2, and E_6+B_3. Those for $U_{1,0}$ seem to be E_8+G_2, $E_8+A_2(1/3)$, E_7+G_2, $E_6+A_2+A_2(1/3)$, and A_8+G_2. (See section 3 Agreement 2 for the notation $A_2(1/3)$.) However, the proof of them contains difficulties which we cannot find in the four cases in this article. (For $S_{1,0}$ we cannot write down the Coxeter-Vinberg graph for Λ_2/P and P' has no nice decomposition. For $U_{1,0}$ the corresponding lattice P has a proper overlattice and because of this reason we have to develope our general theory further. For example, for $U_{1,0}$, we have to introduce the dual extended Dynkin graph 

of type G_2 as well as the ordinary extended Dynkin graph 

of type G_2.)

We will study $S_{1,0}$ and $U_{1,0}$ in a forthcoming article.

As for general elliptic K3 surfaces, we can formulate the corresponding theorem about combinations of singular fibers in them. The basic Dynkin graphs in this case are $2E_8$ and D_{16}. The part [I] in the corresponding theorem is certainly true. The part [II] and [III] also hold. Perhaps there may exist several exceptions for the converse of [II] and [III]. However, anyway, we do not have considered this case closely and we are not sure.

I would like to give a theorem dealing with all elliptic K3 surfaces in a forthcoming article.

Besides there exist similar theorems for 14 exceptional hypersurface singularities with modules number 1 (Arnold [1], [2]). We would also like to deal with them in a forthcoming article.

The plan of this article is like the following. In section 0 we stated the main results. Though many words were necessary to state them, all of the graph-theoretical parts were natural and simple. Once we understand them, they appeal to our intuition. In section 1 we review Looijenga's results. The relation between quadrilateral singularities and elliptic K3 surfaces is explained. Our problem is reduced to the problem on existence of the embedding of lattices with certain conditions. Section 2 is chiefly devoted to the calculation in order to convert Looijenga's condition on lattices into a simpler condition on root systems. In section 3 we first introduce the concept of root modules and develop the general theory of root systems in our situation. Secondly we do the conversion using the results in section 2. Short roots and obstruction components are introduced to represent certain obstructions related to Looijenga's condition. Section 4 is used to develope the theory of elementary transformations and tie transformations. In particular, the theory for obstruction components is developed. The Coxeter-Vinberg graphs associated with hyperbolic spaces are studied in section 5. They are powerful tools to study root systems in quasi-lattices. In section 6, after dealing with conditions on isotropic elements written with the Hasse symbol and the Hilbert norm residue symbol, we collect all ideas in the previous sections in order.

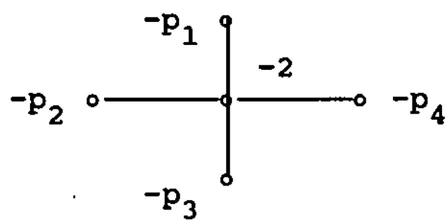
Here I would like to express thanks to Professor E. Brieskorn and Professor F.-J. Bilitewski for useful discussions.

§1. Qudrilateral singularities and elliptic K3 surfaces

The theme of this article is 6 kinds of hypersurface quadrilateral singularities. To each one of the 6 kinds we associate a quadruple (p_1, p_2, p_3, p_4) of integers.

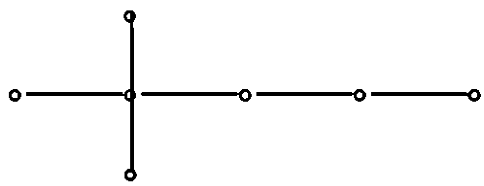
$J_{3,0}(2,2,2,3)$, $Z_{1,0}(2,2,2,4)$, $Q_{2,0}(2,2,2,5)$,
$W_{1,0}(2,2,3,3)$, $S_{1,0}(2,2,3,4)$,
$U_{1,0}(2,3,3,3)$.

The exceptional curve in the minimal resolution of each singularity has 5 irreducible components. Every component is a smooth rational curve. The following dual graph represents how they intersect.



The numbers attached to vertices are the self-intersection numbers of the corresponding components.

Now, apart from the above graph, we consider a combination of $\sum_{i=1}^{4} p_i - 3$ of smooth rational curves on a smooth surface whose mutual intersection is represented by the following dual graph (1). However, here, we represent particularly the dual graph in the case $Z_{1,0}(2,2,2,4)$.



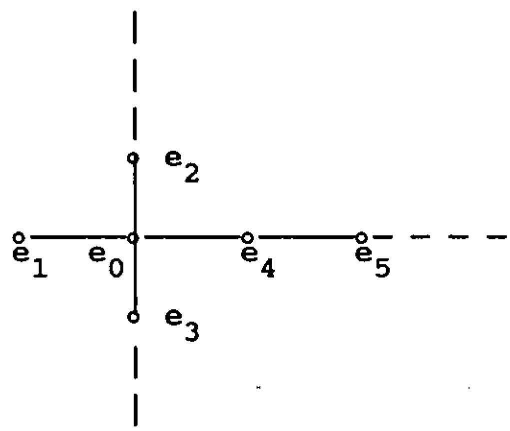
(1)

The 4 arms of the dual graph (1) have p_1, p_2, p_3, p_4 of vertices respectively including the common central one. We define that every curve corresponding to a vertex in the graph has self-intersection number (-2) on the surface. We call this combination <u>the curve at infinity</u> $IF = IF(p_1,p_2,p_3,p_4)$.

Related to IF, we define a lattice $P = P(p_1,p_2,p_3,p_4)$ generated by the basis $B = \{e_0,e_1,\ldots,e_{q-1}\}$ consisting of $q = \sum_{i=1}^{4} p_i - 3$ vectors.

The basis B has one-to-one correspondence with the set of vertices in the dual graph (1) of $IF(p_1,p_2,p_3,p_4)$. The bilinear form on $P = \sum_{e \in B} \mathbb{Z}e$ is defined as follows. For each $e \in B$, $e^2 = e \cdot e = (e,e) = +2$. For two elements $e, e' \in B$, $e \cdot e' = (e,e') = -1$, if the corresponding vertices to e and e' are connected in the graph (1), and $e \cdot e' = (e,e') = 0$ if they are not connected in (1). P is an even lattice with signature $(\sum_{i=1}^{4} p_i - 4, 1)$.

We choose special elements $e_0, e_1, e_2, e_3, e_4, e_5 \in B$ for the convenience of quotation later. Let e_0 be the one corresponding to the central vertex with 4 edges in (1). We assign e_1, e_2, e_3, e_4 to the 4 vertices connected to the central one e_0. We choose e_i $(1 \leq i \leq 4)$ in such a manner that it belongs to the arm with length p_i. In particular, e_4 belongs to the longest arm with length $p_4 \geq 3$. The vector e_5 is assigned to the adjacent vertex to the one associated with e_4 which is not associated with e_0.



Under the above choice, by P_0 we denote the sublattice of P of rank 5 generated by e_0, e_1, e_2, e_3 and e_4. We define isotropic elements $u_0 \in P_0$ and $v_0 \in P$ by

$$u_0 = 2e_0 + e_1 + e_2 + e_3 + e_4$$
$$v_0 = -(u_0 + e_5) = -(2e_0 + e_1 + e_2 + e_3 + e_4 + e_5).$$

One knows $u_0^2 = v_0^2 = 0$, and $u_0 \cdot v_0 = 1$. The sublattice $H_0 = \mathbb{Z}u_0 + \mathbb{Z}v_0$ is isomorphic to <u>the hyperbolic plane</u> H (the even unimodular lattice with signature $(1, 1)$). Let P' denote the orthogonal complement of H_0 in P. One has $P = P' \oplus H_0$ (orthogonal direct sum).

The starting point of this article is following Looijenga's result. This is contained in his paper on triangle singularities (Looijenga [8]).

We fix one of 6 kinds of hypersurface quadrilateral singularities. Let G be a Dynkin graph with components of type A, D or E only. Assume that there exists a K3 surface Z satisfying the following conditions (1) and (2).

(1) Z contains the curve at infinity $IF = IF(p_1, p_2, p_3, p_4)$ corresponding to the quadrilateral singularity as a subvariety.

(2) Let E be the union of all smooth rational curves on Z which do not intersect with the curve IF. Then, the dual graph representing the mutual intersections among the components of E coincides with the graph G.

We define an open variety Y and a lattice embedding associated with Z.

An open variety Y is defined to be the one obtained by contracting each connected component of E to a rational double point and moreover removing the image of IF.

Now, let $\Lambda_3 \cong Q(2E_8) \oplus H \oplus H \oplus H$ denote the even unimodular lattice with signature (19, 3). By Q(G) we denote the positive definite root lattice of type G. Under a choice of an isomorphism $H^2(Z,\mathbf{Z}) \xrightarrow{\sim} \Lambda_3$ which preserves the bilinear form up to sign, we have an embedding of lattices

$$S = P \oplus Q(G) \subset \Lambda_3 ,$$

considering the image of the sublattice of $H^2(Z,\mathbf{Z})$ generated by the classes corresponding to the irreducible components of E and IF.

Theorem 1.1. (Looijenga)(1) There exists a fiber isomorphic to Y in (the non-positive weight part of) the semi-universal deformation family of the quadrilateral singularity.

(2) The above embedding $S \subset \Lambda_3$ satisfies the following conditions (a) and (b). By $\tilde{S}$ we denote the primitive hull of S in Λ_3. $\tilde{S} = \{x \in \Lambda_3 \mid mx \in S$ for some non-zero integer m.$\}$

(a) If an element $\eta \in \tilde{S}$ with $\eta^2 = +2$ is orthogonal to P, then $\eta \in Q(G)$.

(b) If an element $\eta \in \tilde{S}$ with $\eta^2 = +2$ satisfies $\eta \cdot u_0 = 0$, then either $\eta \in P_0$ or η is orthognal to P_0.

Theorem 1.2. (Looijenga) We fix one of 6 kinds of hypersurface quadrilateral singularities. Let G be a Dynkin graph with components of type A, D or E only. By r we denote the number of vertices in G. The following condition are all equivalent.

(1) There exists a fiber Y in the semi-universal deformation family of the quadrilateral singularity such that Y has only rational double points as singularities and such that the combination of rational double points on Y just agrees with G.
(2) There exists a K3 surface Z satisfying the conditions (1) and (2) just before Theorem 1.1.
(2′) There exists a K3 surface Z satisfying the conditions (1) and (2) just before Theorem 1.1 and moreover the Picard number ρ of Z is equal to $\sum_{i=1}^{4} p_i - 3 + r$.
(3) There exists an embedding of lattices $S = P \oplus Q(G) \subset \Lambda_3$ satisfying the conditions (a) and (b) in Theorem 1.1.

When we treat only geometric situations, the arguments become clearer if we assume that the lattice Λ_3 has the opposite signature (3, 19) and $Q(G)$ is negative definite. However, we define the sign of the bilinear form on Λ_3 and $Q(G)$ as above in this article, because we use much algebraic theory on lattices and it is convenient for the use. Therefore, note that the isomorphism $H^2(Z) \xrightarrow{\sim} \Lambda_3$ reverses the sign of the bilinear forms.

In the above item (1) it should be noted that Y is not necessarily in the non-positive weight part of the deformation family. However, if (1) holds, then we can choose another Y' lying in the non-positive weight part satisfying the same condition as Y.

Next, we explain the relation to the theory of elliptic surfaces. Let Z be a K3 surface satisfying the conditions (1) and (2). Let C_i $(0 \leq i < q)$ be components of the curve IF on Z. We assign the number i to C_i in such a way that the vertex on the dual graph (1) corresponding to C_i is associated with the vector e_i with the same number i for $0 \leq i < q$. The divisor $F = 2C_0 + \sum_{i=1}^{4} C_i$ defines a morphism $\Phi : Z \longrightarrow \mathbb{P}^1$ whose general fiber is a smooth elliptic curve. By definition Φ has a singular fiber $\bigcup_{i=0}^{4} C_i$, which is of type I_0^*. If $C_i \cdot F \neq 0$ for $0 \leq i < q$, then $C_i \cdot F = 1$ and C_i is a section of Φ. In particular Φ has the section C_5. If a smooth rational curve D on Z satisfies $D \cdot F = 0$, then D is a component of a singular fiber of Φ. In particular, every connected component of E is contained in a singular fiber.

Some readers might notice that the proof of Lemma (4.6) in Looijenga [8] is incomplete. (He misses treating the case $\alpha' \in B_0$ and $B' \neq B_0$.) However, we can easily complete the proof and the claim itself is true. On the other hand, the claim of Theorem (4.5) in Looijenga [8] is not complete unless we add a certain condition on an isotropic element.

§2. The theory of lattices

In this article we freely use standard terminologies in the theory of lattices i.e. the theory of integral symmetric bilinear forms (Cassels [4], Milnor-Husemoller [9], Serre [11]).

Let L be a free $\mathbb{Z}$-module of finite rank and M be a submodule. We say that M is <u>primitive</u> in L, if the quotient L/M is a free module. An element $x \in L$ is <u>primitive</u> in L, if $\mathbb{Z}x$ is primitive in L. On the other hand, if L/M is finite, L is an <u>over</u> module of M. We denote the <u>primitive hull</u> of M in L by $P(M,L) = \{x \in L \mid mx \in M$ for some non-zero integer $m.\}$ or $\tilde{M}$ when we need not mention L. $P(M,L)$ is the minimal primitive submodule of L containing M.

Moreover, when L has a symmetric bilinear form $(\ ,\ ) : L \times L \longrightarrow \mathbb{Q}$ with values in rational numbers, the pair $(L,(\ ,\ ))$ is called a <u>quasi-lattice</u>. If the values of the bilinear form are all integers, $(L,(\ ,\ ))$ is called a <u>lattice</u>. For two quasi-lattices L and L' we denote the orthogonal direct sum $L \oplus L'$ using the symbol $\oplus$.

Let L be a quasi-lattice and M be a submodule. The orthogonal complement $\{x \in L \mid$ For every $y \in M$ $(x,y) = 0.\}$ of M in L is denoted by $C(M,L)$ or $M^{\perp}$ when we need not mention L. Note that $(M^{\perp})^{\perp} = \tilde{M}$ when L is non-degenerate.

Next, assume that M is non-degenerate and primitive in L. Then, L is an over-quasi-lattice of $M \oplus M^{\perp}$. Choose two elements $\bar{x}$, $\bar{y} \in L/M$ in the quotient module, and choose their representatives $x, y \in L$. We can write them in the form $x = x_1 + x_2$ and $y = y_1 + y_2$ $(x_1, y_1 \in M \otimes \mathbf{Q}$, $x_2, y_2 \in M^{\perp} \otimes \mathbf{Q})$. If we set $(\bar{x},\bar{y}) = (x_2,y_2)$, this rational number depends only on $\bar{x}$ and $\bar{y}$, and does not depend on the choice of representatives $x, y \in L$. Therefore it defines a symmetric bilinear form on L/M with values in $\mathbf{Q}$. In this article we always give the bilinear form in this manner to the quotient module by a primitive non-degenerate submodule.

For simplicity we write $x^2 = (x,x)$. Sometimes we write $(x,y) = x\cdot y$. An element x with $x \neq 0$, $x^2 = 0$ is called an <u>isotropic element.</u>

Let L be a non-degenerate lattice. The dual module $L^* = \mathrm{Hom}(L,\mathbf{Z})$ is identified with the submodule $\{x \in L \otimes \mathbf{Q} \mid (x,y) \in \mathbf{Z}$ for every $y \in L.\}$ in $L \otimes \mathbf{Q}$. L^* becomes a quasi-lattice containing L. Then, the order of the quotient group L^*/L equals to the absolute value of the discriminant $d(L)$ of L. We call the quotient L^*/L the <u>discriminant group</u> of L. The <u>discriminant bilinear form</u>

$$b_L : L^*/L \times L^*/L \longrightarrow \mathbf{Q}/\mathbf{Z}$$

is defined for $x, y \in L^*$ by
$b_L\ (x \bmod L, y \bmod L) \equiv (x,y) \bmod \mathbf{Z}$.

A lattice L is an <u>even</u> lattice, if x^2 is an even integer for every $x \in L$. Otherwise it is <u>odd</u>.

We can define the <u>discriminant quadratic form</u>

$$q_L : L^*/L \longrightarrow \mathbb{Q}/2\mathbb{Z}$$

for a non-degenerate <u>even</u> lattice L. For $x \in L^*$ q_L (x mod L) $\equiv x^2$ mod $2\mathbb{Z}$ (Nikulin [10]).

Next, let L be a positive definite even lattice and $q_L : L^*/L \longrightarrow \mathbb{Q}/2\mathbb{Z}$ be the discriminant quadratic form. By $\pi : L^* \longrightarrow L^*/L$ we denote the canonical surjective morphism. For an element $\bar{x} \in L^*/L$ we define the <u>characteristic number</u> $\upsilon(\bar{x})$ of $\bar{x}$ by

$$\upsilon(\bar{x}) = \min\langle x^2 | x \in L^*, \pi(x) = \bar{x}.\rangle.$$

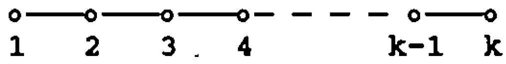
<u>Lemma 2.1.</u> (1) $\upsilon(\bar{x}) \geq 0$, $\upsilon(\bar{x}) = 0 \Leftrightarrow \bar{x} = 0$.

(2) $\upsilon(\bar{x}) \equiv q_L(\bar{x})$ mod $2\mathbb{Z}$.

(3) Let L and L′ be positive definite even lattices. We regard $L^*/L + L'^*/L' = (L \oplus L')^*/L \oplus L'$. Then for $\bar{x} \in L^*/L$ $\bar{y} \in L'^*/L'$ we have $\upsilon(\bar{x} + \bar{y}) = \upsilon(\bar{x}) + \upsilon(\bar{y})$.

Let G be a Dynkin graph with components of type A, D or E only. We can define a lattice associated with G by the same rule as we used to make the lattice P from the graph (1) in section 1. The resulting lattice is the root lattice $Q = Q(G)$ of type G (Bourbaki [3]). The root lattice Q has a basis $\alpha_1, \ldots, \alpha_k$ ($\alpha_1^2 = \ldots = \alpha_k^2 = +2$) associated with vertices of the Dynkin graph. The dual basis of it (a basis of Q^*) is denoted by $\omega_1, \omega_2, \ldots, \omega_k$. We call ω_i the <u>i-th fundamental weight</u>. ($\alpha_i \cdot \omega_j = \delta_{ij}$.)

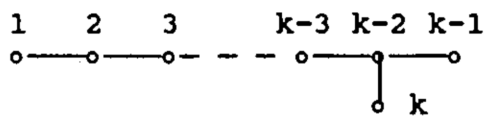
Example 2.2. $L = Q(A_k)$ (The root lattice of type A_k)



We assign numbers 1, 2, 3, ..., k to vertices in the Dynkin graph of type A_k from the end in order. By ω_i we denote the fundamental weight associated with the i-th vertex. $L^*/L \cong \mathbf{Z}/(k+1)$. $\pi(\omega_1)$ is a generator of this cyclic group, and $\pi(\omega_i) = i\pi(\omega_1)$. $\upsilon(\pi(\omega_i)) = \upsilon(i\pi(\omega_1)) = i(k+1-i)/(k+1)$.

Example 2.3. $L = Q(D_k)$ $(k \geq 4)$ (The root lattice of type D_k.)

We assign numbers to the Dynkin graph of type D_k as follows.



We consider the corresponding fundamental weight ω_i and the fundamental root α_i.

$$\omega_i = \alpha_1 + 2\alpha_2 + \dots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \dots + \alpha_{k-2}) + i(\alpha_{k-1} + \alpha_k)/2 \quad (1 \leq i \leq k-2)$$

$$\omega_{k-1} = \{\alpha_1 + 2\alpha_2 + \dots + (k-2)\alpha_{k-2} + k\alpha_{k-1}/2 + (k-2)\alpha_k/2\}/2$$

$$\omega_k = \{\alpha_1 + 2\alpha_2 + \dots + (k-2)\alpha_{k-2} + (k-2)\alpha_{k-1}/2 + k\alpha_k/2\}/2$$

Thus we have $\omega_{k-1} - \omega_k = \alpha_{k-1}/2 - \alpha_k/2$ and $\pi(\omega_1) = \pi(\omega_k) - \pi(\omega_{k-1}) = \pi(\omega_{k-1}) - \pi(\omega_k)$.

If k is even, then $L^*/L \cong \mathbb{Z}/2 + \mathbb{Z}/2$, and $\pi(\omega_{k-1})$ and $\pi(\omega_k)$ are generators of two components.

If k is odd, then $L^*/L \cong \mathbb{Z}/4$, $\pi(\omega_k)$ is a generator, $\pi(\omega_1) = 2\pi(\omega_k)$, and $\pi(\omega_{k-1}) = -\pi(\omega_k)$.

We have $\upsilon(\pi(\omega_1)) = 1$ and $\upsilon(\pi(\omega_{k-1})) = \upsilon(\pi(\omega_k)) = k/4$.

<u>Example 2.4.</u> $L = Q(E_6)$ (The root lattice of type E_6)

$L^*/L \cong \mathbb{Z}/3$. If $\bar{x} \neq 0$, $\upsilon(\bar{x}) = 4/3$.

<u>Example 2.5.</u> $L = Q(E_7)$ (The root lattice of type E_7)

$L^*/L \cong \mathbb{Z}/2$. If $\bar{x} \neq 0$, $\upsilon(\bar{x}) = 3/2$.

<u>Example 2.6.</u> $L = Q(E_8)$ (The root lattice of type E_8)

$L^*/L \cong \langle 0 \rangle$. L is a unimodular even lattice.

<u>Lemma 2.7.</u> Let G be a Dynkin graph with components of type A, D or E only. Assume an element $\xi \in Q(G)^*$ in the dual module of the root lattice of type G satisfies $0 < \xi^2 < 1$. Then, $k = \xi^2/(1-\xi^2)$ is a positive integer, and moreover G contains a component G_0 of type A_k such that ξ is contained in $Q(G_0)^*$ and ξ and $Q(G_0)$ together generate $Q(G_0)^*$. In particular $\xi^2 \geq 1/2$ and $\xi^2 = k/(k+1)$.

<u>Proof.</u> In Example 2.2 $\upsilon(\pi(\omega_i)) = i(k+1-i)/(k+1) \geq 1$ if $2 \leq i \leq k-1$, and $\upsilon(\pi(\omega_1)) = \upsilon(\pi(\omega_k)) = k/(k+1) \geq 1/2$. Besides

in Example 2.3, 2.4, 2.5 and 2.6, if $\overline{x} \in L^*/L$, $\overline{x} \neq 0$, then $\upsilon(\overline{x}) \geq 1$. Our lemma follows from these facts and Lemma 2.1 (3).

Q.E.D.

For each one of 6 kinds of quadrilateral singularities under our consideration the lattice P was defined. It has a decomposition $P = P' \oplus H_0$. P' is an even positive definite lattice. We can define the characteristic number $\upsilon(\overline{x})$ for every element $\overline{x} \in P'^*/P'$.

<u>Definition 2.8.</u> Let $\overline{x}$ be an element in P'^*/P'. By $q = q_{P'}$ we denote the discriminant quadratic form of P'.

(1) We call $\overline{x}$ <u>an element of the first kind</u>, if $q(\overline{x}) \equiv t \bmod 2\mathbf{Z}$ for some number t with $0 \leq t < 1$.

(2) If $q(\overline{x}) \equiv 1 \bmod 2\mathbf{Z}$, we call $\overline{x}$ <u>an element of the second kind</u>. Besides if $\upsilon(\overline{x}) = 1$ and $\overline{x}$ has order 2, we call it <u>a special element of type B.</u>

(3) If for some positive integer k

$$q(\overline{x}) \equiv 1 + (1/(k+1)) \bmod 2\mathbf{Z},$$

$\overline{x}$ is called <u>an element of the third kind</u> and k is called <u>the associated number</u> of $\overline{x}$. Besides if an element of the third kind with the associated number 1 satisfies $\upsilon(\overline{x}) = 3/2$ and if it has order 2, then we call it <u>a special element of type C.</u> If an element of the third kind with the associated number 2 satisfies $\nu(\bar{x}) = 4/3$ and if it has order 3, then we call it _a special element of type G_.

(4) Any element neither of the first kind, of the second kind, nor of the third kind is called _one of the fourth kind_.

(5) The associated numbers of the elements of the third kind in P'^{*}/P' are called the associated numbers of P or P'.

Indeed, special elements are related to short roots in root systems. The above terminologies "type B, C or G" are used to imply this relation.

The following proposition plays a key role later to convert Looijenga's condition (a) and (b) to a simpler condition.

Proposition 2.9. Fixing one of the 6 kinds of hypersurface quadrilateral singularities, we consider the corresponding lattices P' and P_0. Set $P'_0 = P' \cap P_0$.

(1) P' has the following property (G').

(G') If $\eta \in P'$, $\eta^2 = +2$ and $\eta \notin P'_0$, then η is orthogonal to P'_0.

(2) Let $\bar{x} \in P'^{*}/P'$ be an element of the first kind. Let $x \in P'^{*}$ be an element such that $x \bmod P' = \bar{x}$. If $x^2 < 2$, then x is orthogonal to P'_0.

(3) For every element $\bar{x} \in P'^{*}/P'$ of the second or third kind, there exists an element $x_0 \in P'^{*}$ satisfying the following three conditions: $x_0 \bmod P' = \bar{x}$, $x_0^2 < 2$ and x_0 is not orthogonal to P'_0.

(4) Every element of the second kind is a special element of type B.

(5) Every element of the third kind with the associated number 1 (respectively 2) is necessarily a special element of type C (resp. type G).

(6) For the former 5 kinds except the last one $U_{1,0}(2,3,3,3)$ the following assertion holds.

If $q(\bar{x}) \equiv 0 \bmod 2\mathbb{Z}$ for $\bar{x} \in P'^{*}/P'$, then $\bar{x} = 0$.

For $U_{1,0}$ this assertion does not hold.

Corollary 2.10. For every element $\bar{x} \in P'^{*}/P'$ of the second or third kind, $\upsilon(\bar{x}) < 2$.

In the rest of this section we show Proposition 2.9 for each kind of quadrilateral singularity. Recall the following. $P = P' \oplus H_0$. P_0 is the sublattice of P generated by e_0, e_1, e_2, e_3 and e_4. $P_0' = P_0 \cap P'$. P_0' is the sublattice of P' generated by e_0, e_1, e_2 and e_3. $P_0' \cong Q(D_4)$ (the root lattice of type D_4).

The case of $J_{3,0}(2,2,2,3)$.

$P' \cong P_0' \cong Q(D_4)$. Obviously it has the property (G'). By Example 2.3 $(k = 4)$, one knows that any non-zero element in the discriminant group is of the second kind. By Example 2.3 we can check the rest of the proposition. In particular, P has no associated numbers. It has special elements of type B.

The case of $Z_{1,0}(2,2,2,4)$.

Let e_6 be the member of the basis B corresponding to the end of the longest arm of the dual graph (1). Set $e_6' = e_6 - u_0$. $P' = P_0' \oplus \mathbf{Z}e_6'$. ${e_6'}^2 = +2$. Thus if the assumption of (G') is satisfied, then $\eta = \pm e_6'$ and we have the conclusion.

Every element $\bar{x} \in P'^*/P'$ can be uniquely expressed in the form $\bar{x} = \bar{y} + \bar{z}$ where $\bar{y} \in P_0'^*/P_0'$ and $\bar{z} \in (\mathbf{Z}e_6')^*/\mathbf{Z}e_6'$. We can deduce the rest of the proposition by Example 2.2 ($k = 1$) and Example 2.3 ($k = 4$).

Consider the case $\bar{y} = \bar{z} = 0$. The element $\bar{x} = 0$ is of the first kind. However, obviously, if $x \in P'$, $x^2 \equiv 0 \bmod 2\mathbf{Z}$, and $x^2 < 2$, then $x^2 = 0$ and thus $x = 0$. For $\bar{x} = 0$, the assertion (2) holds.

If $\bar{y} = 0$ and $\bar{z} \neq 0$, then $q(\bar{x}) \equiv 1/2 \bmod 2\mathbf{Z}$ and $\bar{x}$ is of the first kind. If $x \in P^*$, $x \bmod P' = \bar{x}$, and $x^2 < 2$, then $x = \pm e_6'/2$ and thus x is orthogonal to P_0'. The assertion (2) holds.

When $\bar{y} \neq 0$ and $\bar{z} = 0$, $q(\bar{x}) \equiv 1 \bmod 2\mathbf{Z}$. The element $\bar{x}$ is of the second kind. By Example 2.3 one knows the assertion (3). By (3) $\upsilon(\bar{x}) = 1$. Since $\bar{x}$ has order 2, it is a special element of type B.

When $\bar{y} \neq 0$ and $\bar{z} \neq 0$, $q(x) \equiv 3/2 \bmod 2\mathbf{Z}$. The element $\bar{x}$ is of the third kind with the associated number 1. We have an element $y \in P_0'$ such that $y \bmod P_0' = \bar{y}$ and $y^2 = 1$. Setting $x_0 = y + (e_6'/2)$, one knows this x_0 satisfies the assertion (3). In particular, $\upsilon(\bar{x}) = 3/2$. Since $\bar{x}$ has order 2, it is a special element of type C.

The assertion (6) is obvious by the above.

P has the associated number 1 and has special elements of type B and of type C.

The case of $Q_{2,0}(2,2,2,5)$.

We associate vectors e_0, e_4, e_5, e_6, e_7 in order from the central one with 4 edges, with the 5 vertices on the longest arm of the dual graph (1). Set $e_6' = e_6 - u_0$ and $T = \mathbf{Z}e_6' + \mathbf{Z}e_7$. One has $P' = P_0' \oplus T$, and $T \cong Q(A_2)$ (the root lattice of type A_2). (G′) follows easily from this decomposition. Write an element $\bar{x} \in P'^*/P'$ in the form $\bar{x} = \bar{y} + \bar{z}$ where $\bar{y} \in P_0'^*/P_0'$ and $\bar{z} \in T^*/T$. We can apply Example 2.2 $(k = 2)$ and Example 2.3 $(k = 4)$.

If $\bar{y} = \bar{z} = 0$, $\bar{x} = 0$. Then $\bar{x}$ is of the first kind. The assertion (2) holds.

When $\bar{y} = 0$ and $\bar{z} \neq 0$, $q(\bar{x}) \equiv 2/3 \bmod 2\mathbf{Z}$ and $\bar{x}$ is of the first kind. Let $x \in P'^*$ be an element satisfying $x \bmod P' = \bar{x}$ and $x^2 < 2$. We can write it in the form $x = y + z$ where $y \in P_0'^*$ and $z \in T^*$. One has $z^2 \geq 0$. On the other hand $y \in P_0'$, since $y \bmod P_0' = \bar{y} = 0$. If $y \neq 0$, then $y^2 \geq 2$ and thus $x^2 = y^2 + z^2 \geq 2$, which is a contradiction. Thus $y = 0$ and x is orthogonal to P_0'. One knows that the assertion (2) holds in this case.

If $\bar{y} \neq 0$ and $\bar{z} = 0$, then $q(\bar{x}) \equiv 1 \bmod 2\mathbf{Z}$ and $\bar{x}$ is of the second kind. By Example 2.3 one knows that the assertion (3) holds in this case. In particular, $\upsilon(\bar{x}) = 1$.

Since $\bar{x}$ has order 2, it is a special element of type B. The assertion (4) also holds.

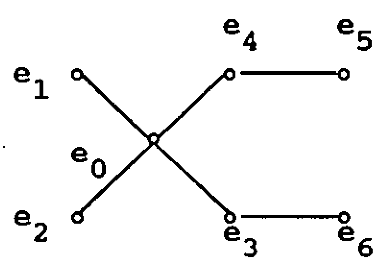
When $\bar{y} \neq 0$ and $\bar{z} \neq 0$, $q(\bar{x}) \equiv 5/3 \bmod 2\mathbb{Z}$, and $\bar{x}$ is of the fourth kind.

By the above one knows the assertion (6), too.

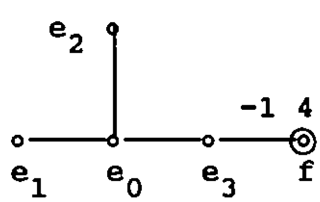
P has no associated numbers. It has special elements of type B.

The case of $W_{1,0}(2,2,3,3)$.

We associate the vectors $e_0, e_1, \ldots, e_6$ with the vertices in the dual graph like the following.



Set $f = e_6 - u_0 + v_0 = e_6 - e_5 - 2u_0$. We have $P = P' \oplus H_0$ and $P' = \sum_{i=0}^{3} \mathbb{Z}e_i + \mathbb{Z}f$. Moreover $f^2 = 4$, $f \cdot e_0 = f \cdot e_1 = f \cdot e_2 = 0$, and $f \cdot e_3 = -1$. The dual graph of the lattice P' is like the following.



Set

$$w_0 = (14e_0 + 7e_1 + 7e_2 + 8e_3 + 2f)/6$$
$$w_1 = (14e_0 + 13e_1 + 7e_2 + 8e_3 + 2f)/12$$
$$w_2 = (14e_0 + 7e_1 + 13e_2 + 8e_3 + 2f)/12$$
$$w_3 = (4e_0 + 2e_1 + 2e_2 + 4e_3 + f)/3$$
$$z = (2e_0 + e_1 + e_2 + 2e_3 + 2f)/6.$$

They are elements in $P' \otimes \mathbb{Q}$. We can check $w_i \cdot e_j = \delta_{ij}$, $z \cdot e_j = 0$, $w_i \cdot f = 0$ and $z \cdot f = 1$. Thus $w_0, \ldots, w_3, z$ is a basis of the dual module P'^*. Note that the coefficient of e_i in w_j equals to $w_i \cdot w_j$, the coefficient of e_j in z equals to $w_j \cdot z$, and by the same reason $z^2 = 1/3$. Moreover, $w_0 - 2w_1$, $w_2 + 5w_1$, $w_3 + 4w_1$, and $z - 2w_1$ belong to P'. Thus one knows the following. For an element $x \in P'^*$ we denote $\bar{x} = x \bmod P' \in P'^*/P'$.

<u>Proposition 2.11.</u> In the case of $W_{1,0}(2,2,3,3)$, the discriminant group $P^*/P \cong P'^*/P'$ is a cyclic group of order 12. We can take $\bar{w}_1$ or $\bar{w}_2$ as its generator. We have $\bar{w}_0 = 2\bar{w}_1$, $\bar{w}_2 = -5\bar{w}_1$, $\bar{w}_3 = -4\bar{w}_1$, and $\bar{z} = 2\bar{w}_1$. For the discriminant quadratic form q,

$$q(\bar{w}_1) \equiv q(\bar{w}_2) \equiv 13/12 \bmod 2\mathbb{Z}.$$

We continue to check Proposition 2.9. First we show the property (G'). Assume $\eta \in P'$, $\eta^2 = +2$ and $\eta \notin P'_0$. Set $\eta = \sum_{i=0}^{3} a_i e_i + bf$ with integers a_i and b. $b \neq 0$. Corresponding to η, we set $\tilde{\eta} = \sum_{i=0}^{3} a_i e_i + b e_6$. Since $e_i \cdot f = e_i \cdot e_6$ $(0 \leq i \leq 3)$ and $f^2 = e_6^2 + 2$, we have $\eta^2 = \tilde{\eta}^2 + 2b^2$. Since e_0, e_1, e_2, e_3 and e_6 generate a root lattice of type D_5 and since $\tilde{\eta} \neq 0$, one has $\tilde{\eta}^2 \geq 2$. It implies $2 = \eta^2 = \tilde{\eta}^2 + 2b^2 \geq 4$, which is a contradiction. An element η satisfying the assumption of (G′) never exists. Thus (G′) holds.

Next, we show the assertions (2)-(5). Let $\bar{x} \in P'^*/P'$ be an element. We deal with each case separately.

(1) $\bar{x} = 0$. The zero element is of the first kind. The assertion (2) holds in the case.

(2) $\bar{x} = \pm\bar{w}_1$ or $\pm 5\bar{w}_1$. $q(\bar{x}) \equiv 13/12 \bmod 2\mathbf{Z}$. This element $\bar{x}$ is of the third kind with the associated number 11. An element $x_0 = \pm w_1, \pm w_2$ satisfies $x_0^2 = 13/12$, and it also satisfies the assertion (3) in the proposition.

(3) $\bar{x} = \pm 2w_1$. $q(\bar{x}) \equiv 1/3 \bmod 2\mathbf{Z}$. This $\bar{x}$ is of the first kind. Note that $2\bar{w}_1 = \bar{z}$. We would like to show the assertion (2) for $\bar{x}$. To show it it suffices to see that if $\eta \in P'$ and $(z+\eta)^2 < 2$, then $\eta = 0$, because z is orthogonal to P_0'. Corresponding to $\eta = \sum_{i=0}^{3} a_i e_i + bf$, set $\tilde{\eta} = \sum_{i=0}^{3} a_i e_i + b e_6$. We have $(z+\eta)^2 = (1/3) + 2b(b+1) + \tilde{\eta}^2 < 2$. Since b is an integer, $b(b+1) \geq 0$. If $\tilde{\eta} \neq 0$, then $\tilde{\eta}^2 \geq 2$ and we have $(1/3) + 2 < 2$, which is a contradiction. Thus $\tilde{\eta} = 0$ and $\eta = 0$.

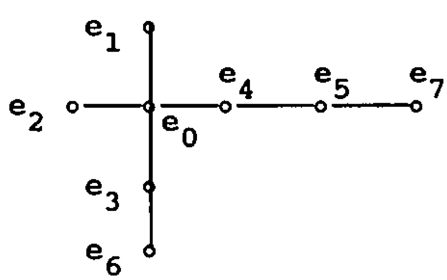
(4) $\bar{x} = \pm 3\bar{w}_1$. $q(\bar{x}) \equiv 7/4 \bmod 2\mathbb{Z}$. This is of the fourth kind.
(5) $\bar{x} = \pm 4\bar{w}_1$. $q(\bar{x}) \equiv 4/3 \bmod 2\mathbb{Z}$. This is of the third kind with the associated number 2. Since $\bar{w}_1 = -\bar{w}_3$, and since $w_3^2 = 4/3$, the element $x_0 = \pm w_3$ satisfies the assertion (3). Besides $\pm 4\bar{w}_1$ has order 3 and the assertion (5) holds, too.
(6) $\bar{x} = 6\bar{w}_1$. $q(\bar{x}) \equiv 1 \bmod 2\mathbb{Z}$. This is of the second kind. Set $z_1 = (e_1-e_2)/2$. Since $z_1 \cdot e_0 = 0$, $z_1 \cdot e_1 = z_1 \cdot e_2 = 1$, and $z_1 \cdot e_3 = z_1 \cdot f = 0$, $z_1 \in P'^*$. Besides $\bar{z}_1 = 6\bar{w}_1$, since $z_1 - 6w_1 = -(7e_0 + 6e_1 + 4e_2 + 4e_3 + f)$. One knows that the assertion (3) holds, since $z_1^2 = 1$ and z_1 is not orthogonal to P_0. The assertion (4) holds, too, since $6\bar{w}_1$ has order 2.

By the above one sees that the assertion (6) holds, too.

P has the associated numbers 2 and 11, and has special elements of type B and of type G.

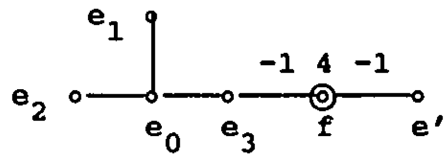
The case of $S_{1,0}(2,2,3,4)$.

We associate vectors $e_0, e_1, \ldots, e_7$ with vertices in the graph as follows.



Set $e' = u_0-e_7$ and $f = e_6-e_5-2u_0$. The orthogonal complement P' of $H_0 = \mathbb{Z}u_0 + \mathbb{Z}v_0$ in P is spanned by

e_0, e_1, e_2, e_3, f and e'. We have $f \cdot e_0 = f \cdot e_1 = f \cdot e_2 = 0$, $f \cdot e_3 = -1$, $e' \cdot e_i = 0$ $(0 \leq i \leq 3)$ and $e' \cdot f = -1$. Thus the dual graph of the lattice P' is like the following.



Set

$$
\begin{aligned}
w_0 &= (12e_0+6e_1+6e_2+7e_3+2f+e')/5 \\
w_1 &= (12e_0+11e_1+6e_2+7e_3+2f+e')/10 \\
w_2 &= (12e_0+6e_1+11e_2+7e_3+2f+e')/10 \\
w_3 &= (14e_0+7e_1+7e_2+14e_3+4f+2e')/10 \\
z &= (2e_0+e_1+e_2+2e_3+2f+e')/5 \\
w_4 &= (2e_0+e_1+e_2+2e_3+2f+6e')/10.
\end{aligned}
$$

We can check the following.

$$
\begin{aligned}
& w_i \cdot e_j = \delta_{ij} \ (0 \leq i, j \leq 3), \quad w_i \cdot f = w_i \cdot e = 0 \quad (0 \leq i \leq 3), \\
& z \cdot e_j = 0 \quad (0 \leq j \leq 3), \quad z \cdot f = 1, \quad z \cdot e' = 0 \\
& w_4 \cdot e_j = 0 \quad (0 \leq j \leq 3), \quad w_4 \cdot f = 0, \ w_4 \cdot e' = 1.
\end{aligned}
$$

Thus $w_0, \ldots, w_4$, z is a basis of the dual module P'^*. We have the following proposition. We denote

$\bar{x} = x \bmod P' \in P'^*/P'$ for $x \in P'^*$.

Proposition 2.12. In the case of $S_{1,0}(2,2,3,4)$, the discriminant group $P'^*/P' \cong P^*/P$ is the direct sum of three cyclic groups, each of which has order 5, 2, 2 respectively. The first direct summand is generated by $\bar{w}_0 = \bar{z} = 2\bar{w}_1 = 2\bar{w}_2 = -4\bar{w}_3 = 2\bar{w}_4$. The second is generated by $\bar{g}_1 = 5\bar{w}_1$ and the third by $\bar{g}_2 = 5\bar{w}_2$. For the discriminant quadratic form q

$$q(a\bar{z}+b_1\bar{g}_1+b_2\bar{g}_2) \equiv \tfrac{2}{5}a^2 - \tfrac{1}{2}(b_1^2+b_2^2) \bmod 2\mathbb{Z}.$$

Besides $\bar{w}_0 = \bar{z}$, $\bar{w}_1 = -2\bar{z}+\bar{g}_1$, $\bar{w}_2 = -2\bar{z}+\bar{g}_2$, $\bar{w}_3 = \bar{z}+\bar{g}_1+\bar{g}_2 = \bar{w}_1+\bar{w}_2$ and $\bar{w}_4 = -2\bar{z}+\bar{g}_1+\bar{g}_2$.

We check Proposition 2.9. First we show (G′). Assume that $\eta \in P'$, $\eta^2 = +2$ and $\eta \notin P'_0$. Corresponding to $\eta = \sum_{i=0}^{3} a_i e_i + bf + ce'$, set $\tilde{\eta} = \sum_{i=0}^{2} a_i e_i + a_3 e_4 + b e_5 + c e_7$. This $\tilde{\eta}$ is an element in the root lattice of type D_6 generated by e_0, e_1, e_2, e_4, e_5 and e_7. Since $\tilde{\eta} \neq 0$, $\tilde{\eta}^2$ is a positive even integer. Since $2 = \eta^2 = \tilde{\eta}^2 + 2b^2 \geq \tilde{\eta}^2 \geq 2$, we have $b = 0$. Thus $2 = (\sum_{i=0}^{3} a_i e_i)^2 + 2c^2$. If $c = 0$, then $\eta = \sum_{i=0}^{3} a_i e_i \in P'_0$, which contradicts the assumption. Therefore $c \neq 0$, and thus $\sum_{i=0}^{3} a_i e_i = 0$ and $\eta = \pm e'$. Consequently η is orthogonal to P'_0.

Next, we check the assertions (2)-(5). Let $\overline{x} \in P'^{*}/P'$ be an element.

(1) $\overline{x} = 0$. The zero element is of the first kind and satisfies the assertion (2).

(2) $\overline{x} = \pm\overline{z}$. $q(\overline{x}) \equiv 2/5 \bmod 2\mathbf{Z}$. This $\overline{x}$ is of the first kind. To show the assertion (2) it suffices to see that if $\eta \in P'$ and $(\eta+z)^2 = z^2 = 2/5$, then $\eta = 0$. Set $\tilde{\eta} = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_4 + be_5 + ce_7$, corresponding to $\eta = \sum_{i=0}^{3} a_ie_i + bf + ce'$. If $\eta \neq 0$, then $\tilde{\eta}^2 > 0$ and $2/5 = (\eta+z)^2 \cong \tilde{\eta}^2 + 2b(b+1) + (2/5) > 2/5$, which is a contradiction. Thus $\eta = 0$.

(3) $\overline{x} = \pm 2\overline{z}$, $q(\overline{x}) \equiv 8/5$. This is of the fourth kind.

(4) $\overline{x} = \overline{g}_1$ or $\overline{g}_2$. $q(\overline{x}) \equiv -1/2 \equiv 3/2$. This $\overline{x}$ is of the third kind with the associated number 1. We see the assertion (3). Indeed, set $x_0 = (e_1+e_3+e')/2$. Since $5w_1-x_0 = 6e_0+5e_1+3e_2+3e_3+f$, we have $\overline{x}_0 = \overline{g}_1$. This x_0 is not orthogonal to P'_0, since $x_0\cdot e_1 \neq 0$. Besides $x_0^2 = 3/2 < 2$. When we treat the element $\overline{g}_2$, we can consider the element $(e_2+e_3+e')/2$ instead. Lastly one sees that $\overline{x}$ is a special element of type C, since it has order 2.

(5) $\overline{x} = \pm\overline{z} + \overline{g}_1$ or $\pm\overline{z}+g_2$. $g(\overline{x}) \equiv 19/10$. This is of the fourth kind.

(6) $\overline{x} = \pm(2\overline{z}+\overline{g}_1)$ or $\pm(2\overline{z}+\overline{g}_2)$. $q(\overline{x}) \equiv 11/10$. This $\overline{x}$ is of the third kind with the associated number 9. We have to show the assertion (3). For $\pm(2\overline{z}+\overline{g}_1)$, set $x_0 = \pm w_1$. For $\pm(2\overline{z}+\overline{g}_2)$, set $x_0 = \pm w_2$. For the both cases, this x_0 satisfies the condition.

(7) $\bar{x} = \bar{g}_1+\bar{g}_2$. $q(\bar{x}) \equiv 1$. This is of the second kind. The element $x_0 = (e_1+e_2)/2$ satisfies the condition in the assertion (3). It is also a special element of type B.

(8) $\bar{x} = \pm\bar{z}+\bar{g}_1+\bar{g}_2$. $q(\bar{x}) \equiv 7/5$. It is of the fourth kind.

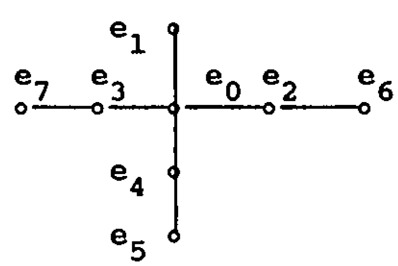
(9) $\bar{x} = \pm 2\bar{z}+\bar{g}_1+\bar{g}_2$. $q(\bar{x}) \equiv 3/5$. This is of the first kind. Set $x_0 = \pm w_4$. One sees that $\bar{x}_0 = \bar{x}$, $x_0^2 = 3/5$ and this x_0 is orthogonal to P_0'. Thus to show the assertion (2), it suffices to see that if $\eta \in P'$ and $(\eta+w_4)^2 = 3/5$, then η belongs to $\mathbb{Z}e'$. Assume that $\eta^2+2\eta\cdot w_4 = 0$. Corresponding to $\eta = \sum_{i=0}^{3} a_i e_i+bf+ce'$, set $\bar{\eta} = \sum_{i=0}^{2} a_i e_i + a_3 e_4 + be_5$. $\bar{\eta}^2$ is an even non-negative integer. Set $B = b^2-bc+c^2+c$. One sees $\bar{\eta}^2+2B = \eta^2+2\eta\cdot w_4 = 0$. On the other hand, it is easy to see that $B \geq 0$ for integers b and c. Thus we have $\bar{\eta}^2 = 0$ and $B = 0$. It implies that $\eta = 0$ or $\eta = -e'$.

The assertion (6) is obvious.

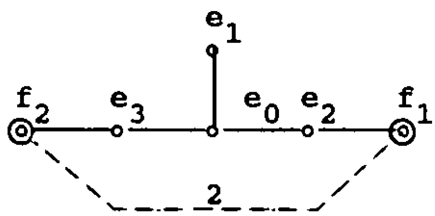
P has the associated numbers 1, 9, and has special elements of type B and of type C.

The case of $U_{1,0}(2,3,3,3)$.

We assign numbers to the basis of P corresponding to the dual graph as follows.



Set $f_1 = e_6-e_5-2u_0$ and $f_2 = e_7-e_5-2u_0$. The orthogonal complement P' of $H_0 = \mathbb{Z}u_0+\mathbb{Z}v_0$ in P is spanned by e_0, e_1, e_2, e_3, f_1 and f_2. One has $f_1 \cdot e_i = 0$ $(i = 0, 1, 3)$, $f_1 \cdot e_2 = -1$, $f_2 \cdot e_i = 0$ $(i = 0, 1, 2)$, $f_2 \cdot e_3 = -1$ and $f_1 \cdot f_2 = 2$. Thus the dual graph of P' is like the following.



We define a basis of the dual module P'^* as follows

$$
\begin{aligned}
w_0 &= (22e_0+11e_1+12e_2+12e_3+2f_1+2f_2)/9 \\
w_1 &= (11e_0+10e_1+6e_2+6e_3+f_1+f_2)/9 \\
w_2 &= (4e_0+2e_1+4e_2+2e_3+f_1)/3 \\
w_3 &= (4e_0+2e_1+2e_2+4e_3+f_2)/3 \\
z_1 &= (2e_0+e_1+3e_2+4f_1-2f_2)/9 \\
z_2 &= (2e_0+e_1+3e_3-2f_1+4f_2)/9.
\end{aligned}
$$

We can check that $w_i \cdot e_j = \delta_{ij}$, $w_i \cdot f_j = 0$, $z_i \cdot e_j = 0$ and $z_i \cdot f_j = \delta_{ij}$. On the other hand, $w_0 - (w_3 - 4z_1)$, $w_1 + (w_3+2z_1)$, $w_2+(w_3-3z_1)$ and $z_2-(w_3+4z_1)$ are elements in P'. Thus one knows the following proposition. For an element $x \in P'^*$ we denote $\bar{x} = x \bmod P' \in P'^*/P'$.

<u>Proposition 2.13.</u> In the case of $U_{1,0}(2,3,3,3)$, the discriminant group $P^*/P \cong P'^*/P'$ is the direct sum of the cyclic

group order 3 generated by $\bar{w}_3$ and the cyclic group of order 9 generated by $\bar{z}_1$. It can be also represented as the direct sum of the cyclic group of order 3 generated by $\bar{w}_2$ and the cyclic group of order 9 generated by $\bar{z}_2$. For the discriminant quadratic form q, we have

$$q(a\bar{w}_3+b\bar{z}_1) \equiv \frac{4}{3}a^2+ \frac{2}{9}b^2 \bmod 2\mathbb{Z}.$$

Besides, $\bar{w}_0 = \bar{w}_3 - 4\bar{z}_1$, $\bar{w}_1 = -\bar{w}_3 - 2\bar{z}_1$, $\bar{w}_2 = -\bar{w}_3 + 3\bar{z}_1$ and $\bar{z}_2 = \bar{w}_3 + 4\bar{z}_1$.

We check Proposition 2.9. First we show (G′). Assume that $\eta \in P'$, $\eta^2 = +2$ and $\eta \notin P'_0$. Set

$\tilde{\eta} = \sum_{i=0}^{3} a_i e_i + b_1 e_6 + b_2 e_7$, corresponding to

$\eta = \sum_{i=0}^{3} a_i e_i + b_1 f_1 + b_2 f_2$. This $\tilde{\eta}$ is an element in the root lattice generated by e_0, e_1, e_2, e_3, e_6 and e_7. Since $\tilde{\eta} \neq 0$, $\tilde{\eta}^2$ is a positive even integer. Since $2 = \eta^2 = \tilde{\eta}^2+2(b_1^2+b_1b_2+b_2^2) \geq \tilde{\eta}^2 \geq 2$, we have $b_1 = b_2 = 0$. Thus $\eta \in P'_0$, which contradicts the assumptions. There exists no η satisfying the condition, and (G′) holds.

Next, we show the assertions (2)-(5) for each element $\bar{x} \in P'^*/P'$.

(1) $\bar{x} = 0$. This $\bar{x}$ is of the first kind. The assertion (2) holds.

(2) $\bar{x} = \pm\bar{z}_1$. $q(\bar{x}) \equiv 4/9$. This $\bar{x}$ is of the first kind. To show the assertion (2) it suffices to see that if $(\eta+z_1)^2 = z_1^2 = 4/9$ for $\eta \in P'$, then $\eta = 0$, because z_1 is orthogonal to P_0'. Consider
$B = b_1^2 + b_1b_2 + b_2^2 + b_1 = (b_2+(b_1/2))^2 + b_1((3b_1/4)+1)$, in preparation to see this. We can show $B \geq 0$. Now, set

$$\tilde{\eta} = \sum_{i=0}^{3} a_ie_i + b_1e_6 + b_2e_7,$$ corresponding to

$$\eta = \sum_{i=0}^{3} a_ie_i + b_1f_1 + b_2f_2 \in P'.$$ Assume that

$(\eta+z_1)^2 = 4/9$ and $\eta \neq 0$. Since $\tilde{\eta}^2 > 0$, we have
$4/9 = 4/9 + \tilde{\eta}^2 + B > 4/9$, a contradiction.

(3) $\bar{x} = \pm 2\bar{z}_1$. $q(\bar{x}) \equiv 16/9$. This is of the fourth kind.

(4) $\bar{x} = \pm 3\bar{z}_1$. $q(\bar{x}) \equiv 0$. This is of the first kind. However, $\upsilon(\bar{x}) \geq 2$ by Lemma 2.1 (1), (2). Since the assumption of the assertion (2) is never satisfied, (2) holds.

(5) $\bar{x} = \pm 4\bar{z}_1$. $q(\bar{x}) \equiv 10/9$. This is of the third kind with the associated number 8. Set $x_0 = w_3 - z_2$. One has $\bar{x}_0 = -4\bar{z}_1$. This x_0 is not orthogonal to P_0', since $x_0 \cdot e_3 \neq 0$. On the other hand $x_0^2 = w_3^2 - 2w_3 \cdot z_2 + z_2^2 = (4/3)-(2/3)+(4/9) = 10/9 < 2$. One sees that the assertion (3) holds.

(6) $x = \pm\bar{w}_3$. $q(\bar{x}) \equiv 4/3$. This is of the third kind with the associated number 2. Since w_3 is not orthogonal to P_0' and since $w_3^2 = 4/3$, one knows the assertion (3). Besides $\bar{x}$ has order 3 and is a special element of type G.

(7) $\bar{x} = \pm\bar{w}_3 \pm \bar{z}_1$. $q(\bar{x}) \equiv 16/9$. This is of the fourth kind.

(8) $\bar{x} = \pm(\bar{w}_3+2\bar{z}_1)$. $q(\bar{x}) \equiv 10/9$. This is of the third kind with the associated number 8. Now, $\bar{w}_1 = -(\bar{w}_3+2\bar{z}_1)$

$w_1^2 = 10/9 < 2$, and w_1 is not orthogonal to P_0', since $w_1 \cdot e_1 \neq 0$. The assertion (3) holds.

(9) $\bar{x} = \pm(\bar{w}_3 - 2\bar{z}_1)$. $q(\bar{x}) \equiv 10/9$. This is of the third kind with the associated number 8. Setting $x_0 = w_2 - z_1$, one can show the assertion (3) by the same argument as in the above (5).

(10) $\bar{x} = \pm(\bar{w}_3 + 3\bar{z}_1)$. $q(\bar{x}) \equiv 4/3$. This is of the third kind with the associated number 2. Set $x_0 = w_3 + 3z_1 - (e_0 + e_2 + e_3 + f_1)$. One can show that $x_0^2 = 4/3 < 2$ and $x_0 \cdot e_3 = 2 \neq 0$. One knows the assertion (3). The element $\bar{x}$ has order 3, and one has the assertion (5), too.

(11) $\bar{x} = \pm(\bar{w}_3 - 3\bar{z}_1)$. $q(\bar{x}) \equiv 4/3$. This is of the third kind with the associted number 2. Now, $\bar{w}_2 = -(\bar{w}_3 - 2\bar{z}_1)$, $w_2^2 = 4/3$ and w_2 is not orthogonal to P_0'. One knows the assertion (3). Besides one knows the assertion (5), too, since $\bar{x}$ has order 3.

(12) $\bar{x} = \pm(\bar{w}_3 + 4\bar{z}_1)$. $q(\bar{x}) \equiv 4/9$. This is of the first kind. Here note that $\bar{z}_2 = \bar{w}_3 + 4\bar{z}_1$ and $z_2^2 = 4/9$. Thus one can show the assertion (2) by the same argument as in the above case (2) $\bar{x} = \pm\bar{z}_1$.

(13) $\bar{x} = \pm(\bar{w}_3 - 4z_1)$. $q(\bar{x}) \equiv 4/9$. This is of the first kind. We would like to show the assertion (2). First note that $\overline{z_1 + z_2} = \bar{w}_3 - 4\bar{z}_1$, $(z_1 + z_2)^2 = 4/9$ and $z_1 + z_2$ is orthogonal to P_0'. Thus to show (2) it suffices to see that if $(\eta + z_1 + z_2)^2 = (z_1 + z_2)^2 = 4/9$ for $\eta \in P'$, then $\eta = 0$. Consider $B = b_1^2 + b_1 b_2 + b_2^2 + b_1 + b_2$, in preparation to see this. We can show $B \geq 0$ for integers b_1 and b_2. Then, we can show $\eta = 0$ by the same argument as the above case (2) $\bar{x} = \pm\bar{z}_1$.

As for the assertion (6) in the proposition, it is obvious, since $q(\bar{x}) \equiv 0 \bmod 2\mathbb{Z}$ for $\bar{x} = \pm 3\bar{z}_1 \neq 0$.

In the case of $U_{1,0}(2,3,3,3)$, P has the associated numbers 2 and 8, and has the special elements of type G.

We have established Proposition 2.9. This proposition is the basis of the following arguments.

In the last part of this section, we want to make the meaning of Proposition 2.9 (6) clearer. Consider the last case $U_{1,0}$ in particular. Associated with the basis $e_0, \ldots, e_7$ of P, set $u_1 = 3e_0+2(e_2+e_3+e_4)+e_5+e_6+e_7$. $u_1^2 = 0$, $u_1 \cdot e_i = 0$ $(i \neq 1)$ and $u_1 \cdot e_1 = -3$. Set further

$$y_1 = -e_1 + e_4 + e_5 - u_1/3$$
$$y_2 = -e_1 + e_2 + e_6 - u_1/3$$
$$y_3 = -e_1 + e_3 + e_7 - u_1/3.$$

We can check $y_i^2 = +2$ $(i = 1, 2, 3)$. Since

$$(u_1/3)-3z_1 = 3e_0-e_1+e_2+2e_3+2e_4+e_5-e_6+e_7,$$

$$\bar{y}_1 = \bar{y}_2 = \bar{y}_3 = 3\bar{z}_1 \in P'^*/P' \cong P^*/P.$$

<u>Lemma 2.14.</u> (Nikulin [10]) Let L be a non-degenerate even lattice and K be an even overlattice of L. We can regard $L \subset K \subset K^* \subset L^*$. Set $I = K/L$. We regard I as a subgroup of L^*/L. Set

$$I^{\perp} = \{\bar{x} \in L^{*}/L \mid b_L(\bar{x},\bar{y}) \equiv 0 \bmod \mathbb{Z} \text{ for every } \bar{y} \in I\}.$$

(1) I is an isotropic subgroup of L^{*}/L, i.e., $q_L|I \equiv 0$.

(2) $I^{\perp} = K^{*}/L$.

(3) Associated with the exact sequence

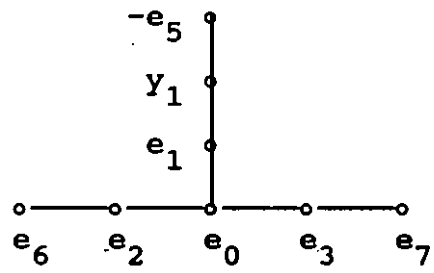
$$0 \longrightarrow I \longrightarrow I^{\perp} \xrightarrow{\ \sigma\ } K^{*}/K \longrightarrow 0,$$

one has $q_K\sigma = q_L|I^{\perp}$.

(4) Conversely, for any isotropic subgroup $I' \subset L^{*}/L$, the inverse image K' of I' by the natural surjective morphism $L^{*} \longrightarrow L^{*}/L$ is an even overlattice of L.

<u>Proposition 2.15.</u> The corresponding lattice P has no even overlattice except P itself in the case of the former 5 kinds $J_{3,0}$, $Z_{1,0}$, $Q_{2,0}$, $W_{1,0}$ and $S_{1,0}$ of quadrilateral singularities.

As for the last case $U_{1,0}$, P has a unique overlattice P_1 except P itself. P_1 has index 3 over P, and $P_1 = P + \mathbb{Z}y_1 = P + \mathbb{Z}y_2 = P + \mathbb{Z}y_3$. Choosing one of y_1, y_2, y_3 corresponds to choosing one of the 3 arms with length 3 in the dual graph of the basis of P. If we choose y_1, the bilinear form on P_1 is described by the following dual graph.



In particular, $P_1 \cong Q(E_6) \oplus H$.

§3. Root modules

We develop general theory of root systems in this section, and we convert Looijenga's condition (a) and (b) into a simpler one by the theory introduced here and by the results in section 2.

We always work fixing arbitrary one of 6 kinds of quadrilateral singularities. By P we denote the corresponding lattice to it. The lattices P', P_0 and P'_0 and the elements u_0, $v_0 \in P$ are also defined.

An embedding $P \subset \Lambda$ into another even lattice Λ is said to be <u>good</u>, if it satisfies the following condition (G) (Looijenga [8]).

(G) Let $\tilde{P} = P(P,\Lambda)$ be the primitive hull of P in Λ. If $\eta \in \tilde{P}$, $\eta^2 = 2$, $\eta \cdot u_0 = 0$ and $\eta \notin P'_0$, then η is orthogonal to P_0.

On the other hand, if the image of P in Λ is primitive in Λ, then the embedding is said to be <u>primitive</u>. By Proposition 2.15, every embedding of P into an even lattice is primitive for the former 5 kinds of singularities.

<u>Proposition 3.1.</u> For an embedding of our lattice P into an even lattice, it is good if and only if it is primitive.

Proof. For the former 5 kinds it suffices to show that a primitive embedding is a good embedding. On the other hand by Proposition 2.9 (1) P has the property (G'). This (G') implies the desired claim.

Next, we consider the case of $U_{1,0}$. Similarly by Proposition 2.9(1), a primitive embedding is a good embedding. Any non-primitive embedding into an even lattice Λ can be factored as $P \subset P_1 \subset \Lambda$. (See Proposition 2.15.) The element $y_1 \in P_1$ satisfies $y_1 \cdot u_0 = 0$ and $y_1 \notin P_0$, but y_1 is not orthogonal to P_0, since $y_1 \cdot e_4 \neq 0$. Thus the embedding is not good.

Q.E.D.

Proposition 3.2 (1) Let Λ_N denote the even unimodular lattice with signature $(16+N, N)$. If $N \geq 1$, then the lattice P has a primitive embedding into Λ_N.
(2) Besides if $N \geq 2$, for any two primitive embeddings $\iota, \iota' : P \subset \Lambda_N$, we have an integral orthogonal transformation $\varphi : \Lambda_N \longrightarrow \Lambda_N$ with $\iota' = \varphi\iota$.

Proof. (1) By $\ell(A)$ we denote the minimum number of generators of an abelian group A. Obviously $\ell(P^*/P) \leq \text{rank } P$. Now, P has signature $(\sum p_i - 4, 1)$. For our 6 cases $\sum p_i - 4 \leq 7$. Comparing the signature of Λ_N, one has:

$$\begin{aligned} (16+N)-(\textstyle\sum p_i-4) &\geq 9+N>0 \\ N-1 &\geq 0. \end{aligned}$$

Comparing the ranks of Λ_N and P, one has:

$$\begin{aligned}\text{rank } \Lambda_N - \text{rank } P &= 16+2N-\sum p_i+3 \\ &\geq 8+2N \geq 2+\text{rank } P \\ &\geq 2+\ell(P^*/P) > \ell(P^*/P).\end{aligned}$$

Appyling Nikulin [10] Theorem 1.12.2, one knows the existence of an embedding.

(2) Besides if $N \geq 2$, one has a stronger inequality $N-1>0$ about the negative signature. Thus by Nikulin [10] Theorem 1.14.4 one has the uniqueness.

Q.E.D.

Remark. When $N = 1$, we cannot assert the uniqueness.

In Urabe [13] we have introduced the concept of root modules. They have been a kind of quasi-lattices satisfying certain conditions. However, the conditions there are too strong for our 6 cases under consideration. We would like to define the concept of root modules again in this article, as a more general concept. In Urabe [13] we have had only irreducible root systems of type A, B, D or E. However, according to the definition here, we have root systems of all types A, B, C, D, E, F or G, and moreover we have non-reduced root systems of type CB (Bourbaki [3]).

In addition to these genralized root modules, we will introduce the concept of obstruction components later.

Now, let L be a quasi-lattice, FL be a submodule of L such that the index $\#(L/FL)$ is finite. We define the set $R = R(L,FL)$ for this pair (L,FL) as follows:

$$R = \{\alpha \in FL \mid \alpha^2 = 2\} \cup \{\beta \in L \mid \beta^2 = 1/2,\ 2/3 \text{ or } 1\}.$$

We call the set R the <u>root system</u> of (L,FL), and call an element in R a <u>root</u>. For any root $\alpha \in R$, $\sqrt{\alpha^2}$ is called the <u>length</u> of α. A root with length $\sqrt{2}$ is called a <u>long root</u>. A root with length $1/\sqrt{2}$, $\sqrt{2/3}$ or 1 is called a <u>short root</u>. Setting $\alpha^\vee = 2\alpha/\alpha^2$ for a root $\alpha \in R$, we call $\alpha^\vee$ the <u>co-root</u> of α. We have $\alpha^\vee \in \mathbb{Z}\alpha$. Consider the following condition (R1)

(R1) $2(x,\alpha)/\alpha^2 = (x,\alpha^\vee)$ is an integer for every $x \in L$ and $\alpha \in R$.

When (R1) is satisfied, for every $\alpha \in R$ we can define an isomorphism $s_\alpha : L \longrightarrow L$ preserving bilinear forms, by setting for $x \in L$

$$s_\alpha(x) = x - 2(x,\alpha)\alpha/\alpha^2 = x-(x,\alpha^\vee)\alpha = x-(x,\alpha)\alpha^\vee.$$

We call s_α the <u>reflection</u> with respect to α. Indeed, on $L \otimes \mathbb{R}$ s_α defines the reflection whose mirror is the hyperplane orthogonal to α. In particular, s_α^2 is the identity and $s_\alpha = s_{-\alpha}$.

(R2) $s_\alpha(FL) = FL$ for every $\alpha \in R$.

(R3) If $\alpha \in R$ and $\alpha^2 = 1/2$, then $2\alpha \in FL$.

Assume that the pair (L,FL) satisfies the three conditions (R1), (R2) and (R3). Then, we call this pair (L,FL) a <u>root module.</u> When L = FL, particularly we say that this root module is <u>regular</u>, and we do not mention FL and abbreviate it. (Sometimes we abbriviate FL for simplicity even if $L \neq FL$.)

Any lattice is a regular root module.

Let (L,FL) be a root module. The root system of the root module satisfies the most important axioms (SR_{II}) and (SR_{III}) of the 4 axioms for root systems in Bourbaki [3] (Chap. VI, n° 1.1, Def. 1 and Résumé in the last part) and thus it suits the name.

For a root $\beta \in R$ with length $1/\sqrt{2}$, $\alpha = 2\beta$ is a long root , $\beta^\vee = 2\alpha^\vee$, and $s_\alpha = s_\beta$ for the reflections. Setting

$$\hat{R} = \hat{R}(L,FL) = \{\alpha \in R \mid \alpha^2 \neq 1/2\},$$

we call $\hat{R}$ the <u>reduced root system</u> of the root module (L,FL). This satisfies the axioms (SR_{II}), (SR_{III}) and the reduced axiom

$$(SR_{IV}) \quad \text{If } \alpha \in \hat{R}, \text{ then } 2\alpha \notin \hat{R}.$$

of the axioms for root systems.

The subgroup generated by all reflections s_α with $\alpha \in R$ in the group of all integral orthogonal transformations on L is called the Weyl group of (L,FL) and is denoted by $W(L,FL)$ or $W(R)$. Since it is obviously equal to the subgroup generated by reflections corresponding to the reduced root system $\hat{R}$, it is also denoted by $W(\hat{R})$.

$W(L,FL) = W(R) = W(\hat{R})$.

The submodule in L generated by R (respectively $\hat{R}$) is denoted by $Q(R)$ (resp. $Q(\hat{R})$) and is called the root quasi-lattice of R (resp. $\hat{R}$). $Q(R) \supset Q(\hat{R})$. These are not necessarily lattices. When it is a lattice, we call it a root lattice. Sometimes we write $Q(L,FL) = Q(R(L,FL))$ for simplicity, which is the submodule of L generated by roots, and we call it the root quasi-lattice of the root module (L,FL).

On the other hand, the submodule generated by all co-roots $\alpha^\vee$ $(\alpha \in R)$ is denoted by $Q(R^\vee)$ and is called the co-root lattice. Indeed $Q(R^\vee)$ is always an even lattice. For $\alpha, \beta \in R$,

$$(\alpha^\vee,\beta^\vee) \in \mathbb{Z}$$
$$(\alpha^\vee,\alpha^\vee) = 4/\alpha^2 \in 2\mathbb{Z}.$$

Let $Q(\hat{R}^\vee)$ be the submodule generated by all co-roots corresponding to roots in the reduced root system $\hat{R}$. Note that $Q(\hat{R}^\vee) = Q(R^\vee)$.

The reflection $s_\alpha (\alpha \in R)$ induces an isomorphism $s_\alpha : Q(R^\vee) \longrightarrow Q(R^\vee)$ of co-root lattices. Indeed, for $\beta \in R$,

$$s_\alpha(\beta^\vee) = \beta^\vee - (\alpha,\beta^\vee)\alpha^\vee \in Q(R^\vee).$$

The Weyl group $W(R)$ acts on $Q(R^\vee)$.

Next, let M be a submodule of L. Then, setting $FM = FL \cap M$, (M,FM) is a root module. A submodule is always regarded as a root module in this manner. In other words, a homomorphism $\varphi : (M,FM) \longrightarrow (L,FL)$ between root modules is <u>defined</u> to be a homomorphism $\varphi : M \longrightarrow L$ of modules which preserves bilinear forms and such that $\varphi^{-1}(FL) = FM$.

<u>Lemma 3.3.</u> Let $\hat{R} = \hat{R}(L,FL)$ be the <u>reduced</u> root system of a root module (L,FL). For every $\alpha \in \hat{R}$,

$$\mathbb{R}\alpha^\vee \cap Q(R^\vee) = \mathbb{Z}\alpha^\vee.$$

<u>Proof.</u> Easy. (See Urabe [13] Lemma 2.2.)

<u>Remark.</u> The above equality does not hold for a short root α with length $1/\sqrt{2}$.

Note that if L is positive definite, then the root system $R(L,FL)$ is a finite set.

In the following we freely use standard concepts and terminologies in the theory of root systems (Bourbaki [3]). Any finite root system is uniquely decomposed into a direct sum of irreducible ones.

Proposition 3.4 (1) Finite irreducible root systems containing a long root are classified into the following types. The lower index represents the rank of the root system.

$A_k (k \geq 1)$, $B_k (k \geq 2)$, $C_k (k \geq 4)$,
$D_k (k \geq 4)$, E_6, E_7, E_8, F_3, F_4, G_2,
$CB_k (k \geq 1)$.

(Sometimes for the one of type F_3 the name of the type is also called C_3.) A root system R of type CB_k is non-reduced, i.e., R does not satisfy the axiom (SR_{IV}). The reduced root subsystem $\hat{R}$ consisting of all long roots and all short roots with length 1 in R has the following type:

A_1 $(k = 1)$, B_2 $(k = 2)$, F_3 $(k = 3)$, C_k $(k \geq 4)$.

Every long root α in R is divisible, i.e., $\alpha/2 \in R$.
(2) Any finite root system of a root module has at most one component containing a short root with length $1/\sqrt{2}$.
(3) Consider a finite root system of a regular root module. Any irreducible component of it is never of type C. If an irreducible component of it is of type CB_k, then $1 \leq k \leq 3$. Besides, it has at most one component containing a short root with length 1.

<u>Proof</u> (1) The main parts follow from Bourbaki [3]. Note that by the axiom (R1) the reduced root system $\hat{R}$ of a non-reduced root system R cannot be of type B_k with $k \geq 3$.

(2) If $\gamma_1^2 = \gamma_2^2 = 1/2$ and $\gamma_1 \cdot \gamma_2 = 0$, then $(\gamma_1+\gamma_2)^2 = 1$. It follows from this fact.

(3) Assume $k \geq 4$. Consider a free module $F = \sum_{i=1}^{k} \mathbf{Z}\epsilon_i$ of rank k. Set $L = \sum_{i=1}^{k} \mathbf{Z}(\epsilon_i/2)$. L is an overmodule of F with index 2^k. We define a bilinear form by $\epsilon_i^2 = 2$ $(1 \leq i \leq k)$, $\epsilon_i \cdot \epsilon_j = 0$ $(i \neq j)$. Then, (L,F) is a root module whose root system is of type CB_k. Set $L' = \{\sum a_i(\epsilon_i/2) \in L \mid \sum a_i$ is an even integer.$\}$. The pair (L',F) is also a root module, whose root system is of type C_k. Set $\beta_1 = (\epsilon_1+\epsilon_2)/2$ and $\beta_2 = (\epsilon_3+\epsilon_4)/2$. Then, we have $\beta_1^2 = \beta_2^2 = 1$, $\beta_1 \cdot \beta_2 = 0$ and $\beta_1, \beta_2 \in L'$. However, $(\beta_1+\beta_2)^2 = 2$ and $\beta_1+\beta_2 \notin F$. By this fact one sees the former half of the assertion. By the same argument as in (2) one has the latter half.

Q.E.D.

Here we introduce three agreements in order to make the following descriptions clearer. Consider the symbols of root systems in Proposition 3.4 (1). In general situations we can use these symbols (in particular, those of type A, D or E) for root systems containing no long root. However, in this article we obey the following agreements.

(Agreement 1) When we use the symbols of irreducible root systems in Proposition 3.4 (1), they imply that the root system contains a long root at the same time.

(Agreement 2) Consider the case where an irreducible root system R contains no long root. Then, we have an irreducible root system R′ with a long root and a positive real number t such that $R = \{t\alpha \mid \alpha \in R'\}$. If R′ is of type X, then we denote that R is of type $X(t^2)$.

(Agreement 3)(Exceptions) A reduced root system $\{\alpha, -\alpha\}$ of rank 1 is defined to be of type B_1 if $\alpha^2 = 1$, and of type G_1 if $\alpha^2 = 2/3$. A reduced irreducible root system of rank 2 consisting of only short roots with length 1 is defined to be of type F_2.

Therefore, $B_1 = A_1(1/2)$, $G_1 = A_1(1/3)$, and $F_2 = A_2(1/2)$.

Let R be a finite root system. We can choose a root basis $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subset R$ ("une base de racines" in Bourbaki [3], Ch. IV, n°1.5. Sometimes it is also called a fundamental system of roots.) when we fix a Weyl chamber. Each fundamental root α_i is indivisible, i.e., $\alpha_i/2 \notin R$. Δ is a basis of $Q(R)$.

We would like to explain the concept of Dynkin graphs here. We can draw a graph G by the following rules, corresponding to the root basis Δ of R.

(1) The vertices in G have one-to-one correspondence with elements in Δ. A vertex has one of four different expressions depending on the length of the corresponding root as follows:

Length:	$\sqrt{2}$	1	$\sqrt{2/3}$	$1/\sqrt{2}$
Expression:	∘	•	◎	⊕

(2) If two roots $\alpha, \beta \in \Delta$ are orthogonal i.e., $\alpha\cdot\beta = 0$, then we do not connect the two vertices corresponding to α and β.

(3) If two roots $\alpha, \beta \in \Delta$ are not orthogonal, then we connect the two vertices corresponding to α and β by an edge which is a single segment. (Note that if moreover α is a long root, then $\alpha\cdot\beta = -1$.)

The resulting graph G is the Dynkin graph of the root system R. It depends only on the isomorphism class of R and does not depend on the choice of Δ, since for another root basis Δ' we have an element $w \in W(R)$ of the Weyl group such that $\Delta' = w(\Delta)$. Non-isomorphic two finite root systems have different Dynkin graphs.

Our Dynkin graphs are slightly different from those used commonly. Next we explain the difference.

Dynkin graphs under common use have only one kind of vertex, but have three different kinds of edges —a single segment, a double one and a triple one—. The difference of the edge indicates the difference of the angle $(\alpha,\beta)/\sqrt{\alpha^2}\sqrt{\beta^2}$ between α and β. The absolute length of each root is ignored, but the mutual difference of the length between two roots is indicated by an arrow on a double or triple edge. Besides customarily we do not associate a Dynkin graph with any non-reduced root system.

Our Dynkin graph of type CB_k is the following.

CB_1 : ⊗

$CB_k(k\geq 2)$: ⊗——•——•–...–•——• (k vertices).

Corresponding to a reduced root system of rank 1, we have three different graphs ∘, •, ⊚ depending on the length of the root, but customarily they are not distinguished and all of them have the same expression ∘. Even in the case of rank 2, our graph of type A_2 ∘——∘ and one of type F_2 •——• are not customarily distinguished and are expressed by the same graph ∘——∘. Besides our Dynkin graph of type G_2 is ∘——⊚, while customarily it is ∘⇛∘.

Consider our Dynkin graph of a reduced irreducible root system which is not of type G_2. If it has a part like ∘——•, we replace it by ∘⇒∘. After that, if we replace all vertices corresponding to short roots by ∘, the obtained graph is the Dynkin graph under common use.

When a finite root system R has a_k of components of type A_k, b_k of components of type B_k, ..., we identify the formal sum $G = \sum a_k A_k + \sum b_k B_k + \ldots$ with the Dynkin graph of R, and we say that R is of type G. We use abbreviations like $R = R(G)$, $Q(R) = Q(G)$, $W(R) = W(G)$ etc.

Next we explain the concept of extended Dynkin graphs ("graphs de Dynkin complété" in Bourbaki [3]).

Let R be an <u>irreducible</u> finite root system and $\Delta \subset R$ be a root basis. The maximal root $\eta \in R$ is uniquely determined depending on Δ. We can write it in the form $\eta = \sum_{\alpha \in \Delta} n_\alpha \alpha$. The coefficient n_α is necessarily a positive integer, and is called the <u>coefficient of the maximal root</u> corresponding to α. The set $\Delta^+ = \Delta \cup \{-\eta\}$ is called the <u>extended root basis</u>. We define the coefficient $n_{-\eta}$ of the maximal root corresponding to $-\eta$ to be $n_{-\eta} = 1$. We have $\sum_{\alpha \in \Delta^+} n_\alpha \alpha = 0$.

Here we assume further that the rank of R is greater than or equal to 2. The <u>extended Dynkin graph</u> of R is the resulting graph when we apply the same rules (1), (2) and (3), which we used to make the Dynkin graph from Δ, to Δ^+ instead of Δ.

We define that the following is the extended Dynkin graph for a finite root system of rank 1.

A_1 : 

B_1 : 

G_1 : 

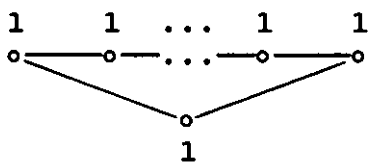
CB_1 : 

The edge in the extended Dynkin graph of rank 1 is a <u>bold</u> segment. (Sometimes we use a single segment $\overset{\infty}{\text{———}}$ accompanied with the mark ∞ instead.) The attached integers in the above extended graph are the coefficients of the maximal root. Even these graphs of rank 1 are drawn basically by applying the rules (1), (2) and (3) to Δ^+. However, in the case of rank 1, two elements in Δ^+ are proportional. To express the fact of proportion we use a bold edge.

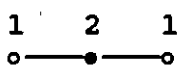
Note in particular that the maximal root in the case of type CB is a long root.

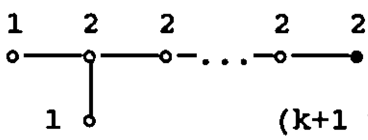
For a reducible finite root system R, the disjoint union of the extended Dynkin graphs of irreducible components is called the extended Dynkin graph of R. The number of vertices minus the number of connected components is called the rank of the extended Dynkin graph. The union of the extended root bases of the irreducible components is called the extended root basis of R.

In the following we show the extended Dynkin graphs for main types. Numbers are the coefficients of the maximal root.

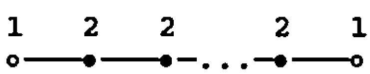
A_k (k≥2) :


(k+1 vertices)

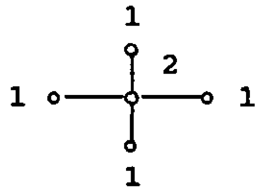
B_2 :


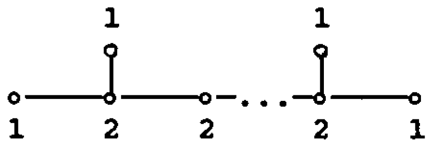
B_k (k≥3) :


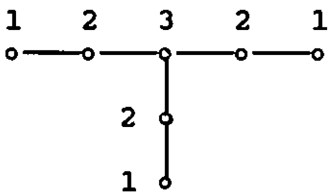
(k+1 vertices)

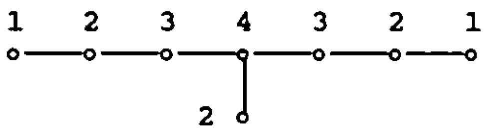
C_k (k≥4) :


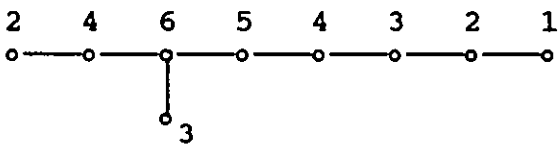
(k+1 vertices)

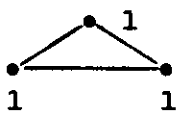
D_4 : 

D_k $(k \geq 5)$:  (k+1 vertices)

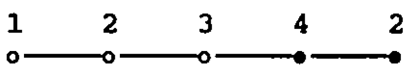
E_6 : 

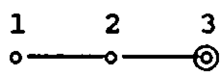
E_7 : 

E_8 : 

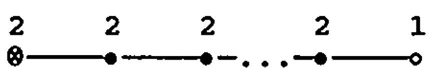
F_2 : 

F_3 : 

F_4 : 

G_2 : 

CB_1 : 

CB_k $(k \geq 2)$:  (k+1 vertices)

When we choose an arbitrary vertex with the attached number 1 in a connected extended Dynkin graph, the graph obtained by erasing out the chosen vertex and the edges issuing from it is the corresponding Dynkin graph.

Lemma 3.5. Let Λ be an even unimodular lattice, L be a non-degenerate primitive sublattice, and $M = C(L,\Lambda)$ be the orthogonal complement of L in Λ.

(1) We define a morphism $\Lambda \longrightarrow \mathrm{Hom}(M,\mathbf{Z}) = M^*$ by associating an element $x \in \Lambda$ with a morphism $M \longrightarrow \mathbf{Z}$ defined by $y \in M \longrightarrow (x,y)$. This induces an embedding of quasi-lattices $M \subset M^*$ and an isomorphism $\Lambda/L \cong M^*$ of quasi-lattices.

(2) The composition $M \subset \Lambda \longrightarrow \Lambda/L$ of the natural morphisms is injective and it induces an embedding $M \subset \Lambda/L$ of quasi-lattices such that the composition $M \subset \Lambda/L \cong M^*$ coincides with the inclusion $M \subset M^*$.

(3) Let $r : L^*/L \longrightarrow M^*/M$ be the canonical isomorphism obtained by composing canonical isomorphisms $(\Lambda/L)/M \cong \Lambda/L \oplus M$, $(\Lambda/M)/L \cong \Lambda/L \oplus M$, $\Lambda/L \cong M^*$ and $\Lambda/M \cong L^*$. Then for discriminant quadratic forms,

$$q_L(\bar{x}) = -q_M(r(\bar{x})) \quad (\bar{x} \in L^*/L).$$

Proposition 3.6. Let P be the lattice corresponding to a fixed one of 6 kinds of hypersurface quadrilateral singularities. Let Λ be an even unimodular lattice. Assume that there exists a primitive embedding $P \subset \Lambda$. Let F denote the orthogonal complement of P in Λ.

(1) When we regard F as a subspace of a quotient quasi-lattice Λ/P,

$$F = (\Lambda/P)^{*} = \{x\in\Lambda/P \mid (x,y)\in\mathbb{Z} \text{ for every } y\in\Lambda/P.\}.$$

(2) For the former 5 kinds except $U_{1,0}$, Λ/P is a regular root module. Besides

$$F = \{x\in\Lambda/P \mid x^2 \text{ is an even integer.}\}.$$

(3) For the last case $U_{1,0}$, the pair $(\Lambda/P,F)$ is a root module.

<u>Proof.</u> (1) follows from Lemma 3.5. First of all we show the latter half of (2). In the following we identify Λ/P and F^{*} via the canonical isomorphism. The canonical surjective morphism $\Lambda/P \longrightarrow (\Lambda/P)/F \cong F^{*}/F$ is denoted by π. The composition $F^{*}/F \xrightarrow{\sim} P^{*}/P \xrightarrow{\sim} P'^{*}/P'$ of the canonical isomorphisms is denoted by $\tilde{r}$. The discriminant quadratic forms of P' and F are denoted by $q_{P'}$ and q_F respectively. Then, the assertion is equivalent to that $q_F^{-1}(0 \bmod 2\mathbb{Z}) = \{0\}$. Since $q_F = -q_{P'}\tilde{r}$, this is equivalent to that $q_{P'}^{-1}(0 \bmod 2\mathbb{Z}) = \{0\}$. However, this is equivalent to Proposition 2.9 (6).

We check the axioms of root modules. First let $\alpha \in \Lambda/P$ be a long root. For the former 5 kinds we have $\alpha \in F$ by the latter half of (2). For $U_{1,0}$ $\alpha\in F$ by definition. By (1) $2(\alpha,x)/\alpha^2 = (\alpha,x)\in\mathbb{Z}$ for every $x\in\Lambda/P$.

Secondly assume $\beta^2 = 1$ for $\beta \in \Lambda/P$. Set $\bar{\beta} = \tilde{r}(\pi(\beta)) \in P'^*/P'$. $q_{P'}(\bar{\beta}) \equiv -q_F(\pi(\beta)) \equiv 1 \bmod 2\mathbf{Z}$. $\bar{\beta}$ is an element of the second kind. By Proposition 2.9 (4) $2\bar{\beta} = 0$. Since $\tilde{r}$ is an isomorphism, $2\pi(\beta) = 0$ and thus $2\beta \in F$. By (1) $2(\beta,x)/\beta^2 = (2\beta,x) \in \mathbf{Z}$ for every $x \in \Lambda/P$.

Thirdly assume $\gamma^2 = 2/3$ for $\gamma \in \Lambda/P$. Set $\bar{\gamma} = \tilde{r}(\pi(\gamma)) \in P'^*/P'$. $q_{P'}(\bar{\gamma}) \equiv -q_F(\pi(\gamma)) \equiv -\gamma^2 \equiv 1+(1/3) \bmod 2\mathbf{Z}$. $\bar{\gamma}$ is an element of the third kind with the associated number 2. By Proposition 2.9 (5), $3\bar{\gamma} = 0$. Thus $3\gamma \in F$. By (1) $2(\gamma,x)/\gamma^2 = (3\gamma,x) \in \mathbf{Z}$ for every $x \in \Lambda/P$.

Lastly assume $\delta^2 = 1/2$ for $\delta \in \Lambda/P$. Set $\bar{\delta} = \tilde{r}(\pi(\delta)) \in P'^*/P'$. $q_{P'}(\bar{\delta}) \equiv -q_F(\pi(\delta)) \equiv -\delta^2 \equiv 1+(1/2) \bmod 2\mathbf{Z}$. $\bar{\delta}$ is an element of the third kind with the associated number 1. By Proposition 2.9 (5), $2\bar{\delta} = 0$. Thus $2\delta \in F$. The axiom (R3) is satisfied. By (1) $2(\delta,x)/\delta^2 = 2(2\delta,x) \in \mathbf{Z}$ for every $x \in \Lambda/P$.

Now, since F is invariant under all integral orthognal transformations on Λ/P by (1), the axiom (R2) is also satisfied in the case of $U_{1,0}$.

Q.E.D.

<u>Definition 3.7</u>. (1). Let (L,FL) be a root module. Let M be a submodule of L, and $\tilde{M} = P(M,L)$ be the primitive hull. We say that M is <u>full</u> in L if $R(M,M\cap FL) = R(\tilde{M},\tilde{M}\cap FL)$ for root systems. An embedding of root modules whose image is full is called a <u>full embedding</u>.

(2) Let k be a positive integer, G be a Dynkin graph. Let $G = G_1+\ldots+G_m$ be the decomposition into components and

$Q(G) = \bigoplus_{i=1}^{m} Q(G_i)$ be the corresponding orthogonal decompositon of root quasi-lattices. Assume that there is an embedding $Q(G) \subset L$ into a root module (L,FL). If a component G_i is of type A_k, and if the index of the primitive hull satisfies $[P(Q(G_i),L) : Q(G_i)] \geq k+1$, then we say that G_i is an <u>obstruction component</u> for this embedding with respect to k.

<u>Lemma 3.8.</u> We consider the situation in Definition 3.7 (2) above. Besides, we assume that L is the root module Λ/P in Proposition 3.6. If a component G_i is of type A_k, the following three conditions are equivalent. Set $Q_i = Q(G_i)$ and $\tilde{Q}_i = P(Q_i, \Lambda/P)$.

(1) G_i is an obstruction component, i.e., $[\tilde{Q}_i : Q_i] \geq k+1$.

(2) $[\tilde{Q}_i : Q_i] = k+1$.

(3) $\tilde{Q}_i = Q_i^*$.

<u>Proof</u>. We use the notation in Proposition 3.6. Since G_i has no vertex corresponding to a short root, $Q_i \subset F$. By Proposition 3.6(1)

$$Q_i^* \supset \tilde{Q}_i \supset Q_i.$$

Since $[Q_i^* : Q_i] = k+1$, the above (1), (2) and (3) are equivalent.

Q.E.D.

Theorem 3.9. Let P be the lattice corresponding to one of 6 kinds of hypersurface quadrilateral singularities. Let Λ be an even unimodular lattice with signature $(16+N, N)$ with $N\geq 1$. Let G be a Dynkin graph without a vertex corresponding to a short root and $Q(G)$ be the root lattice of type G. We consider an embedding $P\oplus Q(G) \subset \Lambda$ and the corresponding induced embedding $Q(G) \subset \Lambda/P$ defined as the composition $Q(G) \subset \Lambda \longrightarrow \Lambda/P$ of natural morphisms. Then, the following (A) and (B) are equivalent.

(A) The embedding $P\oplus Q(G) \subset \Lambda$ satisfies Looijenga's conditions (a) and (b) in Theorem 1.1.

(B) The embedding $Q(G) \subset \Lambda/P$ is full and for every associated number k of P with $k\geq 4$, G has no obstruction component with respect to k.

Proof. Set $F = C(P,\Lambda)$. By $\pi:\Lambda \longrightarrow \Lambda/P$ we denote the canonical surjective morphism.

(1) We will show that $\widetilde{Q(G)} = P(Q(G),\Lambda/P) \subset \Lambda/P$ contains no short root under the condition (b).

Let $\eta \in \widetilde{Q(G)}$ be a short root. We have $\eta^2 = 1,\ 2/3$ or $1/2$. Choose an element $\alpha\in\Lambda$ with $\pi(\alpha) = \eta$. We can write it in the form $\alpha = x+y$ $(x\in P^*,\ y\in F^*)$, since $P \oplus F \subset \Lambda \subset P^* \oplus F^*$. By the definition of the bilinear form on Λ/P, $y^2 = \eta^2 = 1,\ 2/3$ or $1/2$. There is a non-zero integer m with $m\eta \in Q(G)$, since η belongs to the primitive hull. This implies $my \in Q(G)$. We can assume that $mx \in P$ for the same m at the same time and we have $m\alpha \in P \oplus Q(G)$. Thus α belongs to the primitive hull $\widetilde{P\oplus Q(G)}$.

On the other hand, the element $\alpha_1 = \alpha - (\alpha\cdot u_0)v_0 - (\alpha\cdot v_0)u_0$ also satisfies $\pi(\alpha_1) = \eta$. Thus we can assume further that α satisfies $\alpha\cdot u_0 = \alpha\cdot v_0 = 0$. Under this assumption one has $x \in P'^*$. Obviously $\alpha^2 \equiv 0 \bmod 2\mathbf{Z}$ and $x^2 = \alpha^2 - y^2 \equiv -y^2 \equiv -\eta^2 \equiv 1,\ 4/3$ or $3/2 \bmod 2\mathbf{Z}$. The element $x \bmod P' \in P'^*/P'$ is of the second or third kind. By Proposition 2.9(3) we have an element $z \in P'$ such that $(x-z)^2 = 2-\eta^2$ and $(x-z)$ is not orthogonal to P'_0. By exchanging α for $\alpha-z$, we can assume that $x^2 = 2-\eta^2$ and x is not orthogonal to P'_0. Consequently one has an element $\alpha \in \Lambda$ such that $\pi(\alpha) = \eta$, $\alpha^2 = 2$, $\alpha\cdot u_0 = \alpha\cdot v_0 = 0$, $\alpha \in \widetilde{P\oplus Q(G)}$ and α is not orthogonal to P'_0. Then, by the condition (b), $\alpha \in P_0$. This implies $y = 0$, which contradicts the fact $y^2 = \eta^2 \neq 0$. Therefore we have no short root η.

(2) Assume that a component G_0 of type A_k in G is an obstruction component with respect to an associated number $k \geq 4$ of P. We will deduce a contradiction from the condition (b).

Let $\Delta_0 \subset Q(G_0)$ be a root basis of $Q(G_0)$. Let ω be the fundamental weight corresponding to a vertex at one of the two ends of the Dynkin graph of Δ_0. One has $\omega^2 = k/(k+1)$. On the other hand, by Lemma 3.8 and by assumption one has $Q(G_0)^* = P(Q(G_0), \Lambda/P)$, and thus $\omega \in P(Q(G_0), \Lambda/P)$.

Then, we have an element α in the primitive hull of $P\oplus Q(G)$ in Λ such that $\pi(\alpha) = \omega$ and $\alpha\cdot u_0 = \alpha\cdot v_0 = 0$. We can write it in the form $\alpha = x+\omega$ $(x\in P'^{*})$. Since $\alpha^2 \equiv 0 \bmod 2\mathbf{Z}$,

$$x^2 \equiv -\omega^2 \equiv 1+(1/(k+1)) \bmod 2\mathbf{Z}.$$

The element $x \bmod P' \in P'^{*}/P'$ is of the third kind with the associated number k. By Proposition 2.9 (3) we have an element $z \in P'$ such that $(x-z)^2 = 1+(1/(k+1)) = 2-\omega^2$ and $(x-z)$ is not orthogonal to P'_0. Exchanging α by $\alpha-z$, one has an element $\alpha\in\widetilde{P\oplus Q(G)}$ such that $\pi(\alpha) = \omega$, $\alpha^2 = 2$, $\alpha\cdot u_0 = \alpha\cdot v_0 = 0$ and α is not orthogonal to P'_0. By the condition (b), $\alpha\in P'_0$, and $0 = \pi(\alpha) = \omega \neq 0$, which is a contradiction.

(3) We will show that the condition (b) is satisfied, if $\widetilde{Q(G)}$ does not contain a short root, and if G has no obstruction component for any associated number $k\geq 4$ of P.

Let α be an element in the primitive hull of $P\oplus Q(G)$ such that $\alpha\cdot u_0 = 0$, $\alpha^2 = 2$ and α is not orthogonal to P_0. Replacing α by $\alpha-(\alpha\cdot v_0)u_0$ one can assume further that it satisfies $\alpha\cdot v_0 = 0$. We can write it in the form $\alpha = x+y$ $(x\in P'^{*}$, $y\in Q(G)^{*})$. Here x is not orthogonal to P'_0. $2 = \alpha^2 = x^2+y^2$. Since both of P' and $Q(G)$ are positive definite, we have $0\leq y^2\leq 2$. We divide the case into four cases.

<i> $1<y^2\leq 2$.

In this case $0 \leq x^2 < 1$ and $x \bmod P' \in P'^*/P'$ is of the first kind. By Proposition 2.9(2) x is orthogonal to P'_0, which contradicts the choice of α.

<ii> $y^2 = 1$.

The element $\pi(\alpha)$ belongs to the primitive hull of $Q(G)$ in Λ/P. On the other hand, since $\pi(\alpha)^2 = y^2 = 1$, $\pi(\alpha)$ is a short root , which contradicts the assumption.

<iii> $0 < y^2 < 1$.

Since G has no vertex corresponding to a short root, $Q(G) \subset \pi(F)$ and we have $Q(G)^* \supset \widetilde{Q(G)}$ in Λ/P. By Lemma 2.7 $k = y^2/(1-y^2)$ is a positive integer and G has a component G_0 of type A_k such that $\widetilde{Q(G_0)} = \mathbb{Z}\pi(\alpha) + Q(G_0) = Q(G_0)^*$. By Lemma 3.8 G_0 is an obstruction component with respect to k. On the other hand, $x^2 = 2-y^2 = 2-(k/(1+k)) = 1+(1/(1+k))$. The element $x \bmod P' \in P'^*/P'$ is of the third kind with the associated number k, and k is an associated number of P. By assumption $k \leq 3$.

If $k = 3$, then the second fundamental weight $\omega_2 \in Q(G_0)^* = \widetilde{Q(G_0)}$ satisfies $\omega_2^2 = 1$ and it is a short root in the primitive hull. It contradicts the assumption.

If $k = 2$, then $\pi(\alpha)^2 = y^2 = 2/3$ and $\pi(\alpha)$ is a short root with length $\sqrt{2/3}$ in the primitive hull of $Q(G)$, which contradicts the assumption.

If $k = 1$, then $\pi(\alpha)^2 = 1/2$ and $\pi(\alpha)$ is a short root with length $1/\sqrt{2}$ with $\pi(\alpha) \in \widetilde{Q(G)}$, which contradicts the assumption, too.

<iv> $y^2 = 0$

We have $y = 0$ and $\alpha \in P'^*$. By the property (G'), one has $\alpha \in P'_0$. Thus the conclusion of the condition (b) holds.

(4) Under the condition (a), if $\eta \in P(Q(G), \Lambda/P)$ for a long root $\eta \in \Lambda/P$, then $\eta \in Q(G)$.

In the case of $U_{1,0}$ by the definition of a long root we have $\eta \in \pi(F) \subset \Lambda/P$. For the other 5 cases we have $\eta \in \pi(F)$ by Propositon 3.6(2). Thus, anyway, there is an element $\tilde{\eta} \in F \subset \Lambda$ with $\pi(\tilde{\eta}) = \eta$. This $\tilde{\eta}$ is orthogonal to P and it is contained in the primitive hull of Q(G) in Λ. By the condition (a) one has $\tilde{\eta} \in Q(G)$, and thus
$\eta = \pi(\tilde{\eta}) \in \pi(Q(G)) = Q(G)$.

(5) If the condition $\eta \in P(Q(G), \Lambda/P)$ for a long root $\eta \in \Lambda/P$ implies $\eta \in Q(G)$, then the condition (a) holds.

Let α be an element in $P(P \oplus Q(G), \Lambda)$ with $\alpha^2 = 2$ such that it is orthogonal to P. Then, $\pi(\alpha) \in \pi(F)$, $\pi(\alpha)^2 = 2$ and $\pi(\alpha) \in P(Q(G), \Lambda/P)$. Thus $\pi(\alpha)$ is a long root in Λ/P. By the assumption one has $\pi(\alpha) \in Q(G)$. On the other hand, since $\alpha \in F$ and $P(Q(G), \Lambda) \subset F$ and since $\pi|F$ is injective, one has $\alpha \in Q(G)$. Thus (a) holds.

Theorem 3.9 has been shown by the above.

Q.E.D.

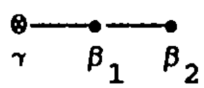
§4. Elementary transformations and tie transformations

By Λ_N we denote the even unimodular lattice with signature $(16+N, N)$ for $N \geq 1$. It is unique up to isomorphisms, and isomorphic to $Q(2E_8) \oplus H^N$ and also to $\Gamma_{16} \oplus H^N$. Here Γ_{16} is the even overlattice with index 2 over the root lattice $Q(D_{16})$ of type D_{16}.

We have defined the concept of elementary transformations for finite root systems and Dynkin graphs in Urabe [13]. The concept of root modules in [13] is more restricted than the concept of root modules in this article. Besides even the concept of Dynkin graphs is slightly different from that in this article.

In spite of such difference, the same definition of elementary transformations as before is effective even in our present situation.

Here we give an example of a non-reduced root system and explain it. Consider a Dynkin graph of type CB_3 and the corresponding root basis $\Delta = \{\gamma, \beta_1, \beta_2\}$.



By definition $\gamma^2 = 1/2$, $\beta_1^2 = \beta_2^2 = 1$. The maximal root η is equal to $2\gamma + 2\beta_1 + 2\beta_2$, which is a long root. Setting $\alpha = -\eta$, we have the extended root basis $\Delta^+ = \{\gamma, \beta_1, \beta_2, \alpha\}$. The irreducible component containing γ of the root system generated by a proper subset of Δ^+ is non-reduced and of type

CB. Note that on the contrary the irreducible component containing α of the root system generated by a proper subset of Δ^+ is reduced and it is of type A_1, B_2 or F_3.

Proposition 4.1. Let (L,FL) be a positive definite root module and M be a submodule of L.

(1) If M is primitive in L, then any root basis of the root system of $(M,M\cap FL)$ can be extended to a root basis of the root system of (L,FL).

(2) If the torsion group of the quotient L/M is cyclic, then the root system of $(M,M\cap FL)$ is obtained from that of (L,FL) by one elementary tranformation.

Proof (1) The Weyl group $W(M,M\cap FL) = W(M)$ of M acts transitivity on the set of all root bases of M, and the action of $W(M)$ on M can be extended naturally to L. Thus it suffices to show that there is a root basis Δ_L for L such that Δ_L contains a root basis Δ_M for M.

By assumption we have a linear mapping $\xi : L \longrightarrow \mathbb{R}$ such that the kernel $\xi^{-1}(0)$ coincides with M. Regarding ξ as an element in $L^* \otimes \mathbb{R}$, we consider the action of the Weyl groups $W(M)$ and $W(L) = W(L,FL)$. Let $C \subset L^* \otimes \mathbb{R}$ be a Weyl chamber for $W(L)$ such that the closure $\overline{C}$ contains ξ. Let Δ_L be a root basis for L corresponding to C. Set $\Delta' = \{\alpha\in\Delta_L | <\alpha,\xi> = 0\} = \{\alpha\in\Delta_L |$ The hyperplane orthogonal to α passes through $\xi.\}$. Let W' be the subgroup of $W(L)$ generated by reflections corresponding to roots in Δ'. If

$\alpha\in\Delta'$, then $\xi(\alpha) = 0$ and thus $\alpha\in M$. One knows $W' \subset W(M)$. Let $I(W(L),\xi) = \{g\in W(L) \mid g(\xi) = \xi\}$ be the isotropic subgroup with respect to ξ. Since for $\alpha \in R(M,M\cap FL)$
$s_\alpha(\xi) = \xi-2<\alpha,\xi>\alpha/\alpha^2 = \xi-2\xi(\alpha)\alpha/\alpha^2 = \xi$, one has
$W(M) \subset I(W(L),\xi)$. Now, by Bourbaki [3] Ch. 5 §3 n°3 Prop. 2, $W' = W(M) = I(W(L),\xi)$. This implies that the Weyl chamber for W' is the Weyl chamber for $W(M)$, and Δ' is a root basis for M.

(2) First of all, note that Lemma 3.3 does not hold for a short root with length $1/\sqrt{2}$.

By assumption we have a linear morphism $\xi : L \longrightarrow \mathbb{R}$ with $\xi^{-1}(\mathbb{Z}) = M$.

Let $\Delta_M \subset R(M,M \cap FL)$ be a root basis for M. If Δ_M contains a short root β with length $1/\sqrt{2}$, then replacing β with 2β we can make the set $\hat{\Delta}_M$. This $\hat{\Delta}_M$ is a root basis of the reduced root system $\hat{R}(M,M\cap FL)$. By Lemma 3.3 and by results in Urabe [13] (Prop. 2.5, Cor. 2.6, Prop. 2.9(4), Lemma 2.10 in [13]) there is a root basis $\hat{\Delta}_L$ of the reduced root system $\hat{R}(L,FL)$ such that $\hat{\Delta}_M \subset \hat{\Delta}_L^+$ for the extended root basis $\hat{\Delta}_L^+$, that is, $\hat{\Delta}_M$ can be obtained from $\hat{\Delta}_L$ by an elementary transformation.

First we consider the case where Δ_M contains no short root with length $1/\sqrt{2}$. Then, $\Delta_M = \hat{\Delta}_M$. If $\hat{\Delta}_L$ does not contain a divisible root, then $\hat{\Delta}_L$ is a root basis for

$R(L,FL) = \hat{R}(L,FL)$ and we have the claim. Thus in the following we consider the case where $\hat{\Delta}_L$ contains a divisible root. By Proposition 3.4 (2) we have a unique component $\hat{\Delta}_1$ of $\hat{\Delta}_L$ containing a divisible root. We can write $\hat{\Delta}_1^+$ in the form $\hat{\Delta}_1^+ = \{2\beta_1,\gamma_1,\ldots,\gamma_{k-1},2\beta_2\}$ with $\beta_1^2 = \beta_2^2 = 1/2$, $\gamma_1^2 = \ldots = \gamma_{k-1}^2 = 1$.

Here assume that $2\beta_1 \in \Delta_M$ and $2\beta_2 \in \Delta_M$. We will deduce a contradiction. Now, the graph of $\hat{\Delta}_1^+$ is the extended Dynkin graph of type A_1, B_2, F_3 or C_k $(k \geq 4)$. The vertices corresponding to $2\beta_1$ and $2\beta_2$ are at the both ends. The subgraph G_1 consisting of vertices corresponding to $\Delta_M \cap \hat{\Delta}_1^+$ does not contain at least one vertex corresponding to a short root under our assumption. Thus $2\beta_1$ and $2\beta_2$ belong to different connected components of G_1. This implies that any root of M is orthogonal to either $2\beta_1$ or $2\beta_2$.

Now, on the other hand, since any two elements in Δ_M are linearly independent, the rank k of $\hat{\Delta}_1$ satisfies $k \geq 2$. Under the assumption $\beta_1, \beta_2 \in L$ and $2\beta_1, 2\beta_2 \in M$. Besides, since Δ_M contains no short root with length $1/\sqrt{2}$, $\beta_1 \notin M$ and $\beta_2 \notin M$. The torsion part of L/M is cyclic and L/M has only one element of order 2. Thus $\beta_1 - \beta_2 \in M$. On the other hand, since $\beta_1+\gamma_1+\ldots+\gamma_{k-1}+\beta_2 = 0$, one knows $\gamma_1+\ldots+\gamma_{k-1}+2\beta_1 \in M$ and $\gamma = \gamma_1+\gamma_2+\ldots+\gamma_{k-1} \in M$. This γ is a short root with length 1, and satisfies $2\beta_1 \cdot \gamma \neq 0$ and $2\beta_2 \cdot \gamma \neq 0$. One has a contradiction.

Therefore either $2\beta_1 \notin \Delta_M$ or $2\beta_2 \notin \Delta_M$ holds. If $2\beta_1 \notin \Delta_M$, set $\Delta_1 = \{\beta_1, \gamma_1, \ldots, \gamma_{k-1}\}$. If $2\beta_2 \notin \Delta_M$, set $\Delta_1 = \{\gamma_1, \ldots, \gamma_{k-1}, \beta_2\}$. Since $\hat{\Delta}_L - \hat{\Delta}_1$ contains no divisible root, $\Delta_L = (\hat{\Delta}_L - \hat{\Delta}_1) \cup \Delta_1$ is a root basis for L and it satisfies $\Delta_M \subset \Delta_L^+$.

Next, we consider the case where Δ_M contains a short root β with length $1/\sqrt{2}$. In this case $\hat{\Delta}_M$, $\hat{\Delta}_L^+$ and $\hat{\Delta}_L$ has a unique component containing a divisible root. Let $\hat{\Delta}_1^+$ be the component of $\hat{\Delta}_L^+$ containing a divisible root. We can write it in the form $\hat{\Delta}_1^+ = \{2\beta, \gamma_1, \ldots, \gamma_{k-1}, 2\beta'\}$, $\beta^2 = \beta'^2 = 1/2$, $\gamma_1^2 = \ldots = \gamma_{k-1}^2 = 1$. Set $\Delta_1 = \{\beta, \gamma_1, \ldots, \gamma_{k-1}\}$ and $\Delta_L = (\hat{\Delta}_L - \hat{\Delta}_1) \cup \Delta_1$. This Δ_L is a root basis of $R(L, FL)$ and it satisfies $\Delta_M \subset \Delta_L^+$.

Q.E.D.

Let $F \subset \Lambda_N/P$ denote the same module as in Proposition 3.6.

<u>Lemma 4.2.</u> (1) For any primitive isotropic element u in Λ_{N+1}/P belonging to F, there exists another isotropic element v in F with $u \cdot v = 1$.

(2) Set $H = \mathbb{Z}u + \mathbb{Z}v$ and $J = C(H, \Lambda_{N+1}/P)$. One has the decomposition $\Lambda_{N+1}/P \cong J \oplus H$. Besides, there is a primitive embedding $P \subset \Lambda_N$ with $\Lambda_N/P \cong J$.

(3) For the former 5 kinds of quadrilateral singularities except $U_{1,0}$, any isotropic element in Λ_{N+1}/P belongs to F.

Proof. (1) By primitiveness we have a homomorphism $f : \Lambda_{N+1}/P \longrightarrow \mathbf{Z}$ with $f(u) = 1$. By Proposition 3.6 (1) we have an element $v' \in F$ with $f(x) = x \cdot v'$ for $x \in \Lambda_{N+1}/P$. In particular $u \cdot v' = 1$. Since F is an even lattice, $v'^2 = 2m$ for some integer m. Setting $v = v' - mu$, we have $v^2 = 0$ and $u \cdot v = 1$.

(2) Since $H \subset F$, $x \cdot u$ and $x \cdot v$ are integers for all $x \in \Lambda_{N+1}/P$ by Proposition 3.6 (1). Thus we can define an isomorphism $\varphi : \Lambda_{N+1}/P \longrightarrow J \oplus H$ by

$$\varphi(x) = (x-(x \cdot v)u-(x \cdot u)v,\ (x \cdot v)u+(x \cdot u)v) \in J \oplus H.$$

Now, let $\tilde{u}$ and $\tilde{v}$ be elements in the orthogonal complement of P in Λ_{N+1} such that their images by the morphism $\Lambda_{N+1} \longrightarrow \Lambda_{N+1}/P$ coincide with u and v respectively. By Lemma 3.5 (2) we have $\tilde{u}^2 = \tilde{v}^2 = 0$, $\tilde{u} \cdot \tilde{v} = 1$. Setting $\tilde{H} = \mathbf{Z}\tilde{u} + \mathbf{Z}\tilde{v}$, $K = C(\tilde{H}, \Lambda_{N+1})$, one knows that K is an even unimodular lattice with signature $(16+N, N)$. The existence of u implies $N \geq 1$. Thus we have an isomorphism $K \cong \Lambda_N$, since indefinite even unimodular lattice is uniquely determined by the signature. The composition $P \subset K \cong \Lambda_N$ is a primitive embedding such that $\Lambda_N/P \cong J$.

(3) It follows from Proposition 3.6(1).

Q.E.D.

By Proposition 4.1 and Lemma 4.2 one sees that the theory of elementary transformations in Urabe [13] is effective even in our general situation.

Theorem 4.3. Assume $N \geq 1$ (1) Assume that for a given primitive embedding $P \subset \Lambda_{N+1}$, and for a positive definite full root submodule $L \subset \Lambda_{N+1}/P$, the orthogonal complement of L contains a primitive isotropic element belonging to F. Then for some primitive embedding $P \subset \Lambda_N$, there is a positive definite root submodule $M_0 \subset \Lambda_N/P$ with the following property. The property: rank M_0 = rank $R(L)$ and for every positive definite full root submodule M with $M_0 \subset M \subset \Lambda_N/P$, the root system of L is obtained from that of M by one elementary transformation. In particular, the Dynkin graph of L is obtained from that of M by one elementary transformation.
(2) Conversely, let $P \subset \Lambda_N$ be a primitive embedding and $M \subset \Lambda_N/P$ be a positive definite full submodule. Let R' be a root system obtained from the root system $R(M)$ of M by one elementary transformation. Then, there is a full embedding $Q(R') \subset \Lambda_{N+1}/P$ of the root quasi-lattice such that the orthogonal complement of the image contains a primitive isotropic element belonging to F.

Let us proceed to the theory of tie transformations. The key parts in the theory of tie transformations in Urabe [14] are the theory of elementary transformations and Fact 1.6 in Urabe [14] section 1. It is easy to check that Fact 1.6 is essentially true even under the general definition of root modules in this article. However, of course, we have to replace the expression

$$\left(\sum_{\alpha\in\Delta^+} m_\alpha \alpha\right)^2 = 1 \quad \text{or} \quad 2$$

in the statement of Fact 1.6 by

$$\left(\sum_{\alpha\in\Delta^+} m_\alpha \alpha\right)^2 = \min\langle \beta^2 \,|\, \beta \in \Delta^+\rangle .$$

We can check that $\gamma \in F$ if $\gamma^2 = 2$ in the proof of Proposition 1.5 in [14].

<u>Theorem 4.4.</u> Assume $N \geq 1$ (1) Assume that a primitive embedding $P \subset \Lambda_{N+1}$ is given. Let $L \subset \Lambda_{N+1}/P$ be a positive definite full submodule satisfying the following conditon <*>.

<*> [For some root basis $\Delta \subset R(L)$, for some long root $\alpha \in \Delta$ and for some isotropic element u belonging to F, $u \cdot \alpha = 1$ and $u \cdot \beta = 0$ for every $\beta \in \Delta$ with $\beta \neq \alpha$.

Then, there are a primitive embedding $P \subset \Lambda_N$ and a positive definite root submodule $M_0 \subset \Lambda_N/P$ with the following property. The property: rank M_0 = rank $R(L) - 1$ and for every positive definite full root submodule M with $M_0 \subset M \subset \Lambda_N/P$, the Dynkin graph of L is obtained from the Dynkin graph of M by one tie transformation.

(2) Conversely, for a primitive embedding $P \subset \Lambda_N$ and for a positive definite full root submodule $M \subset \Lambda_N/P$, if a Dynkin graph G' can be obtained from the Dynkin graph of M by one tie transformation, then there are a primitive embedding $P \subset \Lambda_{N+1}$ and a full embedding $Q(G') \subset \Lambda_{N+1}/P$ such that the above condition <*> is satisfied for $L = Q(G')$.

Obstruction components are a concept which we cannot find in my previous articles [13] and [14]. We would like to show that they behave like the description in Definition 0.4.

In the following we assume that $k \geq 4$ and $H = \mathbf{Z}u+\mathbf{Z}v$ ($u^2 = v^2 = 0$, $u \cdot v = 1$) is a hyperbolic plane.

We first consider the behavior under elementary transformations.

Let G' be a Dynkin graph. First, we consider the case where a full embedding

$$Q(G') \subset (\Lambda_N/P) \oplus H$$

into the orthogonal complement of u is given. Besides, we assume that G' has an obstruction component G'_1 of type A_k. By definition

$$[P(Q(G'_1), (\Lambda_N/P) \oplus H) : Q(G'_1)] = k+1.$$

By $p : (\Lambda_N/P) \oplus H \longrightarrow \Lambda_N/P$ we denote the projection to Λ_N/P. Let $M \subset \Lambda_N/P$ be a positive definite full root submodule containing the image $p(Q(G'))$, and $Q(M)$ be the root quasi-lattice of M.

<u>Lemma 4.5.</u> The component Q_1 of $Q(M)$ containing the image $p(Q(G_1'))$ is also of type A_k and $[P(Q_1,\Lambda_N/P) : Q_1] = k+1$.

<u>Proof</u>. Let $\Delta' \subset Q(G')$ be a root basis, and $\Delta' = \bigcup_{i=1}^{m'} \Delta_i'$ be the decomposition into irreducible components. We assume that the component G_1' corresponds to Δ_1'. Set $\Delta_1' = \{\alpha_1',\alpha_2',\ldots,\alpha_k'\}$. We assign numbers to elements α_i' in Δ_1' from the end of the Dynkin graph in order. Let

$$\omega_1' = \{k\alpha_1'+(k-1)\alpha_2'+\ldots+2\alpha_{k-1}'+\alpha_k'\}/(k+1)$$

be the first fundamental weight. By definition $\omega_1'\cdot\alpha_1' = 1$ and $\omega_1'\cdot\alpha_i' = 0$ for $2 \leq i \leq k$. By assumption $\omega_1' \in P(Q(G_1'), (\Lambda_N/P) \oplus H)$.

Set $\alpha_i = p(\alpha_i')$ and $\omega_1 = p(\omega_1')$. Let $\Delta \subset Q(M)$ be a root basis and $\Delta = \bigcup_{i=1}^{m} \Delta_i$ be the irreducible decomposition. By the theory of elementary transformations we can assume that $p(\Delta')$ is a subset of the extended root basis $\Delta^+ = \bigcup_{i=1}^{m} \Delta_i^+$. Let Δ_1^+ be the component containing $p(\Delta_1')$. One has $\{\alpha_1,\ldots,\alpha_k\} \subset \Delta_1^+$.

Assume that Δ_1^+ is not of type A_k. Since $k \geq 4$, by the classification of finite irreducible root systems with a long root (Proposition 3.4 (1)), one knows that there exist number i with $1 \leq i \leq k$ and a long root $\beta \in \Lambda_1^+$ such that $\beta \cdot \alpha_i = -1$ and $\beta \cdot \alpha_j = 0$ for $j \neq i$, $1 \leq j \leq k$. Then, we have

$$\omega_1 \cdot \beta = -(k+1-i)/(k+1) \notin \mathbb{Z}.$$

This contradicts Proposition 3.6 (1), since $\beta \in F$. Thus Λ_1^+ is of type A_k and the index is $k+1$, since $\omega_1 \in P(Q(G_1), \Lambda_N/P)$. Q.E.D.

Next, we consider the situation when we go up from Λ_N/P to Λ_{N+1}/P.

Let R be a finite root system and $R = \bigoplus_{i=1}^{m} R_i$ be the irreducible decomposition. We assume that R_1 is of type A_k with $k \geq 4$. Assume that a full embedding $Q(G) \subset \Lambda_N/P$ such that $[\tilde{Q}_1 : Q_1] = k+1$ for $Q_1 = Q(R_1)$ and $\tilde{Q}_1 = P(Q_1, \Lambda_N/P)$ is given. Let R' ($\subset R$) be a root system obtained from R by one elementary transformation. We assume moreover that $R' \cap R_1 = R_1$.

<u>Lemma 4.6.</u> Under the above situation there exists a full embedding

$$\varphi : Q(R') \subset (\Lambda_N/P) \oplus H$$

satisfying the following conditions (1), (2) and (3). Besides, there also exists a full embedding satisfying (1), (2) and (4). Here we denote $Q_1' = \varphi(Q_1)$ and $\tilde{Q}_1' = P(Q_1', (\Lambda_N/P) \oplus H)$.

(1) The image Q_1' of φ is orthogonal to u.

(2) The composition of φ and the projection $(\Lambda_N/P)\oplus H \longrightarrow \Lambda_N/P$ coincide with the given embedding $Q(R') \subset Q(R) \subset \Lambda_N/P$.

(3) $[\tilde{Q}_1' : Q_1'] < k+1$

(4) $[\tilde{Q}_1' : Q_1'] = k+1$.

<u>Proof.</u> Let $\Delta \subset R$ be a root basis, $\Delta = \bigcup_{i=1}^{m} \Delta_i$ be the irreducible decomposition, and $\Delta^+ = \bigcup_{i=1}^{m} \Delta_i^+$ be the extended root basis. If we denote the maximal root for Δ_i by η_i, then $\Delta_i^+ = \Delta_i \cup \{-\eta_i\}$.

For every i with $1 \leq i \leq m$ we have a proper subset $\Delta_i' \subset \Delta_i^+$ and R' is the root system generated by $\bigcup_{i=1}^{m} \Delta_i'$. Now, Δ_1 is of type A_k and Δ_1^+ consists of k+1 elements. By assumption Δ_1' is also of type A_k and Δ_1' is a subset of Δ_1^+ consisting of k elements. we have k+1 ways of choosing Δ_1', and under any choice the basis Δ_1' generates the same root system $R' \cap R_1$.

In order to define the embedding φ satisfying (3), we choose Δ_1' in such a way that $-\eta_1 \in \Delta_1'$. To define φ satisfying (4) we choose Δ'_1 with $-\eta_1 \notin \Delta'_1$. $\Delta' = \bigcup_{i=1}^{m} \Delta'_i$ is a free basis of $Q(R')$. We define the embedding φ by setting for $\alpha \in \Delta'_i$

$$\varphi(\alpha) = \begin{cases} \alpha\oplus 0 & (\text{if } \alpha\neq -\eta_i) \\ \alpha\oplus u & (\text{if } \alpha = -\eta_i). \end{cases}$$

Obviously it defines an embedding of quasi-lattices satisfying (1) and (2). Besides, fullness follows from Proposition 4.2 in Urabe [13].

We show the condition (3). Set $\Delta'_1 = \{\alpha'_1, \alpha'_2, \ldots, \alpha'_k\}$. We assume that the numbers are assigned from the end of the Dynkin graph in order. If $-\eta_1 \in \Delta'_1$, then there is a number j $(1\leq j\leq k)$ with $\alpha'_j = -\eta_1$. Now, if $[\tilde{Q}'_1 : Q'_1] \geq k+1$, then for $\omega_1 = \{k\alpha'_1+(k-1)\alpha'_2+\ldots+\alpha'_k\}/(k+1)$ we have $\varphi(\omega_1) \in (\Lambda_N/P) \oplus H$. However, $\varphi(\omega_1) = \omega_1\oplus(k+1-j)u/(k+1)$. Thus $\varphi(\omega_1)\cdot v = (k+1-j)/(k+1) \notin \mathbf{Z}$, which is a contradiction. We have (3).

When $-\eta_1 \notin \Delta'_1$, we have $\varphi(\omega_1) = \omega_1\oplus 0$, $\varphi(\omega_1) \in \tilde{Q}'_1$, and we have (4).

Q.E.D.

By Lemma 4.5 and 4.6, one knows that obstruction components behave like the rule in Definition 0.4 under elementary transformation.

In the following we consider obstruction components and tie transformations.

Let G' be a Dynkin graph and $Q(G') \subset \Lambda_{N+1}/P$ be a full embedding. We assume the following assumptions (O1) and (O2)

(O1) G' has a component G'_1 of type A_k with $k \geq 4$ such that

$$[P(Q(G'_1), \Lambda_{N+1}/P) : Q(G'_1)] = k+1.$$

(O2) For some root basis $\Delta' \subset Q(G')$, for some long root $\alpha \in \Delta'$ and for an isotropic element u belonging to F, $\alpha \cdot u = 1$ and $\beta \cdot u = 0$ for every $\beta \in \Delta$ with $\beta \neq \alpha$.

Now, set $v = u - \alpha$. We have $u^2 = v^2 = 0$ and $u \cdot v = 1$. Setting $H = \mathbf{Z}u + \mathbf{Z}v$, $J = C(H, \Lambda_{N+1}/P)$, one has $\Lambda_{N+1}/P = J \oplus H$. Let $p: J \oplus H \longrightarrow J$ denote the projection. Let $\Delta' = \bigcup_{i=1}^{m'} \Delta'_i$ be the decomposition into irreducible components. We assume that the component G'_1 corresponds to Δ'_1. By T we denote the submodule of $Q(G')$ generated by $\Delta' - \{\alpha\}$.

Let M be a positive definite full root submodule of $J \cong \Lambda_N/P$ containing $p(T)$, and G be the Dynkin graph of M. A root basis Δ of M is decomposed $\Delta = \bigcup_{i=1}^{m} \Delta_i$ into irreducible components. By $\Delta^+ = \bigcup_{i=1}^{m} \Delta_i^+$ we denote the extended root basis. By the theory of elementary transformations, we can assume that $p(\Delta' - \{\alpha\}) \subset \Delta^+$.

<u>Lemma 4.7</u> (1) The element u is necessarily orthogonal to $Q(G_1')$. In particular, $\Delta_1' \subset T$.

(2) Let Δ_1^+ be the component of Δ^+ containing $p(\Delta_1')$. Δ_1 is also of type A_k and for some unique element $\gamma \in \Delta_1^+$,

$$p(\Delta_1') = \Delta_1^+ - \{\gamma\}.$$

<u>Proof.</u> (1) Set $\Delta_1' = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$. We assign numbers of γ_i's from the end of the Dynkin graph in order. Let

$$\omega_1 = (k\gamma_1 + (k-1)\gamma_2 + \ldots + \gamma_k)/(k+1)$$

be the first fundamental weight. $\omega_1 \cdot \gamma_1 = 1$ and $\omega_1 \cdot \gamma_i = 0$ for $2 \leq i \leq k$. By (01) $\omega_1 \in \widetilde{Q(G_1')} = Q(G_1')^*$. Assume that u is <u>not</u> orthogonal to $Q(G_1')$. The element α in (02) belongs to Δ_1'. We have a number j with $\alpha = \gamma_j$, $1 \leq j \leq k$. However,

$$u \cdot \omega_1 = (k+1-j)/(k+1) \notin \mathbf{Z},$$

which is a contradiction.

(2) First, note that $\beta \cdot x \in \mathbf{Z}$ for every long root $\beta \in J$ and for every element $x \in J$ by Proposition 3.6.

Assume that Δ_1 is <u>not</u> of type A_k. By the classification of root systems, one has a long root $\beta \in \Delta_1^+$ and a root $\gamma_i \in \Delta_1'$ $(1 \leq i \leq k)$ such that $\beta \cdot p(\gamma_i) = -1$ and $\beta \cdot p(\gamma_j) = 0$ for $j \neq i$, $1 \leq j \leq k$. By (1) $p(\omega_1) \in J$. However,

$$\beta\cdot p(\omega_1) = -(k+1-i)/(k+1) \notin \mathbb{Z},$$

which is a contradiction. The latter half is obvious, since Δ_1^+ has k+1 elements.

Q.E.D.

Lastly, we consider the case when we go up from Λ_N to Λ_{N+1} by a tie transformation.

Let G be a Dynkin graph and $Q(G) \subset \Lambda_N/P$ be a full embedding. By Δ we denote a root basis of $Q(G)$ and Δ^+ is the extended root basis.

However, this time, we assume that G' is a Dynkin graph obtained from G by one tie transformation and $Q(G') \subset (\Lambda_N/P) \oplus H$ be the full embedding obtained by the transformation (Urabe [14]).

Corresponding to the procedure of the tie transformation, we have subsets $A, B \subset \Delta^+$ with $A \cap B = \phi$ which satisfy the condition on G.C.D. with respect to coefficients of maximal roots. We have

$$Q(G') = \sum_{\alpha\in\Delta^+-(A\cup B)} \mathbb{Z}\alpha + \sum_{\alpha\in B} \mathbb{Z}(\alpha-u)+\mathbb{Z}(u+v),$$

and $\Delta' = [\Delta^+-(A\cup B)] \cup \{\alpha-u|\alpha\in B\} \cup \{u+v\}$ is a root basis for $Q(G')$.

Here assume moreover that G has an obstruction component G_1 of type A_k in Λ_N/P. We assume that the components Δ_1 and Δ_1^+ correspond to G_1.

Lemma 4.8. The following (1) and (2) are equivalent.

(1) In $(\Lambda_N/P) \oplus H$, G' has an obstruction component G'_1 of type A_k such that it contains a vertex corresponding to a root in $\Delta_1^+ - A$.

(2) $\Delta_1^+ \cap B = \phi$ and $\Delta_1^+ \cap A$ consists of a unique element.

Proof. (1) ⇒ (2). It follows from Lemma 4.7. (2) ⇒ (1). Under the assumption $\Delta'_1 = \Delta_1^+ - A$ is a root basis of type A_k and is an irreducible component of Δ'. Let G'_1 be the component of G' corresponding to Δ'_1. Then we have

$$Q(G'_1) = \sum_{\alpha\in\Delta'_1} \mathbb{Z}\alpha = \sum_{\beta\in\Delta_1} \mathbb{Z}\beta .$$

This implies that the embedding $Q(G'_1) \subset (\Lambda_N/P) \oplus H$ coincides with the composition of the identification $Q(G'_1) = Q(G_1)$, the given embedding $Q(G_1) \subset \Lambda_N/P$ and the embedding into the direct summand $\Lambda_N/P \subset (\Lambda_N/P) \oplus H$. Thus, when we compute the index $[P(Q(G'_1), (\Lambda_N/P) \oplus H) : Q(G'_1)]$, erasing out the prime symbols and $\oplus H$ from the expression, we have the same number, which is equal to $k+1$ by assumption.

Q.E.D.

By Lemma 4.7 and 4.8 one knows that obstruction components behave like the rule in Definiton 0.4 under a tie transformation.

§5. Coxeter-Vinberg graphs

First we give a definition. An isotropic element $u \in \Lambda_{N+1}/P$ is said to be in a nice position with respect to a positive definite root submodule $L \subset \Lambda_{N+1}/P$, if either u is orthogonal to all roots in L, or for some root basis $\Delta_L (\subset R(L) \subset L)$ of L and for some long root $\alpha \in \Delta_L$, $\alpha \cdot u = 1$ and $\beta \cdot u = 0$ for every $\beta \in \Delta_L$ with $\beta \neq \alpha$.

Now, if we have an isotropic element in a nice position, and if it is primitive and belongs to F, we can reduce the problem from Λ_{N+1}/P to Λ_N/P by elementary or tie transformations (Theorem 4.3, Theorem 4.4). Therefore existence of such elements comes into question.

In this section we explain an effective method for the reduction from Λ_2/P to Λ_1/P. The main tool in this method is the Coxeter-Vinberg graph (Vinberg [15], [16], [17], [18], Conway-Sloane [5]). It is closely related to the geometry on the hyperbolic space.

Let (L, FL) be a root module with $\ell = \text{rank } L$. We assume that the bilinear form on L is non-degenerate and it has signature $(\ell-1, 1)$. The negative cone $\Sigma_L \subset L \otimes \mathbb{R}$ of L is defined to be

$$\Sigma_L = \{x \in L \otimes \mathbb{R} \mid x^2 < 0\}.$$

The cone Σ_L has two connected components by the assumption on the signature. Choosing one of two and fixing it, we denote it by Σ_+. The other component is $\Sigma_- = -\Sigma_+$. The quotient

$\Sigma_+/\mathbb{R}_+$ by the multiplicative group $\mathbb{R}_+$ of positive real numbers can be regarded as a Lobačevskiĭ space of dimension $\ell-1$. The Weyl group $W = W(L,FL)$ acts properly discontinously on $\Sigma_\pm$ and $\Sigma_+/\mathbb{R}_+$. Let $R = R(L,FL)$ be the root system. We denote the hyperplane in $L \otimes \mathbb{R}$ orthogonal to a root $\alpha \in R$ by $H_\alpha = \{x \in L \otimes \mathbb{R} | (x,\alpha) = 0\}$. A connected component of $\Sigma_+ - \bigcup_{\alpha\in R} H_\alpha$ is called a <u>fundamental polyhedron</u> of W or a <u>Weyl chamber</u> of W. The Weyl group W acts transitively on the set of all fundamental polyhedrons. Choose and fix one fundamental polyhedron C. By $\bar{C}$ we denote the closure of C. Corresponding to the walls of C, we choose a set $\Delta \subset R$ of indivisible roots as follows. (Note that $H_\alpha = H_{-\alpha}$.)

$$\Delta = \{\alpha\in R | \alpha \text{ is indivisible, } H_\alpha \cap \bar{C} \text{ contains an open set of } H_\alpha, \ \alpha \text{ is directed outwards from } C.\}$$

We call an element in Δ a <u>fundamental root</u>. This set Δ is defined depending on C. However, if we choose Δ' depending on another polyhedron C', Δ' and Δ are conjugate with respect to the Weyl group W.

We can draw a graph from Δ following the rules below which are similar to those for Dynkin graphs. The resulting graph is called the <u>Coxeter-Vinberg graph</u> of the root module (L,FL). Indeed, it is defined by (L,FL) and does not depend on the choice of the fundamental polyhedron.

(1) The vertices in the graph have one-to-one correspondence with elements in Δ. A vertex has one of four different expressions depending on the length of the corresponding root as follows.

Length:	$\sqrt{2}$	1	$\sqrt{2/3}$	$1/\sqrt{2}$
Expressions:	○	●	◎	⊗ .

(2) If two roots $\alpha, \beta \in \Delta$ are orthogonal $\alpha \cdot \beta = 0$, then we do not connect the two vertices corresponding to α and β.
(3) If two roots $\alpha, \beta \in \Delta$ are not orthogonal and if the quasi-lattice $\mathbf{Z}\alpha+\mathbf{Z}\beta$ generated by them is positive-definite, then we connect the two vertices corresponding to α and β by a single segment ———.
(4) If the quasi-lattice $\mathbf{Z}\alpha+\mathbf{Z}\beta$ generated by two roots $\alpha, \beta \in \Delta$ is degnerate, then we connect the corresponding two vertices by a bold segment ▬▬.
(5) If the quasi-lattice $\mathbf{Z}\alpha+\mathbf{Z}\beta$ generated by two roots $\alpha, \beta \in \mathbf{Z}$ is non-degenerate and indefinite, then we connect the corresponding two vertices by a dotted segment ······ . Besides, if necessary, we add the intersection number $\alpha \cdot \beta$ to the dotted segment.

As a practical method to construct the set of fundamental roots, we have an algorithm due to Vinberg [16].

To carry out Vinberg's algorithm, at the first step, we choose and fix a vector $v_0 \in \Sigma_+$ called the <u>controlling</u>

vector in the chosen component of the negative cone. Let L_0 be the set of vectors in L orthogonal to v_0. L_0 is a positive definite root module. Let $e_1, e_2, \ldots, e_k$ be a root basis for the root system of L_0.

At the second step we choose $e_\ell \in L$ for an integer $\ell > k$ inductively. Assume that $e_1, \ldots, e_{\ell-1}$ have been chosen. Set

$$R_\ell = \{\alpha \in R \mid (\alpha, e_i) < 0 \quad (1 \leq i < \ell), \quad (\alpha, v_0) \neq 0\}.$$

If R_ℓ is empty, set $\Delta_V = \{e_1, \ldots, e_{\ell-1}\}$. If $R_\ell \neq \phi$, we define e_ℓ to be an element α in R_ℓ which attains the minimal value for $(\alpha, v_0)^2/\alpha^2$ and satisfies $(\alpha, v_0) \leq 0$.

Lastly, when $R_\ell \neq \phi$ for all $\ell > k$, we set $\Delta_V = \{e_1, \ldots, e_k, e_{k+1}, \ldots, e_\ell, \ldots\}$ (an infinite set).

Then, Δ_V coincides with the set Δ of fundamental roots associated with some fundamental polyhedron C (Vinberg [16] Prop. 4.).

Lemma 5.1. Let $M \subset L$ be a positive definite full root submodule. A root basis Δ_M ($\subset R(M) \subset M$) for M is necessarily conjugate to some subset of Δ with respect to the Weyl group $W(L)$ for L. In particular, the Dynkin graph of M is a subgraph of the Coxeter-Vinberg graph of L.

Proof. Note that the real number field $\mathbb{R}$ is a vector space of infinite dimension over the rational number field $\mathbb{Q}$. Because of this reason, we have a vector $v_0 \in \Sigma_+$ in the negative cone such that $\tilde{M} = \{x \in L \mid (x,v_0) = 0\}$. Using v_0 as the controlling vector, we carry out Vinberg's algorithm. The root basis of $\tilde{M}$ becomes a subset of the constructed set Δ_V. By fullness, the root basis of $\tilde{M}$ is a root basis of M, and Δ_V is conjugate to a given set Δ of fundamental roots over W. Thus we get the lemma.

Q.E.D.

In Vinberg's algorithm explained above a vector v_0 with $v_0^2 < 0$ is used as the controlling vector. We have another similar algorithm using an isotropic element as the controlling vector, which is also due to Vinberg ([18] section 1.4).

Let $u \in L$ be a primitive isotropic element. Set $I = \mathbb{Z}u$ and $I^\perp = \{x \in L \mid (x,u) = 0\}$. The pair $(I^\perp/I,\ (FL \cap I^\perp)+I/I)$ can be regarded as a positive definite root module. By $\rho : I^\perp \longrightarrow I^\perp/I$ we denote the canonical surjective morphism. Let $\Delta_I \subset I^\perp/I$ be a root basis for $I^\perp/I$ and $\Delta_I = \bigcup_{i=1}^{m} \Delta_i$ be the irreducible decomposition. We assume that Δ_I generates $I^\perp/I \otimes \mathbb{Q}$ over $\mathbb{Q}$. (This condition is called the compactness property. See Vinberg [18] section 1.3.) For each root $\alpha \in \Delta_I$ we choose a root $\tilde{\alpha} \in I^\perp \cap R(L,FL)$ with $\rho(\tilde{\alpha}) = \alpha$. Set $\tilde{\Delta}_i = \{\tilde{\alpha} \mid \alpha \in \Delta_i\}$ and $\tilde{\Delta}_I = \bigcup_{i=1}^{m} \tilde{\Delta}_i$. $\tilde{\Delta}_i \subset I^\perp$ is an irreducible root basis for $1 \leq i \leq m$. Here we have 2 cases.

(Case 1) The case where $\tilde{\Delta}_i$ contains a short root, the maximal root for $\tilde{\Delta}_i$ is a long root, and $u \notin FL$.

Let θ_i be the maximal <u>short</u> root with respect to $\tilde{\Delta}_i$. Set $\tilde{\Delta}_i^* = \tilde{\Delta}_i \cup \{-(\theta_i+u)\}$.

(Case 2) Otherwise.

Let η_i be the maximal root with respect to $\tilde{\Delta}_i$. Set $\tilde{\Delta}_i^* = \tilde{\Delta}_i \cup \{-(\eta_i+\epsilon_0 u)\}$, where ϵ_0 is the minimum positive integer ϵ such that $\eta_i+\epsilon u$ is a root. (If η_i is a short root, $\epsilon_0 = 1$. If η_i is a long root, ϵ_0 is equal to the minimum positive integer ϵ such that $\epsilon u \in FL$.)

By the above we have $\tilde{\Delta}_i^*$ for $1 \leq i \leq m$. Let $e_1, e_2, \ldots, e_k$ be all the members in $\bigcup_{i=1}^{m} \tilde{\Delta}_i^*$.

Next, we choose $e_\ell \in L$ for $\ell > k$ inductively. Assume that we have chosen $e_1, \ldots, e_{\ell-1}$. If the set of roots

$$R_\ell = \{\alpha \in R(L,FL) \mid (\alpha,e_i) \leq 0 \ (1 \leq i < \ell),\ (\alpha,u) \neq 0\}$$

is empty, set $\Delta_V = \{e_1,e_2,\ldots,e_{\ell-1}\}$. If $R_\ell \neq \phi$, we define e_ℓ to be an element α in R_ℓ which attains the minimal value for $(\alpha,u)^2/\alpha^2$ and satiesfies $(\alpha,u) < 0$.

When $R_\ell \neq \phi$ for all $\ell > k$, we set

$\Delta_V = \{e_1,\ldots,e_k,e_{k+1},\ldots,e_\ell,\ldots\}$ (an infinite set).

Then, even under this definition Δ_V coincides with the set Δ of fundamental roots associated with some fundamental polyhedron C (Vinberg [16], [18]).

One knows the following. In the following the graph corresponding to a root basis plus (-1)-times the maximal short root is also called the extended Dynkin graph.

Lemma 5.2. The obtained set $\bigcup_{i=1}^{m} \tilde{\Delta}_i^*$ by the above construction is conjugate to a subset of the set Δ of fundamental roots for L over the Weyl group W. In particular, the extended Dynkin graph corresponding to the Dynkin graph for $I^{\perp}/I$ is a subgraph of the Coxeter-Vinberg graph for L.

Proposition 5.3. (Vinberg [18] section 2.4 and section 3.2) The following four conditions are equivalent.

(1) The Weyl group $W(L,FL)$ has finite index in the group of all integral orthogonal transformations on L.

(2) A set Δ of fundamental roots of (L,FL) is finite.

(3) The polyhedron $C/\mathbb{R}_+$ in the Lobačevskiǐ space $\Sigma_+/\mathbb{R}_+$ associated with the fundamental polyhedron C has finite volume.

(4) There are a finite number of vectors $v_1, \ldots, v_\ell$ in the closure of Σ_+ such that the fundamental polyhedron C coincides with the interior of the minimum convex body containing the set $\bigcup_{i=1}^{\ell} \mathbb{R}_+ v_i$.

When we carry out Vinberg's algorithm, we need further some practical method to determine whether the obtained set $\{e_1, \ldots, e_\ell\}$ equals to Δ_V or does not. For this purpose we have the following (Vinberg [16] Prop. 11).

<u>Proposition 5.4.</u> Consider one of the above 2 kinds of Vinberg's algorithm. Let Θ be a subset of Δ_V spanning $L \otimes \mathbb{Q}$ over $\mathbb{Q}$. Assume that we have obtained a graph G applying the above rules (1)-(5) to Θ instead of Δ. If the graph G satisfies the following conditions <a> and <b>, then $\Theta = \Delta_V$. In particular, (L,FL) satisfies the equivalent conditions in Proposition 5.3.

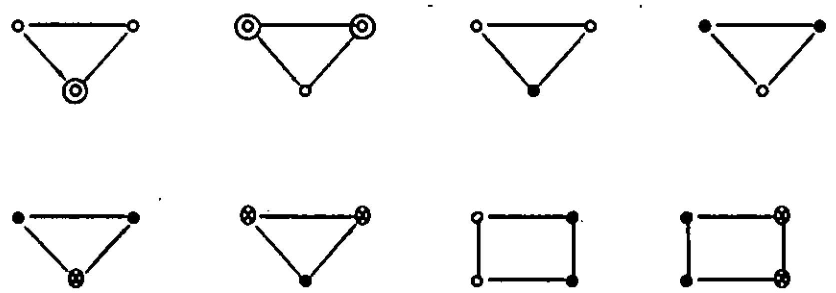
Conversely, if (L,FL) satisfies a condition in Proposition 5.3, the Coxeter-Vinberg graph G of (L,FL) satisfies the following <a> and <b> for $\Theta = \Delta_V$.

<a> If a subgraph S of G is an extended Dynkin graph, then we can find a subgraph T of G containing S such that T is an extended Dynkin graph whose rank equals to rank $L-2$.

<b> Let $\Theta(T)$ denote the subset of Θ corresponding to the vertices in a subgraph T of G. Let S be an arbitrary subgraph of G such that S is isomorphic to one of the following indefinite critical graphs. Let $x \in L\otimes\mathbb{R}$ be an element. If $(x,\alpha) = 0$ for every $\alpha\in\Theta(S)$ and if $(x,\beta) \leq 0$ for every $\beta \in \Theta-\Theta(S)$, then $x = 0$.

Indefinite critical graphs:

$*_1 \cdots\cdots *_2$ $(*_1, *_2 = \circ, \bullet, ⊚$ or $\otimes$.)



<u>Remark.</u> The first one is a graph with only two vertices and with a dotted edge. The latter 11 kinds of the indefinite critical graphs except the first one belong to the class of Lannér graphs (Vinberg [16] Table 3). In spite that there are other kinds of Lannér graphs, only the above 11 can appear in our problem. This is because the angle between two roots $\alpha, \beta \in \Lambda$ is either $\pi/2$, $2\pi/3$, $3\pi/4$ or $5\pi/6$ in the case where the quasi-lattice $\mathbb{Z}\alpha+\mathbb{Z}\beta$ is positive definite, and the angle uniquely determines the ratio of the length of roots in that case.

We will write down the Coxeter-Vinberg graph for the root module Λ_2/P in the case of $J_{3,0}$, $Z_{1,0}$ and $W_{1,0}$. We will discuss the case of $Q_{2,0}$, too.

The case of $J_{3,0}(2,2,2,3)$.

In this case $P \cong Q(D_4) \oplus H$. Let F denote the orthogonal complement of P in Λ_2. We have $F \cong C(Q(D_4),\Lambda_1)$ and $\Lambda_1 \cong \Gamma_{16} \oplus H$. Γ_{16} is the even overlattice over $Q(D_{16})$ with index 2. An embedding $Q(D_4) \subset \Gamma_{16}$ is unique up to orthogonal transformations and $C(Q(D_4), \Gamma_{16}) \cong Q(D_{12})$. Thus we have $F \cong Q(D_{12}) \oplus H$ and $\Lambda_2/P \cong F^* \cong Q(D_{12})^* \oplus H$.

Now, let K be the odd unimodular lattice with signature $(13, 1)$. We can write it in the form $K = \sum_{i=0}^{13} \mathbb{Z}v_i$, where $v_0^2 = -1$, $v_i^2 = +1$ $(1 \leq i \leq 13)$, and $(v_i, v_j) = 0$ $(i \neq j)$. We define the elements w, f_1, ..., f_{12}, g, h as follows:

$$w = v_0+v_1+\ldots+v_{13}, \quad g = v_0+v_{13}, \quad h = -(v_0+v_{12}),$$
$$f_i = -v_i+v_{i+1} \quad (1 \leq i \leq 10),$$
$$f_{11} = v_0-v_{11}+v_{12}+v_{13}, \quad f_{12} = -(v_0+v_{11}+v_{12}+v_{13}).$$

Set $M = \langle x \in K \mid (x,w) \equiv 0 \pmod 2 \rangle = \langle x = \sum x_i v_i \mid \sum x_i \equiv 0 \pmod 2 \rangle$. The elements f_1, ..., f_{12}, g, h are a basis for M. The elements f_1, ..., f_{12} is a root basis of type D_{12}, and g and h generate a hyperbolic plane orthogonal to f_i $(1 \leq i \leq 12)$. Thus we have $M \cong Q(D_{12}) \oplus H$ and

$$\Lambda_2/P \cong M^* = K + \mathbb{Z}(w/2).$$

One knows that applying Vinberg's algorithm to the quasi-lattice $K+\mathbb{Z}(w/2)$, we can draw the Coxeter-Vinberg graph of Λ_2/P. We carry out the algorithm with the controlling vector v_0. As the root basis for the orthognal complement of v_0, we take

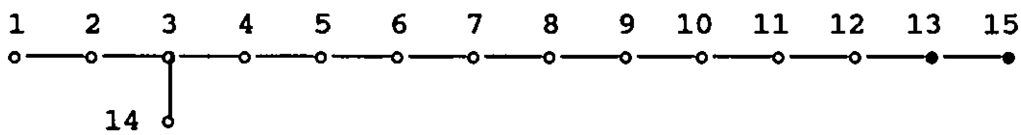
$$e_i = -v_i + v_{i+1} \quad (1 \leq i \leq 12),$$
$$e_{13} = -v_{13}.$$

Succeedingly we get

$$e_{14} = v_0 + v_1 + v_2 + v_3$$

$$e_{15} = (3v_0 + \sum_{i=1}^{13} v_i)/2$$

Drawing the graph for these 15 vectors, we get:



This contains no indefinite critical subgraph. By Proposition 5.4, the above is the Coxeter-Vinberg graph for Λ_2/P.

Corollary 5.5. Let P be the lattice defined for the case $J_{3,0}(2,2,2,3)$. For every positive definite full root submodule $L \subset \Lambda_2/P$, there exists a primitive isotropic element $u \in \Lambda_2/P$ in a nice position for L such that the root system of the positive definite root module $(\mathbb{Z}u)^{\perp}/\mathbb{Z}u$ is of type E_8+F_4.

Proof. Considering conjugation over the Weyl group $W(\Lambda_2/P)$, we can assume that the root basis Δ_L for L is a subset of above $\{e_1,\ldots,e_{15}\}$. On the other hand, setting

$$u = -(e_{10}+2e_{11}+3e_{12}+4e_{13}+2e_{15}),$$

one has $(u,e_9) = 1$ and $(u,e_i) = 0$ for $i\neq 9$, $1\leq i\leq 15$. Thus u is in a nice position for L. Assume that we can write it in the form $u = aw$ $(a\in\mathbb{Z},\ w\in\Lambda_2/P)$. Since $w^2 = 0$, one has $(e_9,w) \in \mathbb{Z}$ by Proposition 3.6 (1), (2). One has $a = \pm 1$ since $a(e_9,w) = (e_9,u) = 1$. Thus u is primitive. One can read off from the above graph that the root system of $(\mathbb{Z}u)^{\perp}/\mathbb{Z}u$ is of type E_8+F_4, since vertices except 9 form the extended Dynkin graph of type E_8+F_4.

Q.E.D.

Set $u' = -(e_2+2\sum_{i=3}^{13} e_i + e_{14})$. This u' is a primitive isotropic element and the root system of $(\mathbb{Z}u')^{\perp}/\mathbb{Z}u'$ is of type B_{12}. The above Coxeter-Vinberg graph contains only two types — $E_8 + F_4$ and B_{12} — of extended Dynkin graphs of rank 12. Combining this with Lemma 4.2, one gets the following.

<u>Corollary 5.6.</u> Let P be the lattice associated with $J_{3,0}$. The root system of the quotient quasi-lattice Λ_1/P is of type G for some primitive embedding $P \subset \Lambda_1$, if and only if $G = E_8+F_4$ or B_{12}.

The case of $Z_{1,0}(2,2,2,4)$.

In this case $P = P_0' \oplus T \oplus H_0$, $P_0' \cong Q(D_4)$ and $T \cong Q(A_1)$. Let F be the orthogonal complement of P in Λ_2 for a fixed embedding $P \subset \Lambda_2$. Since every embedding $P \subset \Lambda_2$ is equivalent by Proposition 3.2, we can choose a convenient one for our purpose. Regarding $\Lambda_2 = \Gamma_{16} \oplus H \oplus H$, we can take the direct sum of the embeddings for each component $P_0' \subset \Gamma_{14}$, $T \subset H$, $H_0 \subset H$. Since $C(P_0',\Gamma_{16}) \cong Q(D_{12})$, $C(T,H) \cong \mathbf{Z}v_0$ $(v_0^2 = -2)$, we have $F \cong Q(D_{12}) \oplus \mathbf{Z}v_0$ $(v_0^2 = -2)$.

On the other hand, let K be the root lattice of type B_{12}. We can write it in the form $K = \sum_{i=1}^{12} \mathbf{Z}v_i$ where $v_i^2 = 1$ $(1 \leq i \leq 12)$, $(v_i,v_j) = 0$ $(i \neq j)$. Set $w = v_1+v_2+\ldots+v_{12} \in K$. The root lattice of type D_{12} can be identified with the sublattice $\langle x \in K \mid (x,w) \equiv 0 \pmod 2\rangle$ of K with index 2. Thus we have

$$\Lambda_2/P \cong F^* = \mathbf{Z}(v_0/2) \oplus [K + \mathbf{Z}(w/2)].$$

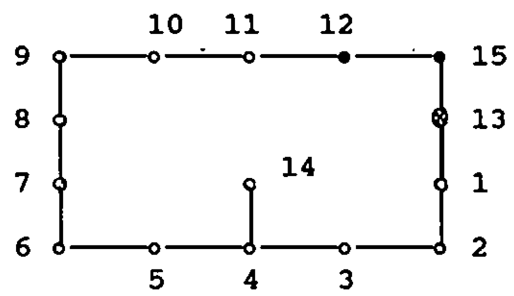
Using the expression in the right-hand side, we can draw the Coxeter-Vinberg graph. We use v_0 as the controlling vector. As the root basis orthogonal to v_0, we take the following:

$$e_i = -v_i + v_{i+1} \quad (1 \leq i \leq 11)$$
$$e_{12} = -v_{12} .$$

At the second step we get vectors:

$$e_{13} = v_0/2 + v_1$$
$$e_{14} = v_0 + v_1 + v_2 + v_3 + v_4$$
$$e_{15} = v_0 + (v_1 + v_2 + \ldots + v_{12})/2.$$

Drawing the graph for these 15 vectors, we get the following one.



By Proposition 5.4 this is the Coxeter-Vinberg graph for Λ_2/P in the case of $Z_{1,0}$.

Corollary 5.7. Let P be the lattice corresponding to $Z_{1,0}(2,2,2,4)$. For every positive definite full root submodule $L \subset \Lambda_2/P$ there exists a primitive isotropic element $u \in \Lambda_2/P$ in a nice position with respect to L such that the root system for $(\mathbb{Z}u)^{\perp}/\mathbb{Z}u$ is either of type $E_7 + F_4$ or of type $E_8 + CB_3$.

Proof. We can regard that the root basis Δ_L for L is a subset of the above system of 15 vectors $e_1, \ldots, e_{15}$. The graph made from Δ_L is a Dynkin graph and it has no bold edge. Thus either $e_1 \notin \Delta_L$ or $e_{13} \notin \Delta_L$.

First consider the case where $e_1 \notin \Delta_L$. Set

$$u_1 = -(e_{11}+2e_{12}+2e_{13}+2e_{15}).$$

This u_1 is a primitive isotropic element and $(\mathbb{Z}u_1)^{\perp}/\mathbb{Z}u_1$ has the root system of type E_8+CB_3.

$$(u_1,e_1) = 2, \quad (u_1,e_{10}) = 1, \quad \text{and}$$
$$(u_1,e_i) = 0 \quad (1\leq i\leq 15,\ i\neq 1,\ 10).$$

The vector e_{10} is a long root. Thus u_1 is in a nice position.

In the case where $e_{13} \notin \Lambda_L$, consider

$$u_2 = -(e_9+2e_{10}+3e_{11}+4e_{12}+2e_{15}).$$

This u_2 is also a primitive isotropic element and $(\mathbb{Z}u_2)^{\perp}/\mathbb{Z}u_2$ has the root system of type E_7+F_4.

$$(u_2,e_8) = 1, \quad (u_2,e_{13}) = 1,$$
$$(u_2,e_i) = 0 \quad (1\leq i\leq 15,\ i\neq 8,\ 13).$$

Thus u_2 is in a nice position in this case.

Q.E.D.

Corollary 5.8. Let P be the lattice corresponding to the case $Z_{1,0}(2,2,2,4)$. The root system of the quotient quasi-lattice Λ_1/P is of type G for some embedding $P \subset \Lambda_1$ if and only if $G = E_7+F_4,\ E_8+CB_3,$ or $B_{10}+CB_1$.

The case of $Q_{2,0}(2,2,2,5)$.

The quotient quasi-lattice Λ_2/P does not satisfy the equivalent conditions in Proposition 5.3. Thus we cannot write down the Coxeter-Vinberg graph.

<u>Lemma 5.9.</u> Let Λ be an even unimodular lattice, and Π and T be non-degenerate primitive sublattice. Assume that Π and T are orthogonal to each other. Set $M = P(\Pi \oplus T,\Lambda)$, $L = C(\Pi\oplus T,\Lambda)$ and $\Xi = C(T,\Lambda)$. Ξ is the primitive hull of $L \oplus \Pi$ and we have the induced embedding $\Xi/\Pi \subset \Lambda/\Pi$. On the other hand we can regard T as a submodule in Λ/Π via the composition $T \subset \Lambda \longrightarrow \Lambda/\Pi$ of natural morphisms.

(1) Then, in Λ/Π, $P(T,\Lambda/\Pi) = M/\Pi$ and Ξ/Π are the orthogonal complements for each other.

(2) The restriction of the natural surjective morphism $\Lambda/\Pi \longrightarrow \Lambda/M$ to Ξ/Π is an isomormphism onto the image R/M which preserves the bilinear forms. Here $R = (\Xi \oplus T) + M$, and thus $R/\Xi \oplus T \cong M/\Pi\oplus T$.

Now, set $I = M/\Pi \oplus T$ and $I^{\perp} = M^*/\Pi\oplus T \subset (\Pi\oplus T)^*/(\Pi\oplus T) = (\Pi^*/\Pi) \oplus (T^*/T)$. We have the following exact sequence:

$$0 \longrightarrow I \longrightarrow I^{\perp} \xrightarrow{\sigma} M^*/M \longrightarrow 0.$$

On the other hand, we can regard that $L \subset \Xi/\Pi$. Thus we have the inclusion relations

$$L \subset \Xi/\Pi = \Xi/M \cap \Xi \subset \Lambda/M \cong L^*,$$

and we can regard $(\Xi/\Pi)/L$ as a subset of L^*/L. By $r : L^*/L \longrightarrow M^*/M$ we denote the canonical isomorphism.
(3) An element $\bar{x} \in L^*/L$ belongs to $(\Xi/\Pi)/L$ if and only if $r(\bar{x}) \in M^*/M$ belongs to the image of $I^{\perp} \cap ((\Pi^*/\Pi) \oplus \langle 0 \rangle)$ by σ.

Proof. Easy.

In our case the lattice P has the following decomposition:

$$P = P_0' \oplus T \oplus H_0, \quad P_0' \cong Q(D_4), \quad T \cong Q(A_2),$$

and $P' = P_0' \oplus T$. We denote $P_1 = P_0' \oplus H_0$. The discriminant group P'^*/P' has elements of the second kind (special elements of type B), but it has no elements of the third kind. Thus it is enough to deal with only short roots with length 1. We need not consider obstruction components.

Corollary 5.10. Assume that $N \geq 1$. We fix an embedding $P \subset \Lambda_N$. Set $\Xi = C(T,\Lambda_N)$. We identify T and the image of T in Λ_N/P_1. Then, T is primitive in Λ_N/P_1 and the restriction of the canonical surjective morphism $\pi : \Lambda_N/P_1 \longrightarrow \Lambda_N/P$ to Ξ/P_1 is injective, and the image $\pi(\Xi/P_1) = (\Xi \oplus T)/P$ has index 3 in Λ_N/P. For the root system $R(\Lambda_N/P) \subset \pi(\Xi/P_1)$.

Proof. Note that in the discriminant group $P'^*/P' = P_0'^*/P_0' \oplus T^*/T$, any element of the second kind is contained in the direct summand $P_0'^*/P_0'$. The inclusion relation for the root system at the last part follows from this fact. Other parts are easy.

Q.E.D.

Lemma 5.11. Let P be the lattice defined in the case for $Q_{2,0}(2,2,2,5)$. The root system of the quotient Λ_1/P is of type G for some embedding $P \subset \Lambda_1$ if and only if $G = E_6+F_4$, E_8+F_2 or B_9.

Proof. By Corollary 5.10 we have only to consider what the root system of the orthogonal complement of T in Λ_1/P_1 is. By Corollary 5.6 the root system of Λ_1/P_1 is either of type E_8+F_4 or of type B_{12}.

Consider the case where Λ_1/P_1 has the root system of type E_8+F_4. If the root system of T (It is of type A_2.) is contained in the component of type E_8, then $G = E_6 + F_4$, and if it lies in the component of type F_4, then $G = E_8 + F_2$.

Next, we consider the case where the root system of Λ_1/P_1 is of type B_{12}. Let $Q \subset \Lambda_1/P_1$ be the root lattice of type B_{12}. We have $T \subset Q$. It is easy to see that the orthogonal complement S of T in Q contains a root lattice Q_1 of type B_9. Thus $S = \mathbf{Z}\xi \oplus Q_1$ for some element $\xi \in S$. Since $\xi^2 = d(S) = d(T) = 3$, the root system of S is of type B_9.

Conversely, we can construct an embedding $P \subset \Lambda_1$ which realizes each case $G = E_6+F_4$, E_8+F_2, B_9.

Q.E.D.

Lemma 5.12. Let P be the lattice defined in the case of $Q_{2,0}(2,2,2,5)$. For every positive definite full root submodule $L \subset \Lambda_2/P$, we have a primitive isotropic element $u \in \Lambda_2/P$ in a nice position with respect to L such that the root system of $(\mathbb{Z}u)^{\perp}/\mathbb{Z}u$ is either of type E_6+F_4 or of type E_8+F_2.

Proof. We use notations in Corollary 5.10 assuming $N = 2$.

By $Q(L)$ we denote the sub-quasi-lattice of L generated by roots. By Corollary 5.10, $Q(L) \subset \pi(\Xi/P_1)$. Let $\rho : \pi(\Xi/P_1) \longrightarrow \Xi/P_1$ denote the inverse morphism of π. Set $Q = \rho(Q(L))$. Let Q' be the sub-quasi-lattice of $P(T \oplus Q, \Lambda_2/P_1)$ generated by roots in it. Q' is an overlattice of $T\oplus Q$. Note that $T\oplus Q$ is generated by roots in

$$P(T\oplus Q,\Lambda_2/P_1) \cap (T\oplus(\Xi/P_1)) = P(T\oplus Q,T\oplus(\Xi/P_1)).$$

Let $Q' = \bigoplus_{i=1}^{m} Q'_i$ be the irreducible decomposition of Q'. We assume that $T \subset Q'_1$. Then we have $Q = (Q \cap Q'_1) \oplus (\bigoplus_{i=2}^{m} Q'_i)$. On the other hand,

$$P(T \oplus Q,\Lambda_2/P_1)/P(T\oplus Q,T\oplus(\Xi/P_1)) = P(Q',\Lambda_2/P_1)/P(T\oplus Q,T\oplus(\Xi/P_1))$$

is isomorphic to a subgroup of $\Lambda_2/T\oplus\Xi \cong \mathbb{Z}/3$ and it is a cyclic group. By Proposition 4.1 one knows that the root system of $T\oplus Q$ is obtained from that of Q' by one elementary transformation. Excluding common components $\bigoplus_{i=2}^{m} Q'_i$, the root basis Δ_1 of $T\oplus(Q\cap Q_1')$ is obtained from the root basis Δ_1' of Q_1' by one elementary transformation. Note that Δ_1 and Δ_1' contains the same number of elements, and Δ_1 has a component of type A_2 corresponding to T. We never have a short root with length $1/\sqrt{2}$ or $\sqrt{2/3}$ in our case, and one knows that the irreducible root basis Δ_1' is of type either F_4, E_6, E_7, E_8 or A_2. According as the type of Δ_1', Δ_1 is of type F_2+A_2, $3A_2$, A_5+A_2, E_6+A_2 or A_2. Let Δ_i' be a root basis for Q_i' for $2\leq i\leq m$. $\Delta' = \bigcup_{i=1}^{m} \Delta_i'$ is a root bais of Q'.

Now, our P_1 is isomorphic to P in the case $J_{3,0}(2,2,2,3)$. Thus considering a conjugate one, we can regard that Δ' is a subset of the system of 15 vectors just before Corollary 5.5. Here we would like to show that there exists a primitive isotropic element $u' \in \Lambda_2/P_1$ satisfying either the following (0)′ and (1)′ or the following (0)′ and (2)′.

(0)′ The root system of $(\mathbb{Z}u')^{\perp}/\mathbb{Z}u'$ is of type $E_8 + F_4$.

(1)′ u' is orthogonal to all elements in Δ'.

(2)′ There exists a long root $\alpha \in \Delta' - \Delta_1'$ such that $(\alpha,u') = 1$ and $(\beta,u') = 0$ for every $\beta \in \Delta'$ with $\beta \neq \alpha$.

If $e_9 \notin \Delta_1'$, then $u_0 = -(e_{10}+2e_{11}+3e_{12}+4e_{13}+2e_{15})$ satisfies the desired condition.

Assume $e_9 \in \Delta_1'$. The graph of Δ_1' is a Dynkin subgraph containing the vertex 9 in the Coxeter-Vinberg graph for $J_{3,0}$. On the other hand any Dynkin subgraph containing the vertex 9 in the Coxeter-Vinberg graph for $J_{3,0}$ is never of type F_4 or E. Thus $Q\cap Q_1' = 0$, $T = Q_1'$, and $\Delta_1' = \langle e_8,e_9\rangle$ or $\Delta_1' = \langle e_9,e_{10}\rangle$.

Consider the case $\Lambda_1' = \langle e_8, e_9 \rangle$. By assumption e_7, $e_{10} \notin \Lambda'$. Set

$$u_1 = -(e_7+e_8+e_9+2e_{11}+3e_{12}+4e_{13}+2e_{15})$$
$$= s_{e_7} s_{e_8} s_{e_9}(u_0). \text{ (s_α stands for a reflection.)}$$

One can check that $(e_i, u_1) = 0$ $(i \neq 6, 7, 10, \ 1 \leq i \leq 15)$ and $(e_6, u_1) = 1$. This u_1 satisfies the condition.

Next, we consider the case $\Lambda_1' = \langle e_9, e_{10} \rangle$. By assumption e_8, $e_{11} \notin \Lambda'$. Set

$$u_2 = -(e_7+2(e_8+e_9+e_{10}+e_{11})+3e_{12}+4e_{13}+2e_{15})$$
$$= s_{e_8} s_{e_9} s_{e_{10}}(u_1).$$

$(u_2, e_i) = 0$ $(i \neq 6, 8, 11, \ 1 \leq i \leq 15)$ and $(u_2, e_6) = 1$. This u_2 satisfies the desired condition.

We have shown the existence of u'.

Set $I' = \mathbf{Z}u'$. By the condition we have always $T \subset I'^{\perp}$. Set $u = \pi(u') \in \Lambda_2/P$ and $I = \mathbf{Z}u$. By Corollary 5.10, I is primitive in $\pi(\Xi/P_1) = (\Xi \oplus T)/P$. If I is not primitive in Λ_2/P, we can write $u = aw$. $(a \in \mathbf{Z},\ w \in \Lambda_2/P,\ w \notin (\Xi \oplus T)/P)$. We have $w^2 = 0$. By Proposition 2.9(6) and Lemma 3.5(1) one knows that w belongs to the image in Λ_2/P of the orthogonal complement F of P. In particular, we have $w \in \Xi \oplus T/P$, which is a contradiction. Thus I is primitive even in Λ_2/P. By Corollary 5.9, the set of all roots in $I'^{\perp}$ orthogonal to T has one-to-one correspondence with the set of all roots in I via π.

Thus the root system of $I^{\perp}/I$ is equal to the root system consisting of all roots in $I'^{\perp}/I'$ orthogonal to T. Here we have identified T and the image of T in $I'^{\perp}/I'$. The root system of $I'^{\perp}/I'$ is of type E_8+F_4. Depending on which one of two components contains T, the root system of $I^{\perp}/I$ is of type either E_6+F_4 or E_8+F_2.

Q.E.D.

The case of $W_{1,0}(2,2,3,3)$.

First, recall the decomposition $P = P' \oplus H$. The discriminant group P'^{*}/P' has elements of the second kind (special elements of type B). Elements of the third kind in it has only the associated number 2 or 11. Thus it is enough to consider only roots with length 1 or $\sqrt{2/3}$ as short roots. However, we have to count obstruction components of type A_{11}.

The discriminant group $P^{*}/P \cong P'^{*}/P'$ is a cyclic group of order 12. Let g be a generator. We can assume that for the discriminant quadratic form q_P, $q_P(g) \equiv 13/12 \bmod 2\mathbf{Z}$.

On the other hand, for the root lattice $Q = Q(A_{11})$ of type A_{11}, $(Q\oplus H)^{*}/(Q\oplus H) \cong Q^{*}/Q$ is also a cyclic group of order 12. It has a generator h with $q_{Q\oplus H}(h) \equiv 11/12 \bmod 2\mathbf{Z}$, for the discriminant quadratic form $q_{Q\oplus H}$. Thus $q_P = -q_{Q\oplus H}$.

This implies that there exists a primitive embedding $P \subset \Lambda_2$ such that the orthogonal complement F of P in Λ_2 is isomorphic to $Q\oplus H$ (Lemma 2.14, Lemma 3.5). Thus by

Proposition 3.2(2), for <u>every</u> primitive embedding $P \subset \Lambda_2$, the orthogonal complement F of P in Λ_2 is isomorphic to $Q \oplus H$.

Now, let $K = \sum_{i=0}^{13} \mathbb{Z}v_i$ be the odd unimodular lattice with signature (13, 1). We assume that the basis satisfies $v_0^2 = -1$, $v_i^2 = +1$ $(1 \leq i \leq 13)$, $(v_i, v_j) = 0$ $(i \neq j)$. Setting $w = v_0+v_1+\ldots+v_{13}$, we define M to be the orthogonal complement of $\mathbb{Z}w$ in K. Set

$$g_1 = v_0+v_{13}, \quad g_2 = -(v_0+v_{12}),$$
$$f_i = -v_i+v_{i+1} \quad (1 \leq i \leq 10), \text{ and } \quad f_{11} = v_0-v_{11}+v_{12}+v_{13}.$$

Elements $g_1, g_2, f_1, \ldots, f_{11}$ are a basis for M. $H = \mathbb{Z}g_1+\mathbb{Z}g_2$ is a hyperbolic plane, since $g_1^2 = g_2^2 = 0$ and $(g_1, g_2) = 1$. Elements $f_1, f_2, \ldots, f_{11}$ form a root basis of type A_{11}, which is orthogonal to g_1 and g_2. Thus one knows $M \cong Q(A_{11}) \oplus H$.

Let $p : K \otimes \mathbb{Q} \longrightarrow M \otimes \mathbb{Q}$ be the orthogonal projection. By definition $p(x) = x-(x,w)w/12$, and $p(x) = x$ if and only if $x \in M \otimes \mathbb{Q}$. For every $x, y \in K \otimes \mathbb{Q}$, $(p(x), p(y)) = (x, p(y))$ $= (p(x), y)$.

Using the projection, set

$$r = p(v_0) = (13v_0+v_1+\ldots+v_{13})/12.$$

$r^2 = -13/12$ and $r \in M \otimes \mathbb{Q}$.

Here note that g_1 and $e_i = -v_i+v_{i+1}$ $(1 \leq i \leq 12)$ are also a basis for M. $(r,g_1) = (v_0,g_1) = -1$. $(r,e_i) = (v_0,e_i) = 0$ $(1 \leq i \leq 12)$. Thus $M^* \supset M + \mathbf{Z}r$. On the other hand, $[M^* : M] = |d(M)| = 12$ and $[M+\mathbf{Z}r : M] = 12$. One knows $M^* = M+\mathbf{Z}r$. Since $\Lambda_2/P \cong F^*$ and $F \cong Q(A_{11}) \oplus H \cong M$, one has an isomorphism of quasi-lattices

$$\Lambda_2/P \cong M+\mathbf{Z}r.$$

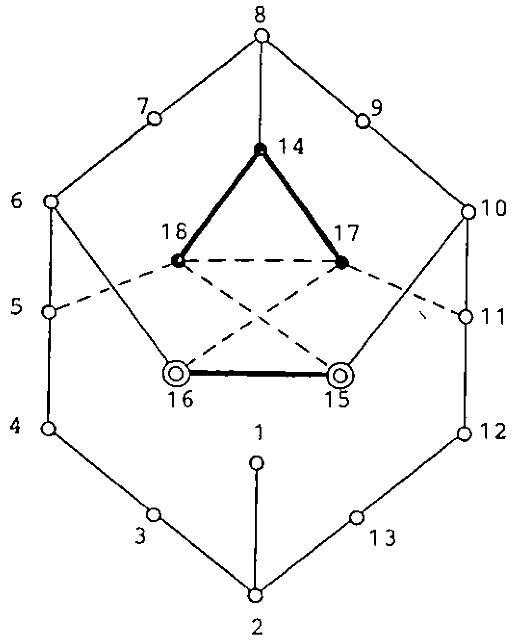
By using the expression on the right-hand side, we carry out Vinberg's algorithm with the controlling vector r. As the root basis orthogonal to r, we take

$$e_i = -v_i+v_{i+1} \quad (1 \leq i \leq 12).$$

By the algorithm we get succeedingly

$$e_{13} = v_0+v_1+v_2-v_{13}$$
$$e_{14} = \{3v_0+(v_1+\ldots+v_8)-(v_9+\ldots+v_{13})\}/2$$
$$e_{15} = \{4v_0+(v_1+\ldots+v_{10})-2(v_{11}+v_{12}+v_{13})\}/3$$
$$e_{16} = \{5v_0+2(v_1+\ldots+v_6)-(v_7+\ldots+v_{13})\}/3$$
$$e_{17} = \{5v_0+(v_1+\ldots+v_{11})-3(v_{12}+v_{13})\}/2$$
$$e_{18} = \{7v_0+3(v_1+\ldots+v_5)-(v_6+\ldots+v_{13})\}/2.$$

Drawing the graph for this system of 18 vectors, we get the following.



We would like to apply Proposition 5.4 to this graph. It has many dotted edges and we have to check the condition <b> in Proposition 5.4. Owing to the following lemma also due to Vinberg, we can check it easily.

<u>Lemma 5.13</u> (Vinberg [16] Proposition 2) We use the notations in Proposition 5.4. Let S be an indefinite critical subgraph of G. Let T be the subgraph of G consisting of vertices not connected with any vertex in S by an edge and not belonging to S. If the following condition on an element $y \in L\otimes\mathbb{R}$ is satisfied, then the condition <b> in Proposition 5.4 for $\theta(S)$ is also satisfied. The condition: If $(y,\alpha) = 0$ for every $\alpha \in \theta(S) \cup \theta(T)$ and if $(y,\beta) \leq 0$ for every $\beta \in \theta - (\theta(S) \cup \theta(T))$, then $y = 0$.

Let S be the subgraph of the above graph consisting of the vertices 17 and 18. The unique edge in S is a dotted one, and S is an indefinite critical subgraph. The corresponding T is the subgraph consisting of the vertices 6, 7, 8, 9, 10 and 1, 2, 3, 4, 12, 13, and T is a Dynkin graph of type A_5+E_6. In this case $\theta(S) \cup \theta(T)$ consists of 13 linearly independent vectors, and $13 = \text{rank}\ \Lambda_2/P$. Thus the condition $(y,\alpha) = 0$ for $\alpha \in \theta(S) \cup \theta(T)$ implies $y = 0$. By the above lemma, one knows that Proposition 5.4 <b> holds for S.

The reasoning is the same even for other indefinite critical subgraphs. By Proposition 5.4 the above is the Coxeter-Vinberg graph in the case of $W_{1,0}(2,2,3,3)$.

Corollary 5.14. Let P be the lattice defined in the case of $W_{1,0}(2,2,3,3)$ and $L \subset \Lambda_2/P$ be a positive definite full root submodule. Assume that the Dynkin graph of L does not contain a component of type B_1. Then, there is a primitive isotropic element $u \in \Lambda_2/P$ in a nice position for L such that the root system of $(\mathbf{Z}u)^{\perp}/\mathbf{Z}u$ is of type $E_8+B_1+G_2$, B_9+G_2, $E_7+B_3+G_1$ or A_{11}.

Proof. We can assume that a root basis Δ_L for L is a subset of the above $\{e_1,\dots,e_{18}\}$. The graph for Δ_L is a Dynkin graph and it does not contain a dotted edge, a bold edge, or an extended Dynkin graph. In particular, either $e_{17} \notin \Delta_L$ or $e_{18} \notin \Delta_L$. By symmetry of the graph we can consider only the case $e_{18} \notin \Delta_L$. If $e_{18} \notin \Delta_L$ and $e_{17} \in \Delta_L$, we have further that $e_{11}, e_{14}, e_{16} \notin \Delta_L$, since the graph has no dotted edge and no bold edge. If $e_{18}, e_{17} \notin \Delta_L$, then either $e_{15} \notin \Delta_L$ or

$e_{16} \notin \Delta_L$, since the graph has no bold edge. It suffices to consider only the case $e_{16} \notin \Delta_L$ by symmetry. Next, consider elements e_{10}, e_{11}, e_{15}. These form an extended Dynkin graph of type G_2 and at least one of them does not belong to Δ_L. We have four cases to be considered.

Case (1) : $e_{11}, e_{14}, e_{16}, e_{18} \notin \Delta_L$
Case (2) : $e_{11}, e_{16}, e_{17}, e_{18} \notin \Delta_L$
Case (3) : $e_{10}, e_{16}, e_{17}, e_{18} \notin \Delta_L$
Case (4) : $e_{15}, e_{16}, e_{17}, e_{18} \notin \Delta_L$.

Case (1) and (2).

In these cases $e_{11}, e_{16}, e_{18} \notin \Delta_L$. Consider $u_1 = -(e_{14}+ e_{17})$. $u_1^2 = 0$ and $(u_1,e_i) = 0$ for $i \neq 8, 11, 16, 18$, $1 \leq i \leq 18$. This u_1 is a primitive isotropic element with $(e_8,u_1) = 1$ and $(\beta,u_1) = 0$ for $\beta \in \Delta_L - \langle e_8 \rangle$.

Consider the subgraph in the above Coxeter-Vinberg consisting of all the vertices not connected to either the vertex 14 or 17, plus the vertices 14 and 17 themselves. This subgraph is the extended Dynkin graph of type $E_8+B_1+G_2$. It follows that the root system of $(\mathbf{Z}u_1)^{\perp}/\mathbf{Z}u_1$ is of type $E_8+B_1+G_2$.

Case (3).

Consider $u_2 = -(e_{15}+e_{16})$. $u_2^2 = 0$. $(u_2,e_i) = 0$ for $i \neq 6, 10, 17, 18$, $1 \leq i \leq 18$, and thus $(\beta,u_2) = 0$ for every

$\beta \in \Delta_L - \langle e_6 \rangle$. Besides $(e_6, u_2) = 1$. This u_2 is a primitive isotropic element in a nice position with respect to L and the root system of $(\mathbb{Z}u_2)^{\perp}/\mathbb{Z}u_2$ is of type $E_7+B_3+G_1$.

Case (4).

The four elements e_7, e_8, e_9, e_{14} form the extended Dynkin graph of type B_3. Thus one of them does not belong to Δ_L. Depending on one of four, we have four subcases in case (4).

Case (4.1). $e_7, e_{15}, e_{16}, e_{17}, e_{18} \notin \Delta_1$.

Consider $u_3 = -(e_5 + 2e_6 + 3e_{16})$. $u_3^2 = 0$. $(u_3, e_i) = 0$ for $i \neq 4, 7, 15, 17, 18$, $1 \leq i \leq 18$. Thus for every $\beta \in \Delta_L - \langle e_4 \rangle$, $(\beta, u_3) = 0$. Besides $(e_4, u_3) = 1$. This u_3 is primitive and the root system of $(\mathbb{Z}u_3)^{\perp}/\mathbb{Z}u_3$ is of type B_9+G_2.

Case (4.2). $e_9, e_{15}, e_{16}, e_{17}, e_{18} \notin \Delta_L$.

By symmetry of the graph, the reasoning is the same as in (4.1). We can consider the element $u_3' = -(2e_{10} + e_{11} + 3e_{15})$.

Case (4.3). $e_{14}, e_{15}, e_{16}, e_{17}, e_{18} \notin \Delta_L$.

Consider $u_4 = -(e_2+e_3+\ldots+e_{13})$. $u_4^2 = 0$. $(u_4, e_i) = 0$ $(2 \leq i \leq 13)$, and thus for every $\beta \in \Delta_L - \langle e_1 \rangle$, $(\beta, u_4) = 0$. Besides $(e_1, u_4) = 1$. This u_4 is primitive and the root system of $(\mathbb{Z}u_4)^{\perp}/\mathbb{Z}u_4$ is of type A_{11}.

Case (4.4). $e_8, e_{15}, e_{16}, e_{17}, e_{18} \notin \Delta_L$.

This is the last remaining case. If $e_{14} \in \Delta_L$ in this case, then $\{e_{14}\}$ is an irreducible component of Δ_L of type B_1, which contradicts the assumption. Thus $e_{14} \notin \Delta_L$, and case (4.4) is reduced to the above case (4.3).

Q.E.D.

Corollary 5.15. Let P be the lattice associated with the case $W_{1,0}(2,2,3,3)$. The root system of the quotient quasi-lattice Λ_1/P is of type G for some embedding $P \subset \Lambda_1$ if and only if $G = E_8+B_1+G_2$, $E_8+B_3+G_1$, B_9+G_2 or A_{11}.

Besides, if the root system of Λ_1/P is of type A_{11}, then Λ_1/P is isomorphic to the dual quasi-lattice Q^* of the root lattice $Q = Q(A_{11})$ of type A_{11}. In particular, for any full embedding $Q(A_{11}) \subset \Lambda_1/P$, the component A_{11} is an obstruction one.

§6. The Hasse symbol and the Hilbert norm residue symbol

In the beginning part of this section we explain the arithmetic conditions in our main theorems (Cassels [4], Serre [11]).

Proposition 6.1. Let S and Λ be non-degenerate quasi-lattices. We call the following claim $I(S,\Lambda)$.

$I(S,\Lambda)$: For every embedding $S \subset \Lambda$, the orthogonal complement of S in Λ contains an isotropic element.

(1) Assume that for some embedding $S \subset \Lambda$, the orthgonal complement of S in Λ contains an isotropic element. Then, $I(S,\Lambda)$ holds.

(2) Let Λ_N denote an even unimodular lattice with signature (16+N, N). If there exists an embedding $S \subset \Lambda_N$ such that $I(S,\Lambda_N)$ holds, then there exists an embedding $S \subset \Lambda_{N-1}$.

(3) Assume that S is a non-degenerate lattice with signature (s, 1) and that there exists an embedding $S \subset \Lambda_3$. Then, both $I(S,\Lambda_3)$ and $I(S,\Lambda_2)$ hold if and only if the following claim $J(S)$ for S holds.

$J(S)$: One of the following conditions <1>, <2>, <3>, <4> holds.

<1> $s = 17$, $-d(S)$ is a square number, and for every prime number p $\epsilon_p(S) = 1$.

<2> $s = 16$, and for every prime number p $\epsilon_p(S) = 1$

<3> $s = 15$, and for every prime number p

$d(S) \notin \mathbb{Q}_p^{*2}$ or $\epsilon_p(S) = 1$.

<4> $s \leq 14$.

(4) Under the same assumptions as in (3), $I(S,\Lambda_3)$ holds if and only if the following claim $K(S)$ for S holds.

$K(S)$: One of the following conditons <1>, <2>, <3> holds.

<1> $s = 18$, and for every prime number p

$\epsilon_p(S) = (-1,-d(S))_p$.

<2> $s = 17$, and for every prime number p

$-d(S) \notin \mathbb{Q}_p^{*2}$ or $\epsilon_p(S) = 1$.

<3> $s \leq 16$.

Proof. (1) By $\varphi : S \subset \Lambda$ we denote an embedding. The condition that the orthogonal complement $T(\varphi)$ of $\varphi(S)$ in Λ contains an isotropic element can be expressed by three invariants; the signature of $T(\varphi)$, the equivalence class of the discriminant of $T(\varphi)$ modulo $\mathbb{Q}^{*2}$, and the Hasse symbol for $T(\varphi)$. However, these three invariants do not depend on the choice of φ and they depend only on S and Λ. (In fact, they depend only on $S\otimes\mathbb{Q}$ and $\Lambda\otimes\mathbb{Q}$.)

(2) Let $u \in \Lambda_N$ be an isotropic element orthogonal to S. By exchanging u for the generator of $\mathbb{Q}u \cap \Lambda_N$, we can assume further that u is primitive. Then, we have an element $v' \in \Lambda_N$ with $(u,v') = 1$, since Λ_N is unimodular. Since Λ_N is even, $v'^2 = 2m$ is an even integer. The element $v = v' - mu$ is isotropic and satisfies $(u,v) = 1$. The sub-

lattice $H = \mathbf{Z}u+\mathbf{Z}v$ is isomorphic to the hyperbolic plane and we have a decomposition $\Lambda_N = \Xi \oplus H$, and $\Xi \cong \Lambda_{N-1}$. The composition of the embedding $S \subset \Xi + \mathbf{Z}u$ and the projection $\Xi + \mathbf{Z}u \longrightarrow \Xi$ defines an embedding $S \subset \Xi$.

(3), (4) See Urabe [13].

Q.E.D.

The following table shows the signature, the discriminant $d(P)$, and the Hasse symbol $\epsilon_p(P)$ (Here p is a prime number, or $p = \infty$.) for the lattice P corresponding to 6 kinds of hypersurface quadrilateral singularities.

	signature	$d(P)$	$\epsilon_p(P)$
$J_{3,0}(2,2,2,3)$	(5, 1)	− 4	1
$Z_{1,0}(2,2,2,4)$	(6, 1)	− 8	1
$Q_{2,0}(2,2,2,5)$	(7, 1)	−12	1
$W_{1,0}(2,2,3,3)$	(6, 1)	−12	$(-1,3)_p$
$S_{1,0}(2,2,3,4)$	(7, 1)	−20	$(-2,5)_p$
$U_{1,0}(2,3,3,3)$	(7, 1)	−27	$(-1,3)_p$

Let $Q = Q(G)$ be a positive definite root lattice of type G and set $S = P \oplus Q$. We have $d(S) = d(P)d(Q)$ and $\epsilon_p(S) = \epsilon_p(P)\epsilon_p(Q)(d(P), d(Q))_p$. Therefore we can rewrite $J(P \oplus Q)$ and $K(P \oplus Q)$ by using only the data for Q. The corresponding claim to $J(P \oplus Q)$ is the arithmetic conditions in [I] of Theorem 0.3 and 0.5. The corresponding one to $K(P \oplus Q)$ is the arithmetic conditions in [II].

Our main theorems - Theorem 0.3 and Theorem 0.5. - are the consequence of the above propositions in the previous sections.

Last of all we would like to explain how to deduce Theorem 0.3 for the case of $m = 1$, $J_{3,0}$. The deduction for other cases and that for Theorem 0.5 is similar.

First, assume the condition [I](a) for $m = 1$. We will show the condition [I](b). By Theorem 1.2, we have an embedding $S = P \oplus Q(G) \subset \Lambda_3$ satisfying Looijenga's condition (a) and (b). By Theorem 3.9 the induced embedding $Q(G) \subset \Lambda_3/P$ is full. Besides, since the condition $J(P \oplus Q(G))$ in Proposition 6.1 is satisfied, both $I(S,\Lambda_3)$ and $I(S,\Lambda_2)$ hold. This is equivalent to that $I(Q(G),\Lambda_3/P)$ and $I(Q(G), \Lambda_2/P)$ hold. By Lemma 4.2 and by Theorem 4.3 one knows that there exists a primitive embedding $P \subset \Lambda_1$ such that G is obtained from the Dynkin graph of Λ_1/P by elementary transformations repeated twice. By Corollary 5.6 the Dynkin graph of Λ_1/P is one the basic Dynkin graphs.

Conversely, assume the condition [I](b) for $m = 1$. Let $\hat{G}$ be either E_8+F_4 or B_{12}. We have a full embedding $Q(\hat{G}) \subset \Lambda_1/P$ for some embedding $P \subset \Lambda_1$. If G is a Dynkin graph obtained from $\hat{G}$ by elementary transformations repeated twice, by Theorem 4.3 we have a full embedding $Q(G) \subset \Lambda_3/P$ satisfying $I(Q(G),\Lambda_3/P)$ and $I(Q(G),\Lambda_2/P)$. The induced embedding $S = P\oplus Q(G) \subset \Lambda_3$ satisfies $I(S,\Lambda_3)$ and $I(S,\Lambda_2)$. By Proposition 6.1 we have the arithmetic conditon in [I](a). By Theorem 3.9 $S \subset \Lambda_3$ satisfies Looijenga's conditions (a) and (b), since the lattice P has no associated number in our case. By Theorem 1.2 $G \in PC(J_{3,0})$.

Next, we proceed to the part [II]. Assume the condition [II](A) for $m = 1$. By Theorem 1.2 and by Theorem 3.9 we have a full embedding $Q(G) \subset \Lambda_3/P$. By Proposition 6.1 $I(Q(G),\Lambda_3/P)$ holds. By Lemma 4.2 and by Theorem 4.3 we have a Dynkin graph G' and a full embedding $Q(G') \subset \Lambda_2/P$ such that G is obtained from G' by one elementary transformation. By Corollary 5.5 we have an isotropic element $u \in \Lambda_2/P$ in a nice position with respect to $Q(G')$. Thus there is a primitive embedding $P \subset \Lambda_1$ such that G' is obtained from the Dynkin graph of Λ_1/P by one elementary or tie transformation. Besides by Lemma 4.2 and by Corollary 5.5 the Dynkin graph of $\Lambda_1/P \cong (\mathbb{Z}u)^{\perp}/\mathbb{Z}u$ is of type E_8+F_4, which is the essential basic Dynkin graph. We have the condition [II](B). Note that the procedure of the third kind "tie after elementary" is dispensable.

Conversely assume that the condition [II](B) for $m = 1$. If we apply the first procedure "elementary twice" or the second one "elementary after tie", then reversing the arguments just above, one can deduce the condition [II](A) for $m = 1$. In the case of the third procedure "tie after elementary", by the similar reasoning one gets an embedding $Q(G) \oplus P \subset \Lambda_3$ satisfying Looijenga's (a) and (b). Thus $G \in PC(J_{3,0})$ by Theorem 1.2.

Now, by assumption we have a Dynkin graph G' and a full embedding $Q(G') \subset \Lambda_2/P$ such that G is obtained from G' by a tie transformation. Since we can assume moreover that G' is obtained from the Dynkin graph of Λ_1/P by an elementary transformation, $I(Q(G'), \Lambda_2/P)$ holds. Then, by the definition of a tie transformation, one sees that $I(Q(G),\Lambda_3/P)$ holds. One has also the arithmetic condition by Propositon 6.1.

The part [III] follows from Theorem 1.2, Theorem 3.9, Theorem 4.4 and Corollary 5.6.

As for the proof of Theorem 0.5, note that Corollary 5.14 contains an additional condition "L contains no component of type B_1". However, this causes no problem because of the following reason: Let $G'+B_1$ be a Dynkin graph containing a component of type B_1. Let G be a Dynkin graph obtained from $G'+B_1$ by one tie or elementary transformation. If G contains components of type A, D or E only, G can be obtained even from G' by the same transformation.

In the next article we will show the converse of Theorem 0.3 and 0.5, part [II] and [III].

REFERENCES

1. ARNOLD, V.: Local normal forms of functions. Invent. Math. 35, 87-109 (1976).

2. ARNOLD, V.: Singularity theory. London mathemtical society lecture note series 53. Cambridge: Cambridge Univ. Press 1981.

3. BOURBAKI, N.: Grupes et algèbre de Lie. Chaps. 4-6. Paris: Hermann 1968.

4. CASSELS, J.W.S.: Rational quadratic forms. London: Academic press 1978.

5. CONWAY, J.H., SLOANE, N.J.A.: Leech roots and Vinberg groups. Proc. R. Soc. Lond. A384, 233-258 (1982).

6. DURFEE, A.H.: Fifteen characterization of rational double points and simple critical points. L'engein. Math. II 25, 131-163 (1979).

7. KODAIRA, K.: On compact analytic surfaces II, III. Ann. of Math. 77, 563-626 (1963), 78, 1-40 (1963).

8. LOOIJENGA, E.: The smoothing components of a triangle singularity II. Math. Ann. 269, 357-387 (1984).

9. MILNOR, J., HUSEMOLLER, D.: Symmetric bilinear forms. Berlin-Heidelberg-New York: Springer 1973.

10. NIKULIN, V.V.: Integral symmetric bilinear forms and some of their applications. Mat. USSR Izv. 43 No. 1, (1979) (English translation: Math. USSR Izv. 14 No. 1, 103-167 (1980)).

11. SERRE, J.-P.: Cours d'arithmetique. Paris: Presses Univ. France 1970.

12. URABE, T.: Dynkin graphs and combinations of singularities on quartic surfaces. Proc. Japan Acad., Ser. A 61, 266-269 (1985).

13. URABE, T.: Elementary transformations of Dynkin graphs and singularities on quartic surfaces. Invent. math. 87, 549-572 (1987).

14. URABE, T.: Tie transformations of Dynkin graphs and singularties on quartic surfaces. preprint MPI/87-60, Bonn, Max-Planck-Institut für Mathematik (1987).

15. VINBERG, È.B.: Discrete groups generated by reflections in Lobačevskiǐ spaces. Mat. Sb. 72 No. 3, 471-488 (1967) (English translation: Math. USSR Sb. 1 No. 3, 429-444 (1967)).

16. VINBERG, È.B.: On the groups of unit elements of certain quadratic forms. Mat. Sb. 87 No. 1, 18-36 (1972) (English translation: Math. USSR Sb. 16 No. 1, 17-35 (1972)).

17. VINBERG, È.B.: On unimodular integral quadratic forms. Funkt. Analiz Prilozh 6 No. 2, 24-31 (1968) (English translation: Funct. Analysis Appl. 6, 105-111 (1972)).

18. VINBERG, È.B.: Some arithmetical discrete groups in Lobačevskiǐ spaces. In: Discrete subgroups of Lie groups and applications to moduli, 323-348. Oxford: Oxford Univ. Press 1975.