Max-Planck-Institut für Mathematik
Bonn

# Cyclotomic polynomials with prescribed height and prime number theory

by

Alexandre Kosyak
Pieter Moree
Efthymios Sofos
Bin Zhang

Max-Planck-Institut für Mathematik
Preprint Series 2019 (59)

# Cyclotomic polynomials with prescribed height and prime number theory


Alexandre Kosyak
Pieter Moree
Efthymios Sofos
Bin Zhang

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Institute of Mathematics
Ukrainian National Academy of Sciences
3 Tereshchenkivs'ka Str.
01024 Kyiv
Ukraine

School of Mathematics and Statistics
University of Glasgow
University Place
Glasgow G12 8SQ
United Kingdom

School of Mathematical Sciences
Qufu Normal University
Qufu 273165
P. R. China



MPIM 19-59

# Cyclotomic polynomials with prescribed height and prime number theory

Alexandre Kosyak, Pieter Moree, Efthymios Sofos and Bin Zhang

**Abstract**

Given any positive integer $n$, let $A(n)$ denote the height of the $n^{\text{th}}$ cyclotomic polynomial, that is its maximum coefficient in absolute value. It is well known that $A(n)$ is unbounded. We conjecture that every natural number can arise as value of $A(n)$ and prove this assuming that for every pair of consecutive primes $p \geq 127$ and $q$ we have $q - p \leq \sqrt{p} - 1$. Using a result of Heath-Brown we show unconditionally that every integer $m \leq x$ occurs as $A(n)$ value with at most $O_\epsilon(x^{3/5+\epsilon})$ exceptions. On the Lindelöf Hypothesis we show there are at most $O_\epsilon(x^{1/2+\epsilon})$ exceptions. Finally, we study these exceptions further by using deep work of Bombieri–Friedlander–Iwaniec on the distribution of primes in arithmetic progressions beyond the square-root barrier.

## 1 Introduction

Let $n \geq 1$ be an integer. The $n^{th}$ cyclotomic polynomial

$$\Phi_n(x) = \sum_{j=0}^{\varphi(n)} a_n(j)x^j,$$

is a polynomial of degree $\varphi(n)$, with $\varphi$ Euler's totient function. For $j > \varphi(n)$ we put $a_n(j) = 0$. The coefficients $a_n(j)$ are usually very small. Indeed, in the $19^{th}$ century mathematicians even thought that they are always 0 or $\pm 1$. The first counterexample to this claim occurs at $n = 105$; we have $a_{105}(7) = -2$. The number 105 is the smallest ternary number (see Definition 1) and these will play a major role in this article. Issai Schur proved that every negative even number occurs as a cyclotomic coefficient. Emma Lehmer [21] reproduced his unpublished proof. Schur's argument is easily adapted to show that *every* integer is assumed as value of a cyclotomic coefficient, see Suzuki [28] or Moree and Hommersom [23, Proposition 5]. Let $m \geq 1$ be given. Ji, Li and Moree [19] adapted Schur's argument to show that

$$\{a_{mn}(j) : n \geq 1, j \geq 0\} = \mathbb{Z}. \tag{1}$$

*Mathematics Subject Classification (2000).* 11B83, 11C08

Fintzen [10] determined the set of all cyclotomic coefficients $a_n(j)$ with $j$ and $n$ in prescribed arithmetic progression, thus generalizing (1).

We put

$$A(n) = \max_{k\geq 0} |a_n(k)|, \ \ \mathcal{A} = \cup_{n\in\mathbb{N}} A(n), \ \ A\{n\} = \{a_n(k) : k \geq 0\},$$

in particular $A(n)$ is the height of the cyclotomic polynomial $\Phi_n$.

It is a classical result that if $n$ has at most two distinct odd prime factors, then $A(n) = 1$, cf. Lam and Leung [20]. The first non-trivial case arises where $n$ has precisely three distinct odd prime divisors and thus is of the form $n = p^e q^f r^g$, with $2 < p < q < r$ prime numbers. It is easy to deduce that $A\{p^e q^f r^g\} = A\{pqr\}$ using elementary properties of cyclotomic polynomials (as given for example by [23, Lemma 2]). It thus suffices to consider only the case where $e = f = g = 1$ and so $n = pqr$. This motivates the following definition.

**Definition 1.** *A cyclotomic polynomial $\Phi_n(x)$ is said to be* ternary *if $n = pqr$, with $2 < p < q < r$ primes. In this case we call the integer $n = pqr$* ternary*. The set of all ternary integers we denote by $\mathbb{N}_t$. We put $\mathcal{A}_t = \cup_{n\in\mathbb{N}_t} A(n)$.*

Note that $\mathcal{A}_t \subseteq \mathcal{A}$. In this article we address the question how the sets $\mathcal{A}, \mathcal{A}_t$ and $\mathcal{A}_{opt}$ (see Definition 2) look like.

**Conjecture 1.** *We have $\mathcal{A} = \mathbb{N}$, that is for any given natural number $m$ there is a cyclotomic polynomial having height $m$.*

**Conjecture 2.** *We have $\mathcal{A}_t = \mathbb{N}$, that is for any given natural number $m$ there is a ternary $n$ such that $\Phi_n$ has height $m$.*

The argument of Schur cannot be adapted to resolve Conjecture 1, as it allows one to control only the coefficients in a tail of a polynomial that quickly becomes very large if we want to show that some larger number occurs as a coefficient, and typically will have much larger coefficients than the coefficient constructed. Instead, we will make use of various properties of ternary cyclotomic polynomials. This class of cyclotomic polynomials has been intensively studied as it is the simplest one where the coefficients display non-trivial behavior. For these we still have $\{a_n(j) : n \text{ is ternary}, \ j \geq 0\} = \mathbb{Z}$, as a consequence of the following result.

**Theorem 1** (Bachman, [3])**.** *For every odd prime $p$ there exists an infinite family of polynomials $\Phi_{pqr}$ such that $A\{pqr\} = [-(p-1)/2, (p+1)/2] \cap \mathbb{Z}$ and another one such that $A\{pqr\} = [-(p+1)/2, (p-1)/2] \cap \mathbb{Z}$.*

If $n$ is ternary, then $A\{n\}$ consists of consecutive integers. Moreover, we have $|a_n(j+1) - a_n(j)| \leq 1$ for $j \geq 0$, see Gallot and Moree [14]. Note that for each of the members of the two families the cardinality of $A\{pqr\}$ is $p+1$. This is not a universal property for ternary $n$.

**Definition 2.** *If the cardinality of $A\{pqr\}$ is exactly $p+1$, we say that $\Phi_{pqr}$ is ternary optimal and call $n = pqr$ optimal. We denote by $\mathcal{A}_{opt}$ the set of all $A(n)$, with $n$ optimal.*

This terminology reflects the fact that $\#A\{pqr\} \leq p+1$, by [2, Corollary 3].

The latter bound only involves the smallest prime factor of the ternary integer $pqr$. This also holds for $A(pqr)$, which we know to be bounded above by $p-1$ since the $19^{th}$ century [22].

**Conjecture 3.** *We have* $\mathcal{A}_{opt} = \mathbb{N}\backslash\{1,5\}$.

We will see that this conjecture is closely related to the following prime number conjecture we propose (with $p_n$ the $n^{\text{th}}$ prime number).

**Conjecture 4.** *Let* $n \geq 31$ *(and so* $p_n \geq 127$*). Then*

$$p_{n+1} - p_n \leq \sqrt{p_n} - 1. \tag{2}$$

Although the gaps $d_n := p_{n+1} - p_n$ between consecutive prime numbers have been studied *in extenso* in the literature, this particular conjecture we have not come across. There is a whole range of conjectures on gaps between consecutive primes. The most famous one is Legendre's that there is a prime between consecutive squares is a bit weaker, but for example Firoozbakht's conjecture that $p_n^{1/n}$ is a strictly decreasing function of $n$ is much stronger. Firoozbakht's conjecture implies that $d_n < (\log p_n)^2 - \log p_n + 1$ for all $n$ sufficient large (see Sun [27]), contradicting a heuristic model, see Banks et al. [4], suggesting that given any $\epsilon > 0$ there are infinitely many $n$ such that $d_n > (2e^{-\gamma} - \epsilon)(\log p_n)^2$, with $\gamma$ Euler's constant. A very classical conjecture of Cramér states that $p_{n+1} - p_n = O((\log p_n)^2)$, which if true, clearly shows that the claimed bound in Conjecture 4 holds for all sufficiently large $n$.

A lot of numerical work on large gaps has been done (see the website [25]), and this can be used to deduce the following result.

**Proposition 1** (Tomás Oliveira e Silva [26])**.** *The inequality* (2) *holds whenever* $127 \leq p_n \leq 2 \cdot 10^{18}$.

We denote the set of natural numbers $\leq h$ by $\mathbb{N}_h$.

**Theorem 2.** *Let $h$ be an integer such that* (2) *holds for* $127 \leq p_n < 2h$*, then*

$$\mathbb{N}_h \subseteq \mathcal{A}_t \subseteq \mathcal{A}, \ \ \mathbb{N}_h\backslash\{1,5\} \subseteq \mathcal{A}_{opt}\backslash\{1,5\}.$$

**Corollary 1.** *If Conjecture* 4 *is true, then so are Conjectures* 1, 2 *and* 3.

Theorem 2 is in essence a consequence of a result of Moree and Roşu [24] (Theorem 6 below) generalizing Theorem 1, as we shall see in § 2.

On combining Proposition 1 and Theorem 2 we are led to the following conclusion.

**Theorem 3.** *Every integer up to* $10^{18}$ *occurs as the height of a ternary cyclotomic polynomial.*

The following theorem is the main result of our paper. Its proof rests on combining Lemma 3b), the key lemma used to prove Theorem 2, with deep work by Heath-Brown [18] and Yu [29] on gaps between primes.

**Theorem 4.** *Almost all positive integers occur as the height of an optimal ternary cyclotomic polynomial. Specifically, for any fixed* $\epsilon > 0$*, the number of positive integers* $\leq x$ *that do not occur as a height of an optimal ternary cyclotomic polynomial is* $\ll_\epsilon x^{3/5+\epsilon}$*. Under the Lindelöf Hypothesis this number is* $\ll_\epsilon x^{1/2+\epsilon}$*.*

(Readers unfamiliar with the Lindelöf Hypothesis are referred to the paragraph § 3 before the statement of Lemma 9.) In addition to Conjecture 4, there is a further prime number conjecture (that we have not come across in the literature) of relevance for the topic at hand.

**Conjecture 5.** *Let* $h > 1$ *be odd. There exists a prime* $p \geq 2h - 1$*, such that* $1 + (h-1)p$ *is a prime too.*

The widely believed Bateman–Horn conjecture [1] implies that given an odd $h > 1$, there are infinitely many primes $p$ such that $1 + (h-1)p$ is a prime too, and thus Conjecture 5 is a weaker version of this.

**Theorem 5.** *If Conjecture 5 holds true, then* $\mathcal{A}_t$ *contains all odd natural numbers. Unconditionally* $\mathcal{A}_t$ *contains a positive fraction of all odd natural numbers.*

The first assertion is a consequence of work of Gallot, Moree and Wilms [15] and involves ternary cyclotomic polynomials that are not optimal. The second makes use of deep work of Bombieri, Friedlander and Iwaniec [6] on the level of distribution of primes in arithmetic progressions with fixed residue and varying moduli. The level of distribution that is needed here goes beyond the square root barrier (that is studied in the Bombieri–Vinogradov theorem, for example) and this is due to the condition $p \geq 2h - 1$ in Conjecture 5, see Remark 2 for more details. As far as we are aware of, this is the first time that this kind of level of distribution is used in the subject of cyclotomic coefficients. We would like to point out though that Fouvry [12] has used the classical Bombieri–Vinogradov theorem in a rather different way and context, namely, for studying the number of nonzero coefficients of cyclotomic polynomials $\Phi_n$ with $n$ having two distinct prime factors. The proof of Theorem 5 is based on a second-moment argument and is found in § 4.

That prime numbers play such an important role in our approach is a consequence of working with ternary cyclotomic polynomials. One would want to work with $\Phi_n$ with $n$ having at least four prime factors, however this leads to a loss of control over the behaviour of the coefficients in general and the maximum in particular.

# 2 More on ternary cyclotomic polynomials

The goal of this section is to prove Theorem 2. Set

$$M(p;q) = \max\{A(pqr) : 2 < p < q < r\},$$

where $p$, $q$ are fixed and $r > q$. This quantity was introduced and first studied by Gallot et al. [15]. Put

$$M(p) = \max\{M(p;q) : q > p\}.$$

Since $A(pqr) \le p-1$ both quantities exist. Note that $M(p)$ is the largest height that occurs among the ternary cyclotomic polynomials having $p$ as smallest prime factor. Sister Marion Beiter [5] conjectured in 1968 that $M(p) \le (p+1)/2$ and proved it for $p \le 5$. Zhao and Zhang [30] proved it for $p = 7$ by showing that $M(7) = 4$. Theorem 1 implies that $M(p) \ge (p+1)/2$. Gallot and Moree [13] disproved the Sister Beiter conjecture. They showed that $M(p) > (p+1)/2$ for $p \ge 11$ and that, for $0 < \varepsilon < 2/3$ one has $M(p) > (2/3-\varepsilon)p$ for every $p$ sufficiently large. They conjectured that $M(p) \le 2p/3$. The smallest counterexample to the Sister Beiter conjecture occurs for $n = 17 \cdot 29 \cdot 41$ (see Table 1).

Given any $m \ge 1$, Moree and Roşu [24] constructed optimal ternary infinite families of $\Phi_{pqr}$ such that $A(pqr) = (p+1)/2+m$, provided that $p$ is large enough in terms of $m$.

**Theorem 6** (Theorem 1.1, [24])**.** *Let $p \ge 4m^2 + 2m + 3$ be a prime, with $m \ge 1$ any integer. Then there exists an infinite sequence of prime pairs $\{(q_j, r_j)\}_{j=1}^{\infty}$ with $pq_j < r_j, q_{j+1} > q_j$, such that*

$$A\{pq_jr_j\} = \left\{-\frac{(p-1)}{2} + m, \dots, \frac{p+1}{2} + m\right\}.$$

Put

$$\mathcal{R} = \left\{\frac{p+1}{2} + m : p \text{ is a prime, } m \ge 0,\ 4m^2 + 2m + 3 \le p\right\}. \tag{3}$$

**Lemma 1.** *We have $\mathcal{R} \subseteq \mathcal{A}_{opt}$.*

*Proof.* For the elements of $\mathcal{R}$ with $m = 0$ this follows from Theorem 1, for those with $m \ge 1$ it follows from Theorem 6. □

**Lemma 2.** *If* (2) *holds for $127 \le p_n < 2h$ with $h$ an integer, then we have* $\mathbb{N}_h \backslash \{1, 5, 63\} \subseteq \mathcal{R}$.

The proof is a consequence of part a) of the following lemma and the computational observation that $1, 5$ and $63$ are the only natural numbers $< 64$ that are not in $\mathcal{R}$.

By $\lfloor r \rfloor$ we denote the entire part of a real number $r$.

**Lemma 3.** *Let $n \ge 5$. Denote the interval $[\frac{p_n+1}{2}, \frac{p_{n+1}-1}{2}]$ by $I_n$.*
a) *If* (2) *holds, then $I_n \cap \mathbb{N} \subset \mathcal{R}$.*
b) *If* (2) *does not hold, then there are at most*

$$\lfloor (p_{n+1} - p_n - \sqrt{p_n} + 1)/2 \rfloor \tag{4}$$

*integers in the interval $I_n$ that are not in $\mathcal{R}$.*

*Proof.* The assumption on $n$ implies that $p_n \ge 11$. Put $z_n = (\sqrt{p_n} - 1)/2$. Note that $4z_n^2 + 2z_n + 3 = p_n - \sqrt{p_n} + 3 < p_n$. As $4x^2 + 2x + 3$ is increasing for $x \ge 0$, the inequality $4x^2 + 2x + 3 < p_n$ is satisfied for every real number $0 \le x \le z_n$. In particular it is satisfied for $x = m_n$, with $m_n$ the unique integer in the interval

$[z_n-1, z_n]$. Thus $m_n \geq (\sqrt{p_n}-3)/2$ and $4m_n^2+2m_n+3 \leq p_n$. Using it, we deduce that the numbers

$$(p_n+1)/2, \ldots, (p_n+1)/2+m_n$$

are in $\mathcal{R}$. As $(p_{n+1}+1)/2$ is clearly in $\mathcal{R}$, part a) follows if we can show that the final number $(p_n+1)/2+m_n$ is at least $(p_{n+1}-1)/2$. For this it suffices that

$$\frac{p_n+1}{2}+m_n \geq \frac{p_n+1}{2}+\frac{\sqrt{p_n}-3}{2} \geq \frac{p_{n+1}-1}{2},$$

where the second inequality is implied by the assumption (2). Part b) follows on noting that the number of integers of $\mathcal{R}$ that are not in $I_n$ is bounded above by $(p_{n+1}-p_n)/2-1-m_n$, which using $m_n \geq (\sqrt{p_n}-3)/2$ we see is bounded above by the integer in (4). $\square$

Since we believe that (2) holds for all $p_n \geq 127$, Lemma 2 leads us to make the following conjecture.

**Conjecture 6.** *We have* $\mathcal{R} = \mathbb{N}\backslash\{1, 5, 63\}$.

The numbers $1, 5$ and $63$ are special in our story.

**Lemma 4.** *The integers* 1 *and* 5 *are in* $\mathcal{A}_t \subseteq \mathcal{A}$, *but not in* $\mathcal{A}_{opt}$. *The integer* 63 *is in* $\mathcal{A}_{opt} \subset \mathcal{A}_t \subseteq \mathcal{A}$, *but not in* $\mathcal{R}$.

*Proof.* If $pqr$ is optimal, then $A(pqr) \geq (p+1)/2 \geq 2$ and so $1 \notin \mathcal{A}_{opt}$. It is also easy to see that there is no optimal $pqr$ such that $A(pqr)=5$ (and so $5 \notin \mathcal{A}_{opt}$). If such an optimal $pqr$ would exist, then as $A(pqr) \leq 3$ for $p \leq 5$ and $A(pqr) \geq 6$ for $p \geq 11$ (for an optimal $pqr$), this would force $p=7$ and $A\{7qr\}=[-5,2]\cap\mathbb{Z}$ or $A\{7qr\}=[-2,5]\cap\mathbb{Z}$, contradicting the result of Zhao and Zhang [30] that $M(7)=4$.

The number 63 is in $\mathcal{A}_{opt}$. This follows on applying Theorem 3.1 of [24]. The obvious approach is to consider the largest prime $p$ such that $(p+1)/2<63$, which is $p=113$, and take $l=11$ (here and below we use the notation of Theorem 3.1). For this combination the result does not apply, unfortunately. However, it does for $p=109$ and $l=15$. In this case we obtain $A\{109 \cdot 6803 \cdot 12084113\} = [-46, \ldots, 63]\cap\mathbb{Z}$ (with $q=6803$, $\rho=2870$, $\sigma=62$, $s=46$, $\tau=18$, $w=45$, $r_1=12084113$). $\square$

*Proof of Theorem 2.* Follows on combining Lemmas 1, 2 and 4. $\square$

# 3 Gaps between primes

The goal of this section is to prove Theorem 4. The quantity of central interest, $N(x)$, is defined below.

**Definition 3.** *The number of integers* $\leq x$ *that does not occur as a height of an optimal ternary cyclotomic polynomial is denoted by* $N(x)$.

Recall that $p_n$ denotes the $n^{\text{th}}$ prime and $d_n = p_{n+1} - p_n$. In this notation Conjecture 4 can be reformulated as

$$d_n \leq \sqrt{p_n} - 1, \text{ for } n \geq 31. \tag{5}$$

Unfortunately this conjecture is out of reach, even under the Riemann Hypothesis (RH). Cramér [8] showed in 1920 that under RH we have $d_n = O(\sqrt{p_n} \log p_n)$. He [9] conjectured in 1936 that

$$0 < \liminf_{x\to\infty} \frac{\max\{d_n : p_n \leq x\}}{(\log x)^2} \leq \limsup_{x\to\infty} \frac{\max\{d_n : p_n \leq x\}}{(\log x)^2} < \infty,$$

and gave heuristical arguments in support of this assertion. Cramér's conjecture implies that $d_n = O\left((\log p_n)^2\right)$. Further work on $d_n$ can be found in [4, 11, 16].

If Cramér's conjecture holds true, then the next lemma implies that $N(x) = O(1)$.

**Lemma 5.** *We have $N(x) \leq E(2x)/2 + O(1)$, where*

$$E(x) = \sum_{\substack{p_n \leq x \\ d_n \geq \sqrt{p_n} - 1}} (d_n - \sqrt{p_n} + 1).$$

*Proof.* By Lemma 1 it suffices to bound above the number of integers $\leq x$ that are not in $\mathcal{R}$. By Lemma 3b) this cardinality, on its turn, is bounded above by $E(2x)/2 + O(1)$. □

Heath-Brown [18] recently proved the following result estimating a quantity closely related to $E(x)$.

**Lemma 6** (Heath-Brown)**.** *We have*

$$\sum_{\substack{p_n \leq x \\ p_{n+1} - p_n \geq \sqrt{p_n}}} (p_{n+1} - p_n) \ll_\epsilon x^{3/5+\epsilon}.$$

As a warm-up we show how from this result with minor adaptations we can obtain a non-trivial upper bound for $N(x)$.

**Proposition 2.** *We have $N(x) \ll \frac{x}{\log x}$.*

*Proof.* By Lemma 5 it suffices to show that $E(x) \ll x/\log x$. In the sum defining $E(x)$ there are two kinds of terms, namely those with $d_n \geq \sqrt{p_n}$ and the ones with $\sqrt{p_n} - 1 \leq d_n < \sqrt{p_n}$, for which obviously

$$0 \leq d_n - \sqrt{p_n} + 1 < 1. \tag{6}$$

The first ones can be dealt with directly using Lemma 6, leading to a contribution

$$\sum_{\substack{p_n \leq x \\ d_n \geq \sqrt{p_n}}} (d_n - \sqrt{p_n} + 1) \leq \sum_{\substack{p_n \leq x \\ d_n \geq \sqrt{p_n}}} d_n \ll_\epsilon x^{3/5+\epsilon}.$$

Taking into account the contribution of the second type of terms and (6) leads to

$$\sum_{\substack{p_n \leq x \\ \sqrt{p_n} - 1 \leq d_n < \sqrt{p_n}}} (d_n - \sqrt{p_n} + 1) \leq \sum_{\substack{p_n \leq x \\ \sqrt{p_n} - 1 \leq d_n < \sqrt{p_n}}} 1 \leq \sum_{p_n \leq x} 1 \ll x/\log x.$$

Putting the two estimates together yields the required estimate. □

We recall the following recent result of Heath-Brown [18, Theorem 2].

**Lemma 7** (Heath-Brown)**.** *For any fixed $\epsilon > 0$ the measure of the set of $y$ in $[0, x]$ such that*

$$\max_{0\le h\le\sqrt{y}} \left| \pi(y+h) - \pi(y) - \int_y^{y+h} \frac{\mathrm{d}t}{\log t} \right| \geq \frac{\sqrt{y}}{(\log y)(\log\log y)}$$

*is $O_\epsilon(x^{3/5+\epsilon})$, where $\pi(y)$ denotes the number of primes not exceeding $y$.*

A closer look at the proof of Lemma 6 leads to the following slightly stronger lemma.

**Lemma 8.** *For every fixed $C > 0$ and $\epsilon > 0$ we have*

$$\sum_{\substack{p_n\le x\\ p_{n+1}-p_n\geq C\sqrt{p_n}}} (p_{n+1} - p_n) \ll_{C,\epsilon} x^{3/5+\epsilon}.$$

*Proof.* If $C \geq 1$, the result is a corollary of Lemma 6. Therefore, without loss of generality, we may assume that $0 < C < 1$.

Suppose that $p_n \leq x$ and $d_n \geq C\sqrt{p_n}$. Since $p_{n+1} < 2p_n < 4p_n$ by Bertrand's Postulate, it follows that $\sqrt{p_n} > \frac{1}{2}\sqrt{p_{n+1}}$ and hence $d_n \geq \frac{C}{2}\sqrt{p_{n+1}}$. This shows that if $y \in J_n := (p_n, p_{n+1} - \frac{C}{2}\sqrt{p_{n+1}})$, then

$$y + \frac{C}{2}\sqrt{y} < p_{n+1} - \frac{C}{2}\sqrt{p_{n+1}} + \frac{C}{2}\sqrt{y} < p_{n+1} - \frac{C}{2}\sqrt{p_{n+1}} + \frac{C}{2}\sqrt{p_{n+1}} = p_{n+1},$$

so that $\pi(y + \frac{C}{2}\sqrt{y}) = \pi(y)$. Note that, for $y$ sufficiently large,

$$\int_y^{y+\frac{C}{2}\sqrt{y}} \frac{\mathrm{d}t}{\log t} \geq \frac{\sqrt{y}}{(\log y)(\log\log y)}.$$

Since by assumption $C < 1$, the length of the interval $(y, y + \frac{C}{2}\sqrt{y})$ is bounded by the square root of its smallest element and hence Lemma 7 can be applied. Therefore, the contribution of primes $p_n \leq x$ with $d_n \geq C\sqrt{p_n}$ towards the set of $y$ in Lemma 7 is at least

$$\mathrm{meas}(J_n) \;=\; d_n - \frac{C}{2}\sqrt{p_{n+1}} \geq d_n - \frac{C}{2}\sqrt{2p_n} \geq d_n - \frac{d_n}{\sqrt{2}} \geq \frac{d_n}{10},$$

provided that $n$ is large enough. In particular, there exists an absolute positive constant $C'$ such that

$$\sum_{\substack{p_n\le x\\ d_n\geq C\sqrt{p_n}}} d_n \leq C' + 10 \sum_{\substack{p_n\le x\\ d_n\geq C\sqrt{p_n}}} \mathrm{meas}(J_n),$$

which is bounded above by $O_\epsilon(x^{3/5+\epsilon})$ by Lemma 7 and the fact that the intervals $J_n$ are pairwise disjoint. □

*Proof of the unconditional bound of Theorem 4.* ¿From Lemma 8 we can immediately deduce that $N(x) \ll_\epsilon x^{3/5+\epsilon}$. To this end we first note that the inequality $1-\sqrt{p_n} \le 0$ gives

$$\sum_{\substack{p_n \le x \\ d_n \ge \sqrt{p_n}-1}} \left(d_n + 1 - \sqrt{p_n}\right) \le \sum_{\substack{p_n \le x \\ d_n \ge \sqrt{p_n}-1}} d_n.$$

We have $\sqrt{p_n} - 1 \ge \frac{1}{4}\sqrt{p_n}$ and so

$$\sum_{\substack{p_n \le x \\ d_n \ge \sqrt{p_n}-1}} d_n \le \sum_{\substack{p_n \le x \\ d_n \ge \frac{1}{4}\sqrt{p_n}}} d_n.$$

Applying Lemma 8 with $C = \frac{1}{4}$ concludes the proof. □

In order to complete the proof of Theorem 4 we need to improve the exponent 3/5 in Lemma 8 to 1/2, conditionally on the Lindelöf Hypothesis. The Lindelöf Hypothesis states that for all fixed $\epsilon > 0$ we have

$$\zeta(1/2+it) = O_\epsilon\left(t^\epsilon\right), \; t \in \mathbb{R}, \; t > 1,$$

where as usual $\zeta$ denotes the Riemann zeta function. It is well-known that the Riemann Hypothesis implies the Lindelöf Hypothesis, but also that the two conjectures are not equivalent. There is a large body of work concerning the Lindelöf Hypothesis (see, for example, the recent work of Bourgain [7]), however, it is still open.

We will make use of the following result of Yu [29].

**Lemma 9** (Yu)**.** *Fix any $\epsilon > 0$. Under the Lindelöf Hypothesis we have*

$$\sum_{p_n \le x} (p_{n+1} - p_n)^2 \ll_\epsilon x^{1+\epsilon}.$$

¿From it one can easily derive a conditional improvement of Lemma 8.

**Lemma 10.** *Assume the Lindelöf Hypothesis and fix any $C > 0$ and $\epsilon > 0$. Then we have*

$$\sum_{\substack{p_n \le x \\ p_{n+1}-p_n \ge C\sqrt{p_n}}} (p_{n+1} - p_n) \ll_{C,\epsilon} x^{1/2+\epsilon}.$$

*Proof.* For all $C > 0$ one has

$$\sum_{\substack{p_n \le x \\ d_n \ge C\sqrt{p_n}}} d_n \ll (\log x) \max_{1 \le y \le x} \sum_{\substack{y < p_n \le 2y \\ d_n \ge C\sqrt{p_n}}} d_n \le (\log x) \max_{1 \le y \le x} \sum_{\substack{y < p_n \le 2y \\ d_n \ge C\sqrt{p_n}}} \frac{d_n^2}{C\sqrt{p_n}},$$

which by Lemma 9 is at most

$$(\log x) \max_{1 \le y \le x} \frac{1}{C\sqrt{y}} \sum_{y < p_n \le 2y} d_n^2 \ll_{C,\epsilon} x^{1/2+\epsilon}.$$

□

*Proof of the conditional bound of Theorem 4.* Here we use Lemma 10 to prove that under the Lindelöf Hypothesis the number of exceptional positive integers $\le x$ is $\ll_\epsilon x^{1/2+\epsilon}$. This can be done in a manner that is similar to our deduction of the unconditional bound $\ll_\epsilon x^{3/5+\epsilon}$ from Lemma 8 and the proof is left to the reader. □

## 4 A special case of the Bateman–Horn conjecture on average

The goal of this section is to prove Theorem 5.

**Lemma 11.** *Let $h > 1$ be odd. If there exists a prime $p \geq 2h-1$, such that the integer $1+(h-1)p$ is a prime too, then $A(n) = h$ for some ternary $n$.*

*Proof.* This is a consequence of the result of Gallot et al. [15, Theorem 43] that if $q \equiv 1 \pmod{p}$, then

$$M(p;q) = \min\Big\{\frac{q-1}{p}+1, \frac{p+1}{2}\Big\}.$$

The conditions on $p$ and $h$ ensure that $M(p;q) = h$. □

**Example 1.** Using the latter result and [15, Lemma 24], we find that

$$A(131 \cdot 8123 \cdot 25497973) = \frac{8123-1}{131} + 1 = 63$$

and $a_{131\cdot 8123\cdot 25497973}(13459462019674) = -63$.

We define the set $G \subset \mathbb{N}$ as follows,

$$G := \{m \in \mathbb{N} : \exists p > 4m \text{ such that } 1+2mp \text{ is prime}\}.$$

In the remaining part of this section we show that the density of $G$ among all integers is positive, i.e. that there exists $c_0 > 0$ such that

$$\liminf_{M\to+\infty} \frac{\#\{m \in G \cap [1,M]\}}{M} \geq c_0. \tag{7}$$

For any natural number $m$ and any $x \in \mathbb{R}$ we define

$$\pi_m(x) := \#\left\{\tfrac{x}{2} \leq p < x : 1+2mp \text{ is prime}\right\}.$$

**Lemma 12.** *For all $x, M \in \mathbb{R}$ with $x > 8M$ and $M \geq 1$ we have*

$$\#\{m \in G \cap [1,M]\} \sum_{1\leq m\leq M} \pi_m(x)^2 \geq \Big(\sum_{1\leq m\leq M} \pi_m(x)\Big)^2. \tag{8}$$

*Proof.* Put

$$u_m(x) = \begin{cases} 1, & \text{if } \pi_m(x) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Fix $x > 8M$. By Cauchy's inequality we have

$$\begin{aligned}\sum_{1\leq m\leq M} \pi_m(x) &= \sum_{1\leq m\leq M} \pi_m(x)u_m(x) \\ &\leq \#\{1 \leq m \leq M : \pi_m(x) > 0\}^{1/2}\Big(\sum_{1\leq m\leq M} \pi_m(x)^2\Big)^{1/2}.\end{aligned}$$

If $m \leq M$ and $p \geq x/2$, then $4m \leq 4M < x/2 \leq p$, hence

$$\#\{1 \leq m \leq M : \pi_m(x) > 0\} \leq \#\{m \in G \cap [1,M]\},$$

concluding the proof. □

We would like to estimate the sums $\sum_{1\le m\le M}\pi_m(x)$ and $\sum_{1\le m\le M}\pi_m(x)^2$ occuring above. One may easily obtain an upper bound, say $A$, for $\sum_{1\le m\le M}\pi_m(x)^2$ by using standard sieve results. If we get a lower bound $\sum_{1\le m\le M}\pi_m(x)\ge B$, then by (8)

$$\#\{m\in G\cap[1,M]\}\ge\frac{B^2}{A}.$$

A good lower bound for $\sum_m \pi_m(x)$ is however not easy to prove owing to the condition $x>8M$; the way to overcome this is to use deep work of Bombieri–Friedlander–Iwaniec regarding the level of distribution of primes in arithmetic progressions with fixed residue and varying moduli.

We start with $\sum_{1\le m\le M}\pi_m(x)^2$, for which we need the following lemma, which is obtained on putting $b=k=l=1$ in [17, Theorem 3.12].

**Lemma 13.** *Let $a$ be a positive even integer. Then for all $x>1$ we have, uniformly in $a$, that*

$$\#\{p\le x: ap+1 \text{ is prime}\}\le\frac{8C_2x}{(\log x)^2}\prod_{\substack{p\mid a\\ p>2}}\Big(\frac{p-1}{p-2}\Big)\Big\{1+O\Big(\frac{\log\log x}{\log x}\Big)\Big\},$$

*where*

$$C_2=\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big)$$

*is the twin prime constant.*

**Remark 1.** Hardy and Littlewood conjectured based on heuristic reasoning that asymptotically

$$\#\{p\le x: p+2 \text{ is prime}\}\sim 2C_2\frac{x}{(\log x)^2}.$$

A similar heuristic reasoning leads to the conjecture that asymptotically

$$\#\{p\le x: ap+1 \text{ is prime}\}\sim C_2\Bigg(\prod_{\substack{p\mid a\\ p>2}}\Big(\frac{p-1}{p-2}\Big)\Bigg)\frac{x}{(\log x)^2}.$$

Both conjectures are special cases of the Bateman-Horn conjecture, cf. [1].

**Lemma 14.** *Let $x,M$ be any two positive real numbers. Then*

$$\sum_{1\le m\le M}\pi_m(x)^2\le 64C_1C_2^2M\frac{x^2}{(\log x)^4}\left\{1+O\left(\frac{\log\log x}{\log x}+\frac{1}{\sqrt{M}}\right)\right\},$$

*where the implied constant is absolute and*

$$C_1:=\prod_{p>2}\left(1+\frac{2}{p(p-2)}+\frac{1}{p(p-2)^2}\right). \tag{9}$$

*Proof.* By Lemma 13 with $a = 2m$, we get

$$\pi_m(x)^2 \leq 8^2 C_2^2 \frac{x^2}{(\log x)^4} \prod_{\substack{p|2m \\ p>2}} \left(\frac{p-1}{p-2}\right)^2 \left\{1 + O\left(\frac{\log\log x}{\log x}\right)\right\},$$

therefore, we conclude that the sum in our lemma is at most

$$8^2 C_2^2 \frac{x^2}{(\log x)^4} \left\{1 + O\left(\frac{\log\log x}{\log x}\right)\right\} \sum_{1\leq m\leq M} \prod_{\substack{p|m \\ p>2}} \left(\frac{p-1}{p-2}\right)^2.$$

We define the multiplicative function $f$ via

$$f(p^e) := \mathbb{1}_{p>2}(p)\mathbb{1}_{e=1}(e)\left(\frac{2}{p-2} + \frac{1}{(p-2)^2}\right), \quad (e \in \mathbb{N},\, p \text{ prime}).$$

One can then easily verify that

$$\prod_{\substack{p|k \\ p>2}} \left(\frac{p-1}{p-2}\right)^2 = \sum_{d|k} f(d) = \sum_{\substack{d|k \\ 2\nmid d}} f(d)$$

for all non-zero integers $k$. This shows that

$$\begin{aligned}\sum_{1\leq m\leq M} \prod_{\substack{p|2m \\ p>2}} \left(\frac{p-1}{p-2}\right)^2 &= \sum_{\substack{1\leq d\leq M \\ 2\nmid d}} f(d) \sum_{\substack{1\leq m\leq M \\ d|2m}} 1 \\ &= M \sum_{1\leq d\leq M} \frac{f(d)}{d} + O\Big(\sum_{1\leq d\leq M} f(d)\Big),\end{aligned}$$

where we used several times that $f(d) = 0$ if $d$ is even. Noting that $f(p) \leq C/p$ for some absolute constant $C > 0$ yields the bound

$$f(d) \leq \mu(d)^2 \frac{C^{\omega(d)}}{d} \ll \frac{1}{\sqrt{d}}, \quad (d \in \mathbb{N}),$$

which can be used to obtain

$$\sum_{1\leq d\leq M} \frac{f(d)}{d} = \sum_{d=1}^{\infty} \frac{f(d)}{d} + O\left(\sum_{d>M} \frac{1}{d^{3/2}}\right) = C_1 + O\left(\frac{1}{\sqrt{M}}\right)$$

and

$$\sum_{1\leq d\leq M} f(d) \ll \sum_{1\leq d\leq M} \frac{1}{\sqrt{d}} \ll \sqrt{M}.$$

Putting everything together it follows that

$$\sum_{1\leq m\leq M} \prod_{\substack{p|2m \\ p>2}} \left(\frac{p-1}{p-2}\right)^2 = C_1 M \left\{1 + O\left(\frac{1}{\sqrt{M}}\right)\right\},$$

which is sufficient for our purposes. $\square$

We now proceed to evaluate the sum $\sum_{1\leq m\leq M}\pi_m(x)$ appearing in Lemma 12. Writing $n=1+2mp$ we see that the it equals

$$\begin{aligned}
&\sum_{1\leq m\leq M}\sum_{\substack{x/2\leq p<x\\ 1+2mp \text{ prime}}}1=\sum_{x/2\leq p<x}\sum_{\substack{1\leq m\leq M\\ 1+2mp \text{ prime}}}1\\
=&\sum_{x/2\leq p<x}\#\left\{n \text{ prime}: 2<n\leq 1+2Mp,\ n\equiv 1(\bmod p)\right\}\\
\geq&\frac{1}{\log(1+2Mx)}\sum_{x/2\leq p<x}\sum_{\substack{n \text{ prime}\\ 2<n\leq 1+2Mp\\ n\equiv 1(\bmod p)}}\log n,
\end{aligned}\tag{10}$$

where we used that $\log n\leq\log(1+2Mp)\leq\log(1+2Mx)$.

**Remark 2.** One now recognizes the argument in the latter sum as a counting function of primes in an arithmetic progression of varying modulus as $p$ runs through $(x/2,x]$. We would now use the Bombieri–Vinogradov theorem, however, the size of the primes $n$ is of the order of magnitude

$$1+2Mp\approx 2Mx,$$

since the moduli $p$ have typical size $x$. Therefore, owing to the condition $x>8M$, we are counting primes in a progression whose modulus exceeds the square-root of the size of the primes. Therefore, the Bombieri–Vinogradov theorem cannot be applied in our case. To be more precise, it can only be applied when the moduli are bounded by $\sqrt{z}/(\log z)^A$, where $A>0$ and $z$ is the length of the interval $(0,z]$ we are counting primes in. This means that we need

$$p\leq\frac{\sqrt{p2M}}{(\log(p2M))^A},$$

for some fixed $A>0$, and this can only happen when $x=o(M)$. To deal with this problem we shall need a special case (Lemma 15 below), of the work of Bombieri–Friedlander–Iwaniec [6].

As usual let

$$\theta(x;q,a):=\sum_{\substack{p\leq x\\ p\equiv a(\bmod q)}}\log p,\quad \psi(x;q,a):=\sum_{\substack{n\leq x\\ n\equiv a(\bmod q)}}\Lambda(n),$$

with $\Lambda$ the von Mangoldt function.

**Lemma 15** (Bombieri–Friedlander–Iwaniec)**.** *For any $t\geq y\geq 3$ we have*

$$\sum_{\sqrt{ty}/2\leq q<\sqrt{ty}}\left|\psi(t;q,1)-\frac{t}{\phi(q)}\right|\ll t\left(\frac{\log y}{\log t}\right)^2(\log\log t)^B,$$

*where $B$ is an absolute constant and the implied constant is absolute.*

This can be obtained by setting $a=1, x=t, Q=\sqrt{xy}$ in [6, Main Theorem, p. 363].

**Lemma 16.** *For any $t \geq y \geq 3$ with $y \leq t^{1/20}$ we have*

$$\sum_{\substack{q \text{ prime} \\ \sqrt{ty}/2 \leq q < \sqrt{ty}}} \left| \theta(t;q,1) - \frac{t}{\phi(q)} \right| \ll t \left( \frac{\log y}{\log t} \right)^2 (\log \log t)^B,$$

*where $B$ is an absolute constant and the implied constant is absolute.*

*Proof.* Clearly

$$\psi(t;q,1) = \theta(t;q,1) + \sum_{k=2}^{\infty} \sum_{\substack{p \leq t^{1/k} \\ p^k \equiv 1 \pmod{q}}} \log p.$$

The inner sum vanishes if $t^{1/k} < 2$, therefore only the integers $k \leq (\log t)/\log 2$ are to be taken into account. The contribution of all such integers with $k \geq 3$ is $\ll t^{1/3} \log t$, since the sum over $p$ is $\ll t^{1/k}$ by the prime number theorem. The steps so far are the standard arguments that one performs when moving from asymptotics for $\psi$ to asymptotics for $\theta$, however, in our case, owing to the level of distribution being comparable to the square root of the length of the interval, the term $k = 2$ cannot be controlled with the classical arguments. Instead, we use the bound

$$\frac{1}{\log t} \sum_{\substack{p \leq t^{1/2} \\ p^2 \equiv 1 \pmod{q}}} \log p \leq \sum_{\substack{m \leq t^{1/2} \\ m^2 \equiv 1 \pmod{q}}} 1 = \sum_{\substack{m \leq t^{1/2} \\ m \equiv -1 \pmod{q}}} 1 + \sum_{\substack{m \leq t^{1/2} \\ m \equiv 1 \pmod{q}}} 1,$$

where we used the fact that $q$ is prime. Each of the sums in the right side is trivially $\ll t^{1/2}/q + 1$ and therefore

$$\sum_{\substack{p \leq t^{1/2} \\ p^2 \equiv 1 \pmod{q}}} \log p \ll (\log t) \left( \frac{t^{1/2}}{q} + 1 \right).$$

We thus find that

$$\psi(t;q,1) = \theta(t;q,1) + O\left( t^{1/3}(\log t) + \frac{t^{1/2}}{q} \log t \right).$$

This shows that the sum over $q$ in the statement of this lemma is

$$\ll \sum_{\sqrt{ty}/2 \leq q < \sqrt{ty}} \left| \psi(t;q,1) - \frac{t}{\phi(q)} \right| + \sum_{\sqrt{ty}/2 \leq q < \sqrt{ty}} \left( t^{1/3}(\log t) + \frac{t^{1/2}}{q} \log t \right).$$

The first sum can be bounded by Lemma 15. Noting that $\sum_{x/2 < q \leq x} 1/q = O(1)$, cf. (11), we see that the second sum is

$$\ll (ty)^{1/2} t^{1/3} (\log t) + t^{1/2} \log t,$$

which is $\ll t^{19/20} \ll t(\log t)^{-2}$, as $y \leq t^{1/20}$. $\square$

**Lemma 17.** *Let $\psi : (1,\infty) \to (4,\infty)$ be any non-decreasing function satisfying $\psi(M) \le \log M + 4$ and $\lim_{M\to\infty} \psi(M) = \infty$. For any $M > 1$, we let $x = M\psi(M)$ and have*

$$\sum_{1\le m\le M} \pi_m(x) \ge \frac{Mx}{\log(Mx)} \frac{\log 2}{\log x} \left\{1 + O\left(\frac{(\log\log x)^{B+2}}{\log x}\right)\right\},$$

*where $B$ is the absolute constant from Lemma 16.*

*Proof.* By (10) and the inequality $1 + 2Mp \ge 2Mp \ge Mx$ valid for primes $p \ge x/2$, we see that the sum in our lemma is at least

$$\frac{1}{\log(1+2Mx)} \sum_{x/2\le p<x} \sum_{\substack{n \text{ prime} \\ 2<n\le Mx \\ n\equiv 1 (\bmod\ p)}} \log n.$$

Using Lemma 16 with $t = Mx$ and $y = \psi(M)$ shows that this is

$$\frac{Mx}{\log(1+2Mx)} \sum_{x/2\le p<x} \frac{1}{p-1} + O\left(\frac{Mx}{\log(1+2Mx)} \left(\frac{\log\psi(M)}{\log x}\right)^2 (\log\log x)^B\right).$$

Using the standard estimate

$$\sum_{p\le x} \frac{1}{p-1} = \log\log x + C' + O\left(\frac{1}{(\log x)^2}\right),$$

we obtain

$$\sum_{x/2<p\le x} \frac{1}{p-1} = \frac{\log 2}{\log x}\left\{1 + O\left(\frac{1}{\log x}\right)\right\}. \tag{11}$$

It follows that the main term is as claimed in our lemma. Furthermore, on using the bound $\log\psi(M) \ll \log\log M \ll \log\log x$, we see that the error term is

$$\ll \frac{Mx}{\log(Mx)} \frac{(\log\log x)^{B+2}}{(\log x)^2},$$

as required. □

*Proof of Theorem 5.* The first assertion is a corollary of Lemma 11.

The inequalities obtained in Lemmas 14 and 17 in combination with the inequality in Lemma 12 give rise, on choosing $x = M\log M$, to the inequality

$$\#\{m \in G \cap [1,M]\}\, 64C_1C_2^2 M \frac{x^2}{(\log x)^4} \ge \left(\frac{Mx}{\log(Mx)} \frac{\log 2}{\log x}\right)^2 (1 + o(1)).$$

We combine this with the estimate

$$\log(Mx) = \log(x^2/\psi(M)) = 2\log x - \log\psi(M) = 2\log x(1 + o(1)),$$

and conclude that (7) holds with

$$c_0 = \frac{(\log 2)^2}{256C_1C_2^2} > 0.$$

It follows that a positive percentage of all integers $m$ are such that there exists a prime $p > 4m$ with $1 + 2mp$ being a prime. By Lemma 11 for each $m$ with the property that there exists a prime $p > 4m$ with $1 + 2mp$ also a prime, we have $1 + 2m \in \mathcal{A}_t$ and it thus follows that unconditionally $\mathcal{A}_t$ contains a positive fraction of all odd natural numbers. □

**Acknowledgements.** The authors thank Danilo Bazzanella, Tomás Oliveira e Silva, Alberto Perelli and Tim Trudgian for helpful email correspondence. Tomás provided Proposition 1. Olivier Ramaré kindly provided us with a high accuracy evaluation of $C_1$.

In case an integer $h$ is not in $\mathcal{R}$ (defined in (3)), still the work of Moree and Roşu [24] offers some hope to show that $h$ occurs as a height (as we saw in case $h = 63$). To make this more precise involves understanding the distribution of inverses modulo primes. We thank Cristian Cobeli for sharing some observations and numerical experiments on this.

A major part of the third author's work was completed during his Postdoc in the Max Planck Institute for Mathematics in Bonn. The fourth author is supported by National Natural Science Foundation of China (Grant No. 11801303), project ZR2019QA016 supported by Shandong Provincial Natural Science Foundation and Project funded by China Postdoctoral Science Foundation (Grant No. 2018M640617).

# 5 Ternary cyclotomic polynomials of small height

**Table 1:** `Ternary examples with prescribed height`

| height | $p$ | $q$ | $r$ | $k$ | sign | diff. |
|---|---|---|---|---|---|---|
| **1** | 3 | 7 | 11 | 0 | + | 2 |
| **2** | 3 | 5 | 7 | 7 | − | **3** |
| **3** | 5 | 7 | 11 | 119 | − | **5** |
| **4** | 11 | 13 | 17 | 677 | − | 7 |
| **5** | 11 | 13 | 19 | 1008 | − | 9 |
| **6** | 13 | 23 | 29 | 2499 | − | 10 |
| **7** | 17 | 19 | 53 | 6013 | + | 14 |
| **8** | 17 | 31 | 37 | 5596 | − | 14 |
| **9** | 17 | 47 | 53 | 14538 | − | **17** |
| **10** | 17 | 29 | 41 | 4801 | − | **17** |
| **11** | 23 | 37 | 61 | 20375 | − | 16 |
| **12** | 23 | 37 | 41 | 14471 | + | 21 |
| **13** | 31 | 59 | 73 | 58333 | − | 25 |
| **14** | 37 | 53 | 61 | 52286 | + | 27 |
| **15** | 37 | 47 | 61 | 45939 | − | 29 |
| **16** | 41 | 79 | 97 | 133844 | − | 30 |
| **17** | 41 | 43 | 53 | 38240 | + | 33 |
| **18** | 61 | 97 | 103 | 178013 | − | 34 |
| **19** | 43 | 83 | 89 | 101051 | − | 33 |
| **20** | 47 | 83 | 131 | 235842 | + | 37 |
| **21** | 47 | 101 | 109 | 217278 | − | 41 |
| **22** | 53 | 83 | 89 | 165453 | − | 44 |
| **23** | 43 | 71 | 109 | 108355 | + | **43** |
| **24** | 53 | 103 | 109 | 189160 | − | 42 |
| **25** | 61 | 79 | 97 | 224640 | − | 47 |
| **26** | 41 | 71 | 97 | 96529 | − | **41** |
| **27** | 61 | 109 | 113 | 332589 | − | 54 |
| **28** | 53 | 89 | 131 | 186685 | − | **53** |
| **29** | 83 | 109 | 139 | 552035 | − | 58 |
| **30** | 67 | 131 | 137 | 389139 | − | 52 |
| **31** | 83 | 107 | 113 | 444435 | + | 61 |
| **32** | 79 | 149 | 163 | 881529 | + | 63 |
| **33** | 73 | 103 | 113 | 389314 | + | 61 |
| **34** | 71 | 109 | 113 | 409320 | − | 60 |
| **35** | 83 | 103 | 139 | 544198 | − | 69 |
| **36** | 127 | 149 | 151 | 1246462 | − | 72 |
| **37** | 71 | 101 | 239 | 671716 | + | 67 |
| **38** | 127 | 137 | 409 | 3355658 | − | 75 |
| **39** | 83 | 149 | 157 | 941094 | + | 76 |
| **40** | 79 | 233 | 239 | 1624556 | + | **79** |

Table 1 gives the minimum ternary integer $n = pqr$ with $p < q < r$ such that $A(n) = m$ for the numbers $m = 1, \ldots, 40$. The integer $k$ has the property that $a_{pqr}(k) = \pm m$, with the sign coming from the one but last column. The final column records the difference between and the largest and smallest coefficient and is in bold if this is optimal, that is equals $p$ (compare Definition 2).

## References


[1] S. L. Aletheia-Zomlefer, L. Fukshansky and S. R. Garcia, The Bateman-Horn conjecture: heuristics, history and applications, `arXiv:math.NT/1807.08899`.

[2] G. Bachman, On the coefficients of ternary cyclotomic polynomials, *J. Number Theory* **100** (2003), 104–116.

[3] G. Bachman, Ternary cyclotomic polynomials with an optimally large set of coefficients, *Proc. Amer. Math. Soc.* **132** (2004), 1943–1950.

[4] W. Banks, K. Ford and T. Tao, Large prime gaps and probabilistic models, arXiv:1908.08613.

[5] M. Beiter, Magnitude of the coefficients of the cyclotomic polynomial $F_{pqr}(x)$, *Amer. Math. Monthly* **75** (1968), 370–372.

[6] E. Bombieri, J. B. Friedlander and H. Iwaniec, Primes in arithmetic progressions to large moduli. II, *J. Math. Ann.* **277** (1987), 361–393.

[7] J. Bourgain, Decoupling, exponential sums and the Riemann zeta function, *J. Amer. Math. Soc.* **30** (2017), 205–224.

[8] H. Cramér, Some theorems concerning prime numbers, *Arkiv f. Math. Astr. Fys.* **15** (1920), 1–33. [Collected Works **1**, 85–91, Springer, Berlin-Heidelberg, 1994.]

[9] H. Cramér, On the order of magnitude of the difference between consecutive primes, *Acta Arith.* **2** (1936), 23–46. [Collected Works **2**, 871–894, Springer, Berlin-Heidelberg, 1994.]

[10] J. Fintzen, Cyclotomic polynomial coefficients $a(n, k)$ with $n$ and $k$ in prescribed residue classes, *J. Number Theory* **131** (2011), 1852–1863.

[11] K. Ford, B. Green, S. Konyagin, J. Maynard and T. Tao, Long gaps between primes, *J. Amer. Math. Soc.* **31** (2018), 65–105.

[12] É. Fouvry, On binary cyclotomic polynomials, *Algebra Number Theory* **7** (2013), 1207–1223.

[13] Y. Gallot and P. Moree, Ternary cyclotomic polynomials having a large coefficient, *J. reine angew. Math.* **632** (2009), 105–125.

[14] Y. Gallot and P. Moree, Neighboring ternary cyclotomic coefficients differ by at most one, *J. Ramanujan Math. Soc.* **24** (2009), 235–248.

[15] Y. Gallot, P. Moree and R. Wilms, The family of ternary cyclotomic polynomials with one free prime, *Involve* **4** (2011), 317–341.

[16] A. Granville, Harald Cramér and the distribution of prime numbers, *Scand. Actuar. J.* **1** (1995), 12–28.

[17] H. Halberstam and H.-E. Richert, *Sieve methods*, London Mathematical Society Monographs **4**, Academic Press, London-New York, 1974.

[18] D. R. Heath-Brown, The differences between consecutive primes. V, `https://arxiv.org/abs/1906.09555`.

[19] C.-G. Ji, W.-P. Li and P. Moree, Values of coefficients of cyclotomic polynomials II, *Discrete Math.* **309** (2009), 1720–1723.

[20] T.Y. Lam and K.H. Leung, On the cyclotomic polynomial $\Phi_{pq}(X)$, *Amer. Math. Monthly* **103** (1996), 562–564.

[21] E. Lehmer, On the magnitude of the coefficients of the cyclotomic polynomials, *Bull. Amer. Math. Soc.* **42** (1936), 389–392.

[22] A. Migotti, Zur Theorie der Kreisteilungsgleichung, *S.-B. der Math.-Naturwiss. Classe der Kaiserlichen Akademie der Wissenschaften, Wien,* (2) **87** (1883), 7–14.

[23] P. Moree and H. Hommersom, Value distribution of Ramanujan sums and cyclotomic polynomial coefficients, `arXiv:math.NT/0307352`.

[24] P. Moree and E. Roşu, Non-Beiter ternary cyclotomic polynomials with an optimally large set of coefficients, *Int. J. Number Theory* **8** (2012), 1883–1902.

[25] T. R. Nicely, First occurrence prime gaps, web page `http://www.trnicely.net/gaps/gaplist.html`.

[26] T. Oliveira e Silva, E-mail to second author, 24-08-2019.

[27] Z.-W. Sun, On a sequence involving sums of primes, *Bull. Aust. Math. Soc.* **88** (2013), 197–205.

[28] J. Suzuki, On coefficients of cyclotomic polynomials, *Proc. Japan Acad. Ser. A Math. Sci.* **63** (1987), 279–280.

[29] G. Yu, The differences between consecutive primes, *Bull. London Math. Soc.* **28** (1996), 242–248.

[30] J. Zhao and X. Zhang, Coefficients of ternary cyclotomic polynomials, *J. Number Theory* **130** (2010), 2223–2237.

Institute of Mathematics,
Ukrainian National Academy of Sciences,
3 Tereshchenkivs'ka Str., 01024 Kyiv, Ukraine.
e-mail: kosyak02@gmail.com

Max-Planck-Institut für Mathematik,
Vivatsgasse 7, D-53111 Bonn, Germany.
e-mail: moree@mpim-bonn.mpg.de

School of Mathematics and Statistics,
University of Glasgow,
University Place, Glasgow, G12 8SQ, United Kingdom.
E-mail: efthymios.sofos@glasgow.ac.uk

School of Mathematical Sciences,
Qufu Normal University,
Qufu 273165, P. R. China.
E-mail: zhangbin100902025@163.com