Periods and mixed motives

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Contents

1 Introduction

Abstract. We define motivic multiple polylogarithms and prove the double shuffle relations for them. We use this to study the motivic fundamental group $\pi_1^{\mathcal{M}}(\mathbb{G}_m - \mu_N, v_0)$, where μ_N is the group of all N-th roots of unity, and relate its structure to the geometry and topology of modular varieties

$$Y_1(m;N) := \Gamma_1(m;N)\backslash GL_m(\mathbb{R})/\mathbb{R}_+^* \cdot O_m \qquad \text{for } m = 1,2,3,4,....$$

Using this we get several new results about the action of the Galois group on $\pi_1^{(l)}(\mathbb{G}_m - \mu_N)$, and the values of multiple polylogarithms at N-th roots of unity.

To prove these results some new tools are developed, including the following:

1) We construct, under certain assumptions, framed mixed motives corresponding to periods. Here is the main example. Suppose that X is an n-dimensional complex variety, A and B are two divisors on X, $\omega_A \in \Omega^n_{\log}(X-A)$ and Δ_B is a relative cycle providing a class in $H_n(X,B)$. Consider the integral

$$\int_{\Delta_B} \omega_A \tag{1}$$

If it is convergent we construct, under some technical conditions on A and B, a framed mixed motive $m(X;[\omega_A];[\Delta_B])$ whose period is given by integral (1).

2) We prove a specialization theorem: if $(\omega_{A(\varepsilon)}, \Delta_{B(\varepsilon)})$ is a perturbation of the data defining (1) then in the Hodge/l-adic realization, under certain assumptions on $(A(\varepsilon), B(\varepsilon))$, the specialization of $m(X;[\omega_{A(\varepsilon)}];[\Delta_{B(\varepsilon)}])$ at $\varepsilon = 0$ is equivalent (as a framed object) to $m(X;[\omega_A];[\Delta_B])$.

These results allow us to construct the framed mixed Tate motives corresponding to multiple polylogarithms and study the corresponding Hopf algebra.

In general a mixed motive corresponding to a period is not uniquely defined. However the *equivalence class of framed mixed motive* is supposed to be uniquely defined by a period. The specialization theorem is used to prove that different constructions lead to the same equivalence class of framed objects. This plays a decisive role in the proof of the motivic double shuffle relations.

An immediate consequence of these results is a motivic proof of the week version of Zagier's conjecture on $\zeta_F(n)$– see section 10.13. The other proves were given in [BD2] and [DJ].

This paper is the second half of [G7].

1. The double shuffle relations. Recall ([G0-1]) that the multiple polylogarithm functions are defined for $|x_i| < 1$ by the power series

$$\mathrm{Li}_{n_1,...,n_m}(x_1,...,x_m) := \sum_{k_1<...<k_m} \frac{x_1^{k_1}...x_m^{k_m}}{k_1^{n_1}...k_m^{n_m}} \tag{2}$$

They can be written as iterated integrals as follows. Set (assuming $a_i \neq 0$)

$$\mathrm{I}_{n_1,...,n_m}(a_1,...,a_m) := \int_0^1 \underbrace{\frac{dt}{t-a_1}\circ ... \circ \frac{dt}{t}}_{n_1 \text{ times}} \circ \quad ... \quad \circ \underbrace{\frac{dt}{t-a_m}\circ ... \circ \frac{dt}{t}}_{n_m \text{ times}} \tag{3}$$

Then (see s. 12 of [G1] or theorem 2.2 in [G7])

$$\mathrm{Li}_{n_1,...,n_m}(x_1,...,x_m) = (-1)^m \mathrm{I}_{n_1,...,n_m}(a_1,...,a_m) \tag{4}$$

where

$$a_1 := (x_1...x_m)^{-1}, ..., a_m := x_m^{-1} \tag{5}$$

The multiple polylogarithms satisfy two sets of relations, called the double shuffle relations. The first, called below the I-relations, are obtained by applying the shuffle product formula

$$\int_\gamma \omega_1 \circ ...\omega_p \cdot \int_\gamma \omega_{p+1} \circ ...\omega_{p+q} = \sum_{\sigma \in \Sigma_{p.q}} \int_\gamma \omega_{\sigma(1)} \circ ...\omega_{\sigma(p+q)}$$

(where $\Sigma_{p.q}$ is the set of shuffles of $\{1, ..., p\}$ and $\{p+1, ..., p+q\}$) to the iterated integrals (3). For example

$$\mathrm{I}_1(a_1) \cdot \mathrm{I}_1(a_2) = \mathrm{I}_{1,1}(a_1, a_2) + \mathrm{I}_{1,1}(a_2, a_1) \tag{6}$$

The second set of relations, called below the Li-relations, are obtained by multiplying the power series (2). The simplest of them is this:

$$\mathrm{Li}_1(x) \cdot \mathrm{Li}_1(y) = \mathrm{Li}_{1,1}(x, y) + \mathrm{Li}_{1,1}(y, x) + \mathrm{Li}_2(xy) \tag{7}$$

The integral (3) is divergent if $a_m = 1, n_m = 1$. We regularize it by replacing $\int_0^1$ to $\int_0^{1-\varepsilon}$ and taking the constant term of the asymptotic expansion in $\log(\varepsilon)$. The regularized integrals obviously obey the I-shuffle relations. However to keep the natural form of the Li-shuffle relations one needs to use a different regularization, and then compare the regularizations. For the multiple ζ-values, i.e. when $x_i = 1$, the double shuffle relations were considered by Zagier [Z1].

If $|x_i| > 1$ the double shuffle relations are not even well defined as relations between numbers because of multivaluedness of multiple polylogarithms. What is more important, these relations are only an analytic reflection of deeper properties of the corresponding geometric objects. Here is how we handle them.

2. The story on the Hodge level. Recall the commutative graded Hopf algebra $\mathcal{H}_\bullet$ of the equivalence classes of framed Hodge-Tate structures (see for instance chapter 3 of [G7]). In chapter 5 of [G7] we defined a framed Hodge-Tate structure

$$\mathrm{I}^{\mathcal{H}}_{n_1,...,n_m}(a_1, ..., a_m) \in \mathcal{H}_w; \quad w := n_1 + ... + n_m, \quad a_i \in \mathbb{C}^* \tag{8}$$

whose period is given by the iterated integral (4), regularized if needed. By lemma 6.6 in [G7] elements (8) satisfy the $\mathrm{I}^{\mathcal{H}}$-shuffle relations.

For *generic* parameters $x_i \in \mathbb{C}^*$ we define

$$\widetilde{\mathrm{Li}}^{\mathcal{H}}_{n_1,...,n_m}(x_1, ..., x_m) := \mathrm{I}^{\mathcal{H}}_{n_1,...,n_m}(a_1, ..., a_m)$$

assuming that x_i's and a_i's are related by (5). If we use this definition for all $x_i \in \mathbb{C}^*$, the $\mathrm{Li}^{\mathcal{H}}$-shuffle relations will not hold: for instance (6) is not consistent with (7) for $x_1 = x_2 = 1$. Therefore we use a different regularization, suggested in theorem 7.1 in [G3], see also s. 2.10 in [G7]:

Definition 1.1 *For arbitrary $x_i \in \mathbb{C}^*$ set*

$$\mathrm{Li}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1,...,x_m) := \mathrm{Sp}_{\partial/\partial\varepsilon}\Big(\widetilde{\mathrm{Li}}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1(1-\varepsilon),...,x_m(1-\varepsilon))\Big) \qquad (9)$$

We set $\mathrm{Li}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1,...,x_m) = 0$ if $x_1...x_m = 0$.

Here the expression under the sign $\mathrm{Sp}_{\partial/\partial\varepsilon}$ is understood as a unipotent variation of Hodge-Tate structures on a small disc with coordinate ε, punctured at $\varepsilon = 0$. We apply to it the Verdier specialization functor $\mathrm{Sp}_{\varepsilon=0}$, getting a unipotent variation over the punctured tangent space at $\varepsilon = 0$, and then take its fiber at the tangent vector $\partial/\partial\varepsilon$. The functor $\mathrm{Sp}_{\partial/\partial\varepsilon}$ is the composition of these two functors.

The specialization theorem proved in chapter 3 implies that, assuming (5),

$$\mathrm{Li}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1,...,x_m) = \mathrm{I}^{\mathcal{H}}_{n_1,\ldots,n_m}(a_1,...,a_m) \quad \text{provided } n_m > 1 \text{ or } x_m \neq 1$$

i.e. when the corresponding integral is convergent.

Theorem 1.2 *The elements $\mathrm{Li}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1,...,x_m)$ satisfy the $\mathrm{Li}^{\mathcal{H}}$-shuffle relations for any $x_i \in \mathbb{C}$.*

For the precise form of the Li–shuffle relations see theorem 7.4. It is proved in chapters 7-9. We first prove it for generic parameters x_i. There are two different methods how to do this. We explain these methods in chapter 8. To demonstrate how they work we give in chapter 8 two proofs of the $\mathrm{Li}^{\mathcal{H}}$-shuffle relations for the double logarithm. In this case all essential problems are already present, while the combinatorics is still very simple. In chapter 9 we employ one of the methods to get theorem 1.2 for generic parameters x_i. In the appendix we outline a proof of the same result via the other method. Then using the specialization theorem we deduce from this that theorem 1.2 is valid for all parameters x_i.

To get the double shuffle relations it remains to compare explicitly the $\mathrm{Li}^{\mathcal{H}}$- and $\mathrm{I}^{\mathcal{H}}$-elements when $n_m = 1$ and $x_m = 1$, i.e. when the corresponding integrals are divergent. For this we need the following.

A Hodge version of the asymptotic expansions of divergent integrals. The equivalence classes of framed unipotent variations of Hodge-Tate structures on $\mathbb{C}^*$ form a commutative graded Hopf algebra, denoted $\mathcal{A}^{\mathcal{H}}_{\bullet}(\mathbb{A}^1 - \{0\})$ (see ch. 3 [G7]). Let ε be a natural coordinate on $\mathbb{A}^1 - \{0\}$ and $\log^{\mathcal{H}}\varepsilon$ the canonical variation of framed Hodge-Tate structures on $\mathbb{C}^*$ with the period function $\log\varepsilon$. We can view $\log^{\mathcal{H}}\varepsilon$ as an element of $\mathcal{A}^{\mathcal{H}}_1(\mathbb{A}^1 - \{0\})$. According to lemma 7.1 there is an isomorphism of graded commutative algebras

$$\mathcal{A}^{\mathcal{H}}_{\bullet}(\mathbb{A}^1 - \{0\}) = \mathcal{H}_{\bullet} \otimes_{\mathbb{Q}} \mathbb{Q}[\log^{\mathcal{H}}\varepsilon] \qquad (10)$$

If H_ε is a unipotent variation of framed Hodge-Tate structures on a punctured disc, denote by $[\mathrm{Sp}_{\varepsilon=0}](H_\varepsilon)$ the equivalence class of its specialization $\mathrm{Sp}_{\varepsilon=0}(H_\varepsilon)$. Then, employing the isomorphism (10), we get an element

$$[\mathrm{Sp}_{\varepsilon=0}](H_\varepsilon) = \sum_{i\geq 0} H_{(i)} \cdot (\log^{\mathcal{H}} \varepsilon)^i \in \mathcal{H}_\bullet \otimes_{\mathbb{Q}} \mathbb{Q}[\log^{\mathcal{H}} \varepsilon]$$

It is the Hodge version of the asymptotic expansion of the period function. Its fiber over the tangent vector $\partial/\partial\varepsilon$ coincides with the constant term $H_{(0)}$.

The two regularizations and the comparison theorem. a) Denote by

$$\mathrm{I}^{\mathcal{H}}_{n_1,\ldots,n_m}(a_1 : \ldots : a_{m-1} : a_m : (1-\varepsilon)) \tag{11}$$

the framed Hodge-Tate structure corresponding to the iterated integral (3) where $\int_0^1$ changed to $\int_0^{1-\varepsilon}$. It has been defined in chapter 5 of [G7]. Consider the unipotent variation of framed Hodge-Tate structures over a small punctured disc with coordinate ε, whose fiber at ε is (11). Let

$$\widehat{\mathrm{I}}^{\mathcal{H}}_{n_1,\ldots,n_m}(a_1,\ldots,a_m) \in \mathcal{A}^{\mathcal{H}}_\bullet(\mathbb{A}^1 - \{0\}) \tag{12}$$

be the equivalence class of the framed unipotent variation of Hodge-Tate structures on the punctured tangent space obtained by specialization of the family (11). Its fiber over $\partial/\partial\varepsilon$ is the framed Hodge-Tate structure (8), by the very definition of (8).

b) Set

$$\widehat{\mathrm{Li}}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1,\ldots,x_m) := \tag{13}$$
$$[\mathrm{Sp}_{\varepsilon=0}]\Big(\mathrm{Li}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1(1-\varepsilon),\ldots,x_m(1-\varepsilon))\Big) \in \mathcal{A}^{\mathcal{H}}_\bullet(\mathbb{A}^1 - \{0\})$$

The constant terms of the expansions (12) and (13) are given by the elements (8) and (9) which we want to compare. An explicit formula for the coefficients of expansion (12) via multiple polylogarithms is given by lemma 6.7 or proposition 2.14 in [G7]. So to compute $\mathrm{Li}^{\mathcal{H}}$ via $\mathrm{I}^{\mathcal{H}}$ we need to compare $\widehat{\mathrm{Li}}^{\mathcal{H}}$ with $\widehat{\mathrm{I}}^{\mathcal{H}}$.

Set $\zeta^{\mathcal{H}}(n) := \mathrm{Li}^{\mathcal{H}}_n(1)$ and similarly with hats. For example

$$\widehat{\zeta}^{\mathcal{H}}(1) = \widehat{\mathrm{Li}}^{\mathcal{H}}(1) = \mathrm{Li}^{\mathcal{H}}(1-\varepsilon) = -\log^{\mathcal{H}}(\varepsilon)$$

Definition 1.3 *$\mathbb{L}$ is the $\mathcal{H}_\bullet$-linear endomorphism*

$$\mathbb{L} : \mathcal{H}_\bullet \otimes_{\mathbb{Q}} \mathbb{Q}[\log^{\mathcal{H}} \varepsilon] \longrightarrow \mathcal{H}_\bullet \otimes_{\mathbb{Q}} \mathbb{Q}[\log^{\mathcal{H}} \varepsilon]$$

determined by the following equality of formal power series in u:

$$\sum_{n\geq 0} \mathbb{L}\Big((\log^{\mathcal{H}} \varepsilon)^n\Big) \cdot \frac{u^n}{n!} := \exp\left(-\sum_{n=1}^{\infty} (-1)^n \frac{\widehat{\zeta}^{\mathcal{H}}(n)}{n} u^n\right)$$

Theorem 1.4 *For any $x_i \in \mathbb{C}^*$ one has, assuming (5),*

$$\widehat{\mathrm{Li}}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1, ..., x_m) = \mathbb{L} \circ \widehat{\mathrm{I}}^{\mathcal{H}}_{n_1,\ldots,n_m}(a_1, ..., a_m)$$

Our proof of theorem 1.4 follows the approach developed for the multiple polylogarithms functions in theorem 7.1 in [G3], see also s. 2.10 in [G7].

A similar comparison formula for the multiple ζ-values was obtained by Zagier (it is documented in [IK]) and later independently by Boute de Monville, and J. Ecalle (and documented in [R]). However their approach seems to have no Hodge/motivic interpretation.

Summarizing, combining theorem 3.2 from [G8] (or lemma 6.6 from [G7]) giving the $\mathrm{I}^{\mathcal{H}}$-shuffle relations, theorem 1.2, and theorem 1.4 we get the double shuffle relations for the equivalence classes of framed Hodge-Tate structures corresponding to multiple polylogarithms.

We suggest that when $x_i \in \mu_N$ these relations, combined with the distribution relations, should provide most of (but in general not all) the relations between the multiple polylogarithm Hodge-Tate structures with $x_i \in \mu_N$. In a contrast with this, if $x_i \in \mathbb{C}$ they provide only a little part of the relations.

3. The motivic and étale versions of the story. Let F be a number field. Then there exists an abelian category $\mathcal{M}_T(F)$ of of mixed Tate motives over F (see chapter 5 of [G9] or [L1]). The objects of this category carry canonical weight filtration $W_\bullet$. The category $\mathcal{M}_T(F)$ is equipped with canonical fiber functor

$$\omega : X \longrightarrow \oplus_n \mathrm{Hom}(\mathbb{Q}(-n), \mathrm{Gr}^W_{2n} X)$$

This fiber functor provides an equivalence of the category $\mathcal{M}_T(F)$ with the category of finite dimensional graded comodules over the fundamental Hopf algebra $\mathcal{A}_\bullet(F)$. Here $\mathcal{A}_\bullet(F)$ is a commutative Hopf algebra over $\mathbb{Q}$ graded by the integers $n \geq 0$. The elements of $\mathcal{A}_n(F)$ can be understood as the equivalence classes of the framed objects in the category $\mathcal{M}_T(F)$, see ch. 3 of [G7].

Suppose that $a_i \in F^*$ and n_i are arbitrary positive integers. We define geometrically framed mixed Tate motives over F

$$\mathrm{I}^{\mathcal{M}}_{n_1,\ldots,n_m}(a_1, ..., a_m) \in \mathcal{A}_w(F); \quad w := n_1 + ... + n_m \tag{14}$$

and

$$\mathrm{Li}^{\mathcal{M}}_{n_1,\ldots,n_m}(x_1, ..., x_m) \in \mathcal{A}_w(F); \tag{15}$$

For any complex embedding $\sigma : F \hookrightarrow \mathbb{C}$ the Hodge realization of (14) is given by $\mathrm{I}^{\mathcal{H}}_{n_1,\ldots,n_m}(\sigma(a_1), ..., \sigma(a_m))$, and similarly for the elements (15). We prove that the motivic multiple polylogarithms satisfy the double shuffle relations. The proof of the $\mathrm{I}^{\mathcal{H}}$-relations given in [G7] is motivic. For generic x_i the proof of the Li-relations given in chapter 9 is also motivic. However in some special cases the result is deduced from the Hodge realization using the injectivity of regulators.

Theorem 1.4 provides an explicit formula expressing $\mathrm{Li}^{\mathcal{H}}$- via the $\mathrm{I}^{\mathcal{H}}$-elements. It implies, using the in jectivity of regulators, the same kind of formula relating the $\mathrm{Li}^{\mathcal{M}}$- and $\mathrm{I}^{\mathcal{M}}$-elements.

The Li-shuffle relations are obvious for the power series (2), but deliver surprisingly nontrivial information on the motivic level. For example if x_i are N-th roots of unity they provide the most sofisticated information about the action of the motivic Galois group on the motivic fundamental group $\pi_1^{\mathcal{M}}(\mathbb{G}_m - \mu_N)$. In the l-adic realization they provide constraints on the image of the absolute Galois group acting by automorphisms of $\pi_1^{(l)}(\mathbb{G}_m - \mu_N)$.

Suppose now that F is *an arbitrary field such that* $\mu_{l^\infty} \notin F^*$. Then we define framed l-adic mixed Tate $\mathrm{Gal}(\overline{F}/F)$-modules

$$\mathrm{Li}^{\mathrm{et}}_{n_1,\ldots,n_m}(x_1, ..., x_m) \tag{16}$$

related to multiple polylogarithms and prove the l-adic version of the double shuffle relations for them. When x_i are elements of a number field F the framed $\mathrm{Gal}(\overline{F}/F)$-modules (16) are given by the l-adic realization of their motivic counterparts (15).

4. The higher cyclotomy and cyclotomic Hopf algebras. Let $\zeta_N := \exp(2\pi i/N)$. We define $\mathcal{Z}_w^{\mathcal{M}}(\mu_N)$ as the $\mathbb{Q}$-vector subspace of $\mathcal{A}_w(\mathbb{Q}(\zeta_N))$ generated by the elements

$$\mathrm{Li}^{\mathcal{M}}_{n_1,\ldots,n_m}(\zeta_N^{\alpha_1}, ..., \zeta_N^{\alpha_m}) \in \mathcal{A}_w(\mathbb{Q}(\zeta_N)), \quad w = n_1 + ...n_m \tag{17}$$

and set $\mathcal{Z}_\bullet^{\mathcal{M}}(\mu_N) := \oplus_{w\geq 0}\mathcal{Z}_w^{\mathcal{M}}(\mu_N)$.

Theorem 1.5 *$\mathcal{Z}_\bullet^{\mathcal{M}}(\mu_N)$ is a graded Hopf algebra.*

Recall

$$S_N := \mathrm{Spec}(\mathbb{Z}[\zeta_N][\frac{1}{N}])$$

By the very definition $\mathcal{Z}_1^{\mathcal{M}}(\mu_N)$ is generated by cyclotomic N-units $1 - \zeta_N^\alpha$, so

$$\mathcal{Z}_1^{\mathcal{M}}(\mu_N) \cong \mathcal{O}^*(S_N) \otimes \mathbb{Q} \tag{18}$$

We call $\mathcal{Z}_\bullet^{\mathcal{M}}(\mu_N)$ the *level N motivic cyclotomic Hopf algebra*, and suggest its study as the goal for higher analog of the theory of cyclotomic units.

Recall the commutative Hopf algebra $\mathcal{A}_\bullet(S_N)$ classifying the abelian category $\mathcal{M}_T(S_N)$ of mixed Tate motives over the scheme S_N (see chapter 3 of [G7] or [DG]).

Theorem 1.6 *$\mathcal{Z}_\bullet^{\mathcal{M}}(\mu_N)$ is a Hopf subalgebra of $\mathcal{A}_\bullet(S_N)$.*

Theorem 1.6 provides a strong bound from above on the size of the cyclotomic Hopf algebra $\mathcal{Z}_\bullet^{\mathcal{M}}(\mu_N)$. Indeed, the Hopf algebra $\mathcal{A}_\bullet(S_N)$ has the following structure. Consider the free graded Lie algebra generated by the vector spaces $K_{2n-1}(S_N) \otimes \mathbb{Q}$ sitting in the degree $-n$, when $n = 1, 2, 3, ...$ Then $\mathcal{A}_\bullet(S_N)$ is isomorphic to the graded dual to the universal enveloping algebra of this graded Lie algebra. In particular its algebra structure is given by

Proposition 1.7 *The algebra $\mathcal{A}_\bullet(S_N)$ is isomorphic to the tensor algebra generated by the graded $\mathbb{Q}$-vector space*

$$\oplus_{n=1}^{\infty} K_{2n-1}(S_N) \otimes \mathbb{Q} \tag{19}$$

and equipped with commutative multiplication provided by the shuffle product.

By Borel's theorem

$$\dim K_{2w-1}(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} 0 & w\text{: even} \\ 1 & w > 1\text{: odd} \end{cases}$$

$$\dim K_{2w-1}(S_2) \otimes \mathbb{Q} = \begin{cases} 0 & w\text{: even} \\ 1 & w\text{: odd} \end{cases}$$

and for $N > 2$:

$$\dim K_{2w-1}(S_N) \otimes \mathbb{Q} = \begin{cases} \frac{\varphi(N)}{2} & w > 1 \\ \frac{\varphi(N)}{2} + p(N) - 1 & w = 1 \end{cases}$$

where $p(N)$ is the number of prime factors of N.

Recall ([G7]) the $\mathbb{Q}$-vector space $\widetilde{\mathcal{Z}}_w(\mu_N)$ spanned over $\mathbb{Q}$ by $(2\pi i)^{-w}$ times the values of weight w multiple polylogarithms at N-th roots of unity. (We take into account all brunches of the corresponding multivalued analytic functions). Then $\widetilde{\mathcal{Z}}(\mu_N) := \cup \widetilde{\mathcal{Z}}_w(\mu_N)$ is a commutative algebra filtered by the weight.

Theorem 1.8 *a) There exists canonical surjective algebra homomorphism*

$$\mathcal{Z}_\bullet^{\mathcal{M}}(\mu_N) \longrightarrow \mathrm{Gr}_\bullet^W \widetilde{\mathcal{Z}}(\mu_N) \tag{20}$$

b) The algebra $\mathrm{Gr}_\bullet^W \widetilde{\mathcal{Z}}(\mu_N)$ is a subquotient of the algebra $\mathcal{A}_\bullet(S_N)$.

The part b) follows immediately from the part a) and theorem 1.6.

A considerable part of the proof of this theorem is contained in [G7]. In particular we constructed there the Hodge version of homomorphism (20), with $\mathcal{Z}_\bullet^{\mathcal{M}}(\mu_N)$ replaced by $\mathcal{Z}_\bullet^{\mathcal{H}}(\mu_N)$. Combined with the given above explicit description of the algebra $\mathcal{A}_\bullet(S_N)$, theorem 1.8 implies strong estimates from above for $\dim \mathrm{Gr}_w^W \widetilde{\mathcal{Z}}(S_N)$ discussed in [G5]. In particular in the case $N = 1$ this completes the proof of theorem 1.2 in [G5]. Another proof in the $N = 1$ case was suggested by T. Terasoma [T].

5. The cyclotomic and dihedral Lie algebras. The space of indecomposables

$$\mathcal{C}_{\bullet}^{\mathcal{M}}(\mu_N) := \frac{\mathcal{Z}_{>0}^{\mathcal{M}}(\mu_N)}{\mathcal{Z}_{>0}^{\mathcal{M}}(\mu_N) \cdot \mathcal{Z}_{>0}^{\mathcal{M}}(\mu_N)}$$

has a natural structure of a graded Lie coalgebra over $\mathbb{Q}$. The cyclotomic Lie algebra $\mathrm{C}_{\bullet}^{\mathcal{M}}(\mu_N)$ is defined as its graded dual.

There is a depth filtration $F_{\bullet}^{D}$ on the cyclotomic Hopf algebra such that the depth $\leq d$ part is generated by the elements (17) with $m \leq d$. It induces the depth filtration on the cyclotomic Lie coalgebra and algebra. The depth filtration is compatible with the weight grading. Taking the associate graded for the depth filtration we get bigraded Lie coalgebra and Lie algebra over $\mathbb{Q}$:

$$\mathcal{C}_{\bullet,\bullet}^{\mathcal{M}}(\mu_N) := \mathrm{Gr}^{D}\left(\mathcal{C}_{\bullet}^{\mathcal{M}}(\mu_N)\right), \quad \mathrm{C}_{\bullet,\bullet}^{\mathcal{M}}(\mu_N) := \mathrm{Gr}^{D}\left(\mathrm{C}_{\bullet}^{\mathcal{M}}(\mu_N)\right)$$

Recall the dihedral Lie algebra $\mathrm{D}_{\bullet,\bullet}(G)$ of a commutative group G. It was defined [G3] or chapter 4 of [G4] as the graded dual to the dihedral Lie coalgebra $\mathcal{D}_{\bullet,\bullet}(G)$. This Lie coalgebra is generated by the symbols

$$\mathrm{I}_{n_1,\ldots,n_m}(g_1,\ldots,g_m), \quad g_i \in G, \quad n_i > 0$$

subject to certain relations. The relations reflect the double shuffle relations projected onto the space of decomposables and considered modulo the lower depth terms, and the distribution relations. Let us define a map $\mathcal{D}_{\bullet,\bullet}(\mu_N) \longrightarrow \mathrm{C}_{\bullet,\bullet}^{\mathcal{M}}(\mu_N)$ by setting ($a_i \in \mu_N$)

$$\mathrm{I}_{n_1,\ldots,n_m}(a_1,\ldots,a_m) \longmapsto \tag{21}$$

$$\mathrm{I}_{n_1,\ldots,n_m}^{\mathcal{M}}(a_1,\ldots,a_m) \quad \text{modulo depth} < m \text{ terms and products}$$

where the right hand side is defined via (14).

Theorem 1.9 *Formula (21) provides a well defined surjective homomorphism of bigraded Lie coalgebras*

$$\nu_{\bullet,\bullet}^{\mathcal{M}}(\mu_N) : \mathcal{D}_{\bullet,\bullet}(\mu_N) \longrightarrow \mathcal{C}_{\bullet,\bullet}^{\mathcal{M}}(\mu_N)$$

This theorem reflects the following two basic facts:

a) we have the motivic double shuffle and distribution relations,

b) the cobracket in the dihedral Lie coalgebra is compatible with the coproduct of the elements (8) explicitly computed in [G8], an earlier version is in [G7]. (In fact our definition of the cobracket in [G3] was suggested by that calculation).

Theorem 1.10 *There are canonical isomorphisms*

$$\mathcal{D}_{w,1}^{\mathcal{M}}(\mu_N) = K_{2w-1}(\mathbb{Q}(\zeta_N)) \otimes \mathbb{Q} = \mathcal{C}_{w,1}^{\mathcal{M}}(\mu_N)$$

For the first isomorphism see in s. 7.2 of [G4]. The second follows from the results of [BD2], or easily deduced from the results of this paper, see section 10.12.

Combining theorem 1.10 with theorem 1.6 and description of the fundamental Hopf algebra $\mathcal{A}_\bullet(S_N)$, we conclude that the Lie algebra $\mathrm{C}_\bullet^{\mathcal{M}}(\mu_N)$ is generated by its depth one component given by the graded vector space (19).

Conjecture 1.11 *The map $\nu_{\bullet,\bullet}^{\mathcal{M}}(\mu_N)$ is an isomorphism for either $N=1$, or $N=p$ is a prime and $w=m$.*

Theorem 1.12 *Conjecture 1.11 is true for $m \leq 3$. Thus the double shuffle relations provide all the relations between the elements $\mathrm{Li}^{\mathcal{M}}_{n_1,...,n_m}(\zeta_p^{\alpha_1},...,\zeta_p^{\alpha_m})$ for $m \leq 3$ if $N=1$, or if $N=p$ is a prime and $n_i=1$.*

This follows from the results of [G4] and chapter 10 below. There is a strong evidence for conjecture 1.11 for $m=4$, see [G11]

6. The cyclotomic Lie algebras and geometry of modular varieties. Theorem 1.9 is the crucial step on the way to a mysterious correspondence between the structure of the cyclotomic Lie algebras and geometry and topology of modular varieties. We defer the detailed discussion of the geometric aspects of this correspondence to [G11]. Several constructions have been described in [G3], [G4] and especially [G5]. Here is a very brief outlook.

The rank m modular complex $M^\bullet_{(m)}$ is a complex of $GL_m(\mathbb{Z})$-modules

$$M^1_{(m)} \xrightarrow{\partial} M^2_{(m)} \xrightarrow{\partial} ... \xrightarrow{\partial} M^m_{(m)}$$

which we defined in [G3], see also s. 2.5 of [G4]. If $m=2$ it is isomorphic to the chain complex of the classical modular triangulation of the hyperbolic plane.

Denote by $\Lambda^*_{(m,w)}\mathcal{G}_{\bullet\bullet}$ the bidegree (m,w) part of the standard cochain complex of a bigraded Lie (co)algebra $\mathcal{G}_{\bullet\bullet}$. Set

$$\widehat{\mathcal{C}}^{\mathcal{M}}_{\bullet,\bullet}(\mu_p) := \mathcal{C}^{\mathcal{M}}_{\bullet,\bullet}(\mu_p) \oplus \mathbb{Q}_{1,1}$$

where $\mathbb{Q}_{1,1}$ is the one dimensional Lie coalgebra of the weight-depth $(1,1)$.

Theorem 1.13 *For $m>1$ there exists canonical surjective map of complexes*

$$\mu^*_{m;w}(N) : M^*_{(m)} \otimes_{\Gamma_1(m;N)} S^{w-m}\mathrm{V}_m \longrightarrow \Lambda^*_{(m,w)}\widehat{\mathcal{C}}^{\mathcal{M}}_{\bullet,\bullet}(\mu_N)\,.$$

This theorem is an immediate consequence of theorem 1.9 and the following result proved in [G3], see also [G10]: for $m>1$ there is canonical surjective map of complexes

$$\eta^*_{m;w}(N) : M^*_{(m)} \otimes_{\Gamma_1(m;N)} S^{w-m}\mathrm{V}_m \longrightarrow \Lambda^*_{(m,w)}\widehat{\mathcal{D}}_{\bullet\bullet}(\mu_N)$$

It is an isomorphism when $N=1$, or when $N=p$ is a prime and $w=m$.

Conjecture 1.14 *Let $N = 1$, or $N = p$ is a prime and $w = m$. Then the map $\mu^*_{m;w}(N)$ is an isomorphism.*

Let $\Gamma_1(m;p) \subset GL_m(\mathbb{Z})$ be the subgroup stabilizing the vector $(0,...,0,1)$ modulo N. Denote by ε_m the one dimensional $GL_m(\mathbb{Z})$-module given by the determinant. The results above allow to replace $\mathcal{G}^{(l)}_{\bullet,\bullet}(\mu_N)$ by $\mathrm{C}^{\mathcal{M}}_{\bullet,\bullet}(\mu_N)$ everywhere in [G5] where the structure of the Galois Lie algebra $\mathcal{G}^{(l)}_{\bullet,\bullet}(\mu_N)$ was related to modular varieties. Here is a sample.

Theorem 1.15 *One has for a prime $p \geq 5$*

$$H^i_{(2,2)}(\widehat{\mathrm{C}}^{\mathcal{M}}_{\bullet,\bullet}(\mu_p)) = H^{i-1}(\Gamma_1(2;p), \varepsilon_2); \quad i = 1,2$$

$$H^i_{(3,3)}(\widehat{\mathrm{C}}^{\mathcal{M}}_{\bullet,\bullet}(\mu_p)) = H^{i}(\Gamma_1(3;p), \mathbb{Q}); \quad i = 1,2,3$$

7. The motivic torsor of path $\mathcal{P}^{\mathcal{M}}(\mathbb{G}_m - \mu_N; v_0, v_1)$ and the cyclotomic Hopf algebra. Let t be the canonical coordinate on $P^1 - \{0,1,\infty\}$ and v_0, v_1 the corresponding tangential base points at 0 and 1. Denote by v_∞ the tangential base point at infinity provided by the local parameter t^{-1}. The motivic torsor of path $\mathcal{P}^{\mathcal{M}}(\mathbb{G}_m - \mu_N; v_0, v_1)$ is a pro-object in the abelian category of mixed Tate motives over S_N. It is defined in [DG]. Its more explicit geometric construction is given in section 6.3 below. Namely, consider the dual to the associate graded for the weight filtration

$$\Big(\mathrm{Gr}^W \mathcal{P}^{\mathcal{M}}(\mathbb{G}_m - \mu_N; v_0, v_1)\Big)^\vee \tag{22}$$

It provides an Ind-object in the category of pure Tate motives. Recall that $\mathcal{A}_\bullet(S_N)$ is a Hopf algebra in the same category. In section 6.3 we define explicitly a coaction of the Hopf algebra $\mathcal{A}_\bullet(S_N)$ on (22). Therefore we get an Ind-object in the category of mixed Tate motives over S_N. This construction immediately implies the following important result.

Theorem 1.16 *The object (22) is a comodule over the cyclotomic Hopf subalgebra $\mathcal{Z}^{\mathcal{M}}_\bullet(\mu_N) \subset \mathcal{A}_\bullet(S_N)$. Moreover $\mathcal{Z}^{\mathcal{M}}_\bullet(\mu_N)$ acts cofreely on (22).*

The grading of $\mathcal{Z}^{\mathcal{M}}_\bullet(\mu_N)$ provides an action of $\mathbb{G}_m$ on the prounipotent algebraic group scheme $\mathrm{Spec}\mathcal{Z}^{\mathcal{M}}_\bullet(\mu_N)$. Theorem 1.16 just means that the image of the motivic Galois group acting on $\mathcal{P}^{\mathcal{M}}(\mathbb{G}_m - \mu_N; v_0, v_1)$ is given by the semidirect product of $\mathbb{G}_m$ and $\mathrm{Spec}\mathcal{Z}^{\mathcal{M}}_\bullet(\mu_N)$.

Theorem 1.17 *a) The motivic Galois group acts on $\mathcal{P}^{\mathcal{M}}(\mathbb{G}_m - \mu_N; v_0, v_1)$ and on $\pi_1^{\mathcal{M}}(\mathbb{G}_m - \mu_N; v_\varepsilon)$, where $\varepsilon \in \{0,1,\infty\}$, via the same quotient.*

b) This quotient is the semidirect product of $\mathbb{G}_m$ and $\mathrm{Spec}\mathcal{Z}^{\mathcal{M}}_\bullet(\mu_N)$.

The part b) of this theorem follows from the part a) and theorem 1.16.

8. Applications to the Galois action on $\pi_1^{(l)}(\mathbb{G}_m - \mu_N; v_\infty)$. Theorem 1.17 in the l-adic realization provides an important new information about the action of the absolute Galois group on the pro-l completion $\pi_1^{(l)}(\mathbb{G}_m - \mu_N; v_\infty)$ of the fundamental group. This combined with the results outlined in s. 1.6 allows to relate the structure of the Galois module $\pi_1^{(l)}(\mathbb{G}_m - \mu_N; v_\infty)$ with the geometry of modular varieties.

More precisely, let $\mathbb{L}_N^{(l)}$ be the l-adic pro-Lie algebra corresponding to $\pi_1^{(l)}(\mathbb{G}_m - \mu_N, v_\infty)$ via Maltsev's theory. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N}))$ acts by automorphisms of $\pi_1^{(l)}$ and hence of $\mathbb{L}_N^{(l)}$, providing a homomorphism

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N})) \longrightarrow \mathrm{Aut}\mathbb{L}_N^{(l)}$$

Let $\mathcal{G}_N^{(l)}$ be the linearization of the image of this map:

$$\mathcal{G}_N^{(l)} := \underleftarrow{\lim}\, \mathrm{Lie}\left(\mathrm{Im}\left(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N})) \longrightarrow \mathrm{Aut}\mathbb{L}_N^{(l)}[m]\right)\right)$$

where $L[m]$ stays for the quotient of a Lie algebra L by the m-th lower central series ideal. Therefore

$$\mathcal{G}_N^{(l)} \subset \mathrm{Der}\mathbb{L}_N^{(l)}$$

The natural weight filtration on $\mathbb{L}_N^{(l)}$ coincides with the lower central series filtration. It induces the weight filtration $W_\bullet$ on $\mathrm{Der}\mathbb{L}_N^{(l)}$, and hence on $\mathcal{G}_N^{(l)}$. The weight filtration admits a splitting compatible with the depth filtration (see [G4]). There is also the depth filtration on $\mathbb{L}_N^{(l)}$ and hence on $\mathcal{G}_N^{(l)}$.

The l-adic realization of $\pi_1^{\mathcal{M}}(\mathbb{G}_m - \mu_N, v_\infty)$ is given by the Galois module $\mathbb{L}_N^{(l)}$.

Theorem 1.18 *a) There is canonical isomorphism of Lie algebras*

$$\mathrm{Gr}^W_\bullet \mathcal{G}_N^{(l)} = \mathrm{C}^{\mathcal{M}}_\bullet(\mu_N) \otimes \mathbb{Q}_l$$

b) This isomorphism is compatible with the depth filtration.

It follows from theorem 1.16, the Tannakian formalism and general properties of the l-adic realization functor on the category $\mathcal{M}_T(F)$.

Let $\mathcal{G}^{(l)}_{\bullet,\bullet}(\mu_N)$ be the associate graded for the depth and weight filtrations. Theorem 1.18 implies an isomorphism

$$\mathcal{G}^{(l)}_{\bullet,\bullet}(\mu_N) = \mathrm{C}^{\mathcal{M}}_{\bullet,\bullet}(\mu_N) \otimes \mathbb{Q}_l$$

Combining this with theorem 1.9 we get the following result stated in conjecture 1.1 in [G4]:

Theorem 1.19 *There is canonical inclusion*

$$\mathcal{G}^{(l)}_{\bullet,\bullet}(\mu_N) \hookrightarrow \mathrm{D}_{\bullet,\bullet}(\mu_N) \otimes \mathbb{Q}_l$$

Now we turn to more technical aspects of the paper.

9. Periods and framed mixed motives. The integral (1) is a typical example of a period. Here are the simplest examples:

$$\log(2) = \int_1^2 \frac{dt}{t}; \quad \zeta(2) = \int_{0 \leq t_1 \leq t_2 \leq 1} \frac{dt_1}{1-t_1} \wedge \frac{dt_2}{t_2}$$

A period is supposed to come from the Hodge realization of a uniquely defined equivalence class of framed mixed motive.

Suppose that the data defining integral (1) satisfies the following conditions:

$$\text{the divisors } A \text{ and } B \text{ have no common irreducible components} \tag{23}$$

and the divisor $A \cup B$ is a normal crossing divisor. Then we define the corresponding mixed motive by

$$H^n(X - A, B - (B \cap A)) \tag{24}$$

It has a natural framing coming from classes $[\omega_A], [\Delta_B]$. Denote the corresponding framed mixed motive as well as its equivalence class by $m(X; [\omega_A]; [\Delta_B])$.

In general construction of the corresponding framed object is neither unique nor obvious even the Hodge realization - see the next section for discussion of $\zeta(2)$. Here is a general method.

Let $\pi : \widehat{X} \longrightarrow X$ be a blow up of X transforming the divisor $A \cup B$ in X to a normal crossing divisor $\widehat{D}$ in $\widehat{X}$. It provides an isomorphism

$$\pi^0 : \widehat{X} - \widehat{D} \xrightarrow{\sim} X - A \cup B$$

Let $\partial\Delta_B$ be the boundary of Δ_B, so $\partial\Delta_B \subset B(\mathbb{C})$. Then the open part $\Delta_B^0 := \Delta_B - \partial\Delta_B$ sits inside of the set of complex points of $X - A \cup B$. Using the isomorphism π^0 we identify Δ^0_B with an open cell in the set of complex points of $\widehat{X} - \widehat{D}$. Denote by $\overline{\Delta^0_B}$ its topological closure there. The Zariski closure of the boundary $\partial\overline{\Delta^0_B}$ is contained in $\widehat{D}$. Let $\widehat{B}$ be the smallest union of the irreducible components of $\widehat{D}$ containing it.

The form ω_A is a regular differential form on $X - A$ with logarithmic singularities at A. The isomorphism π^0 identifies ω_A it with a regular differential form $\widehat{\omega}_A$ on $\widehat{X} - \widehat{D}$. It automatically has logarithmic singularities along the divisor $\widehat{D}$. Let $\widehat{A}$ be the actual divisor of singularities of $\widehat{\omega}_A$. Suppose that integral (1) is convergent. Then one can show that the crucial condition (23) is satisfied. A detailed elaboration of this construction in the case of the multiple zeta values see in [GM]. It works equally well for all multiple polylogarithms.

However in this paper, aiming to the specialization theorem, we choose a different approach. It uses perverse sheaves on X instead of blow ups of X to construct Hodge/étale realizations of the mixed motive related to a period. Here is a more detailed account.

In chapter 2, for so-called admissible pair of divisors (A, B), we construct a perverse sheaf of geometric origin $\mathcal{F}^*_{A,B}$ on X such that

$$H^0(X, \mathcal{F}^*_{A,B})$$

is equipped with a natural framing corresponding to the given cohomology classes $[\omega_A] \in H^n(X - A)$ and $[\Delta_B] \in H_n(X; B)$. We define $\mathcal{F}^*_{A,B}$ as H^0 for the perverse t-structure of an object $\mathcal{F}^\bullet_{A,*,B}$ of the appropriate derived category of sheaves on X.

Let $\pi : X \longrightarrow \mathrm{Spec}(F)$ be the canonical projection. In chapter 3 we show that, assuming a condition on $A \cup B$ valid in all examples through the paper, the object $\mathrm{R}\pi_* \mathcal{F}^\bullet_{A,*,B}$ is a realization of a certain object of the triangulated category $\mathcal{DM}_F$ of mixed motives over F constructed in [V] and [L].

If F is a number field and the divisor $A \cup B$ provides a Tate stratification of X we define $H^0(X; \mathcal{F}^*_{A,B})$ as an object of the abelian category $\mathcal{M}_T(F)$ of mixed Tate motives over F constructed in chapter 5 of [G9].

Often there are many constructions of framed objects with the same period, and it is important to know that they lead to the same equivalence class of framed objects. Here is how we can face such a situation. Suppose that we have a deformation $(\omega_{A(\varepsilon)}, \Delta_{B(\varepsilon)})$ such that $A(\varepsilon) \cup B(\varepsilon)$ for $\varepsilon \neq 0$ is a normal crossing divisor satisfying (23). Suppose that as $\varepsilon \to 0$ the limit of the data $(A(\varepsilon), B(\varepsilon), [\omega_{A(\varepsilon)}], [\Delta_{B(\varepsilon)}])$ exists and coincides with the one $(A, B, [\omega_A], [\Delta_B])$. So in particular

$$\lim_{\varepsilon \to 0} \int_{\Delta_{B(\varepsilon)}} \omega_{A(\varepsilon)} = \int_{\Delta_B} \omega_A \tag{25}$$

Then the specialization $\mathrm{Sp}_{\partial/\partial\varepsilon} m(X; [\omega_{A(\varepsilon)}]; [\Delta_{B(\varepsilon)}])$ is a framed object whose period is given by (25). Taking different perturbations we get different mixed objects. In chapter 4 we prove that nevertheless, under certain assumptions, the equivalence classes of the obtained framed objects are the same. (In fact there is no need to assume that $A(\varepsilon) \cup B(\varepsilon)$ is a normal crossing divisor for $\varepsilon \neq 0$.)

We construct the motives corresponding to periods by resolving singularities and then using construction (24).

Our specialization theorem provides an efficient way to compute the coproduct for the framed mixed object corresponding to a degenerate admissible configuration (A, B): take a generic deformation $(A(\varepsilon), B(\varepsilon))$, compute explicitly the coproduct for $\varepsilon \neq 0$, and take the specialization when $\varepsilon \to 0$.

Here are some applications. In chapter 5 we provide a motivic treatment of Aomoto polylogarithms. In chapter 6 we use the results of chapters 2-4 to study the torsor of path between the tangential base points on a curve. In particular we define the framed mixed Tate motives corresponding to the classical polylogarithms and compute the coproduct for framed these objects, see section 10.12. This implies the weak version of Zagier's conjecture on $\zeta_F(n)$.

10. An example: the motivic double logarithm. To explain the nature of the problem we discuss below the simplest nontrivial example: construction of the framed mixed Tate motive corresponding to double logarithm

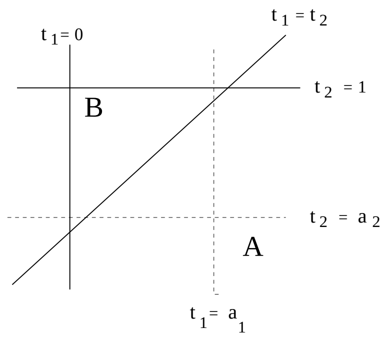
$$\mathrm{I}_{1,1}(a_1, a_2) := \int_{0\leq t_1\leq t_2\leq 1} \frac{dt_1}{t_1 - a_1} \wedge \frac{dt_2}{t_2 - a_2} \tag{26}$$

i) Let us suppose first that

$$a_1, a_2 \notin \{0, 1\} \tag{27}$$

Consider the following two divisors in the plane X with coordinates (t_1, t_2):

$$A := \{t_1 = a_1\} \cup \{t_2 = a_2\}; \qquad B := \{t_1 = 0\} \cup \{t_1 = t_2\} \cup \{t_2 = 1\}$$



Let $B_A := B - B \cap A$. Set $m(0; a_1, a_2; 1) := H^2(X - A, B_A)$. Then, if F is a number field and $a_1, a_2 \in F$, this is a mixed Tate motive over F with a natural framing. One component of the framing is provided by

$$[\frac{dt_1}{t_1 - a_1} \wedge \frac{dt_2}{t_2 - a_2}] : \mathbb{Q}(-2) \longrightarrow \mathrm{Gr}_4^W H^2(X - A) = \mathrm{Gr}_4^W H^2(X - A, B_A)$$

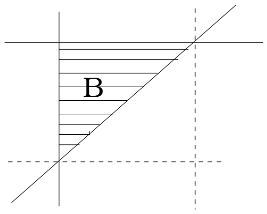
The other is given by canonical isomorphisms

$$\mathbb{Q}(0) = \mathrm{Gr}_0^W H_2(X, B_A) = \mathrm{Gr}_0^W H_2(X - A, B_A)$$

In the Betti realization it looks as follows. Choose an embedding $F \hookrightarrow \mathbb{C}$, so we may assume $a_1, a_2 \in \mathbb{C}$. Let $\gamma : [0, 1] \to \mathbb{C} - \{a_1, a_2\}$ be a path between 0 and 1. Then $\gamma(\Delta_2) \subset \mathbb{C}^2 - A(\mathbb{C})$ provides a relative homology class $[\gamma(\Delta_2)]$ generating $\mathrm{Gr}_0^W H_2(\mathbb{C}^2 - A(\mathbb{C}), B_A(\mathbb{C}))$.

ii) Now suppose that $a_1 \neq 0$; $\quad a_2 \neq 1$. This is precisely the convergence condition for the integral (26).

Observe that $H^2(X - A, B_A)$ is always a mixed Tate motive. However it has $\mathrm{Gr}_0^W = 0$ if $a_1 = 0$ or $a_2 = 1$, e.g. for $\zeta(2)$. Indeed, in these cases one of the vertices of the triangle $\gamma(\Delta_2)$ has been removed, so there is no relative cycle needed as a frame component.



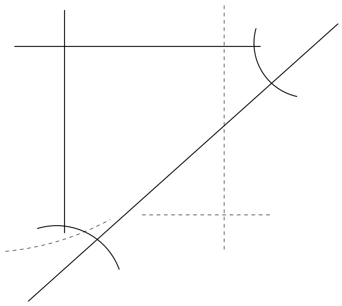
The ζ(2) picture: the shaded B−triangle
does not provide a relative class in $H^2(X-A, B_A)$

In any realization one can overcome this problem by setting

$$m(0; a_1, a_2; 1) := H^2(X, \mathcal{F}_{A,B}) \tag{28}$$

where $\mathcal{F}_{A,B}$ is an object of the corresponding derived category of sheaves on X defined as follows. Take the constant sheaf $\mathbb{Q}(0)$ on $X - (A \cup B)$ and extend it first by full direct image into $A - (A \cap B)$, and after that by $Rj_!$ into B, where $j : X - B \hookrightarrow X$. Then we get a framed mixed object in any realization which is isomorphic to the realization $m(0; a_1, a_2; 1)$ if (27) is valid. However since so far there is no theory of motivic sheaves we can not interpret the right hand side of (28) directly as an object of the triangulated category of motives over F.

Therefore we choose a different strategy. Let $\widehat{X}$ be the blow up of the plane X at the vertices $(0,0)$ and $(1,1)$ of the triangle B. The preimage $\widehat{B}$ of the triangle B is a divisor having shape of the pentagon.



Denote by $\widehat{A}$ the strict preimage of A on the blow up. Then $\widehat{A}$ does not contain any vertex of the algebraic pentagon $\widehat{B}$. It follows that the mixed Tate motive

$$m(0; a_1, a_2; 1) := H^2(\widehat{X} - \widehat{A}, \widehat{B} - (\widehat{B} \cap \widehat{A}))$$

does have a natural framing. It is canonically isomorphic to (28).

In particular when $a_1 = 1, a_2 = 0$ we constructed a motivic avatar for $\zeta(2)$.

iii) When $a_1 = 0$ or $a_2 = 1$ we define the Hodge or l-adic realization of $m(0; a_1, a_2; 1)$ by using the specialization of $m(0; x, y; 1)$ when $x \to 0$ or $y \to 1$. The computation of the specialization given in lemma 6.7 in [G7] (see also propositions 2.14 and 2.15 loc. cit.) dictates the formulae

$$m(0; 0, 1; 1) = m(0; 0, 0; 1) = m(0; 1, 1; 1) = 0,$$

$$m(0; 0, a_2; 1) = -m(0; a_2, 0; 1); \quad m(0; a_1, 1; 1) = -m(0; 1, a_1; 1)$$

We use them as a *definition* of the framed mixed Tate motives $m(0; a_1, a_2; 1)$ in the case $a_1 = 0$ or $a_2 = 1$.

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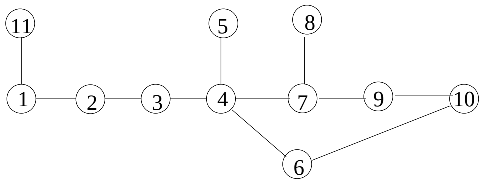


2 Framed perverse sheaves related to periods

1. The language of perverse sheaves [BBD]. Let X be a variety over a filed F. Below $D^b_{\rm Sh}(X)$ stays either for Saito's bounded derived category $D^b_{\mathcal{H}}(X)$ of mixed Hodge sheaves on X (when $F = \mathbb{C}$), or for the bounded derived category $D^b_{et}(X)$ of constructible étale l-adic sheaves on X. For a regular irreducible variety Z denote by $\delta_Z(k)$ the perverse sheaf $\mathbb{Q}_Z[{\rm dim}Z](k)$ where $\mathbb{Q}_Z = p^*\mathbb{Q}$ (or $p^*\mathbb{Q}_l$ in $D^b_{et}(X)$). Here $*(k)$ means the Tate twist. For a map $f : X \longrightarrow Y$ we denote by $f_!, f_*, f^!, f^*$ the functors between the derived categories $D^b_{\rm Sh}(X)$ and $D^b_{\rm Sh}(Y)$. The notation $R^n f_! \mathcal{F}$ is used for the n-th cohomology group of $f_!\mathcal{F}$. Denote by $\mathcal{P}_{\rm sh}(X)$ the category of perverse sheaves in $D^b_{\rm Sh}(X)$.

Let $i : Y \hookrightarrow X$, $U := X - Y$ and $j : U \hookrightarrow X$. Then there are standard exact triangles

$$\mathcal{G}^\bullet \longrightarrow j_*j^*\mathcal{G}^\bullet \longrightarrow i_!i^!\mathcal{G}^\bullet[1] \tag{29}$$

$$i_*i^*\mathcal{G}^\bullet[-1] \longrightarrow j_!j^!\mathcal{G}^\bullet \longrightarrow \mathcal{G}^\bullet \tag{30}$$

Now let Y be a codimension m regular subvariety of a regular variety X. Then

$$i^!\mathbb{Q}_X = \mathbb{Q}_Y[-2m](-m), \qquad i^*\mathbb{Q}_X = \mathbb{Q}_Y \tag{31}$$

So

$$i^!\delta_X = \delta_Y[-m](-m), \qquad i^*\delta_X = \delta_Y[m] \tag{32}$$

Therefore there are exact triangles

$$\delta_X \longrightarrow j_*\delta_U \longrightarrow \delta_Y[-m+1](-m) \tag{33}$$

$$\delta_Y[m-1] \longrightarrow j_!\delta_U \longrightarrow \delta_X \tag{34}$$

In particular if Y is a regular divisor in X there are short exact sequences of perverse sheaves

$$0 \longrightarrow \delta_X \longrightarrow j_*(\delta_U) \longrightarrow \delta_Y(-1) \longrightarrow 0 \tag{35}$$

$$0 \longrightarrow \delta_Y \longrightarrow j_!(\delta_U) \longrightarrow \delta_X \longrightarrow 0 \tag{36}$$

The perverse mixed Hodge sheaves are equipped with canonical weight filtration $W_\bullet$. We use the standard convention for the weights: $\mathrm{w}(X[n]) = \mathrm{w}(X) + n$. By purity for a regular projective variety P and a pure perverse sheaf $\mathcal{F}_a$ of weight a the weight of $H^i(P, \mathcal{F}_a)$ is $i + a$. The weight of $\delta_Y(-k)$ is $\dim(Y) + 2k$.

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two sets of isomorphism classes of objects of a triangulated category. We define a new set $\mathcal{O}_1 * \mathcal{O}_2$ of isomorphism classes of objects so that $Y \in \mathcal{O}_1 * \mathcal{O}_2$ if and only if there exists an exact triangle $X \longrightarrow Y \longrightarrow Z$ with $X \in \mathcal{O}_1$ and $Z \in \mathcal{O}_2$. We say that an object is glued from the set of objects $\{A_i\}$ if it is isomorphic to an object from $\{A_i\} * ... * \{A_i\}$.

Lemma 2.1 *Suppose that a perverse étale l-adic sheaf $\mathcal{F}$ on X is glued from the objects $\delta_{C_i}(m)$ where X and C_i's are regular and defined over a field F. Then $\mathcal{F}$ has a natural weight filtration.*

Proof. If X is defined over a finite field F_q there is a weight filtration on perverse sheaves, see [BBD]. We use the reduction to a finite field to deduce the lemma from this. One can find a model of X and C_i's over a ring $\mathcal{O}$ which is finitely generated over $\mathbb{Z}$. Choose a generic F_q-point $i_\varepsilon : \mathrm{Spec} F_q \hookrightarrow \mathrm{Spec}\mathcal{O}$. Denote by $\overline{X}, \overline{C}_i$ the reduction of the schemes X, C_i modulo the maximal ideal of $\mathcal{O}$ corresponding to i_ε. For generic F_q-points ε the restriction functor i^*_ε commutes with the standard functors. In particular this implies that

$$Ext^1_X(\delta_Y(n), \delta_Z(m)) = Ext^1_{\overline{X}}(\delta_{\overline{Y}}(n), \delta_{\overline{Z}}(m))$$

The group on the right vanishes when $\mathrm{w}(\delta_Y(n)) \leq \mathrm{w}(\delta_Z(m))$. It remains to use lemma 5.3.6 in [BBD]. The lemma follows.

2. Perverse sheaves corresponding to an admissible pair of divisors. *The stratification of X defined by a divisor D.* Let D be a divisor whose irreducible components D_i, $i \in \Lambda$, are divisors. For each point $x \in X$ let I_x be the set parametrizing all irreducible components of D containing x. The subvariety formed by all $x \in X$ with given $I_x = I$ is called the *I-stratum* defined by D.

Regular stratifications. We say that the stratification defined by the divisor D is *regular* if all I-strata are irreducible, and the closure of any stratum is a regular subvariety. The closure of the I-stratum is $D_I := \cap_{i \in I} D_i$.

Admissible pairs of divisors and their faces. Let X be a regular projective variety over a field F of dimension n, A and B are two divisors. Let

$$A = A_1 \cup ... \cup A_s; \quad B = B_1 \cup ... \cup B_t$$

be the decomposition of A and B into irreducible components. For subsets

$$I = \{i_1, ..., i_k\} \subset \{1, ..., s\}; \quad J = \{j_1, ..., j_l\} \subset \{1, ..., t\}$$

we set $A_I = A_{i_1} \cap ... \cap A_{i_k}$ and $B_J = B_{j_1} \cap ... \cap B_{j_l}$ and call them the *A- and B-faces*.

Definition 2.2 *A pair (A, B) is admissible if the divisor $A \cup B$ provides a regular stratification of X, and no A- and B-faces coincide.*

We allow the A-faces to be subvarieties of B-faces of positive codimension, and vice versa. We assume that the pair (A, B) is admissible.

Pure A and B- strata. Set

$$A_I^0 := A_I - \{\text{union of all codimension} > 0 \text{ strata of } A_I\}$$

and similarly B_J^0. We say that A_I^0 and B_J^0 are *pure A- and B- strata.*

Observe that in general A- or B- faces are not strata. They are closures of the corresponding pure A- and B- strata.

Let $U := X - (A \cup B)$. We define an object $\mathcal{F}^{\bullet}_{A,B} \in D^b_{\mathrm{Sh}}(X)$ as a certain extension of δ_U to X. This is done by an inductive procedure consisting of $n = \mathrm{dim} X$ steps. Namely, consider open subsets

$$U(k) := X - \text{all codimension} > k \text{ strata},$$

so

$$U = U(0) \hookrightarrow U(1) \hookrightarrow ... \hookrightarrow U(n-1) \hookrightarrow U(n) = X$$

Then $U(k) - U(k-1)$ is the disjoint union of the codimension k strata of three types: pure A-strata, pure B-strata, and mixed strata.

We define the *extension to codimension k strata* functor

$$E_k^* : D^b_{\mathrm{Sh}}(U(k-1)) \longrightarrow D^b_{\mathrm{Sh}}(U(k))$$

as the extension by j_* to all pure codimension k A-strata and mixed strata, followed by the extension by $j_!$ to all codimension k pure B-strata. Precisely, consider an open subset

$$U_k^A := U(k-1) \cup \text{all codimension } k \text{ pure } A\text{-strata and mixed strata} =$$

$$U(k) - \text{all codimension } k \text{ pure } B\text{-strata}$$

Then we have

$$U(k-1) \quad \overset{j_k^A}{\hookrightarrow} \quad U_k^A \quad \overset{j_k^B}{\hookrightarrow} \quad U(k); \qquad E_k^* := j_{k!}^B j_{k*}^A$$

Set

$$\mathcal{F}^\bullet_{A,*,B} := \quad E_n^* ... E_2^* E_1^* \delta_U \quad = \quad j_{n!}^B j_{n*}^A ... j_{1!}^B j_{1*}^A \delta_U$$

and consider the perverse sheaf

$$\mathcal{F}^*_{A,B} := H^0_\tau \mathcal{F}^\bullet_{A,*,B}$$

where H^*_τ are the cohomology groups with respect to the t-structure on $D^b_{\mathrm{Sh}}(X)$ providing the abelian category of perverse sheaves on X.

One can define a similar functor $E_k^!$ by using the $j_!$-extension to the mixed strata. In particular set

$$\mathcal{F}^!_{A,B} := H^0_\tau \Big(E_n^! ... E_2^! E_1^! \delta_U \Big)$$

More generally, on can define similar objects by using either $E_k^!$ or E_k^* for each given k.

Remark. Since the strata of different types are disjoint, the extensions by $j_!$ and by j_* commute. Therefore

$$*\mathcal{F}^!_{A,B} = \mathcal{F}^*_{B,A}$$

where $*$ is the duality functor.

The frame morphisms. Let $j_A : X - A \hookrightarrow X$, $j_B : X - B \hookrightarrow X$. Set

$$\mathcal{F}_A := j_{A*} \delta_{X-A}; \qquad \mathcal{F}_B := j_{B!} \delta_{X-B}$$

Lemma 2.3 *There are canonical morphisms*

$$\mathcal{F}_B \xrightarrow{\beta} \mathcal{F}^!_{A,B} \xrightarrow{c} \mathcal{F}^*_{A,B} \xrightarrow{\alpha} \mathcal{F}_A \tag{37}$$

Proof. Let us define the map α. We define by induction the maps

$$\alpha_k : \mathrm{Res}_{U(k)} \mathcal{F}^\bullet_{A,*,B} \longrightarrow \mathrm{Res}_{U(k)} \mathcal{F}_A \tag{38}$$

and set $\alpha := H^0_\tau \alpha_n$. The map α_0 is the obvious isomorphism. The map α_k is constructed by combining the following two general observations, which will also play an important role below (especially in chapter 3).

i) Suppose that $\mathcal{G}^\bullet \in D^b_{\mathrm{sh}}(X)$ and there is a map

$$\mathrm{Res}_{U(k-1)}\mathcal{G}^\bullet \longrightarrow \mathrm{Res}_{U(k-1)}\mathcal{F}_A \tag{39}$$

Then we have a map

$$j^A_{k*}\mathrm{Res}_{U(k-1)}\mathcal{G}^\bullet \longrightarrow \mathrm{Res}_{U^A(k-1)}\mathcal{F}_A \tag{40}$$

given by applying j^A_{k*} to (39):

$$j^A_{k*}\mathrm{Res}_{U(k-1)}\mathcal{G}^\bullet \longrightarrow j^A_{k*}\mathrm{Res}_{U(k-1)}\mathcal{F}_A = \mathrm{Res}_{U^A(k-1)}\mathcal{F}_A$$

ii) Suppose that $\mathcal{G}^\bullet \in D^b_{\mathrm{sh}}(X)$ and there is a map

$$\mathrm{Res}_{U^A(k-1)}\mathcal{G}^\bullet \longrightarrow \mathrm{Res}_{U^A(k-1)}\mathcal{F}_A \tag{41}$$

Then we have a map

$$j^B_{k!}\mathrm{Res}_{U^A(k-1)}\mathcal{G}^\bullet \longrightarrow \mathrm{Res}_{U(k)}\mathcal{F}_A \tag{42}$$

given as a composition

$$j^B_{k!}\mathrm{Res}_{U^A(k-1)}\mathcal{G}^\bullet \xrightarrow{(41)} j^B_{k!}\mathrm{Res}_{U^A(k-1)}\mathcal{F}_A = j^B_{k!}j^{B!}_k\mathrm{Res}_{U(k)}\mathcal{F}_A \longrightarrow \mathrm{Res}_{U(k)}\mathcal{F}_A$$

The last map is the adjunction morphism.

Combining the constructions i) and ii) for $\mathcal{G}^\bullet = \mathcal{F}^\bullet_{A,*,B}$ we get the map α_k.

The map β is defined using the duality $*$ interchanging the role of A and B divisors. The map c is defined using the canonical morphism $j_! \longrightarrow j_*$ for each extension to a mixed stratum. The lemma is proved.

3. The structure of $H^0(X, \mathcal{F}^*_{A,B})$. We are going to study the following perverse sheaves:

$$\mathcal{F}^*_{A,B}, \mathcal{F}^!_{A,B}, \mathcal{F}_A, \mathcal{F}_B \tag{43}$$

Proposition 2.4 *Suppose that (A,B) is an admissible pair of divisors on X. Then each of the objects (43) is glued from $\delta_C(-m)$, where C runs through the strata of the stratification defined by $A \cup B$, and $0 \le m \le \mathrm{codim}_X C$.*

Proof. If $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a triangulated functor between triangulated categories, an object X is glued from the objects $\{A_i\}$, and every object $F(A_i)$ is glued from the set of objects $\{B_j\}$, then $F(X)$ is glued from $\{B_j\}$. So using (33)-(34) we see that each of the objects (43) is glued from $\delta_C(-m)[p]$ where $0 \le m \le \mathrm{codim}_X C$. Further, one easily proves by induction that if an object of an abelian category $\mathcal{A}$ is glued from the objects $\{A_i[p_i]\}$ of the derived category

$D^b(\mathcal{A})$ where $A_i \in \mathcal{A}$ then it is glued from those of these objects which belong to $\mathcal{A}$, i.e. have $p_i = 0$. Since objects (43) are from the abelian category of the perverse sheaves the proposition follows.

Let us set

$$\mathrm{dw}(\delta_C(-m)) = \dim_X C + m = \mathrm{w}(\delta_C(m)) - m$$

Proposition 2.5 *Suppose that Y and Z are two closed irreducible regular subvarieties of a regular variety X. Then in $\mathcal{P}_{\mathrm{Sh}}(X)$ one has*

$$Hom_X(\delta_Y(n), \delta_Z(m)) = 0 \quad \textit{unless } Y = Z,\ n = m \tag{44}$$

$$Ext^1_X(\delta_Y(n), \delta_Z(m)) = 0 \quad \textit{if } \mathrm{dw}(\delta_Y(n)) < \mathrm{dw}(\delta_Z(m))$$

$$Ext^1_X(\delta_Y(-n), \delta_Z(-m)) = 0 \quad \textit{if } m > n$$

Proof. The objects $\delta_Y(n)$ are irreducible, thus we have the first statement. The case $Y = Z$ is clear from the weight considerations. So we may assume $Y \neq Z$. We claim that

$$Ext^1_X(\delta_Y(n), \delta_Z(m)) = \begin{cases} 0 & \text{if } |d_{Y,Z}| > 1 \\ 0 & \text{if none of the varieties } Y, Z \text{ contains the other} \end{cases}$$

The second claim is obvious, so we may assume without loss of generality that $Z \subset Y$. Let $d_W = \dim W$ and $d_{Y,Z} := d_Y - d_Z$. Then one has

$$Ext^i_X(\delta_Y(n), \delta_Z(m)) = Ext^i_X(\delta_Z(-m), \delta_Y(-n + d_{Y,Z})) = H^{i - d_{Y,Z}}(Z, \mathbb{Q}(m-n))$$

Indeed, to compute the first Ext observe that

$$Hom_X(\mathbb{Q}_Y[d_Y](n), \mathbb{Q}_Z[d_Z + i](m)) =$$

$$Hom_Y(\mathbb{Q}_Y, \mathbb{Q}_Z[d_Z - d_Y + i](m-n)) = H^{i - d_{Y,Z}}(Z, \mathbb{Q}(m-n))$$

The second equality follows by the duality since

$$*\delta_C(-m) = \delta_C(m + \dim C) \tag{45}$$

So to prove the proposition it remains to consider the case when $|d_{Y,Z}| = 1$. The weight considerations shows that

$$Ext^1_X(\delta_Y(n), \delta_Z(m)) = 0 \quad \text{if } d_{Y,Z} = 1 \text{ and } m - n < 0 \tag{46}$$

and

$$Ext^1_X(\delta_Y(n), \delta_Z(m)) = 0 \quad \text{if } d_{Y,Z} = -1 \text{ and } m - n \leq 0 \tag{47}$$

Indeed, under these assumptions

$$\mathrm{w}(\delta_Y(n)) - \mathrm{w}(\delta_Z(m)) = d_{Y,Z} + 2(m-n) < 0$$

So the Ext^1-groups vanish. Observe also that (47) can be deduced from (46) by duality thanks to (45). This gives the third statement of the proposition. To check the second notice that by the assumption in the second statement

$$\mathrm{dw}(\delta_Y(n)) - \mathrm{dw}(\delta_Z(m)) = d_{Y,Z} + (m-n) < 0$$

This means that

$$d_{Y,Z} = -1, m-n \leq 0 \quad \text{or} \quad d_{Y,Z} = 1, m-n < -1$$

In the first case Ext^1 vanishes by (47), and in the second by (46). The second statement and hence the proposition are proved.

Corollary 2.6 *Let $\mathcal{M}$ be an object of $\mathcal{P}_{\mathrm{Sh}}(X)$ on a regular variety X. Suppose that $\mathcal{M}$ is glued from the objects $\delta_C(n)$, where C's are regular closed irreducible subvarieties of X. Then $\mathcal{M}$ carries two canonical increasing filtrations indexed by integers:*

a) The dimension-weight filtration $DW_\bullet$ such that $\mathrm{Gr}_k^{DW}\mathcal{M}$ is a direct sum of the objects $\delta_C(-m)$ with $\mathrm{dim}C + m = k$.

b) The filtration $T_\bullet$ by the Tate twist such that $\mathrm{Gr}_k^T\mathcal{M}$ is a direct sum of the objects $\delta_C(-k)$.

c) The duality $*$ interchanges these two filtrations, so that

$$*\mathrm{Gr}_k^T\mathcal{M} = \mathrm{Gr}_{-k}^{DW} * \mathcal{M}$$

Proof. The parts a) and b) follow from proposition 2.5 and lemma 5.3.6 in [BBD]. Namely, for the dimension-weight filtration we use the second statement of the proposition, and for the filtration by the Tate twist the third. The part c) follows from a) and b) using formula (45). The corollary is proved.

Recall that $n = \mathrm{dim}X$.

Proposition 2.7 *The morphisms (37) induce isomorphisms*

$$\mathrm{Gr}_n^{DW}(\mathcal{F}^!_{A,B}) \xrightarrow{\sim} \mathrm{Gr}_n^{DW}(\mathcal{F}^*_{A,B}) \xrightarrow{\sim} \mathrm{Gr}_n^{DW}(\mathcal{F}_A) \tag{48}$$

$$\mathrm{Gr}_0^T(\mathcal{F}_B) \xrightarrow{\sim} \mathrm{Gr}_0^T(\mathcal{F}^!_{A,B}) \xrightarrow{\sim} \mathrm{Gr}_0^T(\mathcal{F}^*_{A,B}) \tag{49}$$

Proof. The second line of isomorphisms follows from the first one by the duality using corollary 2.6.

Let us prove the last isomorphism in (48). Let us show by induction that the maps α_k, see (38), induce isomorphisms on $\mathrm{Gr}_n^{DW} H_\tau^0(-)$. If $k = 0$ this is clear.

For a set $\{\delta_{C_i}(-m_i)[p_i]\}$ let

$$\mathrm{dw}_+(\{\delta_{C_i}(-m_i)[p_i]\}) = \mathrm{max}_i(\mathrm{dim}C_i + m_i) \tag{50}$$

Suppose that $\mathcal{M}$ is glued from the objects $\{\delta_{C_i}(-m_i)[p_i]\}$. There are many sets of such objects generating $\mathcal{M}$ (for instance we can add an object to a list). Let $\mathrm{dw}_+(\mathcal{M})$ be the smallest value invariant (50) can have for such a generating set.

Lemma 2.8 *Let Y be a regular subvariety of a regular variety X, $U := X - Y$, and $j : U \hookrightarrow X$. Suppose that $\mathcal{M}$ is glued from $\delta_C(-m)[p]$ where all subvarieties C and $C \cap Y$ are regular. Then*

$$\mathrm{dw}_+(j_*\mathcal{M}) \leq \mathrm{dw}_+(\mathcal{M}), \quad \mathrm{dw}_+(j_!\mathcal{M}) \leq \mathrm{dw}_+(\mathcal{M})$$

Proof. Follows immediately from (29), (30) and (32). The lemma is proved.

By lemma 2.8 and using (33)-(34) we see by induction on k that $\mathcal{G}^\bullet = \mathcal{G}^\bullet_*$ satisfies the following condition:

for $U = U(k)$ and $U = U^A(k)$, $k \geq 0$, one has

$$\mathrm{dw}_+\Big(H^i_\tau(\mathrm{Res}_U\mathcal{G}^\bullet)\Big) \leq \begin{cases} n & \\ n-1 & \text{for } i < 0 \end{cases} \tag{51}$$

Lemma 2.9 *Let $\mathcal{G}^\bullet \in D^b_{\mathrm{sh}}(X)$. Assume that $\mathcal{G}^\bullet$ is glued from $\delta_C(-m)[p]$ where C are strata of a regular stratification of X and $0 \leq m \leq \mathrm{codim}_X C$. Let us also assume (51). Then*

i) If map (39) induces an isomorphism on $\mathrm{Gr}^{WD}_n H^0_\tau(-)$ then map (40) has the same property.

ii) If map (41) induces an isomorphism on $\mathrm{Gr}^{WD}_n H^0_\tau(-)$ then map (42) has the same property.

Proof. i) We claim that

$$\mathrm{Gr}^{DW}_n H^0_\tau\Big(j^A_{k*}\mathrm{Res}_{U(k-1)}\mathcal{G}^\bullet\Big) \cong \mathrm{Gr}^{DW}_n H^0_\tau\Big(j^A_{k*}H^0_\tau(\mathrm{Res}_{U(k-1)}\mathcal{G}^\bullet)\Big) \tag{52}$$

Indeed, it follows from (51) and lemma 2.8 that the left object in (52) is isomorphic to

$$\mathrm{Gr}^{DW}_n H^0_\tau\Big(j^A_{k*}\tau_{\geq 0}(\mathrm{Res}_{U(k-1)}\mathcal{G}^\bullet)\Big)$$

On the other hand by (51) and (33)-(34)

$$\mathrm{Gr}^{DW}_n H^0_\tau\Big(j^A_{k*}\tau_{\geq 1}(\mathrm{Res}_{U(k-1)}\mathcal{G}^\bullet)\Big) = 0$$

Isomorphism (52) follows from these two facts.

Similarly, by (51) and lemma 2.8 the right object in (52) is isomorphic to

$$\mathrm{Gr}^{DW}_n H^0_\tau\Big(j^A_{k*}\mathrm{Gr}^{DW}_n H^0_\tau(\mathrm{Res}_{U(k-1)}\mathcal{G}^\bullet)\Big)$$

By the induction assumption it is isomorphic to

$$\mathrm{Gr}^{DW}_n H^0_\tau\Big(j^A_{k*}\mathrm{Gr}^{DW}_n \mathrm{Res}_{U(k-1)}\mathcal{F}_A\Big)$$

(Observe that $\mathcal{F}_A$ is a perverse sheaf). By lemma 2.8 this object, and hence the right object in (52), are isomorphic to

$$\mathrm{Gr}^{DW}_n H^0_\tau\Big(j^A_{k*}\mathrm{Res}_{U(k-1)}\mathcal{F}_A\Big)$$

It remains to notice that $j^A_{k*}\mathrm{Res}_{U(k-1)}\mathcal{F}_A = \mathrm{Res}_{U^A(k-1)}\mathcal{F}_A$. The statement i) is proved.

To prove statement ii) we follow the same pattern, changing $U^A(k-1)$ to $U(k-1)$ and j^A_{k*} to $j^B_{k!}$. In the end we need to show in addition that

$$\mathrm{Gr}^{DW}_n H^0_\tau\Big(j^B_{k!}j^{B!}_k\mathrm{Res}_{U(k)}\mathcal{F}_A\Big) = \mathrm{Gr}^{DW}_n H^0_\tau\Big(\mathrm{Res}_{U(k)}\mathcal{F}_A\Big)$$

There is a standard exact triangle where i^B_k is the closed embedding complementary to j^B_k:

$$i^B_{k*}i^{B*}_k\mathrm{Res}_{U(k)}\mathcal{F}_A[-1] \longrightarrow j^B_{k!}j^{B!}_k\mathrm{Res}_{U(k)}\mathcal{F}_A \longrightarrow \mathrm{Res}_{U(k)}\mathcal{F}_A$$

So it is sufficient to show that the left object is glued from the objects $\delta_C(-m)[p]$ with $\mathrm{dim}C+m<n$. Since $\mathcal{F}_A$ and hence $\mathrm{Res}_{U(k)}\mathcal{F}_A$ are glued from such objects with $\mathrm{dim}C+m\leq n$, and the stratification defined by $A\cup B$ is regular, the statement follows from the second part of (32). The lemma is proved.

The proof of the first isomorphism in (48) is even simpler: it uses only the argument similar to the one in the end of the proof of part ii). The proposition is proved.

Let $\mathrm{dp}(A)$ (resp. $\mathrm{dp}(B)$) be the codimension of the smallest pure A-strata (resp. pure B-strata) in X.

Theorem 2.10 *In the category $\mathcal{P}_{\mathrm{Sh}}(X)$*

a) the morphisms (37) induce canonical isomorphisms

$$\mathrm{Gr}^W_{2n}H^*(X,\mathcal{F}^!_{A,B}) \xrightarrow{\sim} \mathrm{Gr}^W_{2n}H^*(X,\mathcal{F}^*_{A,B}) \xrightarrow{\sim} \mathrm{Gr}^W_{2n}H^*(X,\mathcal{F}_A) \qquad (53)$$

$$\mathrm{Gr}^W_0 H^*(X,\mathcal{F}_B) \xrightarrow{\sim} \mathrm{Gr}^W_0 H^*(X,\mathcal{F}^!_{A,B}) \xrightarrow{\sim} \mathrm{Gr}^W_0 H^*(X,\mathcal{F}^*_{A,B}) \qquad (54)$$

b) there are canonical isomorphisms

$$\begin{gathered}\mathrm{Gr}^W_{n+\mathrm{dp}(A)}H^0(X,\mathcal{F}^!_{A,B}) \xrightarrow{\sim} \\ \mathrm{Gr}^W_{n+\mathrm{dp}(A)}H^0(X,\mathcal{F}^*_{A,B}) \xrightarrow{\sim} \mathrm{Gr}^W_{n+\mathrm{dp}(A)}H^0(X,\mathcal{F}_A)\end{gathered} \qquad (55)$$

$$\begin{gathered}\mathrm{Gr}^W_{n-\mathrm{dp}(B)}H^0(X,\mathcal{F}_B) \xrightarrow{\sim} \\ \mathrm{Gr}^W_{n-\mathrm{dp}(B)}H^0(X,\mathcal{F}^!_{A,B}) \xrightarrow{\sim} \mathrm{Gr}^W_{n-\mathrm{dp}(B)}H^0(X,\mathcal{F}^*_{A,B})\end{gathered} \qquad (56)$$

Proof. a) Let us calculate the groups $\mathrm{Gr}^W_w H^p(X,\mathcal{F})$ using the spectral sequence for the weight filtration on the perverse sheaf $\mathcal{F}$. Later on $\mathcal{F}$ will be one of the sheaves (43). The E_1-term of the spectral sequence consists of the groups $H^i(X;\mathrm{Gr}^W_a\mathcal{F})$ with the differential

$$H^i(X;\mathrm{Gr}^W_a\mathcal{F}) \longrightarrow H^{i+1}(X;\mathrm{Gr}^W_{a-1}\mathcal{F}) \qquad (57)$$

Observe that by purity the weight of $H^a(X, \mathrm{Gr}^W_b \mathcal{F})$ is $a+b$. So the higher differentials are zero by the weight considerations. The E_1-term of the spectral sequence consists of complexes sitting on the lines where the weight $a+b$ is a constant. In particular the weight zero and $2n$ parts of this spectral sequence are the complexes formed by the groups

$$H^{-a}(X; \mathrm{Gr}^W_a \mathcal{F}) \qquad \text{and} \quad H^a(X; \mathrm{Gr}^W_{2n-a}\mathcal{F}) \tag{58}$$

respectively, with the differential provided by (57).

If C is a regular subvariety of X then

$$H^a(X, \delta_C(-m)) = 0 \quad \text{for } |a| > \dim C$$

It follows that

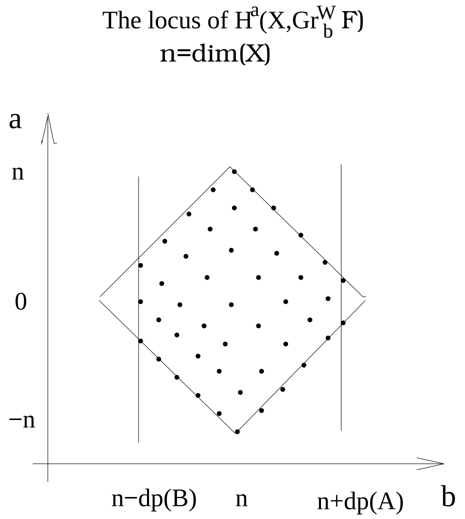
$$H^{-a}(X; \mathrm{Gr}^W_a \mathcal{F}) = H^{-a}(X; \mathrm{Gr}^W_a (T_0\mathcal{F}))$$

Indeed, by proposition 2.4 the object $\mathrm{Gr}^W_a \mathcal{F}$ is a direct sum of $\delta_C(-m)$ where $\dim C + 2m = a$. So if $m > 0$ then $H^{-a}(X; \delta_C(-m)) = 0$. This and proposition 2.7 gives (54). The isomorphism (53) follows by duality. The part a) is proved.

b) *The shape of the E_1-term of the spectral sequence.* If $\mathcal{F}$ is one of the sheaves (43) then

$$H^a(X, \mathrm{Gr}^W_b \mathcal{F}) = 0 \quad \text{if } b \pm a > 2n \text{ or } b \pm a < 0$$

So the E_1-term of the spectral sequence is located inside of the diamond $0 \leq b \pm a \leq 2n$. Moreover, it is bounded by the lines $n - \mathrm{dp}(B) \leq b \leq n + \mathrm{dp}(A)$. A particular interesting case is $\mathrm{dp}(A) = \mathrm{dp}(B) = n$.



It follows that for any perverse sheaf $\mathcal{F}$ glued from $\{\delta_C(-m)\}$ with $0 \leq m \leq \mathrm{codim} C \leq \mathrm{dp}(A)$ one has

$$\mathrm{Gr}^W_{n+\mathrm{dp}(A)} H^0(X, \mathcal{F}) = \mathrm{Gr}^W_{n+\mathrm{dp}(A)} H^0(X, \mathrm{Gr}^{DW}_n \mathcal{F})$$

The isomorphisms (55) follow from this and proposition 2.7. The isomorphisms (56) are obtained by duality. The part b) and hence the theorem are proved.

4. The framing in the Tate case. A regular projective variety X is a *Tate variety* if the motive of X is a direct sum of the Tate motives $\mathbb{Q}(m)$. For example a flag variety G/P where P is a parabolic subgroup of a simple Lie group is Tate. The product of Tate varieties is a Tate variety. If all strata of a divisor D on X (including $D_\emptyset = X$) are Tate varieties we say that D provides a Tate stratification of X. For example a union of hyperplanes provides a Tate stratification of P^n.

If $A \cup B$ provides a Tate stratification of X and (A, B) is an admissible pair then obviously applying the functor $H^0(X; -)$ to any of the perverse sheaves (43) we get a mixed Tate object over the point.

Recall (see, say, [G7] or [G8]) that a framed object in a mixed Tate category $\mathcal{C}$ is defined as a triple (H, v_{2n}, f_0) where $v_{2n} : \mathbb{Q}(-n) \longrightarrow \mathrm{Gr}^W_{2n} H$ and $f : \mathrm{Gr}^W_0 H \longrightarrow \mathbb{Q}(0)$ are non zero morphisms. We define an equivalence of framed mixed Tate objects as the finest equivalence relation such that a morphism compatible with frames is an equivalence. The equivalence classes of objects framed by $\mathbb{Q}(0)$ and $\mathbb{Q}(-n)$ form an abelian group $\mathcal{A}_n(\mathcal{C})$, and $\mathcal{A}_n(\mathcal{C}) := \oplus_{n \geq 0} \mathcal{A}_n(\mathcal{C})$ has a commutative Hopf algebra structure.

Suppose $\mathrm{dp}(A) = \mathrm{dp}(B) = n$. Let us assume that we are given non zero classes

$$[\omega_A] \in \mathrm{Hom}(\mathbb{Q}(-n), \mathrm{Gr}^W_{2n} H^0(X, \mathcal{F}_A)); \qquad [\Delta_B] \in \mathrm{Hom}(\mathrm{Gr}^W_0 H^0(X, \mathcal{F}_B), \mathbb{Q}(0))$$

Then using canonical isomorphisms (55)-(56) we transfer them to distinguished classes

$$[\omega_A] \in \mathrm{Hom}(\mathbb{Q}(-n), \mathrm{Gr}^W_{2n} H^0(X, \mathcal{F}^*_{A,B})) = \mathrm{Hom}(\mathbb{Q}(-n), \mathrm{Gr}^W_{2n} H^0(X, \mathcal{F}^!_{A,B}))$$

and

$$[\Delta_B] \in \mathrm{Hom}(\mathrm{Gr}^W_{2n} H^0(X, \mathcal{F}^*_{A,B}), \mathbb{Q}(0)) = \mathrm{Hom}(\mathrm{Gr}^W_{2n} H^0(X, \mathcal{F}^!_{A,B}), \mathbb{Q}(0))$$

Therefore we get equivalent framed mixed Tate objects

$$(H^0(X, \mathcal{F}^*_{A,B}), [\omega_A], [\Delta_B]) \quad \sim \quad (H^0(X, \mathcal{F}^!_{A,B}, [\omega_A], [\Delta_B]) \tag{59}$$

5. The framing in the general case. The general definition of framed objects in a mixed category see in chapter 2 of [G8]. In this paper we employ an adapted version of this definition sufficient for our goals. The reader is invited to formulate and prove the results of the next chapters using the definition given in [G8].

Let $\mathcal{P}$ be a semisimple category. We assume that each simple object of $\mathcal{P}$ is labelled by an integer called the weight. Let $\mathcal{C}_{\mathcal{P}}$ be a mixed category such that every its object carries canonical weight filtration $W_\bullet$ so that Gr_k^W is a direct sum of weight k objects of $\mathcal{P}$. Let us assume that we are given a fiber functor $F : \mathcal{P} \longrightarrow \mathrm{Vect}$ to the category of finite dimensional vector spaces. Then a framing on a mixed object H is a triple (H, v, f) defined by a pair of non zero vectors

$$v \in F(\mathrm{Gr}_p^W H), \quad f \in F(\mathrm{Gr}_q^W H)^\vee; \qquad p \geq q$$

where $V^\vee$ is the vector space dual to V.

Let us return to our situation. The two categories we have in mind are the categories of mixed Hodge structures and l-adic Galois modules equipped with a weight filtration. The functor F assigns to an object of one of these categories the underlying vector space. We define a framing on $H^0(X, \mathcal{F}^*_{A,B})$ as a triple

$$(H^0(X, \mathcal{F}^*_{A,B}), [\omega_A], [\Delta_B]) \tag{60}$$

where the classes

$$[\omega_A] \in \mathrm{Gr}^W_{n+\mathrm{dp}(A)} H^0(X, \mathcal{F}_A); \qquad [\Delta_B] \in \left(\mathrm{Gr}^W_{n-\mathrm{dp}(B)} H^0(X, \mathcal{F}_B)\right)^\vee$$

are transformed via canonical isomorphisms (55)-(56) to distinguished classes

$$[\omega_A] \in \mathrm{Gr}^W_{n+\mathrm{dp}(A)} H^0(X, \mathcal{F}^*_{A,B}); \qquad [\Delta_B] \in \left(\mathrm{Gr}^W_{n-\mathrm{dp}(B)} H^0(X, \mathcal{F}^*_{A,B})\right)^\vee$$

We define a framing on $H^0(X, \mathcal{F}^!_{A,B})$ using the same two classes. So the map c from lemma 2.3 is an equivalence of the framed objects corresponding to $H^0(X, \mathcal{F}^*_{A,B})$ and $H^0(X, \mathcal{F}^!_{A,B})$.

Remark. Both $X - A$ and $X - B$ are affine varieties, so $H^i_{DR}(X - A) = H^i \Omega^\bullet_{\log}(X - A)$. Thus the class $[\omega]$ is represented by a form $\omega \in \Omega^n_{\log}(X - A)$. So the equivalence class of the framed objects (60) always encodes the properties of the period $\int_\Delta \omega$, where Δ is a relative cycle.

Remark. If X is a regular projective curve then $H^0(X, \mathcal{F}^*_{A,B})$ is a 1-motive in the sense of Deligne [D4].

3 Motivic setting

Let $\pi : X \longrightarrow \mathrm{Spec}(F)$ be the canonical projection. In this section we show that, assuming a condition on D, the objects $\pi_* \mathcal{F}^\bullet_{A,*,B}$ and $\pi_* \mathcal{F}^\bullet_{A,!,B}$ in $D^b_{\mathrm{sh}}(\mathrm{Spec}(F))$ are of geometric origin. If F is a number field and the divisor $D = A \cup B$ provides a Tate stratification of X we define

$$H^0(X; \mathcal{F}^\bullet_{A,*,B}) \quad \text{and} \quad H^0(X; \mathcal{F}^\bullet_{A,!,B})$$

as objects of the abelian category $\mathcal{M}_T(F)$ of mixed Tate motives over F

1. A blow up theorem. Let D be a divisor on X. We assume that all irreducible components of D are divisors, and the stratification defined by D is regular.

By a *sequence of centered at strata blow ups* we understand the following procedure. Blow up the closure of a stratum S of D, getting a divisor D_1 on a variety X_1. Then blow up the closure of a stratum S_1 of D_1, getting a divisor D_2 on X_2, ..., blow up the closure of a stratum S_{m-1} of D_{m-1}, getting a divisor D_m on a variety X_m.

Theorem 3.1 *Suppose that D is a divisor locally isomorphic to a union of hyperplanes. Then there exists a centered at strata sequence of blow ups such that the last blow up delivers a normal crossing divisor.*

Proof. A stratum S of D is called a good stratum if D is a normal crossing divisor near S. Otherwise it is a bad stratum.

Construction 3.2 *Let us blow up all bad vertices of D, then closures of preimages of all bad 1-dimensional strata, after this closures of preimages of all bad 2-dimensional strata, and so on. Let $\widehat{X}$ be the resulting variety and $\widehat{D}$ the corresponding divisor on $\widehat{X}$.*

Lemma 3.3 *$\widehat{D}$ a normal crossing divisor on $\widehat{X}$.*

Proof. We use the induction on $\dim X$. Let $p : \widehat{X} \longrightarrow X$ be the natural projection and $x \in \widehat{X}$. Let k be the dimension of the stratum containing $p(x)$. Suppose that $k > 0$. Since our configuration is locally isomorphic to a family of hyperplanes we have a product situation near $p(x)$. Precisely, there exists a regular divisor D_Y in a variety Y which is locally isomorphic to a family of hyperplanes, such that near $p(x)$ the pair (D, X) is locally isomorphic to the one $(\mathbb{A}^k \times D_Y, \mathbb{A}^k \times Y)$. Moreover the pair $(\widehat{D}, \widehat{X})$ near x is locally isomorphic to $(\mathbb{A}^k \times \widehat{D}_Y, \mathbb{A}^k \times \widehat{Y})$ where $(\widehat{D}, \widehat{Y})$ is obtained from (D, Y) by our construction. Therefore by induction $\widehat{D}$ a normal crossing divisor near such x.

Suppose now that $k = 0$, i.e. $p(x)$ is bad vertex of D. We blow up $p(x)$. Let P be the special fiber of the blow up. It intersects transversally the strict preimages of other strata of D. Intersecting P with them we get a divisor D' in P. It is isomorphic to a family of hyperplanes in P^{n-1}. Now continue our construction, i.e. blow up other bad vertices, then bad edges, and so on, and see how it affects neighborhood of P and its preimages after the blow ups. The preimage $\widehat{P}$ of P i'n $\widehat{X}$ is obtained by applying our construction to the divisor D' in P. Therefore the divisor $\widehat{D}_P$ in $\widehat{P}$ defined as the intersection of $\widehat{P}$ with the closure of $\widehat{D} - \widehat{P}$ is a normal crossing divisor by the induction assumption. The closure of the divisor $\widehat{D} - \widehat{P}$ intersects $\widehat{P}$ transversally, providing a bijective correspondence $S \longrightarrow S \cap P$ between the proper strata of $\widehat{D}_P$ and the ones of $\widehat{D}$ intersecting $\widehat{P}$. It follows that $\widehat{D}$ is a normal crossing divisor near $\widehat{P}$.

The lemma and hence the theorem are proved.

2. A comparison theorem. Suppose that (A,B) is an admissible pair of divisors on X. Let us blow up the closure of a stratum S of $D := A \cup B$. We get a new variety $\widehat{X}$ equipped with a projection $p : \widehat{X} \longrightarrow X$. Then the preimage S_1 of S and strict preimages of the irreducible components of the divisor D are the irreducible components of the divisor $D_1 = p^{-1}D$. Let us assign the A- and B-labels to the irreducible components of D_1, presenting it as $D_1 = A_1 \cup B_1$. The strict preimages of the irreducible components of D inherit their labels from D. So we need only to label S_1. We declare it an A- (resp. B-) component if S is a pure A- (resp. B-) stratum. If S is a mixed stratum we have a freedom to declare it either A- or B-component. It is easy to see that for either choices $(\widehat{A}, \widehat{B})$ is an admissible pair of divisors. If we choose always to declare blow ups of mixed strata as A-components, we denote by $(\widehat{A}^*, \widehat{B}^*)$ the final pair of divisors on $\widehat{X}$. If we always declare blow ups of mixed strata as B-components, denote the final pair of divisors by $(\widehat{A}^!, \widehat{B}^!)$.

Lemma 3.4 *Suppose that $A \cup B$ is a normal crossing divisor. Then there are canonical isomorphisms*

$$\mathcal{F}^\bullet_{A,!,B} \stackrel{\sim}{=} \mathcal{F}^\bullet_{A,*,B} \stackrel{\sim}{=} \mathcal{F}^*_{A,B} \stackrel{\sim}{=} \mathcal{F}^!_{A,B}$$

Proof. It is clear from (35) - (36) that these objects are perverse sheaves. It remains to show that extending to a mixed stratum by $j_!$ we get the same object as when we extend by j_*. Indeed, let A_1 be one of the A-components and $j_1 : X - A_1 \hookrightarrow X$. Then if $\mathcal{F}_{A,B}$ is $\mathcal{F}^!_{A,B}$ or $\mathcal{F}^*_{A,B}$ one has

$$j_{1*}\mathrm{Res}_{X-A_1}\mathcal{F}_{A,B} = \mathcal{F}_{A,B}$$

and a similar statement is valid for $j_{1!}$. The lemma follows by induction. In fact for a normal crossing divisor the extension of the constant sheaf on $X - (A \cup B)$ by j_* to the A-components and by $j_!$ to the B-components does not depend on the order of the extensions.

Let $\widehat{D} = \widehat{A} \cup \widehat{B}$ be a splitting on A- and B-components obtained by the described above procedure, where the blow ups of mixed strata are labeled any way we want. Then $\widehat{A} \subset \widehat{p}^{-1}A$ and $\widehat{B} \subset \widehat{p}^{-1}B$. Therefore there are canonical morphisms

$$\mathcal{F}^*_{\widehat{A}} \longrightarrow \mathcal{F}^*_{\widehat{p}^{-1}A}, \qquad \mathcal{F}^!_{\widehat{p}^{-1}B} \longrightarrow \mathcal{F}^!_{\widehat{B}}$$

(We put ! and $*$ over the $\mathcal{F}$-sheaves to distinguish between the extension functors, i.e. $j_!$ or j_*, used to define them. So $\mathcal{F}^*_A = \mathcal{F}_A$ and $\mathcal{F}^!_A = \mathcal{F}_B$.)

Since $\widehat{p}$ is proper, we get canonical maps

$$H^0(\widehat{X}, \mathcal{F}^*_{\widehat{A}}) \longrightarrow H^0(\widehat{X}, \mathcal{F}^*_{\widehat{p}^{-1}A}) = H^0(X, \mathcal{F}^*_A)$$

$$H^0(\widehat{X}, \mathcal{F}^!_{\widehat{p}^{-1}B}) = H^0(X, \mathcal{F}^!_B) \longrightarrow H^0(X, \mathcal{F}^!_{\widehat{B}})$$

Applying Gr^W, and then dualizing the second map, we get morphisms

$$\alpha_{\widehat{A}} : \mathrm{Gr}^W_{n+\mathrm{dp}(A)}(X, \mathcal{F}^*_{\widehat{A}}) \longrightarrow \mathrm{Gr}^W_{n+\mathrm{dp}(A)} H^0(X, \mathcal{F}^*_A)^\vee$$

$$\beta_{\widehat{B}} : \Big(\mathrm{Gr}^W_{n-\mathrm{dp}(B)}(X, \mathcal{F}^!_{\widehat{B}})\Big)^\vee \longrightarrow \Big(\mathrm{Gr}^W_{n-\mathrm{dp}(B)} H^0(X, \mathcal{F}^!_B)\Big)^\vee$$

Consider a framing on $H^0(X; \mathcal{F}^*_{A,B})$ provided by nonzero classes

$$[\omega_A] \in \mathrm{Gr}^W_{n+\mathrm{dp}(A)} H^0(X, \mathcal{F}_A); \qquad [\Delta_B] \in \Big(\mathrm{Gr}^W_{n-\mathrm{dp}(B)} H^0(X, \mathcal{F}_B)\Big)^\vee$$

Denote by $\mathcal{F}_{A,B}$ the perverse sheaf provided by lemma 3.4. If $\widehat{D} = \widehat{A}^* \cup \widehat{B}^* = \widehat{A}^! \cup \widehat{B}^!$ is a normal crossing divisor then lemma 3.4 provides perverse sheaves $\mathcal{F}_{\widehat{A}^*, \widehat{B}^*}$ and $\mathcal{F}_{\widehat{A}^!, \widehat{B}^!}$.

Theorem 3.5 *i) Assume that construction 3.2 delivers a normal crossing divisor (e.g. D is locally isomorphic to a family of hyperplanes). Then there are canonical isomorphisms*

$$\widehat{p}_* \mathcal{F}_{\widehat{A}^*, \widehat{B}^*} = \mathcal{F}^\bullet_{A,*,B}; \qquad \widehat{p}_* \mathcal{F}_{\widehat{A}^!, \widehat{B}^!} = \mathcal{F}^\bullet_{A,!,B} \tag{61}$$

Moreover the maps $\alpha_{\widehat{A}^*}$, $\beta_{\widehat{B}^*}$, as well as $\alpha_{\widehat{A}^!}$, $\beta_{\widehat{B}^!}$ are isomorphisms.

ii) There is an equivalence of framed objects:

$$\Big(H^0(\widehat{X}; \mathcal{F}_{\widehat{A}^*, \widehat{B}^*}), \alpha^{-1}_{\widehat{A}^*}[\omega_A], \beta^{-1}_{\widehat{B}^*}[\Delta_B]\Big) \sim \Big(H^0(X; \mathcal{F}^*_{A,B}), [\omega_A], [\Delta_B]\Big) \tag{62}$$

iii) There is a similar statement with $*$ changed to $!$.

Proof. i) Thanks to the construction for each bad stratum of D there is a unique irreducible component of the divisor $\widehat{D}$ projecting onto the closure of this stratum. Let us show that if there is an isomorphism over $U(k)$

$$\mathrm{Res}_{U(k)} \widehat{p}_* \mathcal{F}_{\widehat{A}^*, \widehat{B}^*} = \mathrm{Res}_{U(k)} \mathcal{F}^\bullet_{A,*,B}$$

then the same is true over $U(k+1)$. Set $\widehat{X}_k := \widehat{p}^{-1} U(k)$

Let $\{\widehat{D}_i\}$ be the irreducible components of $\widehat{D}$ projecting to the bad strata in $U(k+1) - U(k)$, and $\{\widehat{D}^0_i\}$ their restrictions to $\widehat{X}_{k+1}$. Take one of them, $\widehat{D}^0_1$. Suppose it is an A-component. Let

$$\widehat{j}_1 : \widehat{X}_{k+1} - \widehat{D}^0_1 \hookrightarrow \widehat{X}_{k+1}; \qquad j_1 : U(k+1) - \widehat{p}(\widehat{D}^0_1) \hookrightarrow U(k+1)$$

Since $\widehat{D}$ is a normal crossing divisor we can apply lemma 3.4. So if $\mathcal{F}$ is either $\mathcal{F}_{\widehat{A}^*, \widehat{B}^*}$ or $\mathcal{F}_{\widehat{A}^!, \widehat{B}^!}$ there is an isomorphism

$$\widehat{j}_{1*} \mathrm{Res}_{\widehat{X}_{k+1} - \widehat{D}^0_1} \mathcal{F} = \mathrm{Res}_{\widehat{X}_{k+1}} \mathcal{F}$$

One has $\widehat{p} \circ \widehat{j}_1 = j_1 \circ \widehat{p}$. Thus $\widehat{p}_* \widehat{j}_{1*} = j_{1*} \widehat{p}_*$. Therefore $\widehat{p}_* \mathrm{Res}_{\widehat{X}_{k+1}} \mathcal{F} = j_{1*} \widehat{p}_* \mathrm{Res}_{\widehat{X}_{k+1} - D_1^0} \mathcal{F}$. The case when $\widehat{D}_1^0$ is a B-component is similar. Since the divisors $\widehat{D}_i^0$ are disjoint we immediately get a similar statement for $\cup_i \widehat{D}_i^0$.

The second statement follows from isomorphisms (61) and theorem 2.10b). The part i) of the theorem is proved.

ii) The part i) implies the equivalence

$$\Big(H^0(X; \mathcal{F}^{\bullet}_{A,*,B}), [\omega_A], [\Delta_B]\Big) \sim \Big(H^0(\widehat{X}; \mathcal{F}_{\widehat{A}^*, \widehat{B}^*}), \alpha^{-1}_{\widehat{A}^*}[\omega_A], \beta^{-1}_{\widehat{B}^*}[\Delta_B]\Big)$$

Recall that the frame morphisms for $\mathcal{F}^*_{A,B}$ were constructed by applying the functor H^0_τ to the frame morphisms $\mathcal{F}_B \longrightarrow \mathcal{F}^{\bullet}_{A,*,B} \longrightarrow \mathcal{F}_A$. So the canonical morphism in the derived category $H^0_\tau : \mathcal{F}^{\bullet}_{A,*,B} \longrightarrow \mathcal{F}^*_{A,B}$ provides a commutative diagram where the horizontal arrows are the frame morphisms:

$$\begin{array}{ccccc} \mathcal{F}_B & \longrightarrow & \mathcal{F}^{\bullet}_{A,*,B} & \longrightarrow & \mathcal{F}_A \\ \downarrow = & & \downarrow & & = \downarrow \\ \mathcal{F}_B & \longrightarrow & \mathcal{F}^*_{A,B} & \longrightarrow & \mathcal{F}_A \end{array}$$

Applying the functor $H^0(X; -)$ to it we get an equivalence of framed objects

$$\Big(H^0(X; \mathcal{F}^{\bullet}_{A,*,B}), [\omega_A], [\Delta_B]\Big) \sim \Big(H^0(X; \mathcal{F}^*_{A,B}), [\omega_A], [\Delta_B]\Big)$$

The part ii) is proved. The part iii) is very similar. The theorem is proved.

3. The mixed Tate motive corresponding to $H^0(X; \mathcal{F}_{A,B})$.

Proposition 3.6 *Suppose that F is a number field, and $A \cup B$ is a normal crossing divisor on a regular variety X over F, providing a Tate stratification of X. Then there exists a mixed Tate motive $m(X; A, B)$ whose Hodge and l-adic realizations are given by $H^0(X; \mathcal{F}_{A,B})$.*

Proof. Suppose that X is a regular variety over any field F and Y is a normal crossing divisor on X. Then there is an object $m(X, Y)$ of the triangulated category of mixed motives over F such that realizations of $m^{\bullet}(X, Y)$ are given by the relative cohomology groups $H^*(X, Y)$. (I use a cohomological version of Voevodsky's category $\mathcal{DM}_F$). Namely, let $\{Y_i\}$, $i \in \Lambda$, be the set of all irreducible components of the divisor Y. Set $Y(k) := \cup_{|I|=k} Y_I$ and consider the following complex of regular varieties where X is in the degree zero, and the differentials are the ones in a simplicial resolution:

$$m^{\bullet}(X, Y) := \ldots \longrightarrow Y(2) \longrightarrow Y(1) \longrightarrow X$$

Let $\mathcal{D}_T(F)$ be the full subcategory of $\mathcal{DM}_F$ consisting of the objects glued from the the pure Tate motives $\mathbb{Q}(n)$ and their shifts. If Y provides a Tate stratification of a Tate variety X then $m^{\bullet}(X, Y)$ is an object of $\mathcal{D}_T(F)$. Finally, if F is a number field then there is a canonical t-structure t on the category $\mathcal{D}_T(F)$ providing an abelian category $\mathcal{M}_T(F)$ of mixed Tate motives over F (see ch. 5 of [G9]). Set $m^i(X,Y) := H^i_t(m^\bullet(X,Y))$.

Applying this construction to $Y := B_A := B - B \cap A$ we get mixed Tate motives $m^i(X; A, B_A)$. Then $m^{-\dim X}(X; A, B_A)$ is just what we need. The proposition is proved.

4. Remarks on the frame classes. Recall a splitting on A- and B-components $\widehat{D} = \widehat{A} \cup \widehat{B}$ described in s. 3.2, where the blow ups of mixed strata are labeled any way we want. For applications of theorem 3.5 it is useful to have a more direct description of the classes $\alpha^{-1}_{\widehat{A}}[\omega_A]$ and $\beta^{-1}_{\widehat{B}}[\Delta_B]$.

Lemma 3.7 *Suppose that A is a normal crossing divisor and $\omega_A \in \Omega^n_{\rm log}(X-A)$. Then $\widehat{p}^*\omega_A \in \Omega^n_{\rm log}(\widehat{X} - \widehat{A})$. Therefore $\alpha^{-1}_{\widehat{A}}[\omega_A]$ is represented by $[\widehat{p}^*\omega_A]$.*

Proof. A priori the form $\widehat{p}^*\omega_A$ might have logarithmic singularities at $\widehat{p}^{-1}A$. The following general lemma tells us that $\widehat{p}^*\omega_A$ can have singularities only at the blow ups of pure A-strata.

Lemma 3.8 *i) Let Y be a normal crossing divisor in a regular variety X, and $\omega \in \Omega^n_{\rm log}(X - Y)$. Let $p : \widehat{X} \longrightarrow X$ be the blow up of an irreducible subvariety Z. Suppose that the generic point of Z is different from the generic points of strata of Y. Then $p^*\omega$ does not have a singularity at the special divisor of $\widehat{X}$.*

ii) Let $F \in K^M_n(F(X))$. Assume that F has non zero residues only at the components of a normal crossing divisor Y. Then under the same conditions as in i) the residue of p^*F at the special divisor of the blow up is zero.

Proof. i) Considering the generic point of Z we reduce the problem to the case when Z is a point. So let $x_1, ..., x_n$ be local coordinates such that $\omega = d\log(x_1) \wedge ... \wedge d\log(x_k) \wedge \Omega$ where Ω is a regular form of positive degree near the point $Z = (0, ..., 0)$. Let $u_1 := x_1$, $u_i := x_i/x_1$ be the local coordinates on the blow up. Then $x_1 = 0$ is a local equation of the special divisor. One has

$$p^*\omega = d\log(x_1) \wedge d\log(u_2) \wedge ... \wedge d\log(u_k) \wedge p^*\Omega$$

Since $d\log(x_1) \wedge p^*f \cdot d(x_1 u_{k+1}) = p^*f \cdot dx_1 \wedge du_{k+1}$ where f is regular near Z, the statement follows.

ii) Similar to the proof of i). Lemma 3.8, and hence lemma 3.7 are proved.

Now look at the similar question for the relative cycle Δ_B in the Betti realization. A simple geometric argument shows that there is an open part $\Delta^0_B \subset X - B$ of the cycle Δ_B such that the class $\beta^{-1}_{\widehat{B}}[\omega_B]$ can be represented by the closure of Δ^0_B in $\widehat{X}$. This is a homological version of lemma 3.7.

5. An unramifiedness criteria. Let R be a discrete valuation ring with the fraction field K, $\mathcal{M}$ the maximal ideal in R, k the residue field of R, and $\overline{k}$ its algebraic closure. For a scheme X over a field F set $\overline{X} := X \otimes_F \overline{F}$.

Suppose that D is a proper normal crossing divisor in a proper regular scheme X over R. Let A and B be unions of the irreducible components of D. We will assume that no irreducible components is shared by both of them. Then there is the $\mathrm{Gal}(\overline{K}/K)$-module

$$H^n_{\mathrm{et}}(\overline{X}-\overline{A},\overline{B}_A;\mathbb{Q}_l);\quad B_A:=B-(A\cap B); \tag{63}$$

We want to have a criteria for this module to be unramified.

Definition 3.9 *Let D be a normal crossing divisor in a regular scheme X over R. Assume that the pair (D,X) is proper over R. We say that reduction modulo $\mathcal{M}$ does not change the combinatorics of (D,X) if X and every stratum of D are smooth over R, and the reduction map from the strata of D to ones at the special fiber is a bijection.*

To check that a scheme is smooth over R the following result is useful, see proposition 3.24 in chapter I of [M]. Let Y be an integral scheme over R. Denote by Y^0 and Y^η its special and generic fibers. Then Y is smooth over R if and only if the generic fiber is non empty and $Y^0(\overline{k})$ and $Y^\eta(\overline{K})$ have no singular points,

Proposition 3.10 *Suppose that the reduction modulo $\mathcal{M}$ does not change the combinatorics of (D,X). Then, assuming l is prime to the characteristic of k, there is an isomorphism*

$$H^n_{\mathrm{et}}(\overline{X}-\overline{A},\overline{B}_A;\mathbb{Q}_l)\xrightarrow{\sim}H^n_{\mathrm{et}}(\overline{X^0}-\overline{A^0},\overline{B^0_A};\mathbb{Q}_l) \tag{64}$$

In particular the $\mathrm{Gal}(\overline{K}/K)$-module (63) is unramified.

Proof. By the smooth and proper base change theorem ([M]) for any stratum Y of X, including X itself, there is an isomorphism

$$H^n_{\mathrm{et}}(\overline{Y};\mathbb{Q}_l)\xrightarrow{\sim}H^n_{\mathrm{et}}(\overline{Y^0};\mathbb{Q}_l)$$

and hence $H^n_{\mathrm{et}}(\overline{Y};\mathbb{Q}_l)$ is unramified. We calculate $H^n(X-A,B_A)$ by replacing the pair $(X-A,B_A)$ by its standard ([D4]) simplicial resolution $S_\bullet(X-A,B_A)$. The i-simplices of this simplicial scheme are given by disjoint union of the i-fold intrersections of the irreducible components of B_A. The result follows from the following two observations:

i) There is a well defined reduction modulo $\mathcal{M}$ of the simplicial scheme $S_\bullet(X-A,B_A)$.

ii) The reduction modulo $\mathcal{M}$ induces an isomorphism on the étale cohomology for each of the schemes $S_i(X-A,B_A)$.

Indeed, assuming i), let $S^0_\bullet(X-A,B_A)$ be the simplicial scheme obtained by reduction of the simplicial scheme $S_\bullet(X-A,B_A)$. In particular each component $S^0_i(X-A,B_A)$ is obtained by reduction of the corresponding component $S_i(X-$

$A, B_A)$. Then there is a map of the standard spectral sequence provided by the simplicial resolution which computes the left hand side of (64) to the one computing the right hand side. Thanks to ii) the map induces an isomorphism of the E_1 terms of these spectral sequences, so the proposition follows.

The check the statement i) observe that we can always reduce modulo $\mathcal{M}$ each of the components of the simplicial scheme $S_\bullet(X - A, B_A)$. The maps between them are defined since the reduction modulo $\mathcal{M}$ does not change the combinatorics of D, and in particular no stratum of $\overline{B}_A$ is supposed to be mapped to a stratum which has been removed.

The statement ii) follows from the following lemma

Lemma 3.11 *Let X be a proper scheme over R and B is a proper normal crossing divisor in X. Suppose that reduction modulo $\mathcal{M}$ does not change the combinatorics of B. Then there is an isomorphism*

$$H^n_{\text{et}}(\overline{X} - \overline{B}; \mathbb{Q}_l) \xrightarrow{\sim} H^n_{\text{et}}(\overline{X^0} - \overline{B^0}; \mathbb{Q}_l)$$

Proof. By duality it is sufficient to prove a similar isomorphism for $H^n(X, B)$. Then we replace (X, B) by its simplicial resolution $S_\bullet(X, B)$. Since X is proper this simplicial resolution always has a reduction modulo $\mathcal{M}$ denoted $S^0_\bullet(X, B)$. Observe that in general $H^n(\overline{S^0_\bullet}(X, B); \mathbb{Q}_l)$ does not necessarily isomorphic to $H^n(\overline{X^0}, \overline{B^0}; \mathbb{Q}_l)$ (e.g., take $X = \mathbb{A}^1$, $B = \{0\} \cup \{p\}$, and $\mathcal{M} = (p)$). However we do have the isomorphism if $S^0_\bullet(X, B) = S_\bullet(X^0, B^0)$. This is the case in our situation since the reduction does not change the combinatorics of B. The lemma and hence the proposition are proved.

Example. Let p be a prime number. Then $H^1(P^1 - \{0, \infty\}, \{1, p\})$ is unramified outside of p. The simplicial scheme corrersponding to this cohomology group is $\{1\} \cup \{p\} \longrightarrow P^1 - \{0, \infty\}$. Observe that we can not define its reduction modulo p since $\{p\}$ should be mapped to 0, which is not in $P^1 - \{0, \infty\}$!

Corollary 3.12 *Suppose that we are in the situation described in proposition 3.6. Assume further that (X, A, B) is defined and proper over a ring of S-integers $\mathcal{O}_S$ in the number field F, and $\mathcal{P}$ is a prime ideal of F outside of S. Then if the reduction modulo $\mathcal{P}$ does not change the combinatorics of the divisor $A \cup B$, then the mixed Tate motive $m(X; A, B)$ is unramified at $\mathcal{P}$.*

Proof. A mixed Tate motive over F is unramified at $\mathcal{P}$ if and only if its l-adic realization, where l is prime to $\mathcal{P}$, is unramified at $\mathcal{P}$. So the corollary follows from proposition 3.10.

4 A specialization theorem

1. The specialization functor [Ve]. Let Z be a regular subvariety of a regular variety X. Let $N^0_Z X$ be the normal bundle to Z in X, with the zero section removed. Set $U := X - Z$. Recall the specialization functor

$$\mathrm{Sp} : D^b_{\mathrm{Sh}}(U) \longrightarrow D^b_{\mathrm{Sh}}(N^0_Z X)$$

In fact Verdier defined it for any pair of subvarieties $Z \subset X$, with $N^0_Z X$ replaced by the normal cone, and $D^b_{\mathrm{Sh}}(U)$ by $D^b_{\mathrm{Sh}}(X)$. Extending elements of $D^b_{\mathrm{Sh}}(U)$ by $j_!$ to X and applying the Verdier's definition we come to the functor above.

The functor Sp sends perverse sheaves to perverse sheaves and commutes with the duality functor on D^b_{Sh}.

Since Z is a regular subvariety of a regular variety X the specialization functor Sp provides a Tate functor between the corresponding mixed Tate categories on U and $N^0_Z X$ of unipotent variations of Hodge-Tate structures or lisse Tate l-adic sheaves. Indeed, Sp is an exact tensor functor, and if Z is regular subvariety it obviously transforms the Tate objects, which are the constant sheaves on U tensored by $\mathbb{Q}(n)$ (resp $\mathbb{Q}_l(n)$) and shifted to the left by $\dim U$, to the Tate objects.

2. A specialization theorem. Recall that D_i, $i \in \Lambda$, is the set of irreducible components of the divisor D, and for any subset $I \subset \Lambda$ we set $D_I := \cap_{i\in I} D_i$. In particular $D_\emptyset = X$. The subvarieties $\{D_I\}$ are the closures of the strata of the stratification defined by D.

Excellent variations. We say that $\{D_i(t)\}$ is a smooth variation of regular divisors on X parametrized by a regular base T if there are regular divisors $\mathcal{D}_i \subset X \times T$ smooth with respect to the projection $p : X \times T \longrightarrow T$, so that $D_i(t) = \mathcal{D}_i(t) \cap p^{-1}t$.

Definition 4.1 *A smooth variation of the divisors $\{D_i(t)\}$ is an* excellent *variation if for any $I \subset \Lambda$*

1) $\dim D_I(t)$ is the same for all $t \in T$.

2) $D_I(t)$ are regular irreducible projective subvarieties.

3) The family $\{D_I(t)\}$ extends to a smooth family of regular subvarieties parametrized by a projective variety B_I containing T.

Replacing T by a non empty Zariski open subvariety $T^0 \subset T$ we may always assume that 1).

Example. $X = P^n$; D is a union of hyperplanes. Any base of deformation of hyperplanes D_i satisfying condition 1) satisfy also conditions 2) and 3). Indeed, $D_I(t)$ is a plane in P^n, so one can take B_I to be the corresponding Grassmannian.

Now suppose we have a variation of the divisors $D_i(\varepsilon)$ in X parametrized by a one dimensional regular base Σ containing a point 0. Let $\Sigma^0 := \Sigma - \{0\}$.

Let $\Lambda = I \cup J$. Set

$$A(\varepsilon) := \cup_{i\in I} D_i(\varepsilon), \qquad B(\varepsilon) := \cup_{j\in J} D_j(\varepsilon)$$

and $(A, B) := (A(0), B(0))$. Assume that

i) The pair (A, B) is admissible.

ii) $\{D_i(\varepsilon)\}$ is an excellent variation over Σ^0.

Then replacing the base by a non empty open subset of Σ we may assume that $(A(\varepsilon), B(\varepsilon))$ is an admissible pair for all $\varepsilon \in \Sigma$.

Denote by $\mathcal{A}$ and $\mathcal{B}$ the divisors on $X \times \Sigma^0$ whose fibers over $\varepsilon \in \Sigma^0$ are $A(\varepsilon)$ and $B(\varepsilon)$. So there are the corresponding objects denoted $\widetilde{\mathcal{F}}^{\bullet}_{\mathcal{A},*,\mathcal{B}}$ and $\widetilde{\mathcal{F}}^{*}_{\mathcal{A},\mathcal{B}}$ on $X \times \Sigma^0$. The tilde emphasizes that we are working over Σ^0.

We are going to apply the specialization functor to the divisor

$$i_0 : X \times 0 \hookrightarrow X \times \Sigma \tag{65}$$

Let $v \in T_0(\Sigma) - 0$. Denote by $\mathrm{Sp}_s(\mathcal{G})$ the restriction of $\mathrm{Sp}(\mathcal{G})$ to the section $s := p^{-1}(v)$ of the normal bundle, which is identified with $X \times 0$ in (65).

We claim that in both Hodge and l-adic settings the specialization $\mathrm{Sp}_s\widetilde{\mathcal{F}}^{*}_{\mathcal{A},\mathcal{B}}$ satisfies the same conditions as in proposition 2.4:

Theorem 4.2 *Assume the conditions i), ii) above. Then in both Hodge and l-adic settings $\mathrm{Sp}_s\widetilde{\mathcal{F}}^{*}_{\mathcal{A},\mathcal{B}}$ is glued from the objects $\delta_C(-m)$ where $0 \leq m \leq \mathrm{codim} C$, where C runs through the limiting strata of the stratification $\mathcal{D}_J(\varepsilon)$ as $\varepsilon \to 0$.*

Remark 1. The limiting strata consist of the stratum D_J and perhaps some other subvarieties. For example consider degeneration of a pair (line, plane) in P^3 degenerating to a line sitting in the limiting plane.

Remark 2. A similar result is valid when $\mathcal{F}^{*}_{\mathcal{A},\mathcal{B}}$ is replaced by $H^i(\mathcal{F}^{\bullet}_{\mathcal{A},*,\mathcal{B}})$

Proof. *The Hodge setting.* We have a supply of Hodge sheaves $\delta_{\mathcal{D}_J}(-m)$. For all but finitely many points $\varepsilon \in \Sigma^0(\mathbb{C})$ the restriction to ε commutes with all the standard functors. Let $i_\varepsilon : \{\varepsilon\} \hookrightarrow \Sigma^0(\mathbb{C})$. It follows that for all but finitely many points ε the object $i^*_\varepsilon\widetilde{\mathcal{F}}^{*}_{\mathcal{A},\mathcal{B}}[-1]$ is isomorphic to $\mathcal{F}^{*}_{A(\varepsilon),B(\varepsilon)}$, and hence has all the properties listed in proposition 2.4. Observe that $i^*_\varepsilon\delta_{\mathcal{D}_J}(-m) = \delta_{D_J(\varepsilon)}(-m)[-1]$.

Denote by $\mathcal{P}_{\mathcal{H}}(X)$ the category of perverse Hodge sheaves on X.

Choose a sufficiently small punctured analytic disc $U^* \subset \Sigma^0(\mathbb{C})$ with the puncture at 0. For a finite covering $\pi : \widehat{U}^* \longrightarrow U^*$ consider the map

$$\widehat{\pi} := \mathrm{Id} \times \pi : X \times \widehat{U}^* \longrightarrow X \times U^*$$

Let $\widehat{\mathcal{D}}$ be the variation of the divisors parametrized by $\widehat{U}^*$, obtained by the base change $\pi : \widehat{U}^* \longrightarrow U^*$.

Lemma 4.3 *There exist J, $m \leq \mathrm{codim} D_J$, and a finite covering $\pi : \widehat{U}^* \longrightarrow U^*$ such that there is a non zero element in*

$$\mathrm{Hom}_{\mathcal{P}_{\mathcal{H}}(X \times \widehat{U}^*)}(\widehat{\pi}^*\widetilde{\mathcal{F}}^{*}_{\mathcal{A},\mathcal{B}}, \delta_{\widehat{\mathcal{D}}_J}(-m))$$

Proof. Let $p : X \times U^* \longrightarrow U^*$ be the canonical projection. One has

$$\mathrm{RHom}_{D^b_{\mathcal{H}}(X\times U^*)}(\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}}, \delta_{\mathcal{D}_J}(-m)) = \tag{66}$$

$$\mathrm{RHom}_{D^b_{\mathcal{H}}(X\times U^*)}(\delta_{X\times U^*}, *\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}} \otimes^! \delta_{\mathcal{D}_J}(-m)) =$$

$$\mathrm{RHom}_{D^b_{\mathcal{H}}(U^*)}(\delta_{U^*}, p_*(*\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}} \otimes^! \delta_{\mathcal{D}_J}(-m))) \tag{67}$$

The Hodge sheaves

$$R^i p_*(*\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}} \otimes^! \delta_{\mathcal{D}_J}(-m)) \tag{68}$$

are restrictions to U^* of the similar ones on Σ^0, which are smooth at the generic point of Σ^0 (this follows from the corresponding well known fact about D-modules). Thus removing a finite set of points from Σ^0 we may assume that (68) is a variation of mixed Hodge structures on Σ^0, and hence on U^*.

Recall that $\mathrm{Hom}_{\mathcal{P}_{\mathcal{H}}} = \mathrm{R}^0\mathrm{Hom}_{D^b_{\mathcal{H}}}$. The monodromy around 0 provides an operator N acting on the vector space

$$\mathrm{R}^0\mathrm{Hom}_{D^b_{\mathcal{H}}(\mathbb{C})}(\delta_\varepsilon[-1], i^*_\varepsilon p_*(*\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}} \otimes^! \delta_{\mathcal{D}_J}(-m))), \tag{69}$$

The H^0 of the complex (67) is isomorphic to the invariants N acting on (69). Let us show that this vector space is non zero for some J and $m \le \mathrm{codim} D_J$. Indeed, since restriction to a generic point $\varepsilon \in U^*$ commutes with all the standard functors, we can interchange i^*_ε with p_* and other functors involved in (69). Then proceeding as above we rewrite (69) as a vector space

$$\mathrm{R}^0\mathrm{Hom}_{D^b_{\mathcal{H}}(X\times\{\varepsilon\})}(\mathcal{F}^*_{A(\varepsilon),B(\varepsilon)}, \delta_{D_J(\varepsilon)}(-m)) \tag{70}$$

By proposition 2.4 there exists J and $m \le \mathrm{codim} D_J(\varepsilon)$ such that (70) is non zero.

The regularity theorem for D-modules (see lecture 4 in [Be]) implies that variation (68) on U^* has a quasiunipotent monodromy around 0. Indeed, the variation was obtained as a composition of several functors applied to δ_U. Since δ_U obviously has quasiunipotent spectrum in the terminology of [Be], and the standard functors preserve this property, the statement follows. Therefore the monodromy operator N acting on (69) is quasiunipotent. If N is unipotent then we are done: the subspace of N-invariants on a non zero vector space (69) is non zero. Replacing U^* by its finite cover $\pi : \widehat{U}^* \longrightarrow U^*$ we can make the monodromy N unipotent. The lemma is proved.

Having the lemma we get the theorem as follows. Let us suppose first that the monodromy N is unipotent, so we do not have to pass to a covering $\widehat{U}^*$. Then we find by induction a filtration F of the object $\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}}$ such that its associated graded are isomorphic to $\delta_{\mathcal{D}_J}(-m)$ for some J and $m \le \mathrm{codim}\mathcal{D}_J$. Since $\{D_i(\varepsilon)\}$ is an excellent variation over U^*, each of the families $D_J(\varepsilon)$ extends to a smooth family on $U := U^* \cup 0$. In particular there is a regular reduced subscheme $D_J(0)$, the limit of the family $D_J(\varepsilon)$ as $\varepsilon \longrightarrow 0$. Therefore

$$\mathrm{Sp}_s \delta_{\mathcal{D}_J}(-m) = \delta_{D_J(0)}(-m)$$

Since the specialization is an exact functor we get the Hodge version of theorem when N is unipotent.

Let us deduce the general case from the one when N is unipotent. Consider the diagrams

$$\begin{array}{ccccccc} X \times \{0\} & \stackrel{\widehat{i}}{\hookrightarrow} & X \times \widehat{U} & \qquad & N^0_{X \times \{0\}} X \times \widehat{U} & = & X \times \mathbb{C}^* \\ \downarrow = & & \downarrow \widehat{\pi} & & \downarrow \overline{\pi} & & \downarrow \\ X \times \{0\} & \stackrel{i}{\hookrightarrow} & X \times U & & N^0_{X \times \{0\}} X \times U & = & X \times \mathbb{C}^* \end{array}$$

The fibers of the punctured normal bundle to $X \times \{0\}$ are identified with $\mathbb{C}^*$, and $\overline{\pi}$ is given by $z \longrightarrow z^{\deg \pi}$ on the fibers. Then, denoting by Sp_i (resp. $\mathrm{Sp}_{\widehat{i}}$) the specialization functor for the divisor given by i (resp $\widehat{i}$) we have

$$\mathrm{Sp}_i \circ \widehat{\pi}_* = \overline{\pi}_* \circ \mathrm{Sp}_{\widehat{i}} \tag{71}$$

This follows from the construction of specialization via the nearby cycles functor Ψ, using the fact that Ψ commutes with proper direct images. Observe that $\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}}$ is a direct summand of $\widehat{\pi}_* \widehat{\pi}^* \widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}}$. So $\mathrm{Sp}_i(\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}})$ is a direct summand of

$$\mathrm{Sp}_i(\widehat{\pi}_* \widehat{\pi}^* \widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}}) \stackrel{(71)}{=} \overline{\pi}_* \mathrm{Sp}_{\widehat{i}}(\widehat{\pi}^* \widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}})$$

Let $\pi^{-1}s = \{s_i\}$. Then restriction of the right object to $X \times s$ is a direct sum of the restrictions of $\mathrm{Sp}_{\widehat{i}}(\widehat{\pi}^* \widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}})$ to $X \times s_i$. Each of them satisfies the condition of the theorem. The Hodge version of the theorem is proved.

The l-adic setting. It follows the same pattern as the proof of the Hodge version. There is a perverse l-adic sheaf $\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}}$ over $X \times \Sigma^0$. Choose an algebraic closure $\overline{K}$ of the field of functions $K := \mathbb{Q}(\Sigma^0)$, and an embedding $K \hookrightarrow \overline{K}$ providing a generic geometric point of Σ^0. Denote by $i_{\overline{K}} : X_{\overline{K}} \longrightarrow X \times \Sigma^0$ the corresponding map.

Let y be a generic geometric point of Σ^0 providing map $i_y : X_{\overline{K}} \longrightarrow X \times \Sigma^0$. Then for any perverse sheaf $\mathcal{G}$ on $X \times \Sigma^0$ the (shifted) restriction $i_y^* \mathcal{G}[-1]$ is a perverse l-adic sheaf on $X_{\overline{K}}$.

In particular let $\overline{A} := i^*_{\overline{K}} \mathcal{A}$, $\overline{B} := i^*_{\overline{K}} \mathcal{B}$ and $\overline{D}_J := i^*_{\overline{K}} \mathcal{D}_J$. Then there is a perverse l-adic sheaf $\mathcal{F}^*_{\overline{A},\overline{B}}$ on $X_{\overline{K}}$ which by proposition 2.4 is glued from $\delta_{\overline{D}_J}$. Since all the standard functors commute with restriction to generic geometric point, there is an isomorphism

$$\mathcal{F}_{\overline{K}} := i^*_{\overline{K}} \widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}}[-1] \stackrel{\sim}{=} \mathcal{F}^*_{\overline{A},\overline{B}}$$

So $\mathcal{F}_{\overline{K}}$ is glued from $\delta_{\overline{D}_J}$. Therefore one can find J such that

$$\mathrm{Hom}_{\overline{K}}(\mathcal{F}_{\overline{K}}, \delta_{\overline{D}_J}) \neq 0 \tag{72}$$

Here Hom is in the category of perverse l-adic sheaves over $\overline{K}$. Both perverse sheaves are equipped with an action of the Galois group $\mathrm{Gal}(\overline{K}/K)$. So the Hom has a structure of a finite dimensional l-adic $\mathrm{Gal}(\overline{K}/K)$-module.

Let us show that, twisting (72) by $\mathbb{Q}_l(-m)$ for some $m \leq \mathrm{codim} D_J$, we can find there a non zero $\mathrm{Gal}(\overline{K}/K)$-invariant vector. Indeed, thanks to (44) the $\mathrm{Gal}(\overline{K}/K)$-module (72) is glued from the Tate modules $\mathbb{Q}_l(n)$. It follows that for a certain integer m there is a $\mathrm{Gal}(\overline{K}/K)$-invariant vector in

$$\mathrm{Hom}_{\overline{K}}(\mathcal{F}_{\overline{K}}, \delta_{\overline{D}_J})(-m) \tag{73}$$

Recall that $\mathcal{F}_{\overline{K}}$ and $\delta_{\overline{D}_J}$ are isomorphic to the perverse sheaves obtained by restriction $i^*_{\overline{K}}[-1]$ from the perverse sheaves $\mathcal{F}^*_{\mathcal{A},\mathcal{B}}$ and $\delta_{\mathcal{D}_J}$ defined over K. Therefore the subspace of $\mathrm{Gal}(\overline{K}/K)$-invariants in (73) is identified with

$$\mathrm{Hom}_K(\mathcal{F}^*_{\mathcal{A},\mathcal{B}}, \delta_{\mathcal{D}_J}(-m)) \tag{74}$$

Here Hom is in the category of perverse l-adic sheaves over K. It follows that there is a non zero element in (74). It remains to show that $m \leq \mathrm{codim} D_J$. Unfortunately so far there is no formalism of weights over $\overline{K}$. We will circumvent this problem by using the reduction to a finite field, where we can use the weights. Choose a finitely generated ring $\mathcal{O}$ over $\mathbb{Z}$ such that $X, \{\mathcal{D}_J\}$ are defined over $\mathcal{O}$. The restriction to a generic geometric point $i_\varepsilon : \mathrm{Spec} F_q \longrightarrow \mathrm{Spec}\mathcal{O}$ commutes with all the functors involved in the construction of the object $\mathcal{F}^*_{\mathcal{A},\mathcal{B}}$. So $i^*_\varepsilon \mathcal{F}^*_{\mathcal{A},\mathcal{B}}$ is glued from the object $(i^*_\varepsilon \delta_{D_J})(-m)$ where $m \leq \mathrm{codim} D_J$. So by proposition 2.4 there is a non zero element of

$$\mathrm{Hom}_{\overline{F}_q}(i^*_\varepsilon \mathcal{F}^*_{\mathcal{A},\mathcal{B}} \otimes_{F_q} \overline{F}_q, i^*_\varepsilon \delta_{\mathcal{D}_J}(-m) \otimes_{F_q} \overline{F}_q) \tag{75}$$

invariant with respect to $\mathrm{Gal}(\overline{F}_q/F_q)$. Further, for generic ε the vector spaces (73) is isomorphic to the one (75), so that the isomorphism transforms one Galois action to the other via a surjective projection $\mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Gal}(\overline{F}_q/F_q)$. So we identify the Tate twists in (73) and (75). The l-adic version of the lemma 4.3 is proved.

The rest of the proof of the l-adic version of the theorem copies the end of the proof of the Hodge version. The theorem is proved.

A specialization theorem data. Suppose we have a family of divisors $D(\varepsilon) = A(\varepsilon) \cup B(\varepsilon)$ in X parametrized by $\varepsilon \in \Sigma^0$ and satisfying all the conditions of theorem 4.2. It is given by a divisor $\mathcal{D} = \mathcal{A} \cup \mathcal{B} \subset X \times \Sigma^0$. Since $(\mathcal{A}, \mathcal{B})$ is admissible, there are perverse sheaves

$$\widetilde{\mathcal{F}}_{\mathcal{B}}, \widetilde{\mathcal{F}}^!_{\mathcal{A},\mathcal{B}}, \widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}}, \widetilde{\mathcal{F}}^*_{\mathcal{A}}$$

on $X \times \Sigma^0$ and morphisms

$$\widetilde{\mathcal{F}}_{\mathcal{B}} \xrightarrow{\widetilde{\beta}} \widetilde{\mathcal{F}}^!_{\mathcal{A},\mathcal{B}} \xrightarrow{\widetilde{c}} \widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}} \xrightarrow{\widetilde{\alpha}} \widetilde{\mathcal{F}}^*_{\mathcal{A}} \tag{76}$$

Let us assume in addition that

$$\mathrm{Sp}_s\widetilde{\mathcal{F}}_{\mathcal{A}} = \mathcal{F}_A; \qquad \mathrm{Sp}_s\widetilde{\mathcal{F}}_{\mathcal{B}} = \mathcal{F}_B \tag{77}$$

Corollary 4.4 *Assuming (77), morphisms (76) induce isomorphisms*

$$\mathrm{Gr}^{DW}_n(\mathrm{Sp}_s\widetilde{\mathcal{F}}^!_{\mathcal{A},\mathcal{B}}) \xrightarrow{\sim} \mathrm{Gr}^{DW}_n(\mathrm{Sp}_s\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}}) \xrightarrow{\sim} \mathrm{Gr}^{DW}_n(\mathcal{F}_A) \tag{78}$$

$$\mathrm{Gr}^T_0(\mathcal{F}_B) \xrightarrow{\sim} \mathrm{Gr}^T_0(\mathrm{Sp}_s\widetilde{\mathcal{F}}^!_{\mathcal{A},\mathcal{B}}) \xrightarrow{\sim} \mathrm{Gr}^T_0(\mathrm{Sp}_s\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}}) \tag{79}$$

Proof. Using proposition 2.7 we get a similar isomorphism for the $\widetilde{\mathcal{F}}$-sheaves on $X \times \Sigma^0$. Since the specialization functor is exact we get them for $\mathrm{Sp}_s\widetilde{\mathcal{F}}$-sheaves. It remains to use (77). The corollary is proved.

Suppose now that we are given sections of the following local systems on Σ^0:

$$[\widetilde{\omega}_{\mathcal{A}}] \in \Gamma(\Sigma^0; \mathrm{Gr}^W_{n+\mathrm{dp}(\mathcal{A})}(p_*\widetilde{\mathcal{F}}_{\mathcal{A}})) \tag{80}$$

$$[\widetilde{\Delta}_{\mathcal{B}}] \in \Gamma(\Sigma^0; \mathrm{Gr}^W_{n-\mathrm{dp}(\mathcal{B})}(p_*\widetilde{\mathcal{F}}_{\mathcal{B}})) \tag{81}$$

In this situation there are two different framed objects:

$$(H^0(X, \mathcal{F}^*_{A,B}), [\omega_A], [\Delta_B]) \tag{82}$$

and

$$(H^0(X, \mathrm{Sp}_s\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}}), \mathrm{Sp}_s[\widetilde{\omega}_{\mathcal{A}}], \mathrm{Sp}_s[\widetilde{\Delta}_{\mathcal{B}}]) \tag{83}$$

We will assume that local systems staying on the right of (80)-(81) extend to $0 \in \Sigma$, the sections $[\widetilde{\omega}_{\mathcal{A}}]$ and $[\widetilde{\Delta}_{\mathcal{B}}]$ extend to sections $[\overline{\omega}_{\mathcal{A}}]$ and $[\overline{\Delta}_{\mathcal{B}}]$ over Σ. Then $\mathrm{Sp}_s[\widetilde{\omega}_{\mathcal{A}}]$ and $\mathrm{Sp}_s[\widetilde{\Delta}_{\mathcal{B}}]$ do not depend on the choice of v (and thus s) and can be identified with the values of the sections $[\widetilde{\omega}_{\mathcal{A}}]$ and $[\widetilde{\Delta}_{\mathcal{B}}]$ at zero.

Therefore having (77) it makes sense to ask that these values at 0 coincide with the initially given $[\omega_A]$ and $[\Delta_B]$:

$$[\overline{\omega}_{\mathcal{A}}(0)] = [\omega_A]; \quad [\overline{\Delta}_{\mathcal{B}}(0)] = [\Delta_B] \tag{84}$$

Below we will assume (84).

Observe that (77) implies that $\mathrm{dp}(\mathcal{A}) = \mathrm{dp}(A)$ and $\mathrm{dp}(\mathcal{B}) = \mathrm{dp}(B)$. Therefore using isomorphism (77) and equality (84) we can identify the classes

$$\mathrm{Sp}_s[\widetilde{\omega}_{\mathcal{A}}] \in \mathrm{Gr}^W_{n+\mathrm{dp}(\mathcal{A})} H^0(X; \mathrm{Sp}_s\widetilde{\mathcal{F}}_{\mathcal{A}}) \quad \text{and} \quad [\omega_A] \in \mathrm{Gr}^W_{n+\mathrm{dp}(A)} H^0(X; \mathcal{F}_A)$$

and similarly

$$\mathrm{Sp}_s[\widetilde{\Delta}_{\mathcal{B}}] \in \mathrm{Gr}^W_{n-\mathrm{dp}(\mathcal{B})} H^0(X; \mathrm{Sp}_s\widetilde{\mathcal{F}}_B) \quad \text{and} \quad [\omega_B] \in \mathrm{Gr}^W_{n-\mathrm{dp}(B)} H^0(X, \mathcal{F}_B)$$

Summarizing, we identified the framings of the mixed objects (82) and (83).

Theorem 4.5 *a) The framed objects (82) and (83) are equivalent.*

b) If the divisor $D(\varepsilon)$ provides a Tate stratification for generic $\varepsilon \in \Sigma^0$, then these framed objects are mixed Tate objects, and they are equivalent as mixed Tate objects.

Proof. Let us rewrite the definition of $\mathcal{F}^{\bullet}_{A,*,B}$ in a different way. Recall that

$$U_k^A := U(k-1) \cup \text{all codimension } k \text{ pure } A\text{-strata and mixed strata}$$

Consider the open embeddings

$$q_k^A : U(k-1) \hookrightarrow X; \qquad q_k^B : U_k^A \hookrightarrow X$$

Then, since $q_1^{A*}\delta_X = \delta_U$, one has

$$\mathcal{F}^{\bullet}_{A,*,B} = q^B_{n!} q_n^{B!} q^A_{n*} q_n^{A*} ... q^B_{1!} q_1^{B!} q^A_{1*} q_1^{A*} \delta_X$$

Set

$$Q_k^A := q^A_{k*} q_k^{A*}; \qquad Q_k^B := q^B_{k!} q_k^{B!}$$
$$Q^{C_1,...,C_m}_{k_1,...,k_m} := q^{C_1}_{k_1} \circ ... \circ q^{C_m}_{k_m}; \quad C_i = \quad A \text{ or } B$$

Let $\mathcal{G}^{\bullet} \in D^b_{\mathrm{Sh}}(X)$ be an object whose restriction to U is isomorphic to δ_U. Then one obviously has an isomorphism

$$\mathcal{F}^{\bullet}_{A,*,B} = Q^{B,A,...,B,A}_{n,n,...,1,1} \mathcal{G}^{\bullet} \tag{85}$$

Therefore using the adjunctions $\mathrm{Id}_X \longrightarrow Q_k^A$ and $Q_k^B \longrightarrow \mathrm{Id}_X$ we get a sequence of morphisms:

$$\mathcal{F}^{\bullet}_{A,*,B} = Q^{B,A,...,B,A}_{n,n,...,1,1} \mathcal{G}^{\bullet} \longleftarrow Q^{B,A,...,B}_{n,n,...,1} \mathcal{G}^{\bullet} \longrightarrow Q^{B,A,...,B,A}_{n,n,...,2,2} \mathcal{G}^{\bullet} \longleftarrow$$
$$... \longrightarrow Q^{B,A}_{n,n} \mathcal{G}^{\bullet} \longleftarrow Q^B_n \mathcal{G}^{\bullet} \longrightarrow \mathcal{G}^{\bullet}$$

Applying the functor $H^0_\tau(-)$ to this sequence we get a similar one

$$\mathcal{F}^{*}_{A,B} = H^0_\tau\Big(Q^{B,A,...,B,A}_{n,n,...,1,1} \mathcal{G}^{\bullet}\Big) \longleftarrow H^0_\tau\Big(Q^{B,A,...,B}_{n,n,...,1} \mathcal{G}^{\bullet}\Big) \longrightarrow$$
$$H^0_\tau\Big(Q^{B,A,...,B,A}_{n,n,...,2,2} \mathcal{G}^{\bullet}\Big) \longleftarrow ... \longrightarrow H^0_\tau\Big(Q^{B,A}_{n,n} \mathcal{G}^{\bullet}\Big) \longleftarrow H^0_\tau\Big(Q^{B}_{n} \mathcal{G}^{\bullet}\Big) \longrightarrow \mathcal{G} \tag{86}$$

of perverse sheaves relating $\mathcal{F}^{*}_{A,B}$ and $\mathcal{G} := H^0_\tau \mathcal{G}^{\bullet}$.

A similar treatment for the perverse sheaves $\mathcal{F}^{!}_{A,B}$ looks as follows. There are open embeddings

$$p_k^A = q_k^A : U(k-1) \hookrightarrow X; \qquad p_k^B : V_k^A \hookrightarrow X$$

where

$$V_k^A := U(k-1) \cup \text{all codimension } k \text{ pure } A\text{-strata}$$

Then setting $P_k^A := p_{k*}^A p_k^{A*}$ and $P_k^B := p_{k!}^B p_k^{B!}$ we get

$$\mathcal{F}^!_{A,B} = H^0_\tau\Big(P^{B,A,\dots,B,A}_{n,n,\dots,1,1}\delta_X\Big)$$

Set

$$\mathcal{G}^\bullet_* := \mathrm{Sp}_s\widetilde{\mathcal{F}}^\bullet_{\mathcal{A},*,\mathcal{B}}; \qquad \mathcal{G}^\bullet_! := \mathrm{Sp}_s\widetilde{\mathcal{F}}^\bullet_{\mathcal{A},!,\mathcal{B}}$$

Since specialization sends perverse sheaves to perverse sheaves we have

$$H^0_\tau(\mathcal{G}^\bullet_*) := \mathrm{Sp}_s\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}}; \qquad H^0_\tau(\mathcal{G}^\bullet_!) := \mathrm{Sp}_s\widetilde{\mathcal{F}}^!_{\mathcal{A},\mathcal{B}} \tag{87}$$

Observe that restrictions of these objects to U are isomorphic to δ_U, providing isomorphism (85).

Proposition 4.6 *There are natural maps*

$$\mathcal{F}_B \longrightarrow Q^{B,A,\dots,C}_{n,n,\dots,m}\mathcal{G}^\bullet_! \longrightarrow Q^{B,A,\dots,C}_{n,n,\dots,m}\mathcal{G}^\bullet_* \longrightarrow \mathcal{F}_A \quad \textit{where } C = A \textit{ or } B$$

The first map induces isomorphism on $\mathrm{Gr}^T_0 H^0_\tau(-)$, the second on both $\mathrm{Gr}^{DW}_n H^0_\tau(-)$ and $\mathrm{Gr}^T_0 H^0_\tau(-)$, and the last one induces isomorphism on $\mathrm{Gr}^T_0 H^0_\tau(-)$.

These maps give rise to a commutative diagram

$$\begin{array}{ccccccccc}
\mathcal{F}_A & = & \mathcal{F}_A & = & \dots & = & \mathcal{F}_A & = & \mathcal{F}_A \\
\alpha\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
\mathcal{F}^*_{A,B} & \longleftarrow & H^0_\tau\Big(Q^{B,A,\dots,B}_{n,n,\dots,1}\mathcal{G}^\bullet_*\Big) & \longrightarrow & \dots & \longleftarrow & H^0_\tau(Q^B_n\mathcal{G}^\bullet_*) & \longrightarrow & H^0_\tau(\mathcal{G}^\bullet_*) \\
c\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
\mathcal{F}^!_{A,B} & \longleftarrow & H^0_\tau\Big(P^{B,A,\dots,B}_{n,n,\dots,1}\mathcal{G}^\bullet_!\Big) & \longrightarrow & \dots & \longleftarrow & H^0_\tau(P^B_n\mathcal{G}^\bullet_!) & \longrightarrow & H^0_\tau(\mathcal{G}^\bullet_!) \\
\beta\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
\mathcal{F}_B & = & \mathcal{F}_B & = & \dots & = & \mathcal{F}_B & = & \mathcal{F}_B
\end{array} \tag{88}$$

Thanks to (77) and (87) the right vertical maps are identified with

$$\mathcal{F}_B = \mathrm{Sp}_s\widetilde{\mathcal{F}}_{\mathcal{B}} \longrightarrow \mathrm{Sp}_s\widetilde{\mathcal{F}}^!_{\mathcal{A},\mathcal{B}} \longrightarrow \mathrm{Sp}_s\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}} \longrightarrow \mathrm{Sp}_s\widetilde{\mathcal{F}}_{\mathcal{A}} = \mathcal{F}_A$$

Proof. There is canonical morphism $\mathcal{G}^\bullet_* \longrightarrow \mathcal{F}_A$ given by

$$\mathcal{G}^\bullet_* = \mathrm{Sp}_s(\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}}) \longrightarrow \mathrm{Sp}_s(\widetilde{\mathcal{F}}_{\mathcal{A}}) = \mathcal{F}_A \tag{89}$$

We define the maps

$$Q^{B,A,\dots,B,A}_{n,n,\dots,m,m}\mathcal{G}^\bullet_* = j^B_{n!}j^A_{n*}\dots j^B_{m!}j^A_{m*}\mathrm{Res}_{U(m-1)}\mathcal{G}^\bullet_* \longrightarrow \mathcal{F}_A$$

and

$$Q^{B,A,\dots,B}_{n,n,\dots,m}\mathcal{G}^\bullet_* = j^B_{n!}j^A_{n*}\dots j^B_{m!}\mathrm{Res}_{U^A(m-1)}\mathcal{G}^\bullet_* \longrightarrow \mathcal{F}_A$$

by induction using restrictions of the map (89) to $U(m-1)$ and $U^A(m-1)$ and the two constructions i) and ii) introduced in the proof of lemma 2.3. This generalizes the procedure used in lemma 2.3. The starting point is the isomorphism $\mathrm{Res}_{U(0)}\mathcal{G}^\bullet_* = \mathrm{Res}_{U(0)}\mathcal{F}_A$. After this we apply the construction i) for $U(0)$, then ii) for $U^A(1)$, then i) for $U(1)$, after that ii) for $U^A(1)$, and so on.

To show that the top rectangle in diagram (88) is commutative we need to show that the following diagram is commutative

$$\begin{array}{ccccc} \mathcal{F}_A & = & \mathcal{F}_A & = & \mathcal{F}_A \\ \uparrow & & \uparrow & & \uparrow \\ Q^{B,A,\dots,B}_{n,n,\dots,m-1}\mathcal{G}^\bullet_* & \longrightarrow & Q^{B,A,\dots,B,A}_{n,n,\dots,m,m}\mathcal{G}^\bullet_* & \longleftarrow & Q^{B,A,\dots,B}_{n,n,\dots,m}\mathcal{G}^\bullet_* \end{array}$$

To show that the left square is commutative it is sufficient to show that the diagram

$$\begin{array}{ccc} \mathrm{Res}_{U(m-1)}\mathcal{F}_A & = & \mathrm{Res}_{U(m-1)}\mathcal{F}_A \\ \uparrow (42) & & \uparrow \\ (j^B_{m-1})_!\mathrm{Res}_{U^A(m-2)}\mathcal{G}^\bullet_* & \longrightarrow & \mathrm{Res}_{U(m-1)}\mathcal{G}^\bullet_* \end{array}$$

is commutative. Since the left bottom object in this diagram is isomorphic to

$$(j^B_{m-1})_! j^{B!}_{m-1}\mathrm{Res}_{U(m-1)}\mathcal{G}^\bullet_*$$

this boils down to the fact that the adjunction $j_! j^! \longrightarrow \mathrm{Id}$ is a morphism of functors.

To show that the right square is commutative it is sufficient to show that the diagram

$$\begin{array}{ccc} \mathrm{Res}_{U^A(m-1)}\mathcal{F}_A & = & \mathrm{Res}_{U^A(m-1)}\mathcal{F}_A \\ \uparrow (40) & & \uparrow \\ j^B_{m*}\mathrm{Res}_{U(m-1)}\mathcal{G}^\bullet_* & \longleftarrow & \mathrm{Res}_{U^A(m-1)}\mathcal{G}^\bullet_* \end{array}$$

is commutative. Since the left bottom object in this diagram is isomorphic to

$$j^A_{m*} j^{A*}_m \mathrm{Res}_{U^A(m-1)}\mathcal{G}^\bullet_*,$$

This boils down to the fact that the adjunction $\mathrm{Id} \longrightarrow j_* j^*$ is a morphism of functors.

Therefore we defined the top rectangle of the diagram (88) and proved that all its squares are commutative. Applying the duality $*$ and interchanging the role of A and B we deduce from this a similar statement about the bottom rectangle. The maps involved in the middle rectangle are constructed using the canonical map $j_! \longrightarrow j_*$ for the extensions to the mixed strata. It follows that the middle rectangle is also built from commutative squares. The commutative diagram (88) is constructed.

It remains to prove that the vertical arrows in this diagram induce the isomorphisms on the appropriate pieces of the WD or/and T filtrations. This follows from lemma 2.9. The condition (51) for $i = 0$ is deduced from theorem 4.2. For other i's we use a similar result, see the remark 2 after theorem 4.2.

Applying the duality $*$ and interchanging the role of A and B we deduce the statements about the bottom rectangle from the ones about the top rectangle.

The maps involved in the middle rectangle are constructed using the canonical map $j_! \longrightarrow j_*$ for the extensions to the mixed strata. The proposition is proved.

The specialization theorem follows immediately from this proposition. Indeed, applying the functor $H^0(X, -)$ to the second horizontal sequence of morphisms in (88) we get a sequence of mixed objects and maps between them connecting $H^0(X; \mathcal{F}^*_{A,B})$ and $H^0(X; \mathrm{Sp}_s\widetilde{\mathcal{F}}^*_{\mathcal{A},\mathcal{B}})$. One needs to check that each of these objects is naturally framed, and each of the maps between them respects the frames. The A-parts of the frames on $H^0(X; -)$, where $-$ stays for the objects in the second horizontal line, are provided by the vertical frame maps to $\mathcal{F}_A$. Since the top rectangle of the diagram is commutative and the top vertical arrows induce isomorphisms on Gr_n^{DW}, the maps respect the A-parts of the frame by their very construction, see s. 2.4-2.5.

To handle the B-parts of the frame we employ a similar argumentation for the bottom rectangle in the diagram, and in addition use the fact that the middle vertical maps induce isomorphisms on $\mathrm{Gr}_0^T H^0_\tau(-)$. The part a) of the theorem is proved. Having a), the part b) is straitforward. The theorem is proved.

5 Applications to scissor congruence groups

In this section we apply the results of the sections 2-4 when $D = A \cup B$ is an admissible configuration of hyperplanes in P^n. In particular when A and B are two simplices in P^n we produce a framed mixed Hodge structure of geometric origin whose period is given by the Aomoto polylogarithm. The l-adic version of this result provides a framed mixed Tate l-adic representation of the Galois group $\mathrm{Gal}(\overline{F}/F)$. When F is a number field we produce a mixed Tate motive over F whose Hodge and l-adic realizations are the ones mentioned above.

Similar results are valid for the scissor congruence groups considered in [G9], including the classical scissor congruence groups in the hyperbolic and spherical spaces.

1. The generalized scissor congruence groups and framed Hodge-Tate structures. An algebraic simplex in P^n is a union of $n+1$ hyperplanes. It is called a nondegenerate algebraic simplex if the intersection of these hyperplanes is empty. Let

$$A = A_0 \cup ... \cup A_n \quad \text{and} \quad B = B_0 \cup ... \cup B_n$$

be two nondegenerate algebraic simplices in P^n. Here A_i and B_j are hyperplanes. The pair (A, B) is admissible if and only if $A_I \neq B_J$ if $|I| = |J|$. The stratification defined by any collection of hyperplanes in P^n is always regular.

Let us introduce a $\mathbb{Z}/2\mathbb{Z}$-torsor of orientations of a simplex A. Namely, an ordering of the hyperplanes A_i provides an element of the torsor, and two orderings provide the same element if and only if they differ by an even permutation.

For a field F the generalized scissor congruence group $A_n(F)$ (see [BMS], [BGSV]) is generated by admissible pairs

$$(A; B) = (A_0, ..., A_n; B_0, ..., B_n)$$

of oriented simplices in $P^n(F)$ subject to the following relations:

1) *Nondegeneracy.* $(A; B) = 0$ if one of the simplices A or B is degenerate.

2) *Orientation.* $(A; B)$ changes the sign if we change orientation of A or B.

3) *Additivity.* For any $n + 2$ hyperplanes $A_0, ..., A_{n+1}$ one has

$$\sum_{i=0}^{n+1}(-1)^i(A_0..., \hat{A}_i, ..., A_{n+1}; B_0, ..., B_n) = 0$$

if all the terms are admissible (additivity in A).

We impose a similar additivity in B condition.

4) *Projective invariance.* $(gA; gB) = (A; B)$ for any $g \in PGL_{n+1}(F)$.

Recall the group $\mathcal{H}_n$ of the Hodge-Tate structures framed by $\mathbb{Q}(0)$ and $\mathbb{Q}(-n)$. Its motivic version is the corresponding group $\mathcal{A}_n(F)$ of framed mixed Tate motives over a number field F. The l-adic counterpart $\mathcal{A}_n^{et}(F)$ is the $\mathbb{Q}_l$-vector space of equivalence classes of mixed Tate l-adic Gal$(\overline{F}/F)$-modules framed by $\mathbb{Q}_l(0)$ and $\mathbb{Q}_l(-n)$ ([BD1], see also [G7]). Here we have to assume that F does not contain all l^∞ roots of unity.

Let $\mathcal{F}^*_{A,B}$ be the perverse mixed Hodge sheaf defined in section 2, and by $\mathcal{F}^{et,*}_{A,B}$ its l-adic counterpart.

Theorem 5.1 *a) There is a canonical homomorphism* $h_n : A_n(\mathbb{C}) \longrightarrow \mathcal{H}_n$ *defined on the generators by*

$$h_n^{\mathcal{H}} : (A, B) \longmapsto H^0\Big(\mathbb{C}P^n; \mathcal{F}^*_{A,B}\Big) \tag{90}$$

b) Let F be a field that does not contain all l^∞ roots of unity. Then there is a canonical homomorphism $h_n^{et} : A_n(F) \longrightarrow \mathcal{A}_n^{et}(F)$ defined by

$$h_n^{et} : (A, B) \longmapsto H^0_{et}\Big(P^n \otimes \overline{F}; \mathcal{F}^{et,*}_{A,B}\Big) \tag{91}$$

c) Let F be a number field. Then there is a canonical homomorphism h_n : $A_n(F) \longrightarrow \mathcal{A}_n(F)$ whose Hodge and l-adic realizations provided by a) and b).

Remark. If we restrict our attention to the subgroup generated by pairs $(A; B)$ of simplices in generic position, i.e. $A \cup B$ is a normal crossing divisor, then a) and b) follow from [BGSV], and c) from chapter 5 in [G9].

Conjecture 5.2 *The map constructed in theorem 5.1c) is an isomorphism.*

Proof. a) The right hand side in (91) is obviously a Hodge-Tate structure. Indeed, the perverse sheaf $\mathcal{F}^*_{A,B}$ is glued from $\delta_C(-m)$ where C are planes in $\mathbb{C}P^n$.

Let f_i be a rational function on $\mathbb{C}P^n$ with the divisor is $A_i - A_0$. Set

$$\omega_A = d\log f_1 \wedge ... \wedge d\log f_n \in \Omega^n_{\log}(P^n - A)$$

Let Δ_B be a chain defining a class in $H_n(\mathbb{C}P^n, B(\mathbb{C}); \mathbb{Z})$ corresponding to given orientation of the algebraic simplex B. According to section 2.4 the classes

$$[\omega_A] \in H^n_{\mathrm{DR}}(\mathbb{C}P^n - A) = \mathrm{Gr}^W_{2n} H^n_{\mathrm{DR}}(\mathbb{C}P^n - A) = \mathbb{Z}(-n)$$

$$[\Delta_B] \in H_n(\mathbb{C}P^n, B(\mathbb{C}); \mathbb{Z}) = \mathrm{Gr}^W_0 H_n(\mathbb{C}P^n, B(\mathbb{C}); \mathbb{Z}) = \mathbb{Z}(0)$$

provide a framing of $h^{\mathcal{H}}_n(A, B)$.

It remains to check the relations. The relations 2) and 4) are obvious, and 1) is true by definition. Let us check the additivity in A. We need the following simple general result.

Lemma 5.3 *Let M an object of a mixed Tate category $\mathcal{C}$ (e.g. the abelian category of mixed Tate motives over a number field F, or the category of Hodge-Tate structures). Then*

a) Given non zero maps

$$v_0 : \mathrm{Gr}^W_0 M \longrightarrow \mathbb{Q}(0); \quad f^\alpha_n, f^\beta_n : \mathbb{Q}(-n) \longrightarrow \mathrm{Gr}^W_{2n} M$$

such that $\sum_\alpha f^\alpha_n = \sum_\beta f^\beta_n \neq 0$ one has an equality of framed objects

$$\sum_\alpha (M, v_0, f^\alpha_n) = \sum_\beta (M, v_0, f^\beta_n)$$

b) Similarly if $\sum_\alpha v^\alpha_0 = \sum_\beta v^\beta_0 \neq 0$, then

$$\sum_\alpha (M, v^\alpha_0, f_n) = \sum_\beta (M, v^\beta_0, f_n)$$

Proof. a) It is sufficient to prove that if $f_1, f_2 : \mathbb{Q}(-n) \longrightarrow \mathrm{Gr}^W_{2n} M$ are non zero maps whose sum is also non zero then

$$(M, v_0, f_1) + (M, v_0, f_2) = (M, v_0, f_1 + f_2)$$

By definition the left element is represented by the framed object $(M \oplus M, v_0 + v_0, (f_1, f_2))$. We claim that the natural projection $(\mathrm{id}, \mathrm{id}) : M \oplus M \longrightarrow M$ induces an equivalence of the framed objects

$$(M \oplus M, v_0 + v_0, (f_1, f_2)) \xrightarrow{\sim} (M, v_0, f_1 + f_2)$$

Indeed, there are commutative diagrams

$$\begin{array}{ccc} \mathbb{Q}(-n) & = & \mathbb{Q}(-n) \\ (f_1,f_2)\downarrow & & \downarrow f_1+f_2 \\ \mathrm{Gr}^W_{2n}M\oplus \mathrm{Gr}^W_{2n}M & \stackrel{(\mathrm{id},\mathrm{id})}{\longrightarrow} & \mathrm{Gr}^W_{2n}M \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Gr}^W_{0}M\oplus \mathrm{Gr}^W_{0}M & \stackrel{(\mathrm{id},\mathrm{id})}{\longrightarrow} & \mathrm{Gr}^W_{0}M \\ v_0+v_0\downarrow & & \downarrow v_0 \\ \mathbb{Q}(0) & = & \mathbb{Q}(0) \end{array}$$

The proof of the second statement is similar. The lemma is proved.

More generally, suppose that A and B are unions of hyperplanes in P^n, and the pair $A\cup B$ is admissible. Then there is a Hodge-Tate structure $h^{\mathcal{H}}_n(A,B)$ defined by the same formula (91).

Let $A' := A_0\cup ...\cup A_{n+1}$. Set $A^{(i)} := A_0\cup ...\widehat{A}_i...\cup A_{n+1}$. Then obviously

$$\sum^{*}_{0\leq i\leq n+1}(-1)^i\omega_{A^{(i)}} = 0 \tag{92}$$

Here the summation is over $0\leq i\leq n+1$ such that $[\omega_{A^{(i)}}]\neq 0$.

Lemma 5.4 *Let $A_0,...,A_n$ be any collection of hyperplanes in P^n. Let f_i be a rational function with the divisor $A_i - A_0$. Then*

$$\{f_1,...,f_n\}\in K^M_n(F(P^n))/F^*\cdot K^M_{n-1}(F(P^n))$$

is not zero if and only if the hyperplanes are in generic position.

Proof. If the hyperplanes are in generic position then they are projectively equivalent to the coordinate hyperplanes $z_i = 0$, so our symbol is $\{z_1/z_0,...,z_n/z_0\}$. It is non zero since the homomorphism

$$d\log^n : K^M_n(F(P^n))/F^*\cdot K^M_{n-1}(F(P^n)) \longrightarrow \Omega^n_{\log}((F(P^n));$$

$$\{f_1,...,f_n\}\longmapsto d\log(f_1)\wedge ...\wedge d\log(f_n)$$

maps it to a non zero differential form.

If $g_1,...,g_m$ are elements of a field k such that $g_1+...+g_m = 1$ then it is easy to check by induction that $\{g_1,...,g_m\} = 0$ in $K^M_n(k)$. Let F_i be a linear equation of the hyperplane A_i. If these hyperplanes are not in generic position then, renumbering the hyperplanes, we can find $F_1,...,F_m$ such that

$\lambda_1 F_1 + ... + \lambda_m F_m = 0$, $\lambda_i \neq 0$. Since $\{F_1, ..., F_m\} \sim \{\lambda_1 F_1, ..., \lambda_m F_m\}$ modulo $F^* \cdot K^M_{m-1}(F(P^n))$ the lemma follows.

This lemma implies that the terms omitted during the summation in (92) are precisely the ones which are zero by the non degeneracy relation. Therefore using lemma 5.3 we see that

$$\sum^{*}_{0 \leq i \leq n+1} (-1)^i (h^{\mathcal{H}}_n(A', B), [\omega_{A^{(i)}}], [\Delta_B]) = 0$$

The canonical morphism of perverse sheaves $\mathcal{F}^*_{A^{(i)},B} \longrightarrow \mathcal{F}^*_{A',B}$ provides, assuming $[\omega_{A^{(i)}}] \neq 0$, an equivalence of framed Hodge-Tate structures

$$(h^{\mathcal{H}}_n(A', B), [\omega_{A^{(i)}}], [\Delta_B]) \sim (h^{\mathcal{H}}_n(A^{(i)}, B), [\omega_{A^{(i)}}], [\Delta_B])$$

So the additivity in A is proved. The additivity in B is checked similarly. Or we can use the duality since $*h^{\mathcal{H}}_n(A, B) = h^{\mathcal{H}}_n(B, A)$. The part a) of the theorem is proved.

The part b) is completely similar to a). The only point requiring a comment is construction of the A-part of the framing. Namely, the class $[\omega_A]$ in the l-adic setting is the cohomology class given by the l-adic regulator applied to the symbol $\{f_1, ..., f_n\}$ where f_i are as in lemma 5.4. Then for $n+2$ hyperplanes we have $\sum_i (-1)^i \{f_1, ...\widehat{f_i}, ..., f_{n+1}\} = 0$ providing, together with lemma 5.4, identity (92)

The part c) follows from the results of chapter 3 and, say, a) using the injectivity of the regulators, or can be deduced directly using the same arguments as used in the proof of part a). The theorem is proved.

Remark. A similar result for the scissor congruence groups defined in [G9] is left to the reader as an easy exercise.

6 Applications to motivic torsors of path on curves

In this section we apply the results of chapters 2-4 when $A \cup B$ is a specific configuration of divisors on the n-th power of an arbitrary regular curve X.

Recall that in the Hodge or étale realization the torsor of path $\mathcal{P}(X; v_x, v_y)$ between the tangential base points v_x, v_y was constructed by Deligne [D] as a pro-object in the corresponding category. There is a weight filtration indexed by integers $n \leq 0$ on it, and $\mathrm{Gr}^W_0 \mathcal{P}(X; v_x, v_y) = \mathbb{Q}(0)$ (or $\mathbb{Q}_l(0)$). Let v_0 be the image of $1 \in \mathbb{Q}(0)$ under this isomorphism. Choose a non zero vector $f_k \in (\mathrm{Gr}^W_{-k} \mathcal{P}(X; v_x, v_y))^\vee$. We construct a framed object of geometric origin equivalent to the framed object

$$(\mathcal{P}(X; v_x, v_y), v_0, f_k) \tag{93}$$

The period of its Hodge realization is given by an iterated integral. This construction provides a framed mixed Hodge structure of geometric origin corresponding to any iterated integral between tangential base points on X. In the case of the classical base points this has been done by Beilinson, and our construction in this case leads to the same object.

Let F be a number field and v_x, v_y are non zero tangent vectors at the points $x, y \in \mathbb{P}^1(F)$. Suppose that Λ is a finite subset of $\mathbb{P}^1(F)$, and v_x, v_y are defined over F. It was proved in [DG] that there exists the motivic torsor

$$\mathcal{P}^{\mathcal{M}}(\mathbb{P}^1 - \Lambda; v_x, v_y) \tag{94}$$

of path on $\mathbb{P}^1 - \Lambda$ between the tangential base points v_x and v_y understood as a pro-object in the abelian category $\mathcal{M}_T(F)$ of mixed Tate motives over F. Its Hodge and l-adic realizations are isomorphic to the standard Hodge and l-adic realizations of the torsor of path.

Applying our general construction in the case $X = \mathbb{P}^1 - \Lambda$ we get an independent geometric construction of the motivic torsor of path (94). Namely, we describe it by constructing all its matrix elements. We use the injectivity of the regulators to prove that it has all the needed properties.

1. A framed mixed Hodge structures corresponding to iterated integrals on curves. Suppose that X is a regular projective curve over $\mathbb{C}$ and $\omega_i \in \Omega^1_{\log}(X)$. Let $p_i : X^n \longrightarrow X$ be the projection onto the i-th factor. Choose a path $\gamma : [0,1] \longrightarrow X(\mathbb{C})$ connecting the (tangential) base points v_x, v_y at x and y. Then there is an iterated integral

$$\int_{\gamma; v_x, v_y} \omega_1 \circ \ldots \circ \omega_n := \int_{\gamma(\Delta_n)} p_1^* \omega_1 \wedge \ldots \wedge p_n^* \omega_n \tag{95}$$

Here $\Delta_n = \{0 \leq s_1 \leq \ldots \leq s_n \leq 1\}$ is the standard n-dimensional simplex and

$$\gamma : \Delta_n \longrightarrow X^n(\mathbb{C}), \qquad \gamma(s_1, \ldots, s_n) := (\gamma(s_1), \ldots, \gamma(s_n)) \tag{96}$$

Denote by $\mathrm{Sing}(\omega)$ the singular locus of a form ω. Integral (95) is convergent if and only if

$$x \notin \mathrm{Sing}(\omega_1) \quad \text{and} \quad y \notin \mathrm{Sing}(\omega_n) \tag{97}$$

Our goal is to provide an explicit construction of a framed mixed Hodge structure whose period is given by a convergent iterated integral (95).

Let X be a regular projective curve over a field F and $\omega_i \in \Omega^1_{\log}(X)$. Set

$$A_i := p_i^{-1} \mathrm{Sing}(\omega_i); \qquad A := A_1 \cup \ldots \cup A_n$$

So

$$X^n - A = (X - \mathrm{Sing}(\omega_1)) \times \ldots \times (X - \mathrm{Sing}(\omega_n))$$

Observe that $\mathrm{dp}(A)$ is the number of singular forms among the forms ω_i. Let $(t_1, \ldots, t_n)$ be a point of X^n. Set

$$B := \{x = t_1\} \cup \cup_{i=1}^{n-1} \{t_i = t_{i+1}\} \cup \{t_n = y\};$$

Lemma 6.1 *a) $A \cup B$ defines a regular stratification of X^n.*

b) The pair of divisors (A, B) is admissible if and only if condition (97) is satisfied.

Proof. a) If $\{a_i^k\}$ are given points of X then the stratification defined by the divisors $t_i = a_i^k$ and any collection of diagonals $t_i = t_j$ in X^n is obviously regular.

b) Straitforward check. The lemma is proved.

Below we assume (97). So thanks to lemma 6.1 the construction of section 2 applied to the divisor $A \cup B$ in X^n provides a mixed Hodge structure $H^0(X^n; \mathcal{F}^*_{A,B})$. It is equipped with a framing provided by s. 2.5 by the element

$$[p_1^*\omega_1 \wedge ... \wedge p_n^*\omega_n] \in \mathrm{Gr}^W_{n+\mathrm{dp}(A)} H^n(X^n - A),$$

and the canonical element

$$[\Delta_n] \in \mathrm{Gr}^W_0 H_n(X^n, B) = \mathbb{Z}(0),$$

The sign of the generator here is determined by the natural ordering of irreducible components of divisor B. (If $x = y$ we modify slightly the definition of the group on the right in a similar way as in ch. 4 of [G7]). So we get a framed mixed Hodge structure

$$h(x; \omega_1 \otimes ... \otimes \omega_n; y) := \Big(H^0(X^n; \mathcal{F}^*_{A,B}), [p_1^*\omega_1 \wedge ... \wedge p_n^*\omega_n], [\Delta_n]\Big) \tag{98}$$

If a more restrictive condition

$$x, y \notin \mathrm{Sing}(\omega_i) \quad \text{for } 1 \leq i \leq n \tag{99}$$

is valid then there is *another* framed mixed Hodge structure which has a clear motivic origin, and whose period is given by integral (95). Its construction goes back to Beilinson. Namely set $B_A := B - (B \cap A)$. Consider the mixed motive

$$H^n(X^n - A, B_A) = H^0(X^n, \alpha_* j_! \delta_U) \tag{100}$$

where

$$j : U := X^n - (A \cup B) \hookrightarrow X^n - A; \quad \alpha : X^n - A \hookrightarrow X^n;$$

Both inclusions here are open affine embeddings, so $\alpha_* j_! \delta_U$ is a perverse sheaf.

Lemma 6.2 *Let us assume (99). Then none of the pure B-strata is contained in a pure A-stratum. Therefore we have*

$$\alpha_* j_! \delta_U = \mathcal{F}^*_{A,B} \tag{101}$$

Proof. It is sufficient to check that none of the B-vertices is contained in a pure A-stratum. The B-vertices are given by $t_1 = ... = t_k = x; t_{k+1} = ... = t_n = y$, so it follows immediately from (99). Therefore $A \cup B$ is a union of mixed strata and pure A-strata. This implies that the inductive construction of the object $\mathcal{F}^{\bullet}_{A,*,B}$ can be replaced by a single step construction given by $\alpha_* j_! \delta_U$, i.e. we have an isomorphism $\alpha_* j_! \delta_U = \mathcal{F}^{\bullet}_{A,*,B}$. In particular the latter object is a perverse sheaf. So we get (101). The lemma is proved.

Therefore we have a natural framing on the mixed object (100)

Lemma 6.3 *Assuming 99, integral (95) is a period of the framed mixed Hodge structure (100).*

Proof. The natural framing constructed in chapter 2 admits in this case a more explicit description. Namely, the restriction of the n-form $p_1^*\omega_1 \wedge ... \wedge p_n^*\omega_n$ to B is zero, and it provides a well defined class

$$[p_1^*\omega_1 \wedge ... \wedge p_n^*\omega_n] \in \mathrm{Gr}^W_{n+\mathrm{dp}(A)} H^n(X^n - A, B_A)$$

Further, there is canonical isomorphism

$$\mathbb{Z}(0) \stackrel{\sim}{=} \mathrm{Gr}^W_0 H_n(X^n, B) = \mathrm{Gr}^W_0 H_n(X^n - A, B_A) \qquad (102)$$

providing the second component of the frame. (If $x = y$ we modify the group on the right as in ch. 4 of [G7]). We get a framed mixed Hodge structure.

$$h_B(x; \omega_1 \otimes ... \otimes \omega_n; y) := \Big(H^n(X^n - A, B_A), [p_1^*\omega_1 \wedge ... \wedge p_n^*\omega_n], [\Delta_n] \Big) \qquad (103)$$

A path $\gamma : [0,1] \longrightarrow X(\mathbb{C})$ from x to y provides a lift of the generator of (102) to a relative homology class

$$[\gamma(\Delta_n)] \in H_n^{\mathrm{Betti}}(X^n - A, B_A; \mathbb{Z})$$

It depends only on the homotopy class of γ. The lemma follows.

Applications of the specialization theorem. Let Σ be a curve and $0 \in \Sigma$ is its distinguished point. Let $\{\omega_i(\varepsilon)\}$ be 1-forms on X parametrized by $\varepsilon \in \Sigma$, and $\omega_i(0) = \omega_i$. Suppose that the base points $x(\varepsilon)$ and $y(\varepsilon)$ also depend regularly on $\varepsilon \in \Sigma$, i.e. we are given two maps $x, y : \Sigma \longrightarrow X$ regular near 0, and $x(0) = x, y(0) = y$. Suppose in addition to condition (97) that

$$x(\varepsilon), y(\varepsilon) \notin \mathrm{Sing}(\omega_i(\varepsilon)) \quad \text{for} \quad \varepsilon \neq 0, \quad i = 1, ..., n \qquad (104)$$

Then for $\varepsilon \neq 0$ there are framed objects (103), while at $\varepsilon = 0$ we constructed a framed object (98). The specialization theorem 4.5 in this particular case allows us to compare them:

Theorem 6.4 *Let us assume conditions (97) and (104). Then one has*

$$\mathrm{Sp}_{\varepsilon \to 0} h_B(x(\varepsilon); \omega_1(\varepsilon) \otimes ... \otimes \omega_n(\varepsilon); y(\varepsilon)) = h(x; \omega_1 \otimes ... \otimes \omega_n; y)$$

Proof. It follows immediately from theorem 4.5 thanks to (101), (100) and the very definitions. The theorem is proved.

Here is a version of this result. Let $S := \cup \mathrm{Sing}(\omega_i)$ and $X_S := X - S$. The family of the mixed Hodge structures $H^n(X_S^n, B_S)$ forms a unipotent variation $\mathcal{H}^n$ of mixed Hodge structures over $X_S \times X_S$ (see chapter 4 in [G7]). Combining it with the rigidity theorem of Vologodsky [Vol] one immediately sees that this variation is isomorphic to the one constructed by Hain and Zucker [HZ]. The canonical class $[\Delta_n]$ and the given forms ω_i provide a framing on it. We denote by $\mathcal{H}^n(\omega_1 \otimes ... \otimes \omega_n)$ the corresponding variation of framed mixed Hodge structures. Its fiber over a point $(x, y) \in X_S^2$ is precisely $h_B(x; \omega_1 \otimes ... \otimes \omega_n; y)$.

Corollary 6.5 *Let us assume (97). Then specialization of the variation of framed Hodge-Tate structure $\mathcal{H}^n(\omega_1 \otimes ... \otimes \omega_n)$ at the tangent vector*

$$(v_x, v_y) \in T_{(x,y)}(X \times X); \qquad v_x \neq 0, v_y \neq 0 \tag{105}$$

is equivalent to the framed Hodge-Tate structure $h(x; \omega_1 \otimes ... \otimes \omega_n; y)$. In particular it does not depend on the choice of vectors v_x, v_y.

Proof. Apply twice theorem 6.4, using first specialization with respect to $x \in X$, and then with respect to $y \in Y$. The corollary is proved.

If $x \in \mathrm{Sing}(\omega_1)$ or $y \in \mathrm{Sing}(\omega_n)$ then the integral is divergent, and it has to be regularized. The corresponding framed mixed Hodge structure is *defined* by the specialization of the variation $\mathcal{H}^n(\omega_1 \otimes ... \otimes \omega_n; \gamma)$ at the tangent vector (105), and its period is, by definition, the regularized value of the integral.

Corollary 6.6 *a) Let us assume (97). Then iterated integral (95) is the period of the framed mixed Hodge structure (98).*

b) If $X = \mathbb{P}^1$ then (98) it is a framed Hodge-Tate structure.

Proof. a) Follows, for instance from lemma 6.3 and theorem 6.5. Another way to deduce this result is provided by the results of Sections 3.2 and 3.4. The part b) is obvious. The corollary is proved.

2. Framed mixed Tate motives corresponding to iterated integrals on $\mathbb{A}^1$. Below $X = P^1$.

Theorem 6.7 *Assume that x, y and non zero forms ω_i are defined over a number field F and (97) holds. Then there is a framed mixed Tate motive*

$$m(x; \omega_1 \otimes ... \otimes \omega_n; y) \in \mathcal{A}_n(F)$$

such that for a complex embedding $\sigma : F \hookrightarrow \mathbb{C}$ its Hodge realization is provided by $h(\sigma(x); \sigma(\omega_1) \otimes ... \otimes \sigma(\omega_n); \sigma(y))$.

Proof. We apply the construction of chapter 2 to the divisor $A \cup B \subset X^n$ defined in s. 6.1. Thanks to the results of chapter 3 we get a framed mixed Tate motive over F. Corollary 6.5 provides the comparison of its Hodge realization with the construction used in [G7]. In particular the constructed element (106) does not depend on the choice of tangent vectors v_x, v_y. The theorem is proved.

Suppose that F is a number field, v_x, v_y are non zero tangent vectors at the points $x, y \in \mathbb{P}^1(F)$, and v_x, v_y are defined over F.

Theorem 6.8 *In the above situation for any $a_1, ..., a_n \in F$ there is an element*

$$\mathrm{I}^{\mathcal{M}}(v_x; a_1, ..., a_n; v_y) \in \mathcal{A}_n(F) \tag{106}$$

such that for a complex embedding $\sigma : F \hookrightarrow \mathbb{C}$ its Hodge realization coincides with the defined in [G7] element

$$\mathrm{I}^{\mathcal{H}}(\sigma(v_x); \sigma(a_1), ..., \sigma(a_n); \sigma(v_y)) \in \mathcal{H}_n$$

Proof. If $x \neq a_1$ and $y \neq a_2$ this is given by theorem 6.7.

If $x = a_1$ or $y = a_2$ in the Hodge realization there are explicit formulas provided by lemma 5.2 in [G8] expressing the corresponding element as a product of the elements constructed on the first step. Using these explicit formulas as a definition in the motivic case we define element (106) and at the same time prove that its Hodge realization is as needed. The theorem is proved.

3. A geometric construction of the motivic torsor of path between tangential base points. The fundamental Hopf algebra $\mathcal{A}_\bullet(F)$ can be define as a commutative Hopf algebra $\mathcal{A}_\bullet(F)$ in the tensor category $\mathcal{P}_T$ of pure Tate motives. Indeed, the category $\mathcal{P}_T$ is canonically equivalent to the category of finite dimensional graded $\mathbb{Q}$-vector spaces.

The dual to the motivic torsor of path (94) is an ind-object in $\mathcal{M}_T(F)$. We define it as an ind-object in the category of comodules over $\mathcal{A}_\bullet(F)$. For this one needs

1) to construct a pro-object V in the category of pure Tate motives,

2) to define the coaction map $a : V \longrightarrow V \otimes \mathcal{A}_\bullet(F)$, and

3) to check the coaction axioms for a.

We do this as follows.

1) The $\mathbb{Q}(-n)$-isotipic component V_n of V is given by

$$V_n := \otimes^n H^1(\mathbb{P}^1 - \Lambda)$$

Here, barring the trivial case $\Lambda = \emptyset$, the motive $H^1(\mathbb{P}^1 - \Lambda)$ is a pure Tate motive of weight 2, i.e. a direct sum of copies of $\mathbb{Q}(-1)$. Its rank is $|\Lambda| - 1$. One may assume without loss of generality that $\infty \in \Lambda$. Then there is a basis in $H^1(\mathbb{P}^1 - \Lambda)$ given by the classes

$$\omega_a := d\log(t - a) : \mathbb{Q}(-1) \longrightarrow H^1(\mathbb{P}^1 - \Lambda); \qquad a \in \Lambda - \{\infty\} \tag{107}$$

2) To define the coaction we, given a pair of non zero maps

$$v : \mathbb{Q}(-m) \longrightarrow V_m; \quad f : V_n \longrightarrow \mathbb{Q}(-n),$$

will define a map

$$a(v,f):\mathbb{Q}(n-m)\longrightarrow \mathcal{A}_{m-n}(F) \tag{108}$$

This map is supposed to come as the composition

$$\mathbb{Q}(-m)\xrightarrow{v} V_m \xrightarrow{a} V_n\otimes \mathcal{A}_{m-n}(F)\xrightarrow{f\otimes \mathrm{id}} \mathbb{Q}(-n)\otimes \mathcal{A}_{m-n}(F)$$

Choose basis elements

$$v=\omega_{b_1}\otimes ...\otimes \omega_{b_k}; \qquad f=\omega_{a_1}\otimes ...\otimes \omega_{a_n}; \qquad a_i, b_j\in \Lambda-\{\infty\} \tag{109}$$

Consider an inclusion of ordered subsets

$$\beta:\{b_1,...,b_k\}\hookrightarrow \{a_1,...,a_n\} \tag{110}$$

provided by a sequence $0=i_0<i_1<...<i_k<i_{k+1}=n+1$ such that $b_p=a_{i_p}$.

Definition 6.9 *Assuming (109) we set*

$$a(v,f):=\sum_{\beta}\prod_{p=0}^{k}\mathrm{I}^{\mathcal{M}}(v_{a_{i_p}};a_{i_p+1},...,a_{i_{p+1}-1};v_{a_{i_{p+1}}})\in \mathcal{A}_{n-k}(F)$$

where the sum is over all different inclusions (110).

This definition has been suggested by theorem 6.4 in [G7], see also theorem 1.1 in [G8].

Examples. i) $a(v,f)=0$ if there is no such inclusion β.

ii) Let $v_0:\mathbb{Q}(0)\xrightarrow{=} V_0$ be the canonical isomorphism, and f is as above. In this case $\{b_1,...,b_k\}$ is the empty set, so there is just one inclusion β. Thus

$$a(v_0,f)=\mathrm{I}^{\mathcal{M}}(v_x;a_1,...,a_n;v_y)$$

Theorem 6.10 *a) Let F be a number field. Then the elements $a(v,f)\in A_n(F)$ provide an ind-object in $\mathcal{M}_T(F)$. Its dual is by definition the motivic torsor of path $\mathcal{P}^{\mathcal{M}}(\mathbb{P}^1-\Lambda;v_x,v_y)$.*

b) There is a morphism of pro-objects in $\mathcal{M}_T(F)$, the composition of path,

$$\mathcal{P}^{\mathcal{M}}(\mathbb{P}^1-\Lambda;v_x,v_y)\otimes \mathcal{P}^{\mathcal{M}}(\mathbb{P}^1-\Lambda;v_y,v_z)\longrightarrow \mathcal{P}^{\mathcal{M}}(\mathbb{P}^1-\Lambda;v_x,v_z)$$

c) The Hodge and l-adic realizations of the motivic torsor of path are isomorphic to the ones defined in [D].

Proof. We use the lemma 3.4 from [G7] based on the injectivity of regulators to deduce the parts a) and b) from the corresponding statements in the Hodge setting. Then the Hodge part of c) is then given by the construction.

The Hodge realizations of the motive $H^1(P^1-\Lambda)$ are parametrized by the complex embeddings $\sigma:F\hookrightarrow\mathbb{C}$. The different realizations $H^1(\mathbb{C}P^1-\sigma_i(\Lambda))$

are canonically isomorphic to each other. The isomorphism identifies the basis elements $\omega_{\sigma_i(a)}$ and $\omega_{\sigma_j(a)}$. There is a collection of the Hodge-Tate structures

$$\mathcal{P}^{\mathcal{H}}(\mathbb{C}P^1 - \sigma(\Lambda); \sigma(v_x), \sigma(v_y)); \qquad \sigma : F \hookrightarrow \mathbb{C} \tag{111}$$

such that $\mathrm{Gr}^W_{\bullet}(-)$ are identified for different σ's. Each σ provides a homomorphism of the Hopf algebras $h_\sigma : \mathcal{A}_\bullet(F) \longrightarrow \mathcal{H}_\bullet$. The matrix elements $a(v, f)$ are compatible with the Hodge realization in the following sense.

Lemma 6.11 *For any complex embedding σ the element $h_\sigma(a(v, f))$ coincides with the matrix element of (111) corresponding to v and f (i.e. to the elements of $\mathrm{Gr}^W_{\bullet}(111)$ identified with v and f by the Hodge realization functor).*

Proof. Follows from theorem 6.4 in [G7]. The lemma is proved.

Using lemma 6.11 and lemma 3.4 from [G7] we deduce a) and b) from the corresponding fact in the Hodge realization.

Let us prove the l-adic part of c). In order to follow the described above scheme we need only the l-adic version of theorem 6.4 in [G7]. It has been proved in [G8], theorem 3.2c) where the category $\mathcal{C}$ is the category of mixed l-adic Galois modules, see s. 3.1, iii) loc.cit. The theorem is proved.

7 The motivic shuffle relations

We say that $(x_1, ..., x_m), (n_1, ..., n_m)$ are the parameters of $\mathrm{Li}_{n_1,...,n_m}(x_1, ..., x_m)$. The parameters are admissible if

$$x_m \neq 1 \quad \text{or} \quad n_m > 1 \tag{112}$$

For admissible parameters we have defined in [G7] a framed Hodge-Tate structure corresponding to multiple polylogarithms by

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_m}(x_1, ..., x_m) := (-1)^m \cdot \mathrm{I}^{\mathcal{H}}_{n_1,...,n_m}(a_1, a_2, ..., a_m) \tag{113}$$

$$a_1 := (x_1...x_m)^{-1}, a_2 := (x_2...x_m)^{-1}, ..., a_m := x_m^{-1}$$

where the right hand side is the framed Hodge-Tate structure corresponding to the iterated integral (3). Thanks to the results of previous chapters, e.g. theorem 6.4 and 6.5, in the case of admissible parameters the Hodge-Tate structure $\mathrm{I}^{\mathcal{H}}$ defined in [G7] via the specialization is equivalent to the one defined in chapter 2. In particular it is equivalent to a Hodge-Tate structure of geometric origin.

There are two different ways to extend the definition of the $\mathrm{Li}^{\mathcal{H}}$- and $\mathrm{I}^{\mathcal{H}}$-Hodge-Tate structures to the set of non admissible parameters, keeping the corresponding shuffle relations. In each case the definition is provided by the fiber at the tangent vector $\partial/\partial\varepsilon$ of the specialization functor applied to an appropriate variation of the framed multiple polylogarithm Hodge-Tate structures over a small punctured disc with a natural coordinate ε. For the $\mathrm{I}^{\mathcal{H}}$-elements such a procedure has been worked out in [G7], and called there the canonical regularization. For the $\mathrm{Li}^{\mathcal{H}}$-elements this is done in this section.

It is essentially obvious in the $\mathrm{I}^{\mathcal{H}}$-case, and follows from proposition 7.7 in the $\mathrm{Li}^{\mathcal{H}}$-case, that defined this way $\mathrm{I}^{\mathcal{H}}$- and $\mathrm{Li}^{\mathcal{H}}$-generators satisfy the corresponding shuffle relations for all parameters. However the basic formula (113) comparing the $\mathrm{I}^{\mathcal{H}}$- and $\mathrm{Li}^{\mathcal{H}}$-generators for admissible parameters has to be replaced by a more sophisticated relation - see theorem 7.8. The proof shows that the comparison formula from theorem 7.8 is uniquely determined if we want to keep the shuffle product formula for all parameters.

Our approach to regularization of the Li-generators is contained in the last page of [G3], see also s. 2.10 in [G7]. We show below that, using proposition 7.7, it is easily transformed to the Hodge or étale setting.

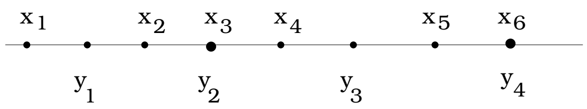
For the multiple ζ-numbers a different regularization of the Li-power series was considered by Zagier and documented by Ihara-Kaneko [IK]. A similar regularization was used by Ecalle, and by Boute de Monville. However it is not clear how to make this regularization Hodge theoretic or motivic.

For multiple polylogarithms, working on the level of numbers, it is not quite clear how even to formulate an explicit version of the comparison problem for two regularizations. Indeed, the values of multiple polylogarithms are not quite well defined numbers since the corresponding functions are multivalued.

1. The power series shuffle product formula. Let us define a generalized shuffle of the two ordered sets

$$\{x_1, ..., x_m\} \quad \text{and} \quad \{y_1, ..., y_n\} \tag{114}$$

Take a string of points on the real line, called slots, and mark every slot either by x_i, or by y_j, or by x_i and y_j, in such a way that x_a (resp. y_a) is on the left of x_b (resp. y_b) if $a < b$. The generalized shuffles are the combinatorial types of configurations of labelled slots obtained this way. An example is presented on the picture.



For instance if $m = n = 1$ there are three generalized shuffles: $(x_1; y_1)$, $(y_1; x_1)$ and (x_1, y_1). The last one has just one slot where both x_1 and y_1 sit, while in the first two there are two different slots.

Let $\overline{\Sigma}_{p,q}$ be the set of all generalized shuffles of the ordered sets $\{1, ..., p\}$ and $\{p+1, ..., p+q\}$. Set

$$\mathbb{Z}^p_{++} := \{(k_1, ..., k_p) \in \mathbb{Z}^p_+ \quad | \quad 0 < k_1 < ... < k_p\},$$

Then there is a natural decomposition of the product of cones $\mathbb{Z}^p_{++} \times \mathbb{Z}^q_{++}$ into a union of the cones Z^σ_{++} parametrized by the generalized shuffles:

$$\mathbb{Z}^p_{++} \times \mathbb{Z}^q_{++} = \cup_{\sigma \in \overline{\Sigma}_{p,q}} Z^\sigma_{++} \tag{115}$$

For example for $p = q = 1$ we have

$$\{k_1 > 0\} \times \{k_2 > 0\} = \{0 < k_1 < k_2\} \cup \{0 < k_1 = k_2\} \cup \{k_1 > k_2 > 0\}$$

For a generalized shuffle $\sigma \in \overline{\Sigma}_{p+q}$ consider the formal power series, convergent if $|x_i| < 1$:

$$\mathrm{Li}^\sigma_{n_1,\ldots,n_{p+q}}(x_1, ..., x_{p+q}) := \sum_{(k_1,\ldots,k_{p+q}) \in Z^\sigma_{++}} \frac{x_1^{k_1} ... x_{p+q}^{k_{p+q}}}{k_1^{n_1} ... k_{p+q}^{n_{p+q}}}$$

Then it follows from (115) that there is an equality of formal power series, the power series shuffle product formula:

$$\mathrm{Li}_{n_1,\ldots,n_p}(x_1, ..., x_p) \cdot \mathrm{Li}_{n_{p+1},\ldots,n_{p+q}}(x_{p+1}, ..., x_{p+q}) = \sum_{\sigma \in \overline{\Sigma}_{p,q}} \mathrm{Li}^\sigma_{n_1,\ldots,n_{p+q}}(x_1, ..., x_{p+q}) \tag{116}$$

Let $\sigma \in \overline{\Sigma}_{p,q}$ be a generalized shuffle. Denote by $\sigma(n_1, ..., n_{p+q})$ the sequence of integers obtained from $n_1, ..., n_{p+q}$ using σ and the following convention: the contribution of a slot of σ into $\sigma(n_1, ..., n_{p+q})$ is the sum of the integers n_i sitting at this slot. We define $\sigma(x_1, ..., x_{p+q})$ similarly using the convention that a slot of σ contributes the product of x_i's sitting at this slot. For example if σ is the generalized shuffle of the sets $\{1\}$ and $\{2\}$ with just one slot then $\mathrm{Li}_{\sigma(1,1)}(\sigma(x_1, x_2)) = \mathrm{Li}_2(x_1 x_2)$.

Using this convention we can write the right hand side of (116) as

$$\sum_{\sigma \in \overline{\Sigma}_{p,q}} \mathrm{Li}_{\sigma(n_1,\ldots,n_{p+q})}(\sigma(x_1, ..., x_{p+q})) \tag{117}$$

2. The category of unipotent variations of Hodge-Tate structures on $\mathbb{C}^*$. Recall that it is canonically equivalent to the category of graded finite dimensional comodules over the corresponding fundamental Hopf algebra, denoted $\mathcal{A}^{\mathcal{H}}_\bullet(\mathbb{A}^1 - \{0\})$, (see for example [G7] ch. 3). There is the canonical element

$$\log^{\mathcal{H}} \varepsilon \in \mathcal{A}^{\mathcal{H}}_1(\mathbb{A}^1 - \{0\}) = \mathrm{Ext}^1_{MHS(\mathbb{A}^1 - \{0\})}(\mathbb{Q}(0), \mathbb{Q}(1))$$

Lemma 7.1 *There is an isomorphism of graded commutative algebras*

$$\mathcal{A}^{\mathcal{H}}_\bullet(\mathbb{A}^1 - \{0\}) = \mathcal{H}_\bullet \otimes_{\mathbb{Q}} \mathbb{Q}[\log^{\mathcal{H}} \varepsilon]$$

Proof. Indeed, let X be a smooth complex algebraic variety and $L_{\bullet}^{\mathcal{H}}(X(\mathbb{C}))$ the fundamental Lie algebra of the category of unipotent variations of Hodge-Tate structures on $X(\mathbb{C})$. Then there is an exact sequence of Lie algebras

$$0 \longrightarrow L^{\mathrm{geom}}(X(\mathbb{C})) \longrightarrow L_{\bullet}^{\mathcal{H}}(X(\mathbb{C})) \longrightarrow L_{\bullet}^{\mathcal{H}}(\mathrm{Spec}(\mathbb{C})) \longrightarrow 0$$

where $L^{\mathrm{geom}}(X(\mathbb{C}))$ is the geometric fundamental Lie algebra. It is isomorphic to the Lie algebra provided by the pronilpotent completion of the topological fundamental group. Therefore in the case $X = \mathbb{G}_m$ we have $L^{\mathrm{geom}}(\mathbb{C}^*) = \mathbb{Q}(1)$. Applying the Poincare-Birkhoff-Witt theorem we get the lemma.

3. The framed Hodge-Tate structures corresponding to multiple polylogarithms. We want to define the framed Hodge-Tate structures $\mathrm{Li}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1, ..., x_m)$ for all x_i and all $n_i > 0$ so that they coincide with (113) for admissible parameters, and satisfy the shuffle relations (116)-(117) for all all parameters.

Lemma 7.2 *Let U^* be a punctured at zero disc. Suppose that $x_i(\varepsilon)$ are holomorphic for $\varepsilon \in U^*$, and $n_m > 1$ or $x_m(\varepsilon) \neq 1$ for small non zero ε. Then for a sufficiently small disc U^* the framed Hodge-Tate structures*

$$\mathrm{Li}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1(\varepsilon), ..., x_m(\varepsilon)); \quad \varepsilon \in U^* \tag{118}$$

are equivalent to fibers of a certain unipotent variation of Hodge-Tate structures over U^*.

Proof. This is a standard general fact. In our case it follows immediately from theorem 5.5 in [G7]. Indeed, translating the assertion of that theorem from the I$-$ to the Li$-$ notation we get the following. Say that $(x_1, ..., x_m)$ and $(x'_1, ..., x'_m)$ are of the same combinatorial type if $x_i...x_j = 1$ if and only if $x'_i...x'_j = 1$. Then the canonical (in the sense of [G7]) Hodge-Tate structures $\mathrm{Li}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1, ..., x_m)$ form a unipotent variation over the space of all configurations of points $(x_1, ..., x_m)$ of given combinatorial type. Making U^* smaller we can assume that $(x_1(\varepsilon), ..., x_m(\varepsilon))$ has the same combinatorial type for all $\varepsilon \in U^*$. The lemma is proved.

If $x_1...x_m \neq 0$, by lemma 7.2 there is a unipotent variation framed Hodge-Tate structures on a little punctured complex disc with a natural parameter ε whose fibers at points ε are

$$\mathrm{Li}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1(1-\varepsilon), ..., x_m(1-\varepsilon)) \tag{119}$$

Observe that the parameters of these elements for small $\varepsilon \neq 0$ are admissible. Its specialization at $\varepsilon = 0$ is a unipotent variation of framed Hodge-Tate structures on the punctured tangent space. It provides the element

$$\widehat{\mathrm{Li}}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1, ..., x_m) \in \mathcal{A}_{\bullet}^{\mathcal{H}}(\mathbb{A}^1 - \{0\})$$

The tangent space is equipped with coordinate ε. By lemma 7.1 we have

$$\widehat{\mathrm{Li}}^{\mathcal{H}}_{n_1,\dots,n_m}(x_1,...,x_m) = \sum_{k\geq 0} \mathrm{Li}^{\mathcal{H}}_{n_1,\dots,n_m}(x_1,...,x_m)_{(k)} \cdot (\log^{\mathcal{H}} \varepsilon)^k \tag{120}$$

We define $\mathrm{Li}^{\mathcal{H}}_{n_1,\dots,n_m}(x_1,...,x_m)$ as the constant term of this expression. We set

$$\widehat{\mathrm{Li}}^{\mathcal{H}}_{n_1,\dots,n_m}(x_1,...,x_m) = 0 \quad \text{if } x_1...x_m = 0$$

Proposition 7.3 *For admissible parameters we have*

$$\widehat{\mathrm{Li}}^{\mathcal{H}}_{n_1,\dots,n_m}(x_1,...,x_m) = \mathrm{Li}^{\mathcal{H}}_{n_1,\dots,n_m}(x_1,...,x_m)$$

Proof. Follows immediately from the specialization theorem 4.5.

Theorem 7.4 *For any $x_i \in \mathbb{C}$ and any positive integers n_i then there is a shuffle product formula*

$$\mathrm{Li}^{\mathcal{H}}_{n_1,\dots,n_p}(x_1,...,x_p) \cdot \mathrm{Li}^{\mathcal{H}}_{n_{p+1},\dots,n_{p+q}}(x_{p+1},...,x_{p+q}) = \tag{121}$$

$$\sum_{\sigma\in\overline{\Sigma}_{p,q}} \mathrm{Li}^{\mathcal{H}}_{\sigma(n_1,\dots,n_{p+q})}(\sigma(x_1,...,x_{p+q}))$$

There is a similar formula with $\mathrm{Li}^{\mathcal{H}}$ replaced by $\widehat{\mathrm{Li}}^{\mathcal{H}}$.

4. The strategy of the proof of theorem 7.4. The proof consists of two steps of different nature.

Step 1.

Theorem 7.5 *Suppose that the parameters $(x_1,...,x_m)$ of each of the terms in (121) satisfy the following condition:*

$$x_i \neq 0, \quad 1 \neq \{x_m^{-1}, \quad (x_{m-1}x_m)^{-1}, \quad ... \quad (x_1...x_m)^{-1}\} \tag{122}$$

Then the shuffle product formula (121) is valid.

We will prove this theorem in chapters 9 below. An essentially different proof will be outlined in chapter 11. Now let us assume theorem 7.5 and proceed to the proof of theorem 7.4.

Corollary 7.6 *The shuffle relations (121) are valid if all the terms in (121) are admissible.*

Proof. Suppose that parameters $(x_1,...,x_m)$ satisfy the condition of the corollary. One can find a little complex curve U passing through the point $(x_1,...,x_m)$ such that the corresponding punctured curve U^* lies in the domain determined by conditions (122). For sufficiently small U there are unipotent variations over U^* corresponding to the terms of (121), and by theorem 7.5 we have identity (121) for the corresponding framed variations. Taking specialization, and using the specialization theorem, we get the corollary. The corollary is proved.

Remark. To get corollary 7.6 it is sufficient to have formula (121) for any non empty Zariski open subset of the parameters set.

Step 2.

Proposition 7.7 *Suppose that $x_i(\varepsilon) \in \mathbb{C}^*$ are holomorphic for small ε, $x_{l+1}(0) = ... = x_m(0) = 1$, and $n_l > 1$ or $x_l(0) \neq 1$. Assume in addition that $x_m(\varepsilon) \neq 1$ for small non zero ε, or $n_m > 1$. Then*

$$[\mathrm{Sp}_{\varepsilon=0}]\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l,1,...,1}(x_1(\varepsilon), ..., x_l(\varepsilon), x_{l+1}(\varepsilon), ..., x_m(\varepsilon)) =$$

$$[\mathrm{Sp}_{\varepsilon=0}]\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l,1,...,1}(x_1(0), ..., x_l(0), x_{l+1}(\varepsilon), ..., x_m(\varepsilon))$$

Proof. Thanks to lemma 7.2 the use of the functor $\mathrm{Sp}_{\varepsilon=0}$ is legitimate.

We prove the proposition by induction on $k := m-l$. If $k = 0$ it follows from the specialization theorem 4.5. The induction step is deduced from theorem 7.5. Namely, consider the shuffle formula for the product

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l}(x_1(\varepsilon), ..., x_l(\varepsilon)) \cdot \mathrm{Li}^{\mathcal{H}}_{1,...,1}(x_{l+1}(\varepsilon), ..., x_m(\varepsilon)), \quad \varepsilon \neq 0 \tag{123}$$

Thanks to the condition in the proposition all the terms in the shuffle product formula are admissible for $\varepsilon \neq 0$, so the shuffle product formula is available by theorem 7.5. This formula is a sum of several terms. One of them is

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l,1,...,1}(x_1(\varepsilon), ..., x_l(\varepsilon), x_{l+1}(\varepsilon), ..., x_m(\varepsilon)) \tag{124}$$

In the rest of the terms $x_{l+1}(\varepsilon)$ stays strictly to the left of $x_l(\varepsilon)$, or appears in the variable $x_l(\varepsilon)x_{l+1}(\varepsilon)$ coupled to the index $n_{l+1} + 1$. For each of these terms our induction invariant k is down at least by one. Thus by the induction assumption applying $\mathrm{Sp}_{\varepsilon=0}$ we can replace all the variables on the left of $x_l(\varepsilon)$ by their values at $\varepsilon = 0$.

On the other hand, thanks to the condition $x_l(0) \neq 1$ or $n_l > 1$, the specialization theorem implies

$$[\mathrm{Sp}_{\varepsilon=0}]\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l}(x_1(\varepsilon), ..., x_l(\varepsilon)) = \mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l,}(x_1(0), ..., x_l(0))$$

So the product (123) equals to

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l}(x_1(0), ..., x_l(0))\mathrm{Li}^{\mathcal{H}}_{1,...,1}(x_{l+1}(\varepsilon), ..., x_m(\varepsilon))$$

One of the terms in the shuffle formula for this product is

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l,1,...,1}(x_1(0), ..., x_l(0), x_{l+1}(\varepsilon), ..., x_m(\varepsilon)) \tag{125}$$

By induction we already know that the specialization of the other terms equal to the specialization of the ones in the shuffle formula for (123). Thus

$$[\mathrm{Sp}_{\varepsilon=0}](124) = [\mathrm{Sp}_{\varepsilon=0}](125)$$

The proposition is proved.

Now we can finish the proof of theorem 7.4. The case when $x_1...x_m = 0$ is trivial since all the terms in (121) are zero. So we may assume $x_1...x_m \neq 0$. Then by theorem 7.5 we have the shuffle product formula

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_p}(x_1(1-\varepsilon),...,x_p(1-\varepsilon)) \cdot \mathrm{Li}^{\mathcal{H}}_{n_{p+1},...,n_{p+q}}(x_{p+1}(1-\varepsilon),...,x_{p+q}(1-\varepsilon)) =$$

$$\sum_{\sigma\in\overline{\Sigma}_{p,q}} \mathrm{Li}^{\mathcal{H}}_{\sigma(n_1,...,n_{p+q})}(\sigma(x_1(1-\varepsilon),...,x_{p+q}(1-\varepsilon))) \tag{126}$$

Applying to it the functor $[\mathrm{Sp}_{\varepsilon=0}]$ we get a valid formula. It remains to match the terms of that formula with the ones of (121). For the left hand side of (126), as well as for the terms in the right hand side of depth $p+q$, this is clear from the very definition. For the depth $< p+q$ terms in the right hand side this is deduced from proposition 7.7. Indeed, there exists a slot of a given generalized shuffle where two indices sit: n_i and n_j. Take the very right such a slot. Since $n_i + n_j > 1$ we may replace the corresponding variables $x_s(1-\varepsilon)^{a(s)}$ at this slot and all the slots to the left of it by x_s, as well as by $x_s(1-\varepsilon)$. Theorem 7.4 is proved assuming theorem 7.5.

5. The canonical specialization and the comparison theorem. Consider the unipotent variation framed Hodge-Tate structures over a small punctured complex disc with the coordinate ε, whose fiber at ε is

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_m}(x_1,...,x_{m-1},x_m(1-\varepsilon)) \tag{127}$$

Observe that

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_m}(x_1,...,x_{m-1},x_m(1-\varepsilon)) = \tag{128}$$

$$\mathrm{I}^{\mathcal{H}}_{n_1,...,n_m}(0;(x_1...x_m)^{-1},...,x_m^{-1};(1-\varepsilon))$$

So according to [G7] the fiber of its specialization at $\partial/\partial\varepsilon$ serves as the definition of the Hodge-Tate structure $\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_m}(x_1,...,x_m)$ for arbitrary set of parameters. This procedure was called in [G7] the canonical regularization.

The specialization of the family (127) provides the element

$$\widehat{\mathrm{I}}^{\mathcal{H}}_{n_1,...,n_m}(a_1,...,a_m) \in \mathcal{A}^{\mathcal{H}}_{\bullet}(\mathbb{A}^1 - \{0\})$$

Theorem 7.8 *For any $x_i \in \mathbb{C}$ one has*

$$\widehat{\mathrm{Li}}^{\mathcal{H}}_{n_1,...,n_m}(x_1,...,x_m) = \mathbb{L} \circ \widehat{\mathrm{I}}^{\mathcal{H}}_{n_1,...,n_m}(a_1,...,a_m)$$

Proof. Let us establish first the case when $n_i = 1$ and $x_i = 1$ for all i. Observe that (see [G7])

$$\widehat{\mathrm{Li}}^{\mathcal{H}}_{1,...,1,1}(\underbrace{1,...,1,1-\varepsilon}_{n}) = \frac{(\log^{\mathcal{H}}(\varepsilon))^n}{n!}$$

Lemma 7.9

$$\sum_{n\geq 0} \widehat{\mathrm{Li}}^{\mathcal{H}}_{1,...,1}(\underbrace{1,...,1}_{n}) \cdot u^n = \exp\left(-\sum_{n=1}^{\infty}(-1)^n \frac{\widehat{\zeta}^{\mathcal{H}}(n)}{n} u^n\right) \tag{129}$$

Proof. Applying $\frac{d}{du}$ to both parts of to (129) and replacing the exponential factor by the left hand side of (129) we get the following corollary of (129)

$$\sum_{k,l\geq 0} (-1)^k \widehat{\mathrm{Li}}^{\mathcal{H}}_{k+1}(1)\cdot \widehat{\mathrm{Li}}^{\mathcal{H}}_{1,...,1}(\underbrace{1,...,1}_{l})u^{k+l} \stackrel{?}{=} \sum_{m\geq 0}(m+1)\widehat{\mathrm{Li}}^{\mathcal{H}}_{1,...,1}(\underbrace{1,...,1}_{m+1})\cdot u^m \tag{130}$$

Arguing by induction on n we see that it is equivalent to the original identity (129). Indeed, all the terms in the exponential have degree at least 1, so we can use the induction assumption to replace the exp-term in the formula for $\frac{d}{du}$(the right hand side of (130)) by the left hand side of (129).

Formula (130) is equivalent to collection of identities, one for each m:

$$\sum_{k=0}^{m}(-1)^k \widehat{\mathrm{Li}}^{\mathcal{H}}_{k+1}(1) \cdot \widehat{\mathrm{Li}}^{\mathcal{H}}_{1,...,1}(\underbrace{1,...,1}_{m-k}) = (m+1)\widehat{\mathrm{Li}}^{\mathcal{H}}_{1,...,1}(\underbrace{1,...,1}_{m+1})$$

Writing the shuffle relations (available by theorem 7.4), for each term in the sum, and taking the sum, we get the last identity. The lemma is proved.

Examples. We have

$$\widehat{\mathrm{Li}}^{\mathcal{H}}_{1}(1) \cdot \widehat{\mathrm{Li}}^{\mathcal{H}}_{1}(1) - \widehat{\mathrm{Li}}^{\mathcal{H}}_{2}(1) = 2\widehat{\mathrm{Li}}^{\mathcal{H}}_{1,1}(1,1)$$

$$\widehat{\mathrm{Li}}^{\mathcal{H}}_{1}(1) \cdot \widehat{\mathrm{Li}}^{\mathcal{H}}_{1,1}(1,1) - \widehat{\mathrm{Li}}^{\mathcal{H}}_{2}(1)\widehat{\mathrm{Li}}^{\mathcal{H}}_{1}(1) + \widehat{\mathrm{Li}}^{\mathcal{H}}_{3}(1) = 3\widehat{\mathrm{Li}}^{\mathcal{H}}_{1,1,1}(1,1,1)$$

So

$$\widehat{\mathrm{Li}}^{\mathcal{H}}_{1,1}(1,1) = -\frac{\mathrm{Li}^{\mathcal{H}}_2(1)}{2} + \frac{(\log\varepsilon)^2}{2}$$

$$\widehat{\mathrm{Li}}^{\mathcal{H}}_{1,1,1}(1,1,1) = \frac{\mathrm{Li}^{\mathcal{H}}_3(1)}{3} + \frac{\mathrm{Li}^{\mathcal{H}}_2(1)\log\varepsilon}{2} - \frac{(\log\varepsilon)^3}{6}$$

Now let us treat the general case. We present $\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_m}(x_1,...,x_m)$ as

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l,1,...,1}(x_1,...,x_l,1,...,1)$$

where $n_l > 1$ or $x_l \neq 1$. We will prove the theorem by induction on $m-l$.

By proposition 7.7 $\widehat{\mathrm{Li}}^{\mathcal{H}}_{n_1,...,n_m}(x_1,...,x_m)$ is the specialization of the variation

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l,1,...,1}(x_1,...,x_l,1-\varepsilon,...,1-\varepsilon)$$

In particular this settles the case $m-l=0$. Lemma 7.9 just means that the theorem is true for $\mathrm{Li}^{\mathcal{H}}_{1,...,1}(1,...,1)$. Consider the shuffle product formulas

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l}(x_1,...,x_l,)\cdot \mathrm{Li}^{\mathcal{H}}_{1,...,1}(1-\varepsilon,...,1-\varepsilon) = \tag{131}$$

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l,1,...,1}(x_1,...,x_l,1-\varepsilon,...,1-\varepsilon) + \text{the rest of the terms} \tag{132}$$

as well as

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l}(x_1,...,x_l,)\cdot \mathrm{Li}^{\mathcal{H}}_{1,...,1}(1,...,1,1-\varepsilon) \tag{133}$$

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l,1,...,1}(x_1,...,x_l,1,...,1,1-\varepsilon) + \text{the rest of the terms} \tag{134}$$

Let us apply the $< \mathrm{Sp}_{\varepsilon=0} >$ functor to these identities. Thanks to the lemma we have (131) $= \mathbb{L} \circ$ (133). By the induction assumption we have

$$[\mathrm{Sp}_{\varepsilon=0}](\text{the rest of the terms in (132)}) =$$

$$\mathbb{L} \circ [\mathrm{Sp}_{\varepsilon=0}](\text{the rest of the terms in (134)})$$

It follows that

$$[\mathrm{Sp}_{\varepsilon=0}]\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l,1,...,1}(x_1,...,x_l,1-\varepsilon,...,1-\varepsilon) =$$

$$\mathbb{L} \circ [\mathrm{Sp}_{\varepsilon=0}]\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_l,1,...,1}(x_1,...,x_l,1,...,1,1-\varepsilon)$$

The theorem is proved.

Corollary 7.10 *There exists an explicit formula expressing* $\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_m}(x_1,...,x_m)$ *via* $\mathrm{Li}^{\mathcal{H}}_{*}(-)$ *with admissible parameters.*

Proof. Theorem 7.8 (and a shuffle product formula) provides a formula expressing $\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_m}(x_1,...,x_m)$ as $\mathbb{Q}$-linear combinations of $\mathrm{I}^{\mathcal{H}}_{*}(-)$ with not necessarily admissible parameters. Non admissible $\mathrm{I}^{\mathcal{H}}_{*}(-)$'s are expressed as linear combinations of admissible ones using explicit formulas from lemma 5.2 in [G8] (or formulas from proposition 2.14 and 2.15 in [G7]). It remains to use formula (113). The corollary follows.

6. The shuffle relations on the motivic level.

Theorem 7.11 *Let F be a number field. Suppose that $x_i \in F$. Then*

a) There exists a framed mixed Tate motive over F

$$\mathrm{Li}^{\mathcal{M}}_{n_1,...,n_m}(x_1,...,x_m) \in \mathcal{A}_w(F) \tag{135}$$

so that for any embedding $\sigma : F \hookrightarrow \mathbb{C}$ its Hodge realization coincides with

$$\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_m}(\sigma(x_1), ..., \sigma(x_m)) \in \mathcal{H}_w$$

b) There is a shuffle product formula

$$\mathrm{Li}^{\mathcal{M}}_{n_1,...,n_p}(x_1, ..., x_p) \cdot \mathrm{Li}^{\mathcal{M}}_{n_{p+1},...,n_{p+q}}(x_{p+1}, ..., x_{p+q}) = \tag{136}$$

$$\sum_{\sigma \in \overline{\Sigma}_{p,q}} \mathrm{Li}^{\mathcal{M}}_{\sigma(n_1,...,n_{p+q})}(\sigma(x_1, ..., x_{p+q}))$$

Proof. a) If the parameters of (135) are admissible the construction is given by the results of chapters 2 and 3. In particular its Hodge realization is the same as the one defined in [G7] thanks to the specialization theorem.

If the parameters are not admissible, we use the explicit formulas provided by corollary 7.10 (with $\mathrm{Li}^{\mathcal{H}}$ changed to $\mathrm{Li}^{\mathcal{M}}$), to define (135). The compatibility with the Hodge realization is then obvious.

b) By theorem 7.4 the shuffle product formula is valid on the Hodge level. Therefore a) and lemma 3.4 from [G7] provide (136). Theorem 7.11 is proved.

7. The shuffle relations in the l-adic setting.

Theorem 7.12 *Let F be a field and $\mu_{l^\infty} \notin F$. Suppose that $x_i \in F$. Then*

a) There exists a framed l-adic mixed Tate $\mathrm{Gal}(\overline{F}/F)$*-module*

$$\mathrm{Li}^{\mathrm{et}}_{n_1,...,n_m}(x_1, ..., x_m) \in \mathcal{A}^{\mathrm{et}}_w(F) \tag{137}$$

b) If F is a number field, the l-adic realization of (135) is given by (137).

c) The elements (137) satisfy the shuffle product formula (136).

Proof. If F is a number field we can apply the l-adic realization functor to the motivic objects from theorem 7.11. In particular this way we can settle the important case when x_i are roots of unity.

In general we proceed just as in the Hodge case, since the proofs of all the results we used there work also in the l-adic situation. More specifically:

a) If the parameters are admissible the part a) is given by the constructions of chapters 2 and 3. In general we define (137) by specialization, via the l-adic analog of the definition of $\widehat{\mathrm{Li}}$. The l-adic version of the specialization theorem 4.5 guarantees that these two approaches give the same result in the case of admissible parameters.

b) In the case of admissible parameters this is true thanks to the results of chapters 2 and 3. For non admissible parameters we work out the l-adic version of corollary 7.10, which follows from the l-adic version of theorem 7.8 and lemma 5.2 in [G8] applied in the l-adic setting, and use it just as we did in the Hodge case.

c) Just as in the Hodge case, it is deduced from the fact that it is true for generic parameters (see the part d) of theorem 9.9, which is the l-adic version of theorem 7.4) and specialization theorem 4.5. The theorem is proved.

8 How to prove identities between periods

Below we discuss two general approaches to prove identities between the framed objects. Then we apply each of them to prove theorem 7.5 in the double logarithm case

The first approach suggested below allows to reduce a proof of any functional equation between the framed mixed Tate objects, e.g. Hodge-Tate structures, to a routine calculation. This method is of algebraic nature, it proceeds by induction, and relies only on the formula for the coproduct of the corresponding framed objects, plus the fact that the identity is true at one particular point.

The second is the well known direct approach to the problem. Any relation between the framed mixed Tate structures of geometric origin is supposed to have a proof of this kind. However such a proof does require an inspiring guess. So, unlike the first method, it is not an algorithmic procedure.

Both methods work equally well in the Hodge or l-adic setting.

1. The Li-shuffle relations for the double logarithm mixed Tate objects. For $xy \neq 0; y \neq 1$ we follow [G7] and define the double logarithm framed Hodge-Tate structures by

$$\mathrm{Li}^{\mathcal{H}}_{1,1}(x,y) := \mathrm{I}^{\mathcal{H}}_{1,1}((xy)^{-1}, y^{-1})$$

They are fibers of a unipotent variation of framed Hodge-Tate structures over a bit smaller domain (see [G10] or [G7]):

$$\{(x,y) \subset \mathbb{C}^* \times \mathbb{C}^* | x \neq 1; y \neq 1; xy \neq 1\} \tag{138}$$

Theorem 8.1 *a) For any (x,y) from the domain (138) one has*

$$\mathrm{Li}^{\mathcal{H}}_{1}(x)\mathrm{Li}^{\mathcal{H}}_{1}(y) - (\mathrm{Li}^{\mathcal{H}}_{1,1}(x,y) + \mathrm{Li}^{\mathcal{H}}_{1,1}(y,x) + \mathrm{Li}^{\mathcal{H}}_{2}(xy)) = 0 \tag{139}$$

b) The l-adic version of this identity holds.

2. Two methods to prove identities between the periods. The first requires to introduce parameters, i.e. interpret the periods as special values of period functions arising from (unipotent) variations of mixed Hodge structures of geometric nature (e.g. multiple polylogarithms). Then we proceed as follows:

1). *Prove that the differential of the suspected identity is zero. So it is valid up to a constant. Specializing to a (degenerate) point check that it is zero.*

This method always works, but sometimes requires a lot of routine labor. The reason it is so efficient is provided by the following basic fact:

$$d(\text{weight } w \text{ unipotent period function }) =$$

$$\sum(\text{weight } w-1 \text{ unipotent period functions}) \cdot d\log f_i$$

where $f_i \in \mathbb{C}(X)^*$ are some rational functions. So taking a basis in $\mathbb{C}(X)^*$ and decomposing f_i in this basis we find out that one needs to prove some identities for the weight $w-1$ period functions. So we proceed by induction on the weight.

Example. To prove formula (7) we use

$$d\mathrm{Li}_{1,1}(x,y) = \log(1-xy)\cdot d\log\frac{x(1-y)}{1-x} + \log(1-y)\cdot d\log(1-x)$$

Then an easy algebraic calculation shows that the differential of (7) is zero. It remains to notice that formula (7) is obvious for $x=y=0$.

Remark. Of course formula (7) is obvious by the power series expansion, but as far as we know this approach has no apparent Hodge/motivic incarnation.

The second approach is this:

2). *Prove an identity between the periods by using identities of algebraic-geometric nature between between the corresponding differential forms and cycles, and the Stokes formula.*

This method does not require to interpret the periods as special values of some period functions.

The Hodge/motivic version of the first method is explained in section 8.3. The second method is manifestly motivic.

3. How to prove identities between the framed objects. The motivic version of the first method is based on the following rigidity lemma. We spell it first the Hodge case. Recall that Δ is the coproduct in the Hopf algebra $\mathcal{H}_\bullet$ of the framed Hodge-Tate structures, and $\Delta' := \Delta - (\mathrm{Id}\otimes 1 + 1\otimes \mathrm{Id})$ is the restricted coproduct. Similarly one can consider the Hopf algebra of unipotent variations of framed Hodge-Tate structures over a base S.

Lemma 8.2 *Suppose that H_S is a unipotent variation of Hodge-Tate structures over a connected manifold S framed by $\mathbb{Q}(0)$ and $\mathbb{Q}(n)$, where $n>1$. Then if Δ' kills H_S then H_S is equivalent to the constant variation over S representing an element of* $\mathrm{Ext}^1_{MHS/S}(\mathbb{Q}(0),\mathbb{Q}(n))$.

In particular if $H(s)$ is the fiber at a point $s\in S$ then it represents an element of $\mathrm{Ext}^1_{MHS}(\mathbb{Q}(0),\mathbb{Q}(n))$ which does not depend on s.

Proof. Recall that $\mathrm{Ker}\Delta' \subset \mathcal{H}_n$ is identified with

$$\mathrm{Ext}^1_{MHS}(\mathbb{Q}(0),\mathbb{Q}(n)) = \frac{\mathbb{C}}{(2\pi i\mathbb{Q})^n} \subset \mathcal{H}_n$$

The Griffiths transversality condition implies that for $n>1$, and a connected smooth base S, one has

$$\mathrm{Ext}^1_{MHS}(\mathbb{Q}(0),\mathbb{Q}(n)) = \mathrm{Ext}^1_{MHS/S}(\mathbb{Q}(0),\mathbb{Q}(n))$$

The lemma is proved.

A general method to prove that an element $\sum_i H_i(s)\in\mathcal{H}_n$ is zero:

1) Find a variation of framed Hodge-Tate structures over a smooth connected base S whose fiber at $s \in S$ is equivalent to $\sum_i H_i(s)$.

2) Check that $\Delta'(\sum_i H_i(s)) = 0$ for all $s \in S$.

3) Prove that it is zero at a single point $s_0 \in S$, or that specialization of the underlying variation at a certain tangential base point at infinity is zero.

Thanks to lemma 8.2 the conditions 1) and 2) guarantee that our variation is constant, and 3) implies that it is zero.

In the l-adic case there is a similar rigidity lemma

Lemma 8.3 *Suppose that L_S is a lisse étale $\mathbb{Q}_l$-mixed Tate sheaf on a connected scheme S, $\mu_{l^\infty} \notin \mathcal{O}_S^*$, framed by $\mathbb{Q}(0)$ and $\mathbb{Q}(n)$, where $n > 1$. Then if $\Delta' L_S = 0$ then L_S is equivalent to the constant variation over S representing an element of* $\mathrm{Ext}^1_S(\mathbb{Q}_l(0), \mathbb{Q}_l(n))$.

Proof. Recall the following rigidity property ([BD], s. 1.7): for $n > 1$ restriction to any point $x : \mathrm{Spec}F \hookrightarrow S$ provides an isomorphism

$$\mathrm{Ext}^1_S(\mathbb{Q}_l(0), \mathbb{Q}_l(n)) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Spec}F}(\mathbb{Q}_l(0), \mathbb{Q}_l(n)) \qquad (140)$$

Since $\mathrm{Ker}\Delta' \cap \mathcal{A}_n^{\mathrm{et}}(X)$ is identified with $\mathrm{Ext}^1_S(\mathbb{Q}_l(0), \mathbb{Q}_l(n))$, the lemma is proved.

4. The first proof of theorem 8.1. We apply the method above to the left hand side of (139). Recall the following formula for the restricted coproduct, see proposition 2.3 in [G10] or ch. 6 in [G8]:

$$\Delta' \mathrm{Li}^{\mathcal{H}}_{1,1}(x,y) = (1-xy) \otimes \frac{x(1-y)}{1-x} + (1-y) \otimes (1-x) \in \mathbb{C}^*_{\mathbb{Q}} \otimes \mathbb{C}^*_{\mathbb{Q}} = \mathcal{H}_1^{\otimes 2}$$

Here $\mathbb{C}^*_{\mathbb{Q}} = \mathbb{C}^* \otimes \mathbb{Q}$, and the last isomorphism is provided by the canonical isomorphisms $\mathcal{H}_1 = \mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), \mathbb{Q}(1)) = \mathbb{C}^*_{\mathbb{Q}}$. Therefore

$$\Delta'(\mathrm{Li}^{\mathcal{H}}_{1,1}(x,y) + \mathrm{Li}^{\mathcal{H}}_{1,1}(y,x)) = (1-xy) \otimes xy + (1-y) \otimes (1-x) + (1-x) \otimes (1-y)$$

and, since $\mathrm{Li}^{\mathcal{H}}_1(z) = -(1-z) \in \mathbb{C}^*_{\mathbb{Q}} = \mathcal{H}_1$,

$$\Delta'(-\mathrm{Li}^{\mathcal{H}}_2(xy) + \mathrm{Li}^{\mathcal{H}}_1(x)\mathrm{Li}^{\mathcal{H}}_1(y)) = (1-xy) \otimes xy + (1-y) \otimes (1-x) + (1-x) \otimes (1-y)$$

Thus Δ' kills the left hand side of (139). Each of the objects from the left hand side of (139) is a fiber of a certain unipotent variation of Hodge-Tate structures over the space (138), framed by $\mathbb{Q}(0)$ and $\mathbb{Q}(2)$. Taking the specialization of the left hand side of (139) at $x = 0$, and after this at $y = 0$ we see that each term in (139) specializes to zero. This implies the Hodge version of the theorem. Therefore we have the motivic version of this result when x, y are from a number field. The l-adic case now follows from lemma 8.3. Theorem 8.1 is proved.

5. The second proof of theorem 8.1. A peculiar property of formula (139) is this. Even for generic x, y, and even on the level of framed Hodge-Tate structures, one of the terms in the formula, $\mathrm{Li}^{\mathcal{H}}_2(xy)$, was defined in [G7] not directly, but using the specialization (= canonical regularization), which is a rather sofisticated functor from the algebraic geometric point of view. So the first thing we might want to do is to use a model where a geometric definition of $\mathrm{Li}_2^{\mathcal{H}}(xy)$ is available. Here is how it works.

Let us blow up the point $(0,0)$ at the (t_1, t_2) plane. The natural coordinates on the blow up are (u_1, u_2) such that $t_1 = u_1 u_2$ and $t_2 = u_2$. The iterated integral for $\mathrm{Li}_{1,1}(x, y)$ on the blow up looks as follows.

$$\mathrm{Li}_{1,1}(x,y) = \int_{0 \le t_1 \le t_2 \le 1} \frac{dt_1}{t_1 - (x_1x_2)^{-1}} \wedge \frac{dt_2}{t_2 - x_2^{-1}} =$$

$$\int_{0 \le u_i \le 1} \frac{d(u_1u_2)}{u_1u_2 - (x_1x_2)^{-1}} \wedge \frac{du_2}{u_2 - x_2^{-1}}$$

Introduce new variables $v_i = x_i u_i$ we arrive to the identity

$$\mathrm{Li}_{1,1}(x,y) = \int_{0 \le v_i \le x_i} \frac{d(v_1v_2)}{v_1v_2 - 1} \wedge \frac{dv_2}{v_2 - 1}$$

Similarly

$$\mathrm{Li}_{1,1}(y,x) = \int_{0 \le v_i \le x_i} \frac{d(v_1v_2)}{v_1v_2 - 1} \wedge \frac{dv_1}{v_1 - 1}$$

and

$$\mathrm{Li}_2(xy) = \int_{0 \le v_i \le x_i} \frac{d(v_1v_2)}{v_1v_2 - 1} \wedge \frac{dv_2}{v_2} = \int_{0 \le v_i \le x_i} \frac{dv_1 \wedge dv_2}{v_1v_2 - 1}$$

On the other hand

$$\mathrm{Li}_1(x)\mathrm{Li}_1(y) = \int_{0 \le v_i \le x_i} \frac{dv_1}{v_1 - 1} \wedge \frac{dv_2}{v_2 - 1}$$

One has an equality of rational differential forms

$$\frac{dv_1}{1-v_1} \wedge \frac{dv_2}{1-v_2} \quad = \quad \frac{d(v_1v_2)}{1-v_1v_2} \wedge \frac{dv_2}{1-v_2} - \frac{d(v_1v_2)}{1-v_1v_2} \wedge \frac{dv_1}{1-v_1} + \frac{dv_1 \wedge dv_2}{1-v_1v_2} \qquad (141)$$

To check it one can develop the denominators in the left and right hand sides into the geometric progression

$$\sum_{0<k_1,k_2} v_1^{k_1} v_2^{k_2} \cdot dv_1 \wedge dv_2 = \Big(\sum_{0<k_1<k_2} + \sum_{k_1>k_2>0} + \sum_{k_1=k_2>0} \Big) v_1^{k_1} v_2^{k_2} \cdot dv_1 \wedge dv_2$$

Let us interpret each of the integrals above as periods of appropriate framed Hodge-Tate structures.

Lifting the real triangle $0 \le t_1 \le t_2 \le 1$ on the blow up we get a square $0 \le u_i \le 1$ denoted C_2. It is just the same as the rectangle $0 \le v_i \le x_i$.

Let $\mathbb{U}_{(2)}$ be the affine chart on the blow up where the coordinates (u_1, u_2) are defined. The square C_2 lies inside of $\mathbb{U}_{(2)}(\mathbb{R})$. Each of the four double integrals above has the form

$$\int_{C_2} \Omega \tag{142}$$

where Ω is one of the 2-forms from (141).

Denote by $C^*_{(2)}$ the restriction to $\mathbb{U}_{(2)}$ of the preimage of the algebraic triangle $B := \{0 = t_1\} \cup \{t_1 = t_2\} \cup \{t_2 = 1\}$. Thus the square provides a generator

$$[C_2] \in H_2(\mathbb{U}_{(2)}(\mathbb{C}), C^*_{(2)}(\mathbb{C}))$$

Let $\mathrm{Sing}(\Omega)$ be the singularity divisor of Ω. Then if $y \neq 1$ and Ω is one of the forms in (141) then $\mathrm{Sing}(\Omega) \cup C^*_{(2)}$ is a normal crossing divisor on $\mathbb{U}_2$.

Remark. For generic x, y a similar pair of divisors on the (t_1, t_2) plane is a normal crossing divisor for $\mathrm{Li}_{1,1}(x, y)$, but not for $\mathrm{Li}_2(xy)$. We had to make the blow up to get a normal crossing divisor for $\mathrm{Li}_2(xy)$.

So setting

$$C^*_{(2),\Omega} := C^*_{(2)} - C^*_{(2)} \cap \mathrm{Sing}(\Omega)$$

we see that integral (142) is a period of the framed Hodge-Tate structure of geometric origin

$$\Big(H^2(\mathbb{U}_{(2)} - \mathrm{Sing}(\Omega), C^*_{(2),\Omega}), [C_2], [\Omega]\Big)$$

It is equivalent (and in fact isomorphic) to the corresponding term in 8.1. Replacing $\mathrm{Sing}(\Omega)$ in $H^2(\mathbb{U}_{(2)} - \mathrm{Sing}(\Omega), C^*_{(2),\Omega})$ by the union of such divisors for all four 2-forms appearing in (141), and cutting down $C^*_{(2),\Omega}$ accordingly, we get a framed object equivalent to the one above. Applying to these objects identity (141) and lemma 5.3 we get theorem 8.1 in the Hodge setting, and hence, as explained in the end of s. 8.3, in the motivic and finally l-adic settings.

6. Calculation of $\mathrm{Li}^{\mathcal{H}}_{1,1}(x, 1)$. By proposition 7.7 for $x \neq 1$ we have

$$\mathrm{Li}^{\mathcal{H}}_{1,1}(x, 1) := \mathrm{Sp}_{\varepsilon \to 0} \mathrm{Li}^{\mathcal{H}}_{1,1}(x, 1 - \varepsilon)$$

If $x \neq 1$ applying the $\mathrm{Sp}_{\varepsilon \to 0}$ functor to

$$\mathrm{Li}^{\mathcal{H}}_1(1-\varepsilon)\mathrm{Li}^{\mathcal{H}}_1(x) = \mathrm{Li}^{\mathcal{H}}_{1,1}(1-\varepsilon, x) + \mathrm{Li}^{\mathcal{H}}_{1,1}(x, 1-\varepsilon) + \mathrm{Li}^{\mathcal{H}}_2((1-\varepsilon)x)$$

we get

$$\widetilde{\mathrm{Li}}^{\mathcal{H}}_{1,1}(x, 1) = -\mathrm{Li}^{\mathcal{H}}_{1,1}(1, x) + \mathrm{Li}^{\mathcal{H}}_2(x)$$

This identity (as we already know) remains valid fop $x = 1$.

9 A proof of theorem 7.5

In this chapter we give an algebraic-geometric proof of the shuffle product formula from theorem 7.5. The subtlety of this problem is explained by the following. Even for generic x_i, and even on the Hodge level, the depth $< p+q$ terms in (121) were defined in [G7] by using the specialization functor, which is a rather sofisticated functor from the motivic point of view. So we wanted to find a model where a direct geometric definition of all terms is available. Below we use a sequence of blow ups which transforms the algebraic simplex to the algebraic cube. After this we use arguments similar to the one used in chapter 5.

1. The set up. Let t be the canonical coordinate on $X := \mathbb{A}^1$, and and $t_1, ..., t_n$ the coordinates on X^n. Recall the standard algebraic simplex B in X^n:

$$B := \quad \{0 = t_1\} \cup \{t_1 = t_2\} \cup \{t_2 = t_3\} \cup ... \cup \{t_{n-1} = t_n\} \cup \{t_n = 1\} \tag{143}$$

Let us use the flag of subvarieties

$$\{0 = t_1 = ... = t_n\} \subset \{0 = t_1 = ... = t_{n-1}\} \subset ... \subset \{0 = t_1 = t_2\}$$

of dimensions $0, 1, ..., n-2$ to construct a sequence of blow ups of X^n. Namely, blow up the zero dimensional subvariety on X^n, then blow up the strict preimage of the one dimensional subvariety of the flag, then blow up the strict preimage of the two dimensional subvariety of the flag, and so on. On the final stage we get a variety denoted $Y_{(n)}$. The preimage of the divisor B in $Y_{(n)}$ is denoted $C_{(n)}$.

Lemma 9.1 *The divisor $C_{(n)}$ has a shape of the n-dimensional cube.*

Proof. After the blow up of the point $\{0 = t_1 = ... = t_n\}$ the preimage of D has the shape of $\Delta^1 \times \Delta^{n-1}$. Then one proceeds by induction.

Another way to see the cube is this. By the very definition of the blow up the divisor $C_{(n)}$ has a shape of the polytop obtained as follows. Take the standard simplex

$$\Delta^n = \{0 \leq t_1 \leq ... \leq t_n \leq 1\} \subset \mathbb{R}^n \tag{144}$$

and cut out the vertex $(0, ..., 0)$ by a hyperplane $t_n = \varepsilon$, $\varepsilon > 0$, then (the remaining of) the edge $(0, ..., 0, t)$ by $t_{n-1} = \varepsilon$, then cut out the 2-face $(0, ..., 0, t, t)$ and so on. The polytop we get is a cube. The lemma is proved.

There is a natural coordinate system $(u_1, ..., u_n)$ on $Y_{(n)}$ such that

$$t_1 = u_1 ... u_n; \quad t_2 = u_2 ... u_n; \quad ... \quad ; t_{n-1} = u_{n-1} u_n; \quad t_n = u_n \tag{145}$$

so that

$$u_1 = \frac{t_1}{t_2}, \quad u_2 = \frac{t_2}{t_3}, \quad ... \quad u_{n-1} = \frac{t_{n-1}}{t_n}, \quad u_n = t_n$$

Then the standard simplex (144) is transformed to the standard cube

$$C_n = \{(u_1, ..., u_n) \subset \mathbb{R}^n \quad | \quad 0 \le u_i \le 1\}$$

Let $w := n_1 + ... + n_m$. Let us write integral representation given by (3) - (4) on the blow up $Y_{(w)}$ using the variables u_i. Then introduce another variables v_j:

$$v_j := \begin{cases} u_j & \text{if } j \neq n_1 + ... + n_k \quad \text{for some } k \\ x_p u_{n_1 + ... + n_k} & \text{if } j = n_1 + ... + n_k \quad \text{for some } k \end{cases}$$

The cube C_w looks in this coordinates as the cube

$$C_{n_1, ..., n_m}(x_1, ..., x_m) := \tag{146}$$

$$\{(v_1, ..., v_w) \quad | \quad 0 \le v_{n_1 + ... + n_p} \le x_p, \quad 0 \le v_j \le 1 \quad \text{otherwise}\}$$

Let us split the sequence of variables $v_1, ..., v_w$ into m consecutive segments of lengths $n_1, ..., n_m$:

$$S_1 := \underbrace{v_1, ..., v_{n_1}}_{n_1 \text{ terms}}; \quad S_2 := \underbrace{v_{n_1+1}, ..., v_{n_1+n_2}}_{n_2 \text{ terms}}; \quad ... \quad ; S_w := \underbrace{v_{w-n_m+1}, ..., v_w}_{n_m \text{ terms}} \tag{147}$$

and, using this splitting, define the following differential form on $Y_{(w)}$:

$$\Omega_{n_1, ..., n_m}(x_1, ..., x_m) :=$$

$$\underbrace{\frac{d(v_1 ... v_w)}{1 - v_1 ... v_w} \wedge \frac{d(v_2 ... v_w)}{v_2 ... v_w} \wedge ... \wedge \frac{d(v_{n_1} ... v_w)}{v_{n_1} ... v_w}}_{n_1 \text{ factors}} \wedge ... \wedge \tag{148}$$

$$\underbrace{\frac{d(v_{w-n_m+1} \cdot ... \cdot v_w)}{1 - v_{w-n_m+1} \cdot ... \cdot v_w} \wedge ... \wedge \frac{d(v_{w-1} v_w)}{v_{w-1} v_w} \wedge \frac{dv_w}{v_w}}_{n_m \text{ factors}}$$

For example

$$\Omega_{2,1}(x_1, x_2) := \frac{d(v_1 v_2 v_3)}{1 - v_1 v_2 v_3} \wedge \frac{d(v_2 v_3)}{v_2 v_3} \wedge \frac{dv_3}{v_3}$$

Let us stress that it indeed depends on the parameters x_i since the variables v_j were defined using x_j's and the natural coordinates u_j on $Y_{(w)}$. On the other hand the cube (146) on $Y_{(w)}$ does not depend on x_i's.

To clarify the algebraic nature of this differential form recall the map

$$d \log^{\wedge n} : \Lambda^n F(Y)^* \longrightarrow \Omega^n_{\log}(F(Y)) \tag{149}$$

$$f_1 \wedge ... \wedge f_n \longmapsto d \log(f_1) \wedge ... \wedge d \log(f_n)$$

The form (148) is obtained by applying such a map to an element

$$F_{n_1,...,n_m}(x_1,...,x_m) \in \Lambda^w F(Y_{(w)})^*; \qquad F = \mathbb{Q}(x_1,...,x_m)$$

Precisely, $F_{n_1,...,n_m}(x_1,...,x_m) = (-1)^w f_1 \wedge ... \wedge f_w$ where

$$f_j := \begin{cases} \prod_{i \geq j} v_i & \text{if } j \neq n_1 + ... + n_k + 1 \quad \text{for some } k \\ 1 - \prod_{i \geq j} v_i & \text{if } j = n_1 + ... + n_k + 1 \quad \text{for some } k \end{cases} \tag{150}$$

For example

$$F_{2,1}(x_1,x_2) = -(1 - v_1v_2v_3) \wedge (v_2v_3) \wedge v_3$$

Lemma 9.2 *One has*

$$\mathrm{Li}_{n_1,...,n_m}(x_1,...,x_m) = \int_{C_w} \Omega_{n_1,...,n_m}(x_1,...,x_m)$$

Proof. Here is how it works in the simplest cases. For $\mathrm{Li}_{1,1}(x,y)$ see ch. 8.

Example 1. In the case of $\mathrm{Li}_{2,1}(x,y)$ the computation looks as follows:

$$\mathrm{Li}_{2,1}(x,y) = (-1)^2 \int \frac{d(u_1u_2u_3)}{u_1u_2u_3 - (xy)^{-1}} \wedge \frac{d(u_2u_3)}{u_2u_3} \wedge \frac{du_3}{u_3 - y^{-1}}; \qquad 0 \leq u_i \leq 1$$

Changing the variables $v_1 = xu_1, v_2 = u_2, v_3 = yu_3$ we get

$$\int_{C_{2,1}(x,y)} \frac{d(v_1v_2v_3)}{1 - v_1v_2v_3} \wedge \frac{d(v_2v_3)}{v_2v_3} \wedge \frac{dv_3}{1 - v_3} \quad = \quad \int_{C_{2,1}(x,y)} \frac{v_3dv_1 \wedge dv_2 \wedge dv_3}{(1 - v_1v_2v_3)(1 - v_3)}$$

Example 2. For $\mathrm{Li}_3(x)$ one has:

$$\mathrm{Li}_3(x) = -\int \frac{d(u_1u_2u_3)}{u_1u_2u_3 - x^{-1}} \wedge \frac{d(u_2u_3)}{u_2u_3} \wedge \frac{du_3}{u_3}; \qquad 0 \leq u_i \leq 1$$

Changing the variables $v_1 = u_1x, v_2 = u_2, v_3 = u_3$ we get

$$\int_{C_3(x)} \frac{d(v_1v_2v_3)}{1 - v_1v_2v_3} \wedge \frac{d(v_2v_3)}{v_2v_3} \wedge \frac{dv_3}{v_3} \quad = \quad \int_{C_3(x)} \frac{dv_1 \wedge dv_2 \wedge dv_3}{1 - v_1v_2v_3}$$

In general the proof follows the same pattern. The lemma is proved.

2. A framed mixed Tate motive corresponding to multiple polylogarithms with general arguments. Let $\mathbb{U}_{(w)}$ be the affine chart of $Y_{(w)}$ were the coordinates $u_1, ..., u_w$ are defined. So there is an isomorphism $(u_1, ..., u_w) : \mathbb{U}_{(w)} \xrightarrow{=} \mathbb{A}^w$. Observe that $C_w \subset \mathbb{U}_{(w)}(\mathbb{R})$.

Let Ω be a differential w-form on $\mathbb{U}_w$ with logarithmic singularities at the divisor $\mathrm{Sing}(\Omega) \subset \mathbb{U}_w$. We will assume that Ω is in the image $d\log^{\wedge n}$ map, see (149). In particular it is of weight $2w$. Let us suppose in addition that

$$\mathrm{Sing}(\Omega) \text{ does not contain any face of the algebraic cube } C_{(w)} \tag{151}$$

We set

$$C^*_{(w)} := C_{(w)} \cap \mathbb{U}_w; \quad C^*_{(w),\Omega} := C^*_{(w)} - (C^*_{(w)} \cap \mathrm{Sing}(\Omega))$$

Then there is a mixed motive

$$H^w(\mathbb{U}_{(w)} - \mathrm{Sing}(\Omega), C^*_{(w),\Omega}) \tag{152}$$

as well as its Hodge and l-adic realizations.

Lemma 9.3 *Let us assume (122). Then the form*

$$\Omega := \Omega_{n_1,\ldots,n_m}(x_1, ..., x_m) \tag{153}$$

satisfies condition (155).

Proof. It is sufficient to prove that $\mathrm{Sing}(\Omega)$ does not contain any vertex of the cube, which is clear from a computation in the coordinates u_i.

Thanks to lemma 9.3 the object (152) is canonically isomorphic to $H^0(\mathbb{U}_{(w)}, \mathcal{F}^*_{A,B})$ for $A := \mathrm{Sing}(\Omega)$ and $B := C_{(w)}$. If x_i are elements of a number field satisfying (122) then (152) is defined as a mixed Tate motive.

The constructions of chapter 2 provide a natural frame on the object (152). Its weight $2w$ component is provided by the cohomology class $[\Omega]$. The weight 0 component is provided by the relative homology class

$$[C_w] \in H_w(\mathbb{U}_{(w)}, C^*_{(w)}) \tag{154}$$

of the cube C_w.

Proposition 9.4 *Let us assume (122). Suppose that Ω is given by (153). Then there is an equivalence of framed Hodge-Tate structures*

$$\mathrm{Li}^{\mathcal{H}}_{n_1,\ldots,n_m}(x_1, ..., x_m) \sim \Big(H^w(\mathbb{U}_{(w)} - \mathrm{Sing}(\Omega), C^*_{(w),\Omega}), [C_w], [\Omega]\Big)$$

If x_i are elements of a number field then there is a similar equivalence of framed mixed Tate motives. If $x_i \in F$ and $\mu_{l^\infty} \neq F$ then there is a similar statement in the l-adic case.

Proof. This is a very particular case of the situation considered in chapters 2 and 3. The proposition is proved.

3. Some algebraic identities between the differential forms. Set

$$s_j := \prod_{v_i \in S_j} v_i = \prod_{n_j \leq i \leq n_{j+1}-1} v_i \quad = \quad x_j \prod_{n_j \leq i \leq n_{j+1}-1} u_i \tag{155}$$

Lemma 9.5 *One has an identity of formal power series*

$$\Omega_{n_1,\ldots,n_m}(x_1, ..., x_m) := \Big(\sum_{0<k_1<\ldots<k_m} s_1^{k_1} ... s_m^{k_m}\Big) \cdot dv_1 \wedge ... \wedge dv_w$$

The right hand side is convergent if $x_i \in \mathbb{C}$ and $|x_i| < 1$.

Proof. One has

$$\Omega_{n_1,...,n_m}(x_1,...,x_m) =$$
$$\frac{\prod_{j=2}^m s_j}{1-\prod_{j=1}^m s_j}\cdot\frac{\prod_{j=3}^m s_j}{1-\prod_{j=2}^m s_j}\cdot ... \cdot\frac{1}{1-s_j}\cdot dv_1\wedge ...\wedge dv_w =$$
$$\Big(\sum_{0<k_1<...<k_m} s_1^{k_1}...s_m^{k_m}\Big)\cdot dv_1\wedge ...\wedge dv_w$$

The power series are convergent on the cube C_w if $|x_i| < 1$. The lemma is proved.

Let $n_1, ..., n_{p+q}$ be positive integers and $w = n_1 + ... + n_{p+q}$. For a generalized shuffle $\sigma \in \overline{\Sigma}_{p,q}$ consider the form with formal power series coefficients

$$\Omega^{\sigma}_{n_1,...,n_{p+q}}(x_1,...,x_{p+q}) := \Big(\sum_{(k_1,...,k_{p+q})\in Z^{\sigma}_{++}} s_1^{k_1}...s_{p+q}^{k_{p+q}}\Big)\cdot dv_1\wedge ...\wedge dv_w \quad (156)$$

This power series are convergent on the cube C_w for $|x_i| < 1$.

Let $w_1 = n_1 + ... + n_p$ and $w_2 = n_{p+1} + ... + n_{p+q}$. Consider the natural projections $p_i : \mathbb{U}_{(w_1+w_2)} \longrightarrow \mathbb{U}_{(w_i)}$ given by

$$p_1 : (v_1, ..., v_{w_1+w_2}) \longrightarrow (v_1, ..., v_{w_1}); \quad p_2 : (v_1, ..., v_{w_1+w_2}) \longrightarrow (v_{w_1+1}, ..., v_{w_1+w_2})$$

Then

$$p_1^*\Omega_{n_1,...,n_p}(x_1,...,x_p) = \Big(\sum_{0<k_1<...<k_p} s_1^{k_1}...s_p^{k_p}\Big)\cdot dv_1\wedge ...\wedge dv_{w_1} \quad (157)$$

and

$$p_2^*\Omega_{n_{p+1},...,n_{p+q}}(x_{p+1},...,x_{p+q}) = \quad (158)$$
$$\Big(\sum_{0<k_{p+1}<...<k_{p+q}} s_{p+1}^{k_{p+1}}...s_{p+q}^{k_{p+q}}\Big)\cdot dv_{w_1+1}\wedge ...\wedge dv_{w_1+w_2}$$

Lemma 9.6

$$p_1^*\Omega_{n_1,...,n_p}(x_1,...,x_p)\wedge p_2^*\Omega_{n_{p+1},...,n_{p+q}}(x_{p+1},...,x_{p+q}) =$$
$$\sum_{\sigma\in\overline{\Sigma}_{p,q}}\Omega^{\sigma}_{n_1,...,n_{p+q}}(x_1,...,x_{p+q})$$

Proof. This is obviously true on the formal power series level thanks to decomposition (115). The lemma is proved.

Let us define forms (156) as rational forms with logarithmic singularities on $Y_{(w)}$. Just as in the definition of the form (147), given positive integers

$n_1, ..., n_{p+q}$ we split the sequence of variables $v_1, ..., v_w$ into $p+q$ consecutive segments of lengths $n_1, ..., n_{p+q}$:

$$S_1 := \underbrace{v_1, ..., v_{n_1}}_{n_1 \text{ terms}}; \quad ... \quad ; \quad S_p := \underbrace{v_{n_1+...+n_{p-1}+1}, ..., v_{n_1+...+n_p}}_{n_p \text{ terms}} \tag{159}$$

$$S_{p+1} := \underbrace{v_{n_1+...+n_p+1}, ..., v_{n_1+...+n_{p+1}}}_{n_{p+1} \text{ terms}}; \quad ... \quad ; \quad S_{p+q} := \underbrace{v_{w-n_{p+q}+1}, ..., v_w}_{n_{p+q} \text{ terms}} \tag{160}$$

Given a generalized shuffle $\sigma \in \overline{\Sigma}_{p,q}$ we define a new sequence of segments by shuffling, according to σ, the segments (159) and (160). If a segment S_i from (159) and a segment S_j from (160) correspond to the same slot of σ we put S_iS_j as a new segment corresponding to this slot. The new sequence of segments determines a new ordered sequence $(v_1', ..., v_w')$ of v_i's. Denote by $\widetilde{\sigma}$ the permutation $(v_1, ..., v_w) \longrightarrow (v_1', ..., v_w')$.

Let us cook up the differential form corresponding to the new sequence of segments following the rule (147), and multiply it by $(-1)^{|\widetilde{\sigma}|}$. It is easy to see that the form we get admits a power series decomposition identical with the right hand side of (156). Thus

$$\Omega^{\sigma}_{n_1,...,n_{p+q}}(x_1, ..., x_{p+q}) = d\log^{\wedge w}(F^{\sigma}_{n_1,...,n_{p+q}}(x_1, ..., x_{p+q}))$$

for an element $F^{\sigma}_{n_1,...,n_{p+q}}(x_1, ..., x_{p+q}) = (-1)^{|\widetilde{\sigma}|} f_1^{\sigma} \wedge ... \wedge f_w^{\sigma}$ defined via the rule (150) applied to the new sequence of segments.

Lemma 9.7 *The formula in lemma 9.6 remains valid for the corresponding rational differential forms.*

Proof. Clear.

Remark. One can prove a stronger statement: the following equality holds in the Milnor K-group $K_w^M(F(Y_{(w)}))$

$$p_1^* F_{n_1,...,n_p}(x_1, ..., x_p) \wedge p_2^* F_{n_{p+1},...,n_{p+q}}(x_{p+1}, ..., x_{p+q})$$

$$\sum_{\sigma \in \overline{\Sigma}_{p,q}} F^{\sigma}_{n_1,...,n_{p+q}}(x_1, ..., x_{p+q}) =$$

A path γ be between 0 and 1 provides a relative cycle modulo the algebraic simplex B. Let $\gamma(C_w)$ be the closure of its lifting on $Y_{(w)}(\mathbb{C})$.

Lemma 9.8 *Define* $\mathrm{Li}^{\sigma}_{n_1,...,n_{p+q}}(x_1, ..., x_{p+q})$ *using an integral presentation like (4) written for an arbitrary path* γ *between* 0 *and* 1 *instead of the path* $[0,1]$. *Then*

$$\int_{\gamma(C_w)} \Omega^{\sigma}_{n_1,...,n_{p+q}}(x_1, ..., x_{p+q}) = \mathrm{Li}^{\sigma}_{n_1,...,n_{p+q}}(x_1, ..., x_{p+q}) \tag{161}$$

Proof. We work out an example when $p = q = 1$. The general case is completely similar. Consider the shuffle σ such that

$$\mathrm{Li}^{\sigma}_{n_1,n_2}(x_1, x_2) = \mathrm{Li}_{n_2,n_1}(x_2, x_1)$$

Then if $\Omega_{n_1,n_2}(x_1, x_2)$ is defined using the sequence

$$S_1 = v_1, ..., v_{n_1}; \quad S_2 = v_{n_1+1}, ..., v_{n_1+n_2}$$

the form $\Omega^{\sigma}_{n_1,n_2}(x_1, x_2)$ is defined using the sequence

$$S'_1 = v_{n_1+1}, ..., v_{n_1+n_2}; \quad S'_2 = v_1, ..., v_{n_1}$$

So if we make change of variables

$$v'_1 = v_{n_1+1}, ..., v'_{n_2} = v_{n_1+n_2}; \qquad v'_{n_2+1} = v_1, ..., v'_{n_1+n_2} = v_{n_1}$$

then

$$\int_{C_w} \Omega^{\sigma}_{n_1,n_2}(x_1, x_2) = \int_{C'_w} \Omega_{n_2,n_1}(x_2, x_1)$$

where $C'_w = C_{n_2,n_1}(x_2, x_1)$ is the corresponding cube in the v'_i space.

The lemma is proved.

Observe that

$$p_1^* C_{n_1,...,n_p}(x_1, ..., x_p) \times p_2^* C_{n_{p+1},...,n_{p+q}}(x_{p+1}, ..., x_{p+q}) = \tag{162}$$
$$C_{n_1,...,n_{p+q}}(x_1, ..., x_{p+q})$$

It follows from (161) that integrating the left and right parts of the formula in lemma 9.6 over the cycles located in the left and right parts of the identity (162) we get the Li-shuffle relation (116) where all the terms are understood as iterated integrals along a path γ. A proof of the identity (116) for the case of multiple ζ-values using similar ideas was independently found by P. Cartier (unpublished).

4. Proof of theorem 7.5. Let $x_1, ..., x_{p+q} \in \mathbb{C}^*$ are as in (122). Denote by

$$\mathrm{Li}^{\mathcal{H},\sigma}_{n_1,...,n_{p+q}}(x_1, ..., x_{p+q})$$

the equivalence class of the framed mixed Tate motive

$$\Big(\mathrm{Li}^{\mathcal{H}}_{n_1,...,n_{p+q}}(x_1, ..., x_{p+q}), [C_w], [\Omega^{\sigma}_{n_1,...,n_{p+q}}(x_1, ..., x_{p+q})]\Big)$$

where the object itself and the $[C_w]$-component of the frame were defined in proposition 9.4.

Theorem 9.9 *a) Suppose that $x_i \in \mathbb{C}^*$ satisfy condition (122) for all terms of the formula (163) below. Then*

$$\mathrm{Li}^{\mathcal{H}}_{n_1,\dots,n_p}(x_1,...,x_p) \cdot \mathrm{Li}^{\mathcal{H}}_{n_{p+1},\dots,n_{p+q}}(x_{p+1},...,x_{p+q}) = \tag{163}$$

$$\sum_{\sigma \in \overline{\Sigma}_{p,q}} \mathrm{Li}^{\mathcal{H},\sigma}_{n_1,\dots,n_{p+q}}(x_1,...,x_{p+q})$$

b) For a generalized shuffle $\sigma \in \overline{\Sigma}_{p,q}$ one has

$$\mathrm{Li}^{\mathcal{H},\sigma}_{n_1,\dots,n_{p+q}}(x_1,...,x_{p+q}) = \mathrm{Li}^{\mathcal{H}}_{\sigma(n_1,\dots,n_{p+q})}(\sigma(x_1,...,x_{p+q}))$$

c) Suppose that F is a number field and $x_i \in F^*$. Then there are motivic versions of the parts a) and b) where $\mathrm{Li}^{\mathcal{H}}$ is replaced by $\mathrm{Li}^{\mathcal{M}}$

d) If $\mu_{l^\infty} \notin F^*$ then there is the l-adic versions of a) and b).

Proof. a) Proposition 9.4 allows to interpret the framed object in the left hand side of (163) as the object (152) (where $w = n_1 + ... + n_{p+q}$) with the frame provided by the cycle staying on the left of (162), and by the product of forms (157) and (158). Each of the framed objects appearing in the sum on the right hand side of (163) has a similar interpretation as the same object (152) with the frame given by the cycle class on the right hand side of (162) and the forms Ω^σ. Thus the statement follows from (162) and lemmas 9.6 and 5.3.

The part b) is a Hodge version of (161). To prove it one needs to do a change of variables similar to the one which was done in the example following (161).

c) Follows from a) and b) using the injectivity of regulators, or can be deduced the same way we did a) and b).

d) Is completely similarly to a) and b). The theorem 9.9 is proved.

Theorem 7.5 follows immediately from the parts a) and b) of theorem 9.9.

The l-adic and motivic versions of theorem 7.5 follow immediately from the parts d) and c).

10 Proofs of the theorems from Introduction

1. Multiple polylogarithms Hopf and Lie algebras. Let F be a number field and $G \subset F^*$ be a subgroup. Then $\mathcal{Z}^{\mathcal{M}}_w(G)$ is the $\mathbb{Q}$-vector subspace of $\mathcal{A}_w(F)$ generated by the weight w objects

$$\mathrm{Li}^{\mathcal{M}}_{n_1,\dots,n_m}(x_1,...,x_m) \in \mathcal{A}_w(F); \quad x_i \in G$$

and $\mathcal{Z}^{\mathcal{M}}_\bullet(G) := \oplus_{w\geq 0}\mathcal{Z}^{\mathcal{M}}_w(G)$.

Theorem 10.1 *$\mathcal{Z}^{\mathcal{M}}_\bullet(G)$ is a graded Hopf algebra.*

Proof. A similar result in the Hodge setting has been proved in theorem 5.5 of [G8]. Observe that we use the Li-objects, while we have used the I-objects in [G8]. However thanks to theorem 7.8 or corollary 7.10 this does not make any difference. After that lemma 3.4 from [G7] allows to deduce the motivic version from the Hodge one. The theorem is proved.

Recall (see chapter 5 of [G8]) that for any subgroup $G \subset F^*$ for any field F such that $\mu_{l^\infty} \notin F^*$ there is an l-adic Hopf algebra $\mathcal{Z}^{\rm et}_\bullet(G)$. If F is a number field it can be defined as the l-adic realization of the corresponding motivic Hopf algebra.

The depth filtration. It is an increasing filtration $F^D_\bullet$ on the vector space $\mathcal{Z}^{\mathcal{M}}_\bullet(G)$ indexed by non negative integers. $F^D_m\mathcal{Z}^{\mathcal{M}}_\bullet(G)$ is generated by the elements ${\rm Li}^{\mathcal{M}}_{n_1,...,n_k}(x_1, ..., x_k)$ with $k \leq m$. In particular $F^D_0\mathcal{Z}^{\mathcal{M}}_\bullet(G) = \mathbb{Q}$.

Lemma 10.2 *The depth filtration is compatible with the Hopf algebra structure on $\mathcal{Z}^{\mathcal{M}}_\bullet(G)$. A similar result hold for $\mathcal{Z}^{\rm et}_\bullet(G)$.*

Proof. The Hodge and étale version of this result is theorem 5.5 from [G8]. The motivic version is deduced from it using lemma 3.4 in [G7]. The lemma is proved.

Just like in section 1.5 we define for an arbitrary subgroup $G \subset F^*$ the Lie coalgebra $\mathcal{C}^{\mathcal{M}}_\bullet(G)$ and the Lie algebra ${\rm C}^{\mathcal{M}}_\bullet(G)$. We can view them as Lie (co)algebras in the category of pure Tate motives, or as a graded Lie (co)algebra over $\mathbb{Q}$. The grading is provided by the weight.

The depth filtration is compatible with the weight grading. Taking the associate graded for the depth filtration we get the corresponding bigraded Lie coalgebra and Lie algebra over $\mathbb{Q}$:

$$\mathcal{C}^{\mathcal{M}}_{\bullet,\bullet}(G) := {\rm Gr}^D\left(\mathcal{C}^{\mathcal{M}}_\bullet(G)\right); \qquad {\rm C}^{\mathcal{M}}_{\bullet,\bullet}(G) := {\rm Gr}^D\left({\rm C}^{\mathcal{M}}_\bullet(G)\right)$$

Equivalently, we can think about them as of a Lie (co)algebras in the category of pure Tate motives graded by the depth.

For a subgroup $G \subset \mathbb{C}^*$ there are the corresponding Hodge Lie algebras ${\rm C}^{\mathcal{H}}_\bullet(G)$ and ${\rm C}^{\mathcal{H}}_{\bullet,\bullet}(G)$, as well as their l-adic analogs ${\rm C}^{\rm et}_\bullet(G)$ and ${\rm C}^{\rm et}_{\bullet,\bullet}(G)$.

2 Proof of theorem 1.6 . We need to show that if $x_i \in \mu_N$ then the framed mixed Tate motive (135) lies in $\mathcal{A}_w(S_N)$. A priori we know that it lies in $\mathcal{A}_w(\mathbb{Q}(\zeta_N))$. We employ definition 3.4 of $\mathcal{A}_w(S_N)$ from [G7]. It is equivalent to the one provided by [DG]. According to that definition an element $x \in \mathcal{A}_w(\mathbb{Q}(\zeta_N))$ belongs to the subspace $\mathcal{A}_w(S_N)$ if and only if

$$\Delta'(x) := \Delta(x) - (x \otimes 1 + 1 \otimes x) \in \oplus_{0<k<w}\mathcal{A}_k(S_N) \otimes \mathcal{A}_{w-k}(S_N)$$

Then the statement of the theorem follows by induction. Indeed, the isomorphism (18) provides the base of induction, and theorem 10.1 the induction step. The theorem is proved.

A scetch of an alternative proof of theorem 1.6. Recall that a mixed Tate motive over $\mathbb{Q}$ is unramified outside N if and only if its l-adic realization for a prime l such that $(l, N) = 1$ is unramified outside lN. The elements of the Hopf algebra $\mathcal{Z}^{\mathcal{M}}(\mu_N)$ are the matrix elements (in the sence of section 3.2 in [G7]) of the pro-object $\mathcal{P}^{\mathcal{M}}(\mathbb{G}_m - \mu_N; v_0, v_1)$. So the claim follows from the fact that its l-adic realization $\mathcal{P}^{(l)}(\mathbb{G}_m - \mu_N; v_0, v_1)$ is unramified outside of lN. More directly, we can use corollary 3.12 and the geometric construction of the motivic torsor of path given in s. 6.3 to show that $\mathcal{P}^{\mathcal{M}}(\mathbb{G}_m - \mu_N; v_0, v_1)$ is unramified over S_N.

3. Mixed Tate categories and their Galois groups. To prove theorem 1.16 we need to recall below some basic material about the mixed Tate categories and their Galois groups, see section 3.1-3.2 in [G7] for more details.

Let $(\mathcal{M}, K(1))$ be a mixed Tate K–category, where K is a characteristic zero field. It is equipped with canonical fiber functor

$$\Psi : \mathcal{M} \longrightarrow \mathrm{Vect}_\bullet, \quad X \longrightarrow \oplus \Psi_n(X) = \oplus_n \mathrm{Hom}_{\mathcal{M}}(K(-n), \mathrm{Gr}^W_{2n} X)$$

Forgetting the grading we get the fiber functor $\widetilde{\Psi}$. Let X be an object of $\mathcal{M}$. A choice of non zero vectors $v^{(p)} \in \Psi_{2p}(X)$ and $f^{(q)} \in \Psi_{2q}(X)$ provide a framed object $(X, v^{(p)}, f^{(q)})$.

Definition 10.3 *$\mathcal{A}_\bullet(X)$ is the commutative K–algebra generated by the equivalence classes of framed objects*

$$[X, v^{(p)}, f^{(q)}] \in \mathcal{A}_{p-q}(\mathcal{M}); \tag{164}$$

for all pairs of non zero vectors $v^{(p)} \in \Psi_p(X)$ and $f^{(q)} \in \Psi_q(X)$, for all $p \geq q$.

Let us choose for every integer n a basis in $\Psi_n(X)$. It provides the dual basis in $\Psi_n(X)^\vee$. Obviously it is sufficient to consider in (164) the framed objects corresponding to all basis/cobasis vectors only.

Recall that according to the Tannakian formalism the category $\mathcal{M}$ is canonically equivalent to the category of graded finite dimensional K–modules over a proalgebraic unipotent group scheme $U(\mathcal{M}) := \mathrm{Aut}^{\otimes}\widetilde{\Psi}$ over K. The equivalence is given by the fiber functor Ψ. In particular the group scheme $U(\mathcal{M})$ acts on the graded vector space $\Psi(X)$, and this action factorizes through a quotient U^X of $U(\mathcal{M})$, i.e. U^X is the image of the group scheme $U(\mathcal{M})$ acting on $\Psi(X)$. Let $\mathcal{X}$ be the mixed Tate category generated by the object X. Then $U^X = U(\mathcal{X})$.

Lemma 10.4 *a) $\mathcal{A}_\bullet(X)$ is a graded Hopf algebra.*

b) It is canonically isomorphic to the Hopf algebra of regular functions on U^X, so $\mathrm{Spec}\mathcal{A}_\bullet(X) = U^X$.

Proof. a) The graded K-vector space generated by the elements (164) is closed under the coproduct by its very definition. Since $\mathcal{A}_\bullet(\mathcal{M})$ is a Hopf algebra, the statement follows.

b) Recall that the equivalence between the category $\mathcal{M}$ and the category of graded $\mathcal{A}_\bullet(\mathcal{M})$–comodules assignes to an object X the graded comodule $\Psi(X)$ with the $\mathcal{A}_\bullet(\mathcal{M})$–coaction $\Psi(X) \otimes \Psi(X^\vee) \longrightarrow \mathcal{A}_\bullet(\mathcal{M})$ given by

$$v \otimes f \longmapsto \quad \text{the equivalence class of the framed object } [M, v, f]$$

This immediately implies the part b) of the lemma. The lemma is proved.

4. The motivic torsor of path on $\mathbb{G}_m - \mu_N$ and theorems 1.16 and 1.17 . Recall ([DG] or chapter 6 above) the motivic torsor of path

$$\mathcal{P}^{\mathcal{M}}(\mathbb{G}_m - \mu_N; v_0, v_1) \qquad (165)$$

between the canonical tangential base points v_0 and v_1 at 0 and 1. It is a proobject in $\mathcal{M}_T(S_N)$.

Theorem 10.5 *There is canonical isomorphism*

$$\mathcal{A}_\bullet\left(\mathcal{P}^{\mathcal{M}}(\mathbb{G}_m - \mu_N; v_0, v_1)\right) = \mathcal{Z}_\bullet^{\mathcal{M}}(\mu_N)$$

Proof. Follows immediately from section 6.3 and lemma 10.4. The theorem is proved.

Proposition 10.6 *For any $\varepsilon \in \{0, 1, \infty\}$ there is an isomorphism*

$$\mathcal{A}_\bullet\left(\mathcal{P}^{\mathcal{M}}(X_N; v_0, v_1)\right) = \mathcal{A}_\bullet\left(\pi_1^{\mathcal{M}}(X_N; v_\varepsilon)\right)$$

Proof. Let us prove the proposition for $\varepsilon = 0$. We will use a shorthand $X_N := \mathbb{G}_m - \mu_N$. Applying the fiber functor Ψ to the left action of $\pi_1^{\mathcal{M}}(X_N; v_1)$ on $\mathcal{P}^{\mathcal{M}}(X_N; v_0, v_1)$ we get a map of graded $L_\bullet(S_N)$-modules

$$\Psi(\pi_1^{\mathcal{M}}(X_N; v_0)) \otimes \Psi(\mathcal{P}^{\mathcal{M}}(X_N; v_0, v_1)) \longrightarrow \Psi(\mathcal{P}^{\mathcal{M}}(X_N; v_0, v_1))$$

Let $L_\bullet(S_N)$ be the Lie algebra of the pro-group scheme $U(\mathcal{M}_T(S_N))$ and $I_\bullet(X)$ its ideal annihilating $\Psi(X)$. A non zero vector $\gamma_0 \in \Psi_0(\mathcal{P}^{\mathcal{M}})$ provides an isomorphism of $I_\bullet(\mathcal{P}^{\mathcal{M}})$–modules $\Psi(\pi_1^{\mathcal{M}}(X_N, v_0)) \otimes \gamma_0 \longrightarrow \Psi(\mathcal{P}^{\mathcal{M}}(X_N; v_0, v_1))$. Thus

$$I_\bullet(\mathcal{P}^{\mathcal{M}}(X_N; v_0, v_1)) \subset I_\bullet(\pi_1^{\mathcal{M}}(X_N, v_0))$$

Let us prove the opposite inclusion. Since $\mathcal{P}^{\mathcal{M}}(X_N; v_0, v_1)$ is a right $\pi_1^{\mathcal{M}}(X_N; v_1)$–torsor, a choice of an element $\gamma \in \Psi(\mathcal{P}^{\mathcal{M}})$ provides a map from $L_\bullet(S_N)$ to $\pi_1^{\mathcal{M}}(X_N; v_1)$. Namely, if $l \in L_\bullet(S_N)$ and $l(\gamma) = \gamma s_l$ for $s_l \in \pi_1^{\mathcal{M}}(X_N; v_1)$, then $l \longmapsto s_l$. Its restriction to the ideal $I_\bullet(\pi_1^{\mathcal{M}}(X_N, v_0))$ is independent on the choice of γ, and hence provide a graded Lie algebra homomorphism

$$I_\bullet\left(\pi_1^{\mathcal{M}}(X_N; v_0)\right) \longrightarrow \Psi\left(\pi_1^{\mathcal{M}}(X_N; v_1)\right) \qquad (166)$$

By its very definition it induces an injective map

$$\frac{I_\bullet\left(\pi_1^{\mathcal{M}}(X_N; v_0)\right)}{I_\bullet\left(\mathcal{P}^{\mathcal{M}}(X_N; v_0)\right)} \hookrightarrow \Psi\left(\pi_1^{\mathcal{M}}(X_N; v_1)\right)$$

Thus to prove the opposite inclusion we have to show that this map is zero.

Recall the canonical map, "the motivic loop around 1 based at v_1":

$$\alpha_1^{\mathcal{M}} : \mathbb{Q}(1) \longrightarrow \pi_1^{\mathcal{M}}(X_N; v_1)$$

Observe that the ideal $I_\bullet(\pi_1^{\mathcal{M}}(X_N; v_0))$ kills

$$\gamma \circ \Psi(\alpha_1^{\mathcal{M}}(\mathbb{Q}(1)) \circ \gamma^{-1} \subset \pi_1^{\mathcal{M}}(X_N; v_0) \tag{167}$$

Since $L_\bullet(S_N)$ kills the loop $\alpha_1^{\mathcal{M}}$, using this it is easy to see that the image of (166) centralizes $\Psi(\alpha_1^{\mathcal{M}})$. Since $\pi_1^{\mathcal{M}}(X_N; v_1)$ is a free Lie algebra, this implies that the image of (166) is contained in $\Psi(\alpha_1^{\mathcal{M}}(\mathbb{Q}(1)))$. Therefore it lies in the one dimensional subspace of weight -2. So our our claim is reduced to the following elementary fact. Let $W_{[p,q]}(X)$ is the subquotient of X of weights $2p \leq w \leq 2q$). Then

$$\mathcal{A}_{-1}\left(W_{[-2,-1]}\left(\pi_1^{\mathcal{M}}(X_N; v_1)\right)\right) = \mathcal{A}_{-1}\left(W_{[-1,0]}\left(\mathcal{P}^{\mathcal{M}}(X_N; v_0, v_1)\right)\right)$$

Indeed, we know that the right hand side is genertaed as a $\mathbb{Q}$–vector space by the elements $\log^{\mathcal{M}}(1-\zeta_N^{\alpha})$, where $0 < \alpha < N$, and the left hand side is contained in the right hand side. So it remains to check that $\log^{\mathcal{M}}(1-\zeta_N^{\alpha})$ appears in the left hand side. This follows, for instance, from the results of [D], where the polylogarithmic quotient of the Hodge fundamental group $\pi_1^{\mathcal{H}}(P^1 - \{0, 1, \infty\}; x)$ aas well as its pro-l version were explicitly described. In particular it was shown that $\log^{\mathcal{H}}(1-x)$ appears as a matrix element for an appropriate $(\mathbb{Q}(1), \mathbb{Q}(2))$–framing on $\pi_1^{\mathcal{H}}(P^1 - \{0, 1, \infty\}, x)$. Then the map $t \longmapsto t/x$ provides an isomorphism

$$\pi_1^{\mathcal{H}}(P^1 - \{0, 1, \infty\}; x) = \pi_1^{\mathcal{H}}(P^1 - \{0, x^{-1}, \infty\}; 1)$$

The right object is a subobject of $\pi_1^{\mathcal{H}}(X_N; v_1)$. Thus setting $x := \zeta_N^{-\alpha}$ we get the statement.

To prove the proposition for other ε we proceed as follows. There is an isomorphism of $L_\bullet(S_N)$–modules ("reversing the path")

$$\mathcal{P}^{\mathcal{M}}(X_N; v_0, v_1) = \mathcal{P}^{\mathcal{M}}(X_N; v_1, v_0)$$

So interchanging the role of 0 and 1 in the proof above we get the statement for $\varepsilon = 1$. The involution $x \longmapsto x^{-1}$ on P^1 provides an isomorphism of mixed Tate motives

$$\pi_1^{\mathcal{M}}(X_N; v_0) \xrightarrow{\sim} \pi_1^{\mathcal{M}}(X_N; v_\infty)$$

Therefore the statement for $\varepsilon = \infty$ follows from the one for $\varepsilon = 0$. The proposition is proved.

Recall the pro-algebraic group scheme $\mathrm{Spec}\mathcal{Z}_\bullet^{\mathcal{M}}(\mu_N)$ over $\mathbb{Q}$. The grading on $\mathcal{Z}_\bullet^{\mathcal{M}}(\mu_N))$ provides an action of the group $\mathbb{G}_m$ on this group. According to lemma 10.4, theorem 10.5 just means that the motivic Galois group acts on (165) via its quotient given by the semidirect product of $\mathbb{G}_m$ and $\mathrm{Spec}\mathcal{Z}_{\bullet}^{\mathcal{M}}(\mu_N)$.

Proof of theorem 1.17. Follows immeduately from theorem 10.5 and proposition 10.6.

Remark. Below we will use only the l-adic version of proposition 10.6.

5. Proof of theorem 1.18. Let X be a mixed Tate motive over a number field F. Denote by $L_{\bullet}(X)$ the image of the motivic Lie algebra acting on $\Psi(X)$. Let $X_{\mathbb{Q}_l}$ be the l-adic realization of the motive X. Denote by $\mathcal{G}_X^{(l)}$ the Lie algebra of the image of $\mathrm{Gal}(\overline{F}/F(\zeta_{l^\infty}))$ acting on $X_{\mathbb{Q}_l}$.

Lemma 10.7 *There is an isomorphism $\mathcal{G}_X^{(l)} = L_{\bullet}(X) \otimes \mathbb{Q}_l$, and canonical isomorphism of graded Lie algebras*

$$\mathrm{Gr}_{\bullet}^W \mathcal{G}_X^{(l)} = L_{\bullet}(X) \otimes \mathbb{Q}_l$$

Proof. Follows from the content of section 3.7 in [G4], see especially the second half of page 424 loc. cit..

Corollary 10.8 *There is canonical isomorphism of the graded l-adic pro-Lie algebras*

$$\mathrm{Gr}_{\bullet}^W \mathcal{G}_N^{(l)} = L_{\bullet}(\pi_1^{\mathcal{M}}(\mathbb{G}_m - \mu_N; v_\infty) \otimes \mathbb{Q}_l$$

Proof. Follows immediately by going to the projective limit from lemma 10.7 and proposition 10.6.

The part a) of theorem 1.18 follows from corollary 10.8 and theorem 10.5. The part b) is also straitforward.

6. Proof of theorem 1.9: the first part. We study the map

$$\nu_{\bullet,\bullet}^{\mathcal{M}}(\mu_N) : \mathcal{D}_{\bullet,\bullet}(\mu_N) \longrightarrow \mathcal{C}_{\bullet,\bullet}^{\mathcal{M}}(\mu_N)$$

We have to check that this map sends the double shuffle and distribution relations to zero, and is compatible with the cobracket.

The relations. We use the list of the relations is given in section 4.1 of [G4]. Observe that we used in [G4] the homogeneous notations

$$\mathrm{I}_{n_1,...,n_m}(g_1 : ... : g_m : g_{m+1}) := \mathrm{I}_{n_1,...,n_m}(g_{m+1}^{-1}g_1 : ... : g_{m+1}^{-1}g_m)$$

So the relation (i) from [G4] is respected by the definition.

The I-shuffle relations, that is relation (63) in [G4], are valid by theorem 3.2 in [G8].

To get the Li-shuffle relation, that is relation (62) in [G4], we notice that thanks to theorem 7.8 we have for any $x_i \in \mathbb{C}^*$ (with a_i are defined by (5)

$$\mathrm{Li}_{n_1,...,n_m}^{\mathcal{H}}(x_1, ..., x_m) \sim \mathrm{I}_{n_1,...,n_m}^{\mathcal{H}}(a_1, ..., a_m)$$

modulo depth $< m$ terms and products. So using theorem 7.4 we get relation (62) in [G4].

The distribution relations in [G4] are easily checked either using the general method of section 8.2, or directly using the realization described in section 9 and the specialization theorem. The distribution relations are proved in the l-adic setting in [G4], but the proof works in the Hodge setting as well. This gives the third proof.

The normalizing relation $\mathrm{I}_1(e:e)$ is true since $\mathrm{I}^{\mathcal{H}}(1)=0$. So we conclude that the map $\nu^{\mathcal{H}}_{\bullet,\bullet}(\mu_N)$ is a well defined map of the bigraded vector spaces.

7. The map $\nu^{\mathcal{M}}_{\bullet,\bullet}(\mu_N)$ commutes with the cobrackets. The formula for the coproduct Δ in the Hopf algebra $\mathcal{Z}^{\mathcal{H}}(\mathbb{C}^*)$ has been computed explicitly in chapter 6 of [G7], and then reproved in theorem 3.2 in [G8] in a more generial setting. The coproduct Δ induces a cobracket in the corresponding Lie coalgebra. We denote it by $\delta_{\mathcal{H}}$. On the other hand the cobracket δ in the dihedral Lie coalgebra was defined in section 4.4 of [G4]. Let us first recall the definitions of Δ and δ.

The formula for Δ. For convenience of the reader we recall some definitions and results borrowing directly from chapters 5 and 6 of [G8].

We package the framed Hodge-Tate structures $\mathrm{I}^{\mathcal{H}}_{n_1,...,n_m}(a_1,...,a_m)$ into the generating series

$$\mathrm{I}^{\mathcal{H}}(a_1:...:a_m:a_{m+1}|t_1,...,t_m):= \tag{168}$$
$$\sum_{n_i\geq 1}\mathrm{I}^{\mathcal{H}}_{n_1,...,n_m}(\frac{a_1}{a_{m+1}},...,\frac{a_m}{a_{m+1}})t_1^{n_1-1}...t_m^{n_m-1}\in \quad \mathcal{H}_\bullet[[t_1,...,t_m]]$$

Following lemma 5.2 in [G8], it is convenient to introduce the homogeneous variables $(t_0:t_1:...:t_m)$ replacing the variables $(t_1,...,t_m)$:

$$\mathrm{I}^{\mathcal{H}}(a_1:...:a_m:a_{m+1}|t_0:t_1:...:t_m):= \tag{169}$$
$$a_{m+1}^{t_0}\mathrm{I}^{\mathcal{H}}(a_1:...:a_m:a_{m+1}|t_1-t_0,...,t_m-t_0)$$

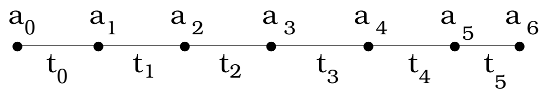
By theorem 5.4 in [G8] we have

$$\Delta\mathrm{I}^{\mathcal{H}}(a_1:...:a_{m+1}|t_0:...:t_m)$$
$$\sum \mathrm{I}^{\mathcal{H}}(a_{i_1}:...:a_{i_k}:a_{m+1}|t_{j_0}:t_{j_1}:...:t_{j_k})\otimes$$
$$\prod_{\alpha=0}^{k}\Big((-1)^{j_\alpha-i_\alpha}\mathrm{I}^{\mathcal{H}}(a_{j_\alpha}:a_{j_\alpha-1}:...:a_{i_\alpha}|-t_{j_\alpha}:-t_{j_\alpha-1}:...:-t_{i_\alpha})\cdot \tag{170}$$
$$\mathrm{I}^{\mathcal{H}}(a_{j_\alpha+1}:...:a_{i_{\alpha+1}-1}:a_{i_{\alpha+1}}|t_{j_\alpha}:t_{j_\alpha+1}:...:t_{i_{\alpha+1}-1})\Big)$$

where the sum is over all special marked colored segment, i.e. over all sequences $\{i_\alpha\}$ and $\{j_\alpha\}$ such that

$$i_\alpha\leq j_\alpha<i_{\alpha+1}\quad \text{for any } 0\leq\alpha\leq k, \qquad j_0=i_0=0, i_{k+1}=i_{m+1} \tag{171}$$

A geometric interpretation of the formula (170) . We picture generating series (169) by a colored segment where $a_0 = 0$:

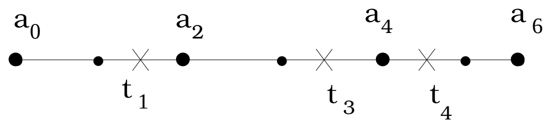


A decorated segment

The terms of this formula correspond to the so-called special marked colored segments:

a) we mark (by making them boldface) the points $a_0; a_{i_1}, ..., a_{i_k}; a_{m+1}$.

b) mark (by cross) segments $t_{j_0}, ..., t_{j_k}$ such that there is just one marked segment between any two neighboring marked points.



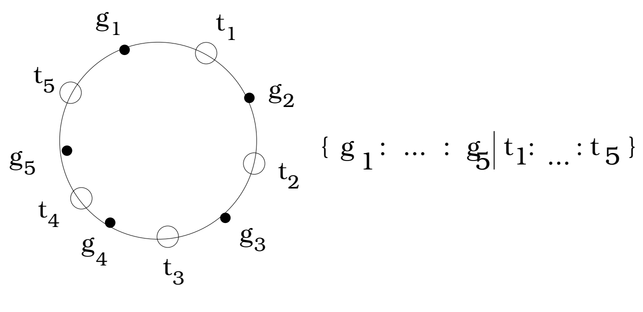
A marked decorated segment

The cobracket $\delta : \mathcal{D}_{\bullet\bullet}(G) \longrightarrow \Lambda^2\mathcal{D}_{\bullet\bullet}(G)$. For the convenience of the reader we reproduce the definition given on the page 434 of [G4]. Consider the formal generating series similar to (168)

$$\{g_1 : ... : g_{m+1} | t_1 : .... : t_{m+1}\} := \tag{172}$$

$$\sum_{n_i > 0} \mathrm{I}_{n_1,...,n_m}(g_1 : ... : g_{m+1})(t_1 - t_{m+1})^{n_1 - 1} ... (t_m - t_{m+1})^{n_m - 1}$$

We picture them on the oriented circle. Namely, the circle has slots, where the g's sit, and in between the consecutive slots, dual slots, where t's sit, see the picture below. The slots are marked by black points, and the dual slots by little circles:



Set

$$\delta\{g_1 : ... : g_{m+1}|t_1 : ... : t_{m+1}\} = \tag{173}$$

$$-\sum_{k=2}^{m} \text{Cycle}_{m+1}\Big(\{g_1 : ... : g_{k-1} : g_k|t_1 : ... : t_{k-1} : t_{m+1}\}\wedge$$

$$\{g_k : ... : g_{m+1}|t_k : ... : t_{m+1}\}\Big)$$

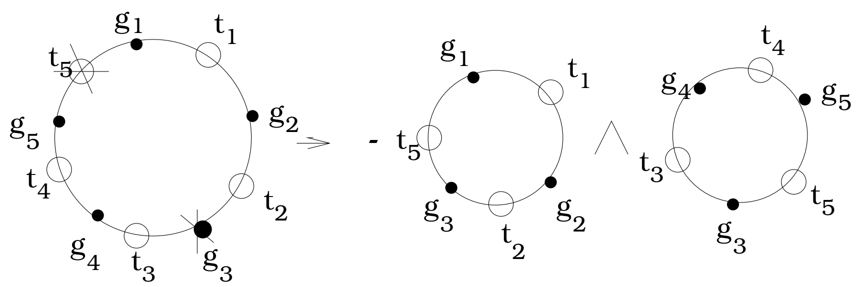
where the indices are modulo $m+1$ and

$$\text{Cycle}_{m+1} f(x_1, ..., x_{m+1}) := \sum_{i=1}^{m+1} f(x_i, ..., x_{m+i})$$

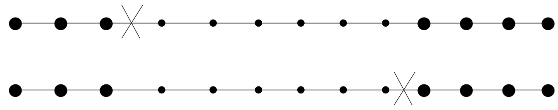
Each term of the formula corresponds to the following procedure: choose a slot and a dual slot on the circle. Cut the circle at the chosen slot and dual slot and make two oriented circles with a dihedral words on each of them out of the initial data. It is useful to think about the slots and dual slots as of little arcs, not points, so cutting one of them we get the arcs on each of the two new circles marked by the corresponding letters. The formula reads as follows:

$$\delta((173)) = -\sum_{\text{cuts}} (\text{start at the dual slot}) \quad \wedge \quad (\text{start at the slot})$$

The only asymmetry between g's and t's is the order of factors.



Comparing $\delta^{\mathcal{H}}$ and δ. We claim that the only terms in the sum which may survive after we kill the products of the weight ≥ 1 terms are the ones corresponding to special marked colored segments shown on the two pictures below

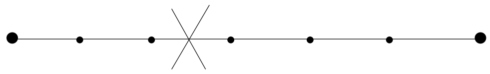


More precisely, these are the special marked colored segments where we mark every point from $a_0 = 0$ to some point a_p, then jump on the right and mark a point a_q and every point after, till we hit the right endpoint a_{m+1}. The cross between a_p and a_q can be either at the very left segment t_p, or at the very right segment t_{q-1}.

To check the claim we observe the following. The factors in the product (170) correspond to "gaps" between the consequetive marked points. We say that such a gap is wide, if it has more then one segment inside, i.e. if there is unmarked point inside the gap, as shown on the picture

A wide gap between two marked points

The factor of the product (170) corresponding to a wide gap is of weight at least one. Thus if there are two such wide gaps on a marked colored segment, then the corresponding term dies after we kill the products. Therefore we allow to have no more then one wide gap.



A cross strictly inside of a gap produces
a product of two terms of positive weight

Now let us make a close look at the cross located inside of the wide gap. If this cross is not in one of the corner segments, we will pick up two terms of weight at least one in the term corresponding to this gap. So the cross must be in one of the corner segments of the gap, and we arrive precisely to the marked colored segments shown on the two pictures above.

Let us show that the contribution of all special marked colored segments having a wide gap matches the formula for δ. Observe that $\mathrm{I}^{\mathcal{H}}(a|t) = a^t$. If there is a wide gap on a special marked colored segment then every other gap contributes just 1 to the product formula. Indeed, its contribution is

$$\mathrm{I}^{\mathcal{H}}(a|t) = a^t = \mathrm{e}^{a\cdot t} = 1 + \log^{\mathcal{H}} a \cdot t + (\log^{\mathcal{H}} a)^2 \cdot t^2/2 + ...$$

but any term of weight bigger then zero will multiply the term corresponding to the wide gap, and thus their product will be killed by going to the Lie coalgebra. Similarly the contribution of the wide gap with the left marked cross is

$$(-1)^{i_{\alpha+1}-1-i_\alpha}\mathrm{I}^{\mathcal{H}}(a_{i_{\alpha+1}-1} : a_{i_{\alpha+1}-2} : ... : a_{i_\alpha} | -t_{i_{\alpha+1}-1} : -t_{i_{\alpha+1}-2} : ... : -t_{i_\alpha}) \tag{174}$$

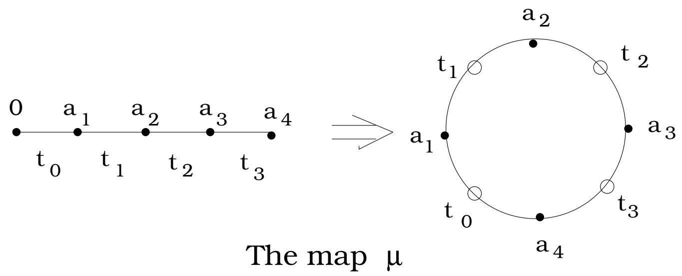
It corresponds to the case $j_\alpha = i_{\alpha+1} - 1$ in (170). The contribution of the wide gap with the right marked cross is

$$\mathrm{I}^{\mathcal{H}}(a_{i_\alpha+1} : a_{i_\alpha+2} : ... : a_{i_{\alpha+1}} | t_{i_\alpha} : t_{i_\alpha+1} : ... : t_{i_{\alpha+1}-1}) \tag{175}$$

It follows from theorem 4.1 and formula (65) in [G4] that, modulo the double shuffle relations, lower depth terms and products the element (174) is equivalent to the element

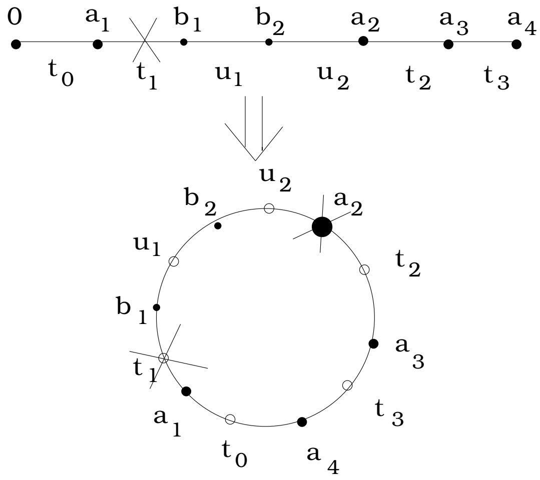
$$-\mathrm{I}^{\mathcal{H}}(a_{i_\alpha} : ... : a_{i_{\alpha+1}-2} : a_{i_{\alpha+1}-1}|t_{i_\alpha+1} : t_{i_\alpha+2} : ... : t_{i_\alpha}) \tag{176}$$

Observe now that the map $\nu^{\mathcal{H}}_{\bullet,\bullet}(\mu_N)$ is described geometrically by the picture

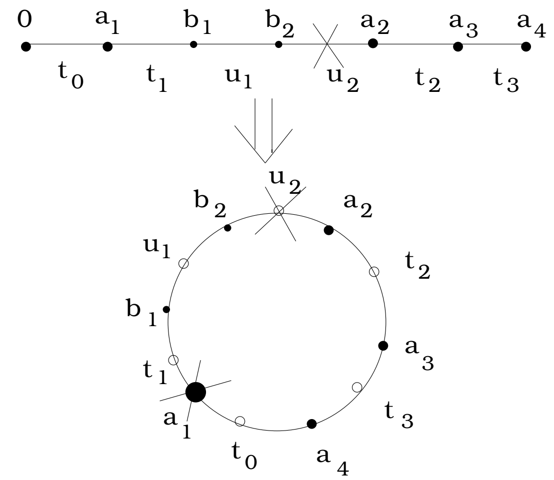


The map μ

Now it is easy to check that the contribution of all special marked colored segments having a wide gap matches the formula for δ. Namely, the two ways to mark by cross the wide gap produce the following two pictures. The first one corresponds to (176):



and the second one to (175):



Observe that the minus sign in (176) comparing to the plus sign in (175) agree with the fact that the order of the terms on the first picture is different from the one prescribed by formula (173). So interchanging the terms we get a minus sign as well, in according with minus sign in front of (173).

So far we did not use the fact that $a_i \in \mu_N$. We proved in chapter 4 of [G8] that the same formulas are valid in the motivic setting if a_i lie in a number field. This is the case when $a_i \in \mu_N$. So if $a_i \in \mu_N$ we can cjange $\mathcal{H}$ to $\mathcal{M}$ in all formulas above in section 7.

Further, let us use $a_i \in \mu_N$ to show that the contribution of the unique special marked colored segment with no wide gaps is zero. Indeed, $\log^{\mathcal{H}} x$ is an N-torsion element if $x^N = 1$. So working over $\mathbb{Q}$ (or at least modulo N-torsion) we have $\log^{\mathcal{H}} x = 0$. Thus

$$x^t := \exp(\log^{\mathcal{M}} x \cdot t) = 1 \quad \text{if } x^N = 1$$

and hence there is no contribution from this term to the restricted coproduct Δ', and hence to $\delta^{\mathcal{M}}$. We proved that the map $\nu^{\mathcal{M}}_{\bullet,\bullet}(\mu_N)$ commutes with the cobrackets, and hence finished the proof of theorem 1.9.

Remark. The very definition of the cobracket in the dihedral Lie algebra given in chapter 4 of [G4] emerged from the calculation of the coproduct for the framed multiple polylogarithm Hodge-Tate structures.

8. Proof of theorem 1.12. We start from the $w = m$ case

Proposition 10.9 *The dual to the map $\nu^{\mathcal{M}}_{\bullet,\bullet}(\mu_N)$ from theorem 1.9 induces isomorphisms*

$$\mathrm{C}^{\mathcal{M}}_{m,m}(\mu_p) = \mathrm{D}_{m,m}(\mu_p) \quad \textit{for } m = 1, 2, 3$$

Proof. For $m = 1$ this is given by the classical Bass theorem on cyclotomic units. The $m = 2$ case can be deduced from lemma 7.11 in [G4], which claims that this is true after tensoring by $\mathbb{Q}_l$. In fact using theorem 1.9 it is easy to translate the the proof of that lemma into motivic setting.

In the $m = 3$ case we observe that it follows from from section 7.7 [G4], see in particular isomorphism (172) there, that the subspace of $\mathrm{C}_{-3,-3}(\mu_p)$ generated by the triple commutators of elements in $\mathrm{C}_{-1,-1}(\mu_p)$ is isomorphic to $\mathrm{D}_{-3,-3}(\mu_p)$. On the other hand by theorem 1.9 there is an inclusion $\mathrm{C}_{-3,-3}(\mu_p) \subset \mathrm{D}_{-3,-3}(\mu_p)$. The proposition is proved.

Theorem 1.12 in the case $N = 1$, $m = 1$ just claims that $\zeta^{\mathcal{M}}(2n) = 0$ and $\zeta^{\mathcal{M}}(2n+1) \neq 0$. In the Hodge realization both assertions are well known: the first follows from the distribution relations (for $l = 2$), and the second from the fact that the real period of $\zeta^{\mathcal{H}}(2n+1)$ is not zero.

In the $N = 1$ theorem 1.12 follows from theorem 2.4 (when $m = 2$) and theorem 2.10 (when $m = 3$) in [G4]. Indeed, those two theorems prove similar results in the l-adic setting, and so thanks to theorem 1.18 imply theorem 1.12 for $N = 1$, $m = 2, 3$ cases. In fact by repeating the arguments used to prove theorems 2.4 and 2.10 in [G4] we get a direct proof of theorem 1.12 in these cases.

9. Proof of theorem 1.15. Combining proposition 10.9 with formulas (169)-(170) in [G4] we get theorem 1.15.

Remark. One should correct the apparent misprints in (169)-(170) in [G4] by changing $H^i_{(w,2)}$ to $H^i_{(2)}$ and $H^i_{(w,3)}$ to $H^i_{(3)}$.

Let $d^*_{w,m}(p) = \dim \mathrm{C}^{\mathcal{M}}_{w,m}(\mu_p)$.

Corollary 10.10 *One has for prime $p \geq 5$*

$$d^*_{2,2}(p) = \frac{(p-1)(p-5)}{12}; \quad d^*_{3,3}(p) = \frac{(p-5)(p^2-2p-11)}{48}$$

Proof. This is the motivic version of the corollary 2.16 refined in the depth three case. The proof goes just like the proof of that corollary in chapter 7 of [G4], but now in the motivic setting. Namely, we calculate the Euler characteristic the (m, m) part of the standard cochain complex for the Lie algebra $\mathrm{C}^{\mathcal{M}}_{\bullet,\bullet}(\mu_p)$ using theorem 1.15.

Observe that for $i = 2, 3$ the group $H^i(\Gamma_1(3;p), \mathbb{Q})$ contain the mysterious cuspidal cohomology subgroup $H^i_{\mathrm{cusp}}(\Gamma_1(3;p), \mathbb{Q})$. There is no closed formula for its dimension, but thanks to the Poincare duality the cuspidal cohomology for $i = 2$ and $i = 3$ cancel each other in the computation of the Euler characteristic. As a result we were able to get a closed formula for $d^*_{3,3}(p)$. The corollary 10.10 is proved.

10. Proof of theorem 1.8. By theorem 1.8 in [G7] there is canonical surjective homomorphism

$$\mathcal{Z}^{\mathcal{H}}_{\bullet}(\mu_N) \longrightarrow \mathrm{Gr}^W \widetilde{\mathcal{Z}}(\mu_N)$$

Let us choose a complex primitive N-th root of unity. Then thanks to theorem 7.11a) the Hodge realization functor provides a surjective homomorphism

$$\mathcal{Z}_{\bullet}^{\mathcal{M}}(\mu_N) \longrightarrow \mathcal{Z}_{\bullet}^{\mathcal{H}}(\mu_N)$$

By theorem 1.6 $\mathcal{Z}_{\bullet}^{\mathcal{M}}(\mu_N)$ is a subalgebra of $\mathcal{A}_{\bullet}^{\mathcal{M}}(S_N)$. The theorem is proved.

11. Understanding $\mathcal{Z}_{\bullet}^{\mathcal{M}}(S_N)$. It would be very interesting to calculate

$$d_w(N) := \dim \mathcal{Z}_w^{\mathcal{M}}(S_N) \quad \text{and} \quad d_{w,m}(N) := \dim \mathrm{Gr}_m^D \mathcal{Z}_w^{\mathcal{M}}(S_N)$$

as functions of N for a given (w, m). According to Deligne [D2-3] $\mathcal{A}_{\bullet}(S_N)$ can not be smaller then $\mathcal{Z}_{\bullet}^{\mathcal{M}}(\mu_N)$ for $N = 2, 3, 4$. This plus theorem 1.6 yields

$$\mathcal{A}_{\bullet}(S_N) = \mathcal{Z}_{\bullet}^{\mathcal{M}}(\mu_N) \quad \text{for } N = 2, 3, 4 \tag{177}$$

According to [G2] or [G4] this is not true for sufficiently big N, e.g. for prime $N \geq 5$. The case $N = 1$ remains open, as well as the case $N = 6$.

Conjecture 10.11 *If p is a prime then $d_w(p)$ and $d_{w,m}(p)$ are polynomials in p.*

For $p = 2, 3$ this follows from (177).

Conjecture 10.11 is equivalent to a similar statement about $d^*_{w,m}(p)$. Corollary 10.10 supports it for $m = w = 2$ and $m = w = 3$. According to the results of [G10] we have the same situation for $m = w = 4$.

12. Proof of theorem 1.10. We need to check the second isomorphism. The crucial fact is the following formula for the coproduct of the classical polylogarithm motive.

Theorem 10.12 *Let F be a number field. Then for any $x \in F$ one has*

$$\Delta \mathrm{Li}_n^{\mathcal{M}}(x) = \mathrm{Li}_n^{\mathcal{M}}(x) \otimes 1 + 1 \otimes \mathrm{Li}_n^{\mathcal{M}}(x) + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}^{\mathcal{M}}(x) \otimes \frac{(\log^{\mathcal{M}} x)^k}{k!}$$

Proof. In the Hodge version this result is well known, see for instance chapter 5 of [G7]. Using lemma 3.4 from [G7] we transfer it to the motivic setting. The theorem is proved.

It follows from this that for any root of unity ζ the element $\mathrm{Li}_n^{\mathcal{M}}(\zeta)$ is primitive (observe that it lies in the $\mathbb{Q}$-vector space). Thus it provides an class

$$\mathrm{Li}_n^{\mathcal{M}}(\zeta) \in \mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(n)) = K_{2n-1}(\mathbb{Q}(\zeta)) \otimes \mathbb{Q}$$

For $n = 1$ theorem 1.10 is classical. Assume $n > 1$. Let α be a residue modulo N. Consider the classes

$$\mathrm{Li}_n^{\mathcal{M}}(\zeta_N^{\alpha}); \qquad (\alpha, N) = 1, \quad 1 \leq \alpha \leq N/2 \tag{178}$$

Using the distribution relations we check that these classes span the space genertaed by $\mathrm{Li}_n^{\mathcal{M}}(\zeta_N^\alpha)$ for all α. We claim that the elements (178) generate $K_{2n-1}(\mathbb{Q}(\zeta)) \otimes \mathbb{Q}$. We need only to check that these elements are linerly independent. Thanks to lemma 3.4 in [G7] this can be done in the Hodge realization, where it is well known and established using Dirichlet's formula for $\zeta_{\mathbb{Q}(\mu_N)}(n)$ plus the fact that this special value is non zero. Theorem 1.10 is proved.

13. The weak version of Zagier's conjecture on the special values of the Dedekind ζ functions at $s = n$. It is deduced from theorem 10.12 using the arguments given in [BD1], or section 12 of [G1]. Another proves has been given in [BD2] and [DJ].

11 Apendix: Li-shuffle Hopf algebras

We define for an arbitrary commutative group G a commutative graded Hopf algebra $Sh_\bullet^{\mathrm{Li}}(G)$. Its generators are the symbols

$$\widehat{\mathrm{Li}}_{n_1,...,n_m}(x_1, ..., x_m); \qquad x_i \in G, \quad n_i \in \mathbb{Z}_+ \tag{179}$$

The relations are given by the Li-shuffle product and the inversion formulas. The formula for the coproduct Δ in this Hopf algebra was suggested by the formula for the coproduct in the multiple polylogarithm Hopf algebra $\mathcal{Z}_\bullet(\mathbb{C}^*)$ obtained in [G7]. The inversion formula reflects the one obtained in s. 2.6 in [G7]. The key result is that the Hopf axiom holds. This enables us to give another proof of the motivic Li-shuffle relations for generic parameters x_i.

1. The Li-shuffle relations revisited. Recall the generating formal power series for multiple polylogarithms:

$$\mathrm{Li}(x_1, ..., x_m | t_1, ..., t_m) := \sum_{n_i > 0} \mathrm{Li}_{n_1,...,n_m}(x_1, ..., x_m) t_1^{n_1 - 1} ... t_m^{n_m - 1}$$

The Li-shuffle relations for these generating series look as follows:

$$\mathrm{Li}(x_1, ..., x_m | t_1, ..., t_m) \cdot \mathrm{Li}(x_{m+1}, ..., x_{m+n} | t_{m+1}, ..., t_{m+n}) = \tag{180}$$
$$\sum_{\sigma \in \overline{\Sigma}_{m,n}} \mathrm{Li}_\sigma(x_1, ..., x_{m+n} | t_1, ..., t_{m+n})$$

We need to make explicit the term $\mathrm{Li}_\sigma(-|-)$ corresponding to a generalized shuffle σ. For example

$$\mathrm{Li}(x_1|t_1) \cdot \mathrm{Li}(x_2|t_2) = \mathrm{Li}(x_1, x_2|t_1, t_2) + \mathrm{Li}(x_2, x_1|t_2, t_1) +$$
$$\frac{1}{t_1 - t_2}\Big(\mathrm{Li}_1(x_1 x_2|t_1) - \mathrm{Li}_1(x_1 x_2|t_2)\Big)$$

Recall that a generalized shuffle of two strings

$$\{x_1, ..., x_m\} \quad \text{and} \quad \{y_1, ..., y_n\} \tag{181}$$

is given by the following data:

i) A pair of subsets I, J where

$$I = \{1 \leq i_1 < ... < i_p \leq m\}, \qquad J = \{1 \leq j_1 < ... < j_p \leq n\}, \quad p \geq 0 \tag{182}$$

providing p special pairs $(x_{i_1}, y_{j_1}), ..., (x_{i_p}, y_{j_p})$. Then each of the complements

$$\{x_1, ..., x_m\} - \{x_{i_1}, ..., x_{i_p}\} \quad \text{and} \quad \{y_1, ..., y_n\} - \{y_{j_1}, ..., y_{j_p}\}$$

is a union of $p+1$ substrings (some of them might be empty).

ii) A choice of $p+1$ permutations $\sigma_0, ..., \sigma_p$, where σ_i shuffles the elements of the i-th substrings of (181).

Consider the operator

$$D_{ij} f(x_i, x_j | t_i, t_j) := \quad \frac{1}{t_i - t_j}\Big(f(x_i x_j | t_i) - f(x_i x_j | t_j)\Big)$$

A generalized shuffle σ provides a well defined usual shuffle σ' obtained by shifting a bit to the right the y variables in each special pair. For a usual shuffle σ' the element $\mathrm{Li}_{\sigma'}(-|-)$ has an obvious meaning Set

$$\mathrm{Li}_\sigma(x_1, ..., x_{m+n} | t_1, ..., t_{m+n}) := D_{i_1 j_1} ... D_{i_p j_p} \mathrm{Li}_{\sigma'}(x_1, ..., x_{m+n} | t_1, ..., t_{m+n})$$

We say that a generalized shuffle has depth $-p$ if $|I| = |J| = p$. Denote the subset of all such shuffles by $\overline{\Sigma}^p_{m,n}$, so $\overline{\Sigma}_{m,n} = \cup_{p \geq 0} \overline{\Sigma}^p_{m,n}$.

2. The Li-shuffle Hopf algebra. Let G be a commutative group. We are going to define a commutative algebra with unit $Sh^{\mathrm{Li}}_\bullet(G)$ the over $\mathbb{Q}$. By definition it is generated by the symbols $\widehat{\mathrm{Li}}_{n_1, ..., n_m}(x_1, ..., x_m)$, as in (179). To present the relations we need to introduce some notations. Let us package the generators into the formal power series in t_i:

$$\widehat{\mathrm{Li}}(x_1, ..., x_m | t_1, ..., t_m) := \sum_{n_i > 0} \widehat{\mathrm{Li}}_{n_1, ..., n_m}(x_1, ..., x_m) t_1^{n_1 - 1} ... t_m^{n_m - 1}$$

Let $x_0 ... x_m = 1$. We will also use notations

$$\widehat{\mathrm{Li}}(*, x_1, ..., x_m | t_0 : ... : t_m) = \widehat{\mathrm{Li}}(x_0, x_1, ..., x_m | t_0 : ... : t_m) := \tag{183}$$

$$\widehat{\mathrm{Li}}(x_1, ..., x_m | t_1 - t_0, ..., t_m - t_0)$$

Let $\widehat{\log}(x) := \widehat{\mathrm{Li}}_1(x^{-1}) - \widehat{\mathrm{Li}}_1(x)$. We employ the notation

$$x^t := \quad e^{t \cdot \widehat{\log}(x)} := \quad 1 + \sum_{n > 0} \widehat{\log}^n(x) \frac{t^n}{n!}; \qquad \widehat{B}(x|t) = x^t / t$$

Relations 1. *The additivity of the formal logarithm.* Then

$$\widehat{\log}(xy) = \widehat{\log}(x) + \widehat{\log}(y) \qquad \text{for any } x, y \in G$$

2. *The* Li*-shuffle product relations.*

$$\widehat{\mathrm{Li}}(*, x_1, ..., x_m | t_0 : t_1 : ... : t_m) \cdot \widehat{\mathrm{Li}}(*, x_{m+1}, ..., x_{m+n} | t_0 : t_{m+1} : ... : t_{m+n}) =$$

$$\sum_{\sigma \in \overline{\Sigma}_{m,n}} \widehat{\mathrm{Li}}_\sigma(*, x_1, ..., x_{m+n} | t_0 : t_1 : ... : t_{m+n}) \tag{184}$$

where σ shuffles the variables $t_1, ..., t_{m+n}$. Set

$$\widehat{B}(x_1, ..., x_m | t_1, ..., t_m) :=$$

$$\sum_{j=1}^{m} (-1)^{j-1} \widehat{\mathrm{Li}}(*, x_{j-1}^{-1}, ..., x_1^{-1} | -t_j : -t_{j-1} : ... : -t_1) \cdot \tag{185}$$

$$\widehat{B}(x_1 ... x_m | t_j) \cdot \widehat{\mathrm{Li}}(*, x_{j+1}, ..., x_m | t_j : t_{j+1} : ... : t_m) \tag{186}$$

Remark. If $G = \{z \in \mathbb{C}^*; |z| = 1\}$ the inversion formula is a formal version the inversion formula derived in section 2.6 of [G7].

3. The inversion formula.

$$\widehat{B}(x_1, ..., x_m | t_1, ..., t_m) =$$

$$\sum_{j=1}^{m} (-1)^j \widehat{\mathrm{Li}}(x_j^{-1}, ..., x_1^{-1} | -t_j, ..., -t_1) \widehat{\mathrm{Li}}(x_{j+1}, ..., x_m | t_{j+1}, ..., t_m) + \tag{187}$$

$$\sum_{j=1}^{m} \frac{(-1)^j}{t_j} \widehat{\mathrm{Li}}(x_{j-1}^{-1}, ..., x_1^{-1} | -t_{j-1}, ..., -t_1) \cdot \widehat{\mathrm{Li}}(x_{j+1}, ..., x_m | t_{j+1}, ..., t_m)$$

Set $X_{a \to b} := \prod_{s=a}^{b-1} x_s$.

Definition 11.1 *We define a linear map* $\Delta : Sh_\bullet^{\mathrm{Li}}(G) \longmapsto Sh_\bullet^{\mathrm{Li}}(G) \otimes Sh_\bullet^{\mathrm{Li}}(G)$ *by*

$$\Delta \widehat{\mathrm{Li}}(x_0, x_1, ..., x_m | t_0 : t_1 : ... : t_m) =$$

$$\sum \widehat{\mathrm{Li}}(X_{i_0 \to i_1}, X_{i_1 \to i_2}, ..., X_{i_k \to m} | t_{j_0} : t_{j_1} : ... : t_{j_k}) \otimes$$

$$\prod_{p=0}^{k} \Big(X_{i_p \to i_{p+1}}^{t_{j_p}} \cdot (-1)^{j_p - i_p} \cdot \widehat{\mathrm{Li}}(*, x_{j_p - 1}^{-1}, ..., x_{i_p}^{-1} | -t_{j_p} : -t_{j_p - 1} : ... : -t_{i_p}) \cdot \tag{188}$$

$$\widehat{\mathrm{Li}}(*, x_{j_p+1}, x_{j_p+2}, ..., x_{i_{p+1}-1} | t_{j_p} : t_{j_p+1} : ... : t_{i_{p+1}-1}) \Big) \tag{189}$$

where the sum is over all sequences $\{i_\alpha\}$ *and* $\{j_\alpha\}$ *satisfying conditions*

$$i_\alpha \le j_\alpha < i_{\alpha+1} \quad \textit{for any } 0 \le \alpha \le k, \qquad j_0 = i_0 = 0 \tag{190}$$

Observe that this map is well defined on the generators. This is not granted for free by the formula for Δ since, according to definition (183), the shift $t_i \longmapsto t_i + t$ does not change the generating series $\widehat{\mathrm{Li}}(x_0, x_1, ..., x_m|t_0 : t_1 : ... : t_m)$. However since $x_0 \cdot ... \cdot x_m = 1$ it also does not change formula (188): one has $\prod_{p=1}^{k} X^t_{i_p \to i_{p+1}} = (x_0 \cdot ... \cdot x_m)^t = 1$.

Theorem 11.2 *Δ is a well defined homomorphism of algebras.*

Proof. This statement is equivalent to a collection of statements, one for each weight w. We will use the induction on w.

Let $\widetilde{Sh}^{\mathrm{Li}}_{\bullet}(G)$ be the graded vector space generated by the symbols (179) subject to the relations 1 and 3. One shows that the formula for Δ provide a well defined map of graded linear vector spaces $\Delta : \widetilde{Sh}^{\mathrm{Li}}_{\bullet}(G) \longmapsto \widetilde{Sh}^{\mathrm{Li}}_{\bullet}(G) \otimes \widetilde{Sh}^{\mathrm{Li}}_{\bullet}(G)$. We left the details to the reader.

Now we need to handle the shuffle relations. This is the complicated part of the proof, and we provide a detailed account of it. We picture a generator $\widehat{\mathrm{Li}}(x_0, ..., x_m|t_0 : ... : t_m)$ by a segment subdivided into $m+1$ arcs labelled by $(x_i|t_i)$. Every term of the formula for the coproduct of this generator is determined by the following geometric procedure: we subdivide this segment into a union of $k+1$ little segments, each being a union of arcs. We mark (by cross) an arc on each of these little segments. We assume in addition that this marking is special, i.e. in the very left segment we always mark the very left arc.

The product

$$\widehat{\mathrm{Li}}(*, x_1, ..., x_m|t_0 : t_1 : ... : t_m) \cdot \widehat{\mathrm{Li}}(*, y_1, ..., y_n|t_0 : s_1 : ... : s_n) \tag{191}$$

is given by the sum of the terms corresponding to the generalized shuffles of symbols $(x_i|t_i)$ and $(y_j|s_j)$. More precisely, a given generalized shuffle of depth $-p$ provides 2^p terms in the formula. Thus applying Δ to this sum we get sum of the terms corresponding to the following data:

a generalized shuffle for (191) plus a choice of one of the 2^p related terms;

a subdivision of the corresponding segment into little marked segments.

Choose integers $1 \leq a \leq b \leq m$ and $1 \leq c \leq d \leq n$. Then there is the following subset of x_i's and y_j's:

$$\{x_a, ..., x_{b-1}\} \cup \{y_c, y_{c+1}, ..., y_{d-1}\} \tag{192}$$

Consider all the terms which correspond to the above data such that the product of x_i's and y_j's on the p-th little segment equals to the product of elements in (192). Then there are four possible types of marking of the p-th little segment:

$$\text{i)} \quad (x_i|t_i), \qquad \text{ii)} \quad (y_j|s_j), \qquad \text{iii)} \quad (x_i y_j|t_i) \qquad \text{iv)} \quad (x_i y_j|s_j)$$

where $a \leq i \leq b, c \leq j \leq d$. The corresponding left factors of the terms in the coproduct look as follows:

$$\widehat{\mathrm{Li}}(..., X_{a \to b} \cdot Y_{c \to d}, ...|... : t_i : ...) \tag{193}$$

$$\widehat{\mathrm{Li}}(..., X_{a\to b}\cdot Y_{c\to d}, ...|... : s_j : ...)$$
$$\widehat{\mathrm{Li}}(..., X_{a\to b}\cdot Y_{c\to d}, ...|... : t_i : ...)\frac{1}{t_i - s_j} \tag{194}$$
$$\widehat{\mathrm{Li}}(..., X_{a\to b}\cdot Y_{c\to d}, ...|... : s_j : ...)\frac{1}{s_j - t_i}$$

Let us assume first that $a < b, c < d$. We claim that then the sum of the terms in the formula for Δ over all generalized shuffles with the markings of p-th little segments of types i) (i.e. in the left factors of these terms are as in (192)) equals to

$$(193)\otimes(X_{a\to b}\cdot Y_{c\to d})^{t_i}\cdot\Big(\sum_{k=c}^{d-1}(-1)^{i+k-a-c+1}\cdot\widehat{\mathrm{Li}}(*, x_{i-1}^{-1}, ..., x_a^{-1}|-t_i : -t_{i-1} : ... : -t_a)\cdot$$
$$\widehat{\mathrm{Li}}(*, y_k^{-1}, ..., y_c^{-1}| - t_i : -s_k : ... : -s_c)\cdot\widehat{\mathrm{Li}}(*, y_{k+1}, ..., y_{d-1}|t_i : s_{k+1} : ... : s_{d-1}) \tag{195}$$
$$\widehat{\mathrm{Li}}(*, x_{i+1}, ..., x_{b-1}|t_i : t_{i+1} : ... : t_{b-1})\Big)\cdot\prod_{q\neq p}(\text{contribution of } q\text{-th little segment})$$

To check this formula at any given weight w we may use the shuffle product relations of weights $< w$. Consider the generalized shuffles related to product (191) where in addition the arcs on the p-th little segment located on the right of $(x_i|t_i)$-arc are labelled by the elements of $\{x_{i+1}, ..., x_{b-1}\}\cup\{y_{k+1}, ..., y_{d-1}\}$. Then using the shuffle product formula we see that contribution of factor (189) to Δ(sum of such generalized shuffles) is (factor 3) $\times$ (factor 4) in the sum (195). The use of the shuffle product formula here is legitimate since we proceed by induction, and we may assume that the left factor has a nonzero weight.

The same kind of arguments, which take into account the signs in (188), show that (188) contributes (factor 1) $\times$ (factor 2) in the sum (195). Formula (195) is proved.

A completely similar considerations for the terms of type iii) lead to a similar to (195) formula where the (factor 2) $\times$ (factor 3) is replaced by

$$\frac{1}{t_i - s_k}\cdot\widehat{\mathrm{Li}}(*, y_{k-1}^{-1}, ..., y_c^{-1}| - t_i : -s_{k-1} : ... : -s_c)\cdot \tag{196}$$
$$\widehat{\mathrm{Li}}(*, y_{k+1}, ..., y_{d-1}|t_i : s_{k+1} : ... : s_{d-1})$$

By the inversion formula (187)

$$\sum_{k=c}^{d-1}\Big((196) + (\text{factor 2})\cdot(\text{factor 3) in (195}\Big) = \widehat{B}(y_c, ..., y_{d-1}|s_c - t_i, ..., s_{d-1} - t_i)$$

Using definition (185) we see that

$$Y_{c\to d}^{t_i}\widehat{B}(y_c, ..., y_{d-1}|s_c - t_i, ..., s_{d-1} - t_i) =$$

$$\sum_{j=c}^{d-1}(-1)^{j-c}\frac{Y^{s_j}_{c\to d}}{s_j-t_i}\cdot \widehat{\rm Li}(y^{-1}_{j-1},...,y_c^{-1}|s_j-s_{j-1},...,s_j-s_c)\cdot$$

$$\widehat{\rm Li}(y_{j+1},...,y_{d-1}|s_{j+1}-s_j,...,s_{d-1}-s_j)$$

Therefore (195) plus a similar expression for the type iii) terms equals to

$$\frac{(193)}{s_j-t_i}\otimes\Big(\sum_{i=a}^{b-1}(-1)^{i-a}X^{t_i}_{a\to b}\cdot\widehat{\rm Li}(x^{-1}_{i-1},...,x_a^{-1}|t_i-t_{i-1},...,t_i-t_a)\cdot \tag{197}$$

$$\widehat{\rm Li}(x_{i+1},...,x_{b-1}|t_{i+1}-t_i,...,t_{b-1}-t_i)\Big)\cdot$$

$$\Big(\sum_{j=c}^{d-1}(-1)^{j-c}Y^{s_j}_{c\to d}\cdot\widehat{\rm Li}(y^{-1}_{j-1},...,y_c^{-1}|s_j-s_{j-1},...,s_j-s_c)\cdot$$

$$\widehat{\rm Li}(y_{j+1},...,y_{d-1}|s_{j+1}-s_j,...,s_{d-1}-s_j)\Big)$$

The remaining two cases, when the labels are of types ii) or iv), are handled the same way.

The computation of the contribution in the case when $a=b$ or $c=d$ is trivial.

Comparing this with

$$\Delta\widehat{\rm Li}(*,x_1,...,x_m|t_0:t_1:...:t_m)\cdot\Delta\widehat{\rm Li}(*,y_1,...,y_n|t_0:s_1:...:s_n)$$

we get the same result. Indeed, this expression lives in $Sh^I_\bullet(G)\otimes Sh^I_\bullet(G)$, and the left factor is

$$\sum\widehat{\rm Li}(X_{i_0\to i_1},...,X_{i_{k-1}\to i_k}|t_{j_0}:...:t_{j_{k-1}})\cdot\widehat{\rm Li}(Y_{i'_0\to i'_1},...,Y_{i'_{l-1}\to i'_l}|s_{j'_0}:...:s_{j'_{l-1}})$$

where the sum is over all sequences i_α, j_α and i'_β, j'_β satisfying the condition (190). Let us write this as a sum over generalized shuffles, and take a term in the sum we get. Then the arcs on the corresponding segment are labelled either by

$$(X_{a\to b}\cdot Y_{c\to d}|t_i)\quad\text{or}\quad(X_{a\to b}\cdot Y_{c\to d}|s_j)\quad\text{where}\quad a\leq i<b,\quad c\leq j<d \tag{198}$$

or by $(X_{a\to b}|t_i)$ or $(Y_{c\to d}|s_j)$. In the first case in (198) the corresponding term in $Sh^{\rm Li}_\bullet(G)\otimes Sh^{\rm Li}_\bullet(G)$ is given by (197). The matching pattern in the second case in (198) is completely similar. The other two cases are trivial. The theorem is proved.

Theorem 11.3 *$Sh^{\rm Li}_\bullet(G)$ is a well defined commutative algebra with the coproduct Δ and the product given by the* Li*-shuffle formula.*

The key ingredient of the proof, the Hopf axiom, is given by theorem 11.2. To finish the proof it remains to check that Δ is coassociative, i.e. $\Delta \circ \Delta = 0$. We leave the details to the reader.

3. Another proof of theorem 7.5. Theorem 11.2 implies that applying the restricted coproduct Δ' to an element presenting the left hand side of the weight w Li-shuffle or inversion relation we get zero modulo the Li-shuffle and inversion relations of weights $\leq w-1$, plus the obvious additivity relation 1. This plus lemma 8.2 immediately imply that in the Hodge or étale setting the corresponding element is a constant. Let us taking specialization along a curve from the generic point to to $x_1 = ... = x_m = 0$. We claim that the first column of the period matrix tends to the column $(1, 0, ..., 0)$ when $x_i \to 0$. Indeed, it follows from the given in chapter 5 of [G7] explicit description of the variation of Hodge-Tate structures related to multiple polylogarithms that the weight w entry of the first column of the period matrix is given by a function of the shape

$$\mathrm{Li}_{k_1,k_2,...}(x_1 \cdot ... \cdot x_{i_1}, x_{i_1+1} \cdot ... \cdot x_{i_1+i_2}, ...) \qquad w = k_1 + ... + k_m$$

Moreover we may assume all of them are given by the convergent power series. Thus if $w > 0$ each of these functions obviously goes to zero as $x_i \to 0$. Thus the equivalence class of the framed object obtained after the specialization is zero. Therefore the constant is zero. So the theorem follows.

There is a natural homomorphism of Hopf $\mathbb{Q}$-algebras $Sh^{\mathrm{Li}}_{\bullet}(\mathbb{C}^*) \longrightarrow \mathcal{H}_{\bullet}$.

4. Reamrks on an integral version of the inversion formula. Our definition of $\widehat{B}(x|t)$ was motivated by Kronecker's formula

$$B(\varphi|t) := \sum_{-\infty<k<\infty} \frac{e^{2\pi i(\varphi k)}}{(k-t)} = -2\pi i \frac{e^{2\pi i\{\varphi\}t}}{e^{2\pi it}-1} = -2\pi i \cdot \sum_{n\geq 0} B_n(\{\varphi\}) \frac{(2\pi it)^{n-1}}{n!}$$

Here $\{\varphi\}$ is the fractional part of φ. However this formula and $\widehat{B}(x|t) := x^t/t$ agree only if we suppress all the Bernoulli numbers except $B_0 = 1$ from Kronecker's formula. The discussion below shows why this definition of $\widehat{B}(x|t)$ is meaningful if we work modulo torsion, and how one can try to make it integral.

The classical formula for the Bernoulli numbers

$$B_{2n} := -(2\pi i)^{-2n} 2(2n)!\zeta(2n)$$

suggests the following definitions

$$B_{2n}^{\mathcal{M}} := -2(2n)!\zeta^{\mathcal{M}}(2n); \qquad B_1^{\mathcal{M}} := \log^{\mathcal{M}}(-1), \quad B_0^{\mathcal{M}} := 1$$

$$B_n^{\mathcal{M}}(\log^{\mathcal{M}}(x)) := \sum_{k=0}^{n} \binom{n}{k} B_k^{\mathcal{M}} (\log^{\mathcal{M}}(x))^{n-k}$$

$$B^{\mathcal{M}}(x|t) := \sum_{n\geq 0} B_n^{\mathcal{M}}(\log^{\mathcal{M}}(x)) \frac{t^{n-1}}{n!}$$

Here the objects we introduce belong to the abelian group of framed mixed Tate motives. This group is supposed to be defined *integrally.* We are not discussing below how to define this group. Its l-adic realization should be given by Galois $\mathbb{Z}_l$-modules. Its Hodge version is given by the equivalence classes of integral Hodge-tate structures. So "the motivic Bernoulli numbers" $B_n^{\mathcal{M}}$ are no longer numbers but rather weight n framed mixed Tate motives, which are torsion elements for $n > 0$.

We claim that one has the inversion formula

$$\mathrm{Li}^{\mathcal{M}}(x|t) - \mathrm{Li}^{\mathcal{M}}(x^{-1}|-t) = -B^{\mathcal{M}}(x|t)$$

For example

$$\mathrm{Li}_1^{\mathcal{M}}(x) - \mathrm{Li}_1^{\mathcal{M}}(x^{-1}) = -\log_1^{\mathcal{M}}(1-x) + \log_1^{\mathcal{M}}(1-x^{-1}) =$$
$$-\log^{\mathcal{M}}(x) + \log_1^{\mathcal{M}}(-1) = -B_1^{\mathcal{M}}(\log^{\mathcal{M}}(x))$$

Further

$$\begin{aligned}\mathrm{Li}_2^{\mathcal{M}}(x) + \mathrm{Li}_2^{\mathcal{M}}(x^{-1}) &= -\frac{1}{2}\Big((\log^{\mathcal{M}}(x))^2 + 2\log^{\mathcal{M}}(-1)\cdot\log^{\mathcal{M}}(x) + B_2^{\mathcal{M}}\Big) = \\ & -\frac{1}{2}\Big((\log^{\mathcal{M}}(x))^2 + B_2^{\mathcal{M}}\Big)\end{aligned}$$

since $2\log^{\mathcal{M}}(-1) = 0$, and so on.

The formula we want to demonstrate has been checked in the example above for $n = 1$. At $x = 1$ this formula is trivial for odd $n > 1$, and boils down to identity

$$\mathrm{Li}_{2n}^{\mathcal{M}}(1) + \mathrm{Li}_{2n}^{\mathcal{M}}(1) \quad = \quad -\frac{1}{(2n)!}B_{2n}^{\mathcal{M}}$$

which is clear by the very definition of the Bernoulli framed motives $B_k^{\mathcal{M}}$. Thus it remains to check that it is killed by the restricted coproduct Δ', which we do by induction using the following formula (see [G7]):

$$\Delta'\mathrm{Li}^{\mathcal{M}}(x|t) = \mathrm{Li}^{\mathcal{M}}(x|t) \otimes (x^t - 1)$$

Namely,

$$\Delta'\Big(\mathrm{Li}^{\mathcal{M}}(x|t) - \mathrm{Li}^{\mathcal{M}}(x^{-1}|-t)\Big) = \Big(\mathrm{Li}^{\mathcal{M}}(x|t) - \mathrm{Li}^{\mathcal{M}}(x^{-1}|-t)\Big) \otimes (x^t - 1)$$

so using the induction assumption we can rewrite this as $-B^{\mathcal{M}}(x|t) \otimes (x^t - 1)$. It remains to notice that observing $\Delta'(B_n^{\mathcal{M}}) = 0$ it is easy to check that

$$\Delta'(-B^{\mathcal{M}}(x|t)) = -B^{\mathcal{M}}(x|t) \otimes (x^t - 1)$$

The theorem is proved.

The following result provides the integral motivic version of the inversion formula (187).

Theorem 11.4 *The inversion formula (187), where* $\widehat{\mathrm{Li}}$ *is changed to* $\mathrm{Li}^{\mathcal{M}}$*, and* $\widehat{B}(x|t) := B^{\mathcal{M}}(x|t)$*, holds for the depth* m *motivic multiple polylogarithms.*

There are three different proofs of this result:

1) Argue as in the proof of the previous theorem.

2) Take the proof of proposition 2.7 in section 2.6 in [G7] and make it motivic following the same ideas as in chapter 9.

3) Follow the proof of theorem 4.1 in [G4] and use the motivic double shuffle relations.

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