Two Remarks On Moishezon Calabi-Yau 3-Folds

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Introduction.

In this paper, we shall prove the following existence theorem concerning with a Calabi-Yau 3-fold, i.e., a 3-dimensional simply connected compact complex manifold with trivial canonical bundle, with 2 extremally distinguished properties each of which never occurs for a projective variety.

Theorem 1. *There exists a Moishezon Calabi-Yau 3-fold Y which satisfies that*

(1) $Pic Y = \mathbb{Z} \cdot L$ *with* $L^3 = 0$, *i.e., the cubic form is identically zero,*
(2) Y *contains an effective algebraic 1-cycle* ℓ *which moves algebraically and sweeps out whole* Y *but* ℓ *itself is homologous to zero.*

This phenomenon is related to Kollár's problem ([Ko, 5.16]) and Nakamura's example ([Na]). We shall construct such Y by taking an elementary transformation, called flop, of a (projective) Calabi-Yau 3-fold X described in the next theorem.

Theorem 2. *There exists a projective Calabi-Yau 3-fold X which satisfies that*

(1) $Pic X = \mathbb{Z} \cdot H$, *where* $H^3 = 8$ *and* $Bs|H| = \emptyset$, *and* $Tor\, H_2(X, \mathbb{Z}) = 0$,
(2) X *contains a smooth rational curve* C *with* $C.H = 2$ *and* $N_{C|X} = \mathcal{O}_C(-1)^{\oplus 2}$.

We shall prove this theorem by modifying Katz's argument on a quintic 3-fold ([Ka]) to that on a complete intersection of a quadratic and a quartic in $\mathbb{P}^5$, or in other words, by showing that a generic complete intersection X of a quadratic and a quartic in $\mathbb{P}^5$ contains a smooth conic C with $N_{C|X} = \mathcal{O}_C(-1)^{\oplus 2}$.

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Proof of Theorem 2.

Let us fix a smooth conic in $\mathbb{P}^5$ defined by

$$C := \{[s^2 : st : t^2 : 0 : 0 : 0]\} \subset \mathbb{P}^5 = \{[x_0 : x_1 : x_2 : x_3 : x_4 : x_5]\},$$

where $[s : t]$ is a homogeneous coordinate of $C = \mathbb{P}^1$. Let us consider a complete intersection $X = F \cap G$ in $\mathbb{P}^5$, where F (resp.G) is defined by the following polynomial f of degree 2 (resp. g of degree 4):

$$f := (x_1^2 - x_0 x_2) + f_3 x_3 + f_4 x_4 + f_5 x_5,\; f_i \in H^0(\mathcal{O}_{\mathbb{P}^5}(1)),$$

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$$g := g_3x_3 + g_4x_4 + g_5x_5,\ g_i \in H^0(\mathcal{O}_{\mathbb{P}^5}(3)).$$

Since C is contained in X, in order to complete the proof, it is enough to show the following claim 1:

Claim 1. *For general f and g, we have,*

(1) *X is non-singular, and*
(2) $N_{C|X} = \mathcal{O}_C(-1)^{\oplus 2}$.

In fact, the other conditions in Theorem 2 are automatically satisfied by Lefschetz Theorem and by the the adjunction formula.

The statement (1) follows from Bertini's argument. Let Λ_1 be the subsystem of $|\mathcal{O}_{\mathbb{P}^5}(2)|$ consisting of f defined above. By taking (f_3, f_4, f_5) as $(0,0,0)$, $(x_3,0,0)$, $(0,x_4,0)$, $(0,0,x_5)$, we see that $Bs\Lambda_1 \subset (x_1^2 - x_0x_2 = 0) \cap (x_3 = 0) \cap (x_4 = 0) \cap (x_5 = 0) = C$, so that $Sing\, F \subset C$ for general f. But, since $(\frac{\partial f}{\partial x_i})|_C = (-t^2, 2st, -s^2, f_3|_C, f_4|_C, f_5|_C)$, F is also non-singular along C. Thus F is non-singular for general f. Let us consider the subsystem Λ_2 of $|\mathcal{O}_F(4)|$ consisting of g defined above on a non-singular F. By taking (g_3, g_4, g_5) as $(x_3^3, 0, 0)$, $(0, x_4^3, 0)$, $(0,0,x_5^3)$, we see that $Bs\Lambda_2 \subset F \cap (x_3 = 0) \cap (x_4 = 0) \cap (x_5 = 0) = C$, so that $Sing\, X \subset C$ for general f and g.

But, since

$$\begin{pmatrix} \frac{\partial f}{\partial x_i}|_C \\ \frac{\partial g}{\partial x_i}|_C \end{pmatrix} = \begin{pmatrix} -t^2 & 2ts & -s^2 & f_3|_C & f_4|_C & f_5|_C \\ 0 & 0 & 0 & g_3|_C & g_4|_C & g_5|_C \end{pmatrix},$$

X is nonsingular along C if $t^2g_3|_C = s^2g_3|_C = t^2g_4|_C = s^2g_4|_C = t^2g_5|_C = s^2g_5|_C = 0$ has no common roots $[s:t]$. But this condition is clearly Zarski open condition and it is satisfied by $(g_3|_C, g_4|_C, g_5|_C) = (s^6+t^6, 2s^6+t^6, s^6+2t^6)$. Thus X is non-singular for general f and g.

We shall prove (2). For a non-singular X, we consider the following 3 standard exact sequences just like as in [Ka, Appendix B]:

(a) $$0 \longrightarrow \mathcal{O}_C \xrightarrow{\varphi_1} \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 6}|_C = \mathcal{O}_C(2)^{\oplus 6} \xrightarrow{\varphi_2} T_{\mathbb{P}^5}|_C \longrightarrow 0$$

(b) $$0 \longrightarrow T_X|_C \xrightarrow{\varphi_3} T_{\mathbb{P}^5}|_C \xrightarrow{\varphi_4} N_{X|\mathbb{P}^5}|_C = \mathcal{O}_C(4) \oplus \mathcal{O}_C(8) \longrightarrow 0$$

(c) $$0 \longrightarrow T_C \longrightarrow T_X|_C \longrightarrow N_{C|X} \longrightarrow 0.$$

Note that every homomorphism above is described by a matrix whose coefficients are in $\oplus H^0(\mathcal{O}_C(a))$, because every vector bundle on $\mathbb{P}^1$ decomposes into a direct sum of line bundles. Since $\varphi_1 \in Hom(\mathcal{O}_C, \mathcal{O}_C(2)^{\oplus 6})$ is described by the matrix $(s^2, st, t^2, 0, 0, 0)^t$ by definition, a matrix representation of $\varphi_2 \in Hom(\mathcal{O}_C(2)^{\oplus 6}, T_{\mathbb{P}^5}|_C)$ is

$$\begin{pmatrix} t & -s & 0 & 0 & 0 & 0 \\ 0 & t & -s & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and we have $T_{\mathbb{P}^5}|_C = \mathcal{O}_C(3)^{\oplus 2} \oplus \mathcal{O}_C(2)^{\oplus 3}$ just as is proved in [Ka, Appendix B, page158]. Now, in order to finish the proof of (2) in claim 1, it is enough to show the next claim 2:

Claim 2. *φ_4 is surjective for general f and g.*

In fact, if claim 2 is true, then we know that $H^1(T_{C|X}) = 0$ by the sequence (b) and by $H^1(T_{\mathbb{P}^5|C}) = H^1(\mathcal{O}_C(3))^{\oplus 2} \oplus H^1(\mathcal{O}_C(2))^{\oplus 3} = 0$. Thus we get $H^1(N_{C|X}) = 0$ by the sequence (c). Since $K_X = 0$, this induces the desired equality $N_{C|X} = \mathcal{O}_C(-1)^{\oplus 2}$.

We shall prove claim 2. Since the map

$$\mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 6} \longrightarrow T_{\mathbb{P}^5} \longrightarrow N_{X|P^5} = \mathcal{O}_X(2) \oplus \mathcal{O}_X(4)$$

is given by

$$(h_0, ..., h_5) \mapsto \sum h_i \frac{\partial}{\partial x_i} \mapsto (\sum h_i \frac{\partial f}{\partial x_i}, \sum h_i \frac{\partial g}{\partial x_i}),$$

the matrix representation

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix}$$

of $\varphi_4 \in Hom(T_{\mathbb{P}^5}|_C, N_{X|\mathbb{P}^5}|_C) = Hom(\mathcal{O}_C(3)^{\oplus 2} \oplus \mathcal{O}_C(2)^{\oplus 3}, \mathcal{O}_C(4) \oplus \mathcal{O}_C(8))$ must satisfies the equality:

$$A \begin{pmatrix} t & -s & 0 & 0 & 0 & 0 \\ 0 & t & -s & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -t^2 & 2ts & -s^2 & \overline{f}_3 & \overline{f}_4 & \overline{f}_5 \\ 0 & 0 & 0 & \overline{g}_3 & \overline{g}_4 & \overline{g}_5 \end{pmatrix},$$

where $\overline{f}_i = f_i|_C$ and $\overline{g}_i = g_i|_C$.

Thus, φ_4 is nothing but the following map:

$$\begin{pmatrix} -t & s & \overline{f}_3 & \overline{f}_4 & \overline{f}_5 \\ 0 & 0 & \overline{g}_3 & \overline{g}_4 & \overline{g}_5 \end{pmatrix} : \mathcal{O}_C(3)^{\oplus 2} \oplus \mathcal{O}_C(2)^{\oplus 3} \to \mathcal{O}_C(4) \oplus \mathcal{O}_C(8).$$

Thus the following 3 conditions (I), (II), (III) are equivalent to each other:

(I) $\varphi_4 : H^0(T_{\mathbb{P}^5}|_C) \longrightarrow H^0(N_{X|P^5}|_C)$ is surjective,

(II) For every $(\varphi, \psi) \in H^0(\mathcal{O}_C(4)) \oplus H^0(\mathcal{O}_C(8))$, there exists an element $(a, b, c, d, e) \in H^0(\mathcal{O}_C(3))^{\oplus 2} \oplus H^0(\mathcal{O}_C(2))^{\oplus 3}$ such that $\varphi = -ta + sb + \overline{f}_3 c + \overline{f}_4 d + \overline{f}_5 e$ and $\psi = \overline{g}_3 c + \overline{g}_4 d + \overline{g}_5 e$,

(III) For every $\psi \in H^0(\mathcal{O}_C(8))$, there exists an element $(c, d, e) \in H^0(\mathcal{O}_C(2))^{\oplus 3}$ such that $\psi = \overline{g}_3 c + \overline{g}_4 d + \overline{g}_5 e$, i.e., the homomorphism

$$(\overline{g}_3, \overline{g}_4, \overline{g}_5) : H^0(\mathcal{O}_C(2))^{\oplus 3} \longrightarrow H^0(\mathcal{O}_C(8))$$

defined by$(c, d, e) \mapsto \overline{g}_3 c + \overline{g}_4 d + \overline{g}_5 e$ is surjective.

Note that the last condition (III) is Zariski open condition for g. On the other hand, since every element $\psi(s,t) = \sum_{i=o}^{8} a_i s^i t^{8-i}$ of $H^0(\mathcal{O}_C(8))$ is written as $\psi(s,t) = s^6(\sum_{i=6}^{8} a_i s^{i-6} t^{8-i}) + s^3t^3(\sum_{i=3}^{5} a_i s^{i-3} t^{5-i}) + t^6(\sum_{i=0}^{2} a_i s^i t^{2-i})$, the last condition of (III) is satisfied by $(g_3, g_4, g_5) = (x_0^3, x_1^3, x_2^3)$.

Now we have just finished the proof of Theorem 2. Q.E.D.

Proof of Theorem 1

Let X be a projective Calabi-Yau 3-fold which satisfies the condition of Theorem 2. Let us take an elementary transformation, or flop, of X along C:

$$C \subset X \xleftarrow{\pi_1} C \times D = E \subset Z \xrightarrow{\pi_2} D \subset Y,$$

where π_1 is the blowing up of X along $C = \mathbb{P}^1$, $E = C \times D (= \mathbb{P}^1 \times \mathbb{P}^1)$ is the exceptional divisor on Z, and π_2 is the contraction of E along C. Since $E|_E = \pi_1^*\mathcal{O}_C(-1) \otimes \pi_2^*\mathcal{O}_C(-1)$ (because, for example, $\pi_1^*\mathcal{O}_C(-2) \otimes \pi_2^*\mathcal{O}_D(-2) = K_E = (\pi_1^*K_X + 2E)|_E = 2E|_E$), and since $X - C \simeq Y - D$, we know that Y is a smooth Calabi-Yau 3-fold with $Pic\, Y = \mathbb{Z}L$, where L is the proper transform of H. Since $C.H = 2$, we have $\pi_1^*H = \pi_2^*L - 2E$. On the other hand, since $(\pi_1^*H)^3 = H^3 = 8$, $(\pi_1^*H)^2.E = H^2.\pi_{1*}E = 0$, $(\pi_1^*H).E^2 = \pi_1^*(H|_C).E|_E = -2$ by $C.H = 2$, and $E^3 = (E|_E)^2 = 2$, we have $L^3 = (\pi_2^*L)^3 = (\pi_1^*H + 2E)^3 = 0$. Thus Y satisfies the condition (1) of Theorem 1. Now, we shall prove that Y satisfies the condition (2) of Theorem 1. Since H is the generator of $Pic\, X$ so that every member of $|H|$ is irreducible, and since $|H|$ is free, we see that $h := H_1 \cap H_2$ is an effective algebraic 1-cycle for every $H_1 \neq H_2$ in $|H|$ and that h moves algebraically and sweeps out whole X. Thus every member of $|L|$ is also irreducible and $\ell = L_1 \cap L_2$ is an effective algebraic 1-cycle on Y for every $L_1 \neq L_2$ in $|L|$ and ℓ moves algebraically and sweeps out whole Y because $X - C \simeq Y - D$ and $Bs|L| = D$. But, since $0 = L^3 = L.\ell$ and since $H^2(Y, \mathbb{Z}) \simeq Pic\, Y = \mathbb{Z}L$, ℓ must be a torsion element in $H_2(Y, \mathbb{Z})$. But, by the property of blowing up and by our assumption, we have $Tor\, H_2(Y, \mathbb{Z}) \simeq Tor H_2(X, \mathbb{Z}) = 0$. Thus ℓ is homologous to zero. This completes the proof of Theorem 1. Q.E.D.

Appendix.

1. There is another example of a Moishezon threefold M with $Pic\,M \cong \mathbb{Z}\cdot\mathcal{O}_M(L)$ and $(L^3)_M = 0$, which was constructed by Nakamura:

(1.1) Example [(3.3), Na]. *There is a smooth Moishezon threefold M which has the following properties:*

(1) $H^1(M;\mathbb{Z}) = 0$.
(2) $H^2(M;\mathbb{Z}) \cong Pic\,M = \mathbb{Z}\cdot\mathcal{O}_M(L)$, *where L is a rational surface with* $(L^3)_M = 0$.
(3) $K_M = -2L$.
(4) $H^i(M;\mathcal{O}_M) = 0$ *for* $1 \le i \le 3$.

From his construction, one sees M is a compactification of $\mathbb{C}^2 \times \mathbb{C}^*$.

On the other hand, Peternell-Schneider (**[pp.131,PS-1]**,**[pp.463,PS-2]**) studied on the projectivity of a Moishezon compactification of $\mathbb{C}^3$ with the second Betti number equal to one. Let (X, Y) be such a Moishezon compactification of $\mathbb{C}^3$ with $b_2(X) = 1$ (i.e., Y is irreducible). Then we have **[BM]**:

(1.a) $H^1(X;\mathbb{Z}) \cong H^1(Y;\mathbb{Z}) = 0$.
(1.b) $H^2(X;\mathbb{Z}) \cong H^2(Y;\mathbb{Z}) = \mathbb{Z}$.
(1.c) $H^i(X;\mathcal{O}_X) = 0$ for $1 \le i \le 3$.
(1.d) $Pic\,X \cong \mathbb{Z}\cdot\mathcal{O}_X(Y)$.
(1.e) $K_X = -dY$ $(d > 0)$.

As a threefold, the above M and X have similar properties, but as a pair, (M, L) and (X, Y) are different. In fact, in the former case, the homomorphism $H^2(M;\mathbb{Z}) \longrightarrow H^2(L;\mathbb{Z})$ is never isomorphic. This suggests the condition $H^2(X;\mathbb{Z}) \cong H^2(Y;\mathbb{Z}) = \mathbb{Z}$ plays a essential role for the projectivity of X. Finally one can prove the following:

Theorem 3 (cf. [PS-1] [PS-2]). *Let (X, Y) be a smooth Moishezon compactification of $\mathbb{C}^3$ with $b_2(X) = 1$. Then X is projective.*

Proof of Theorem 3

2. Let $f : X' \longrightarrow X$ be a birational projectivization of X and $B = \bigcup B_i$ (B_i is a curve or point) the fundamental set of the birational (inverse) map $f^{-1} : X \cdots \to X'$, hence f^{-1} is isomorphic on $X - B$. Let H' be an very ample divisor and put $H = f(H')$. Then H is a Cartier divisor on X. Since $Pic\,X = \mathbb{Z}\cdot\mathcal{O}_X(Y)$ and since both H and Y are non-zero effective divisors, one has $H = kY$ for some positive integer k. Since H is very ample on $X - B$, Y is ample on $X - B$.

(2.1) Lemma [(5.3.8),Ko]. *Let X be a normal proper n-dimensional algebraic space. Let D be a Cartier divisor which is ample in codimension one (i.e., there is a codimension two subset $Z \subset X$ such that $D|_{X-Z}$ is ample). Then we have* $H^{n-1}(X;\mathcal{O}_X(K_X + D)) = 0$.

Since $K_X = -dY$ and since Y is ample in codimension one, by (2.1), we obtain

(2.2) Lemma. $H^2(X;\mathcal{O}_X((t-d)Y)) = 0$ *for any* $t \in \mathbb{Z}$ $(t > 0)$.

(2.3) Corollary.

(1) $H^i(X;\mathcal{O}_X(-Y)) = 0$ *for* $0 \le i \le 2$.
(2) $H^3(X;\mathcal{O}_X(-Y)) = 0$ *if* $d \ge 2$, $= \mathbb{C}$ *if* $d = 1$.
(3) $H^1(Y;\mathcal{O}_Y) = H^2(Y;\mathcal{O}_Y) = 0$ *if* $d \ge 2$ and $H^1(Y;\mathcal{O}_Y) = 0,\ H^2(Y;\mathcal{O}_Y) = \mathbb{C}$ *if* $d = 1$

Proof. Consider an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-Y) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0\ .$$

By (1.c),(2.2) and the Serre duality theorem, one obtains the conclusion. □

(2.4) Lemma. *$H^2(Y;\mathbb{Z}) \cong Pic\,Y = \mathbb{Z}\cdot N_Y$, where $N_Y := \mathcal{O}_Y(Y)$.*

Proof. Consider the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0\ .$$

Since $N_Y \in H^1(Y;\mathcal{O}_Y^*) \neq 0$, by (1.a), (1.b) and (2.3), we have the claim. □

3. We may assume that Y is non-normal (irreducible). In fact, if Y is normal, then the projectivity of X is proved by Brenton-Morrow **[BM]** and Peternel-Schneider [**(1.1), PS-1**], [**PS-2**]). Since X is smooth, Y is Gorenstein. Let K_Y be the canonical (Cartier) divisor. By the adjunction formula, one has $K_Y = (1-d)N_Y$ ($d \ge 1$). Let $\sigma : \overline{Y} \longrightarrow Y$ be the normalization, and $\mathcal{I} \subset \mathcal{O}_Y$ be the conductor of σ defining closed subscheme E in Y and $\overline{E}$ in $\overline{Y}$. Then we have

$$K_{\overline{Y}} = \sigma^* K_Y - \overline{E} = -(d-1)\sigma^* N_Y - \overline{E}$$

(as a Weil divisor) (cf. [**(3.34.2), Mo**]).

(3.1) Lemma. *$H^2(\overline{Y};\mathcal{O}_{\overline{Y}}) = 0$.*

Proof. In the case of $d = 1$, since $K_{\overline{Y}} = -\overline{E}$, one has easily $H^0(\overline{Y};\mathcal{O}_{\overline{Y}}(K_{\overline{Y}})) = 0$. By Serre duality theorem, we have the claim. In the case of $d \ge 2$, since $\overline{E}$ is effective, it is enough to show that $H^0(\overline{Y};\mathcal{O}_{\overline{Y}}(-(d-1)\sigma^* N_Y)) = 0$. In fact, since $H|_Y = kY|_Y = kN_Y$ is an effective divisor for a large integer $k > 0$, one has $H^0(\overline{Y};\mathcal{O}_{\overline{Y}}(-k\sigma^* N_Y)) = 0$. This yields $H^0(\overline{Y};\mathcal{O}_{\overline{Y}}(-(d-1)\sigma^* N_Y)) = 0$, hence $H^0(\overline{Y};\mathcal{O}_{\overline{Y}}(K_{\overline{Y}})) = 0$. □

Let $\mu : \widehat{Y} \longrightarrow \overline{Y}$ be the minimal resolution with the exceptional set $\Delta := \bigcup \Delta_i$ (Δ_i is irreducible) of μ. Since $K_{\widehat{Y}} = \mu^* K_{\overline{Y}} - \Sigma_i m_i \Delta_i$ ($m_i \ge 0$, $m_i \in \mathbb{Z}$, one has $H^0(\widehat{Y};\mathcal{O}_{\widehat{Y}}(mK_{\widehat{Y}}) = 0$ for any $m > 0$. Since Y is Moishezon, $\widehat{Y}$ is projective, indeed, $\widehat{Y}$ is a ruled surface over a smooth algebraic curve Γ of genus $q = h^1(\mathcal{O}_{\widehat{Y}})$. Since $b_3(\overline{Y}) = b_3(\widehat{Y}) = 2q \neq 1$ and since $H^2(\overline{Y};\mathcal{O}_{\overline{Y}}) = 0$ by (3.1), by [**Proposition 7, Br**], one sees $\overline{Y}$ is projective. Since σ is a finite morphism, we have

(3.2) Lemma. *Y is projective.*

4. Finally we shall prove the projectivity of X. Since Y is projective and since $Pic Y = \mathbb{Z} \cdot N_Y$ by (2.4), one sees N_Y is not trivial. Since $X - Y \cong \mathbb{C}^3$, N_Y is not negative line bundle by Grauert. Hence N_Y is positive (= ample) on Y. Since there is no positive dimensional compact analytic subvariety in $X - Y \cong \mathbb{C}^3$, by Nakai-Kleiman's criterion for ampleness, one sees $\mathcal{O}_X(Y)$ is ample. Therefore X is projective. The proof is complete.

(4.1) Remark. It is known that any analytic compactification of $\mathbb{C}^3$ with the second Betti number equal to one is Moishezon (see **[PS-2]**, hence it is projective by Theorem 3. Such a projective compactification of $\mathbb{C}^3$ is classified in **[Fu]**.

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