Non-holonomic Ideals in the Plane and Absolute Factoring

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Abstract

We study *non-holonomic* overideals of a left differential ideal $J \subset F[\partial_x,\ \partial_y]$ in two variables where F is a differentially closed field of characteristic zero. The main result states that a principal ideal $J = \langle P \rangle$ generated by an operator P with a separable *symbol* $symb(P)$, which is a homogeneous polynomial in two variables, has a finite number of maximal non-holonomic overideals. This statement is extended to non-holonomic ideals J with a separable symbol. As an application we show that in case of a second-order operator P the ideal $\langle P \rangle$ has an infinite number of maximal non-holonomic overideals iff P is essentially ordinary. In case of a third-order operator P we give few sufficient conditions on $\langle P \rangle$ to have a finite number of maximal non-holonomic overideals.

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1 Finiteness of the Number of Maximal Non-holonomic Overideals of an Ideal with a Separable Symbol

Let F be a differentially closed field (or universal in terms of [8], [9]) with derivatives $\partial_x,\ \partial_y$ and a linear partial differential operator $P = \sum_{i,j} p_{i,j} \partial_x^i \partial_y^j \in F[\partial_x,\ \partial_y]$ be of order n. Considering e. g. the field of rational functions $\mathbb{C}(x, y)$ as F is a quite different issue. The *symbol* $symb(P) = \sum_{i+j=n} p_{i,j} v^i w^j$ we treat as a homogeneous polynomial in two variables of degree n. We call a left ideal $I \subset F[\partial_x,\ \partial_y]$ *non-holonomic* if the degree of its Hilbert-Kolchin polynomial $ez + e_0$, i.e. if its *differential type* [8], equals 1. We study maximal non-holonomic overideals of the principal ideal $\langle P \rangle \subset F[\partial_x,\ \partial_y]$ (obviously there is an infinite number of maximal *holonomic* overideals of $\langle P \rangle$: for any solution $u \in F$ of $Pu = 0$ we get a holonomic overideal $\langle \partial_x - u_x/u, \partial_y - u_y/u \rangle \supset \langle P \rangle$). We assume w.l.o.g. that $symb(P)$ is not divisible by ∂_y (otherwise one can make a suitable transformation of the type $\partial_x \to \partial_x,\ \partial_y \to \partial_y + b\partial_x,\ b \in F$, in fact choosing b from the subfield of constants of F would suffice).

Clearly, factoring an operator P can be viewed as finding principal overideals of $\langle P \rangle$ and we refer to factoring over a universal field F as *absolute factoring*. We mention also that overideals of an ideal in connection with Loewy and primary decompositions were considered in [6].

Following [4] consider a homogeneous polynomial ideal $symb(I) \subset F[v, w]$ and attach a homogeneous polynomial $g = GCD(symb(I))$ to I. Lemma 4.1 [4] states that $deg(g) = e$ (called also the *typical differential dimension* of I [8]). As above one can assume w.l.o.g. that w does not divide g.

We recall (see [3], [4]) that (Ore [1]) the ring $R = (F[\partial_y])^{-1} F[\partial_x, \partial_y]$ consists of fractions of the form $\beta^{-1}r$ where $\beta \in F[\partial_y], r \in F[\partial_x, \partial_y]$. We also recall that one can represent $R = F[\partial_x, \partial_y] (F[\partial_y])^{-1}$ and two fractions are equal $\beta^{-1}r = r_1\beta_1^{-1}$ iff $\beta r_1 = r\beta_1$ [3], [4].

For a non-holonomic ideal I denote ideal $\overline{I} = RI \subset R$. Since ring R is left-euclidean (as well as right-euclidean) with respect to ∂_x over skew-field $(F[\partial_y])^{-1} F[\partial_y]$, we conclude that ideal $\overline{I}$ is principal, let $\overline{I} = \langle r \rangle$ for suitable $r \in F[\partial_x, \partial_y] \subset R$ (cf. [4]). Lemma 4.3 [4] implies that $symb(r) = w^m g$ for a certain integer $m \geq 0$ where g is not divisible by w.

Now we expose a construction introduced in [4]. For a family of elements $f_1, \dots, f_k \in F$ and rationals $1 > s_2 > \cdots > s_k > 0$ we consider a D-module being a vector space over F with a basis $\{G^{(s)}\}_{s\in\mathbb{Q}}$ where the derivatives of $G^{(s)} = G^{(s)}(f_1, \dots, f_k;\ s_2, \dots, s_k)$ are defined as

$$d_{x_i} G^{(s)} = (d_{x_i} f_1) G^{(s+1)} + (d_{x_i} f_2) G^{(s+s_2)} + \cdots + (d_{x_i} f_k) G^{(s+s_k)}$$

for $i = 1, 2$ using the notations $d_{x_1} = \partial_x, d_{x_2} = \partial_y$.

Next we introduce series of the form

$$\sum_{0\leq i<\infty} h_i G^{(s-\frac{i}{q})} \tag{1}$$

where q is the least common multiple of the denominators of $s_2, \dots, s_k$ (one can view (1) as an analogue of Newton-Puiseux series for *non-holonomic* D-modules).

Theorem 2.5 [4] states that for any linear divisor $v + aw$ of $symb(P)$ and any $f_1 \in F$ such that $(\partial_x + a\partial_y) f_1 = 0$ there exists a solution of $P = 0$ of the form (1) (and conversely, if (1) is a solution of $P = 0$ then $(\partial_x + a\partial_y) f_1 = 0$ for an appropriate divisor $v + aw$ of $symb(P)$). Furthermore, Proposition 4.4 [4] implies that any solution of the form (1) of $r = 0$ such that $(\partial_x + a\partial_y) f_1 = 0$ for suitable $a \in F$ (or equivalently $\partial_y f_1 \neq 0$) is also a solution of ideal I (then the appropriate linear form $v + aw$ is a divisor of g), the inverse holds as well.

In [5] we have designed an algorithm for factoring an operator P in case when $symb(P)$ is separable. In particular, in this case there is only a finite number (less than 2^n) of different factorizations of P. Now we show a more general statement for overideals of $\langle P \rangle$.

Theorem 1 *Let $symb(P)$ be separable. Then there exists at most $n = ord(P)$ maximal non-holonomic overideals of $\langle P \rangle \subset F[\partial_x, \partial_y]$. Moreover, if there exists a non-holonomic overideal $I \supset \langle P \rangle$ with the attached polynomial $g = GCD(symb(I))$ then there exists a unique non-holonomic overideal maximal among ones with the attached polynomial equal g.*

Proof. Let a non-holonomic ideal $I \supset \langle P \rangle$. Then $\beta P = r_1 r$ for suitable $\beta \in F[\partial_y], r_1 \in F[\partial_x, \partial_y]$ and a polynomial $g = GCD(symb(I))$ attached to I is a divisor of $symb(P)$.

We claim that for every pair of non-holonomic ideals $I_1, I_2 \supset \langle P \rangle$ to which a fixed polynomial g is attached, to their sum $I_1 + I_2$ also g is attached. Indeed, any solution of the form (1) of $P = 0$ such that $(v + aw)|g$, is a solution of $r = 0$ as well due to Lemma 4.2 [4] (cf. Proposition 4.4 [4]) taking into account that $symb(P)$ is separable, hence it is also a solution of I as it was shown above and by the same token is a solution of both I_1 and I_2 (in particular $I_1 + I_2$ is also non-holonomic). The claim is established.

Thus among non-holonomic overideals $I \supset \langle P \rangle$ to which a given polynomial $g|symb(P)$ is attached, there is a unique maximal one. Now take two maximal non-holonomic overideals $I, I' \supset \langle P \rangle$ to which polynomials g, g' are attached, respectively. Then g, g' are reciprocately prime. Indeed, if $v + aw$ divides both g, g' then arguing as above one can verify that (1) is a solution of $I + I'$, i. e. the latter ideal is non-holonomic which contradicts to maximality of I, I'. □

Corollary 1 *Let $symb(P)$ be separable. Suppose that there exist maximal non-holonomic overideals $I_1, \dots, I_l \supset \langle P \rangle$ such that for the respective attached polynomials $g_1, \dots, g_l$ the sum of their degrees $deg(g_1) + \cdots + deg(g_l) \geq n$. Then $\langle P \rangle = I_1 \cap \cdots \cap I_l$.*

Proof. As it was shown in the proof of Theorem 1, polynomials $g_j | symb(P),\ 1 \leq j \leq l$ are pairwise reciprocately prime, hence $g_1 \cdots g_l = symb(P)$. Moreover it was established in the proof of Theorem 1 that every solution of $P = 0$ of the form (1) such that $(\partial_x + a\partial_y) f_1 = 0$, is a solution of (a unique) I_j for which $(u + aw)|g_j$, thus every solution of $P = 0$ of the form (1) is also a solution of $I_1 \cap \cdots \cap I_l$. Therefore the typical differential dimension of ideal $I_1 \cap \cdots \cap I_l$ equals n (cf. Lemma 4.1 [4]). On the other hand, any overideal of a principal ideal $\langle P \rangle$ of the same typical differential dimension coincides with $\langle P \rangle$; one can verify it by comparing their Janet bases. □

Remark 1 *One can extend Theorem 1 to non-holonomic ideals J such that homogeneous polynomial $GCD(symb(J))$ is separable: namely, there exists a finite number of maximal non-holonomic overideals $I \supset J$.*

2 Non-holonomic Overideals of a Second-Order Linear Partial Differential Operator

In this section we study the structure of overideals of $\langle P \rangle$ when $n = ord(P) = 2$. The case of a separable $symb(P)$ is covered by Theorem 1. However, it is an open question, whether one can verify existence of a non-holonomic proper overideal of $\langle P \rangle$.

Let $symb(P)$ be non-separable. Then applying a transformation of the type $\partial_x \to b_1\partial_x + b_2\partial_y$, $\partial_y \to b_3\partial_x + b_4\partial_y$ for suitable $b_1, b_2, b_3, b_4 \in F$ one can assume w.l.o.g. that $P = \partial_y^2 + p_1\partial_x + p_2\partial_y + p_3$ (it would be interesting to find out when one can carry out these transformations algorithmically). First let $p_1 = 0$. Then P is essentially ordinary, i.e. it becomes ordinary after a transformation as described above; for any solution $u \in F$ of the equation $P = 0$ we get a non-holonomic overideal $\langle \partial_y - u_y/u \rangle \supset \langle P \rangle$.

Now suppose that $p_1 \neq 0$. Then P is irreducible (see e.g. Corollary 7.1 [4]). Moreover we claim that $\langle P \rangle$ has at most one maximal non-holonomic overideal. Let $I \supset \langle P \rangle$ be a non-holonomic overideal. Choosing arbitrary non-zero elements $b_1, b_2 \in F$ define the derivation $d = b_1\partial_x + b_2\partial_y$. Similar to the proof of Theorem 1 there exists $r \in F[d, \partial_y] = F[\partial_x, \partial_y]$ such that $\langle r \rangle = IR_1 \subset R_1 = (F[d])^{-1}\, F[d, \partial_y]$. Then $\beta P = r_1 r$ for suitable $\beta \in F[d],\ r_1 \in F[d, \partial_y]$ and $symb(r) = (b_1 v + b_2 w)^m g$ for an integer m and $g | w^2$. If $g = 1$ then I cannot be non-holonomic because of Proposition 4.4 [4] (cf. above). If $g = w^2$ then similar to the proof of Corollary 1 one can show that the only non-holonomic overideal of $\langle P \rangle$ among ones to which polynomial w^2 is attached, is just $\langle P \rangle$ itself.

It remains to consider the case $g = w$. Applying the Newton polygon construction from [4] to equation $r = 0$ and a divisor w of $symb(r)$, one obtains a solution of the form (1) of $r = 0$ with $G = G(x)$, thereby it is a solution of $P = 0$. On the other hand, applying the Newton polygon construction from [4] to equation $P = 0$, one gets at its first step $f_1 = x$. At the second step f_2 is obtained which fulfils equation $(\partial_y f_2)^2 + p_1 = 0$, where f_2 corresponds to the edge of the Newton polygon with endpoints $(0, 2),\ (1, 0)$, so its slope is $1/2$. This provides a solution of equation $P = 0$ of the form (1) with $G = G(x, f_2;\ 1/2)$, therefore equation $P = 0$ has no solutions of the form (1) with $G = G(x)$. The achieved contradiction shows that there are no non-holonomic overideals I with attached polynomial w. This completes the proof of the claim.

Summarizing we can formulate (cf. [5] for factoring P over not necessarily differentially closed fields)

Proposition 1 *A principal ideal* $\langle P\rangle$ *for a second-order operator* $P=\partial_y^2+p_1\partial_x+p_2\partial_y+p_3$ *with non-separable* $symb(P)$ *has*

i) no proper non-holonomic overideals in case $p_1\neq 0$*;*

ii) an infinite number of maximal non-holonomic overideals in case $p_1=0$*.*

3 On Non-holonomic Overideals of a Third-Order Operator

Now we study overideals of $\langle P\rangle$ where the order $n=ord(P)=3$ (we mention that in [4] an algorithm is designed for factoring P). Due to Theorem 1 it remains to consider non-separable $symb(P)$. We mention that few explicit calculations for factoring P are provided in [7].

3.1 Symbol with two Different Linear Divisors

First let $symb(P)$ have two linear divisors; therefore one can assume w.l.o.g. (see above) that w is its divisor of multiplicity 2 and v is its divisor of multiplicity 1. One can write

$$P=\partial_y^2\partial_x+p_0\partial_x^2+p_1\partial_y^2+p_2\partial_x\partial_y+p_3\partial_y+p_4\partial_x+p_5.$$

Suppose that $p_0\neq 0$. The Newton polygon construction from [4] applied to equation $P=0$ and the divisor w of $symb(P)$, yields a solution of the form (1) of $P=0$ with $f_1=x$ at its first step. At its second step the construction yields f_2 which fulfils equation $(\partial_y f_2)^2+p_0=0$ and which corresponds to the edge of the Newton polygon with endpoints $(1,2),\ (2,0)$, so with the slope $1/2$. This provides $G=G(x,f_2;\ 1/2)$ in (1).

Let a non-holonomic ideal $I\supset\langle P\rangle$. Choose $d=b_1\partial_x+b_2\partial_y$ for non-zero $b_1,b_2\in F$. As in the previous Section there exists $r\in F[d,\partial_y]$ such that $\langle r\rangle=R_1I\subset R_1=(F[d])^{-1}\ F[d,\partial_y]$. Then $\beta P=r_1r$ for suitable $\beta\in F[d],\ r_1\in F[d,\partial_y]$. Rewrite $symb(r)=(b_1v+b_2w)^m g$ where $g|(vw^2)$. If either $g=w^2$ or $g=v$, one can argue as in the proof of Theorem 1 and deduce that there can exist at most one maximal non-holonomic overideal of $\langle P\rangle$ with the property that the polynomial attached to the overideal is either w^2 or v. Similar to the proof of Corollary 1 one can verify that if there exist maximal non-holonomic overideals $I_2,I_1\supset\langle P\rangle$ with attached polynomials w^2 or v then $\langle P\rangle=I_1\cap I_2$. As in Theorem 1 the existence of a maximal overideal with the attached polynomial w^2 (or v, respectively) follows from the existence of any non-holonomic overideal with the attached polynomial w^2 (or v, respectively).

If either $g=w$ or $g=vw$ then applying the Newton polygon construction from [4] to equation $r=0$ and divisor w of $symb(r)$, one obtains a solution of $r=0$ (and thereby, of $P=0$ due to Lemma 4.2 [4]) of the form (1) with $G=G(x)$ which contradicts to the supposition $p_0\neq 0$ (see above). Thus, in case $p_0\neq 0$ ideal $\langle P\rangle$ has at most two maximal non-holonomic overideals (similar to Theorem 1).

When $p_0=0$ this is not always true, say for $P=(\partial_x+b)(\partial_y^2+b_3\partial_y+b_4)$ (cf. case $n=2$ in the previous Section). It would be interesting to clarify for which P this is still true.

3.2 Symbol with a Unique Linear Divisor

Now we consider the last case when $symb(P)$ has a unique linear divisor with multiplicity 3. As above one can assume w.l.o.g. that $symb(P)=w^3$, so

$$P=\partial_y^3+p_0\partial_x^2+p_1\partial_y^2+p_2\partial_x\partial_y+p_3\partial_y+p_4\partial_x+p_5.$$

Keeping the notations we get $\langle r\rangle=R_1I$ and $\beta P=r_1r$. Then $symb(r)=(b_1v+b_2w)^m g$ where $g|w^3$. If $g=w^3$ then arguing as in the proof of Corollary 1 we deduce that the only non-holonomic overideal of $\langle P\rangle$ to which polynomial w^3 is attached, is just $\langle P\rangle$ itself. Let $g|w^2$. Applying the Newton polygon construction from [4] to equation $r = 0$ and linear divisor w of $symb(r)$ one gets a solution of $r = 0$ (and thereby of $P = 0$) with either $G = G(x)$ or $G = G(x, f_2;\ 1/2)$ where $\partial_y f_2 \neq 0$ (cf. above).

Application of the Newton polygon construction from [4] to equation $P = 0$ (and unique linear divisor w of $symb(P)$) at its first step provides $f_1 = x$. The second step requires a trial of cases. First let $p_0 \neq 0$. Then the second step yields f_2 which fulfils equation $(\partial_y f_2)^3 + p_0 = 0$ and which corresponds to the edge of the Newton polygon with endpoints $(0, 3),\ (2, 0)$, so with the slope $2/3$. Thus we obtain a solution of the form (1) with $G = G(x, f_2, \ldots;\ 2/3, \ldots)$, hence $\langle P\rangle$ in case $p_0 \neq 0$ has no non-holonomic overideals with attached polynomial g being a divisor of w^2 (see above). Now assume that $p_0 = 0$ and $p_2 \neq 0$. Then the second step provides solutions of $P = 0$ of the form (1) with two different possibilities. Either the Newton polygon construction chooses the vertical edge with endpoints $(1, 1),\ (1, 0)$ as a leading edge at the second step, then it terminates at the second step yielding a solution of the form (1) with $G = G(x)$ (we recall that in the construction from Section 2 [4] only edges with non-negative slopes are taken as leading ones and the construction terminates while taking a vertical edge, so with the slope 0, as a leading one, in particular the edge with endpoints $(1, 1),\ (1, 0)$ is taken as a leading one regardless of whether the coefficient at point $(1, 0)$ vanishes). As the second possibility the construction yields a solution of the form (1) with $G = G(x, f_2, \ldots;\ 1/2, \ldots)$ where $f_2 \neq 0$ fulfils equation $(\partial_y f_2)^3 + p_2 \partial_y f_2 = 0$ corresponding to the edge of the Newton polygon with endpoints $(0, 3),\ (1, 1)$, so with the slope $1/2$. One can suppose w.l.o.g. that the Newton polygon construction terminates at its third step (thereby $G = G(x, f_2;\ 1/2)$), otherwise $\langle P\rangle$ cannot have a non-holonomic overideal to which a divisor g of w^2 is attached (see above).

If $g = w^2$ then any solution H_2 of $P = 0$ of the form (1) with $G = G(x, f_2;\ 1/2)$ is a solution of $r = 0$ because otherwise $rH_2 \neq 0$, being also of the form (1) with $G = G(x, f_2;\ 1/2)$, cannot be a solution of $r_1 = 0$ taking into account that $symb(r_1)$ does not divide on w^2 (cf. Lemma 4.2 [4]). Else if $g = w$ then $rH_2 \neq 0$ (again taking into account that $symb(r)$ does not divide on w^2) and therefore $r_1(rH_2) = 0$. Hence for a solution H_1 of $P = 0$ of the form (1) with $G = G(x)$ (see above) we have $rH_1 = 0$ since otherwise rH_1 being also of the form (1) with $G = G(x)$ cannot be a solution of $r_1 = 0$ (again cf. Lemma 4.2 [4]). Then arguing as in the proof of Theorem 1 one concludes that in case $p_0 = 0$ and $p_2 \neq 0$ ideal $\langle P\rangle$ can have at most two maximal non-holonomic overideals (with attached polynomials w and w^2, respectively). Similar to the proof of Corollary 1 (cf. the preceding Subsection) one can verify that if there exist maximal (non-holonomic) overideals $I_1, I_2 \supset \langle P\rangle$ with attached polynomials w and w^2, respectively, then $\langle P\rangle = I_1 \cap I_2$. As in Theorem 1 the existence of a maximal overideal with the attached polynomial w (or w^2, respectively) follows from the existence of any non-holonomic overideal with the attached polynomial w (or w^2, respectively).

Furthermore, let $p_0 = p_2 = 0,\ p_4 \neq 0$. Then as in case $p_0 \neq 0$ we argue that the second step of the Newton polygon construction applied to equation $P = 0$ yields f_2 which fulfils equation $(\partial_y f_2)^3 + p_4 = 0$ and which corresponds to the leading edge of the Newton polygon with endpoints $(0, 3),\ (1, 0)$, so with the slope $1/3$. Thus the Newton polygon construction yields a solution of $P = 0$ of the form (1) with $G = G(x, f_2, \ldots;\ 1/3, \ldots)$ and again $\langle P\rangle$ in case $p_0 = p_2 = 0,\ p_4 \neq 0$ under consideration has no non-holonomic overideals with an attached polynomial being a divisor of w^2.

Finally, when $p_0 = p_2 = p_4 = 0$ the ideal $\langle P = \partial_y^3 + p_1\partial_y^2 + p_3\partial_y + p_5\rangle$ has an infinite number of maximal non-holonomic overideals (similar to the second-order case $P = \partial_y^2 + p_3\partial_y + p_5$, see above). Summarizing we conclude with the following

Proposition 2 *Let P be a third-order operator with non-separable $symb(P)$.*

i) When $symb(P)$ has two different (linear) divisors (one of which of multiplicity 2) then we can assume w.l.o.g. that $P = \partial_y^2\partial_x + p_0\partial_x^2 + p_1\partial_y^2 + p_2\partial_x\partial_y + p_3\partial_y + p_4\partial_x + p_5$. If $p_0 \neq 0$ then $\langle P\rangle$ has at most two maximal non-holonomic overideals. Moreover if there exist two different maximal non-holonomic overideals $I_1, I_2 \supset \langle P\rangle$ then $\langle P\rangle = I_1 \cap I_2$;

ii) when symb(P) has a single linear divisor (of multiplicity 3) then we can assume w.l.o.g. that $P = \partial_y^3 + p_0\partial_x^2 + p_1\partial_y^2 + p_2\partial_x\partial_y + p_3\partial_y + p_4\partial_x + p_5$*. If either* $p_0 \neq 0$*, either* $p_2 \neq 0$ *or* $p_4 \neq 0$ *then* $\langle P\rangle$ *has at most two maximal non-holonomic overideals. Moreover if there exist two different maximal non-holonomic overideals* $I_1, I_2 \supset \langle P\rangle$ *then* $\langle P\rangle = I_1 \cap I_2$*. Otherwise* $\langle P = \partial_y^3 + p_1\partial_y^2 + p_3\partial_y + p_5\rangle$ *has an infinite number of maximal non-holonomic overideals.*

It is a challenge to design an algorithm which produces non-holonomic overideals of a given differential ideal $J \subset F[\partial_x, \partial_y]$.

Appendix. Explicit formulas for Laplace transformation

We exhibit a short exposition and explicit formulas for the Laplace transformation [2].

Let $Q = \partial_{xy} + a\partial_x + b\partial_y + c$ be a second-order operator and $L_n = \sum_{0\le i\le n} l_i\partial_x^i$ a *Laplace divisor* of order n, in particular $Q,\ L_n$ form a Janet basis, hence

$$PQ = (\partial_y + a)L_n \tag{2}$$

for a suitable $P = \sum_{0\le i\le n-1} p_i\partial_x^i$. This form of P is obtained by comparing the highest terms which divide on ∂_x^n in (2)). Comparing the highest terms in (2) which divide on ∂_y, we get that $L_n = P(\partial_x + b)$. Thus

$$PQ = (\partial_y + a)P(\partial_x + b). \tag{3}$$

We have $Q \neq (\partial_y + a)(\partial_x + b)$ iff $0 \neq ab + b_y - c =: K_0$.

Lemma 1 *If* $K_0 \neq 0$ *then there are unique* $B,\ C$ *such that*

$$(\partial_x + B)Q = (d_{xy} + a\partial_x + B\partial_y + C)(\partial_x + b) \tag{4}$$

Proof. (4) is equivalent to a linear algebraic system for B and C.

$$aB - C = b_y + ab - a_x - c, \quad (c - b_y)B - bC = b_{xy} + ab_x - c_x.$$

Therefore (3) holds iff $P = P_1(\partial_x + B)$ by means of dividing P by $\partial_x + B$ with remainder. Substituting the latter equality to (3) and making use of (4) we obtain the equality

$$P_1(d_{xy} + a\partial_x + B\partial_y + C) = (\partial_y + a)P_1(\partial_x + B). \tag{5}$$

Now (5) is similar to (3) but with the order $ord(P_1) = ord(P) - 1 = n - 1$ and a new second-order operator $Q_1 = d_{xy} + a\partial_x + B\partial_y + C$. Continuing this way we get the Laplace transformation with $K_1 = aB + B_y - C$ etc. □

More uniformly denote $b_0 := b,\ c_0 := c$, then $b_1 := B,\ c_1 := C,\ b_2,\ c_2$ etc. obtained from Lemma 1. Denote

$$K_i := ab_i + (b_i)_y - c_i,\ Q_i := d_{xy} + a\partial_x + b_i\partial_y + c_i.$$

Corollary 2 *There exists* L_n *satisfying (2) iff for the minimal* m *such that* $K_m = 0$ *we have* $m \le n$*. In this case*

$$L_n = P_{n-m}(\partial_x + b_{m-1})\cdots(\partial_x + b_0) \tag{6}$$

where $P_{n-m} = \sum_{0\le i\le n-m} p_i\partial_x^i$ *is an arbitrary operator of the order* $n - m$ *which fulfils*

$$P_{n-m}(\partial_y + a) = (\partial_y + a)P_{n-m}. \tag{7}$$

For any order $n - m \ge 0$ *such an operator* P_{n-m} *exists. The pair* $Q,\ L_n$ *constitutes a Janet basis of the ideal* $\langle Q,\ L_n\rangle$*. The ideal* $\langle Q,\ L_m\rangle$ *is the unique maximal non-holonomic overideal of* $\langle Q\rangle$ *which corresponds to a divisor* y *of* $symb(Q) = xy$ *(see Theorem 1).*

Proof. Applying Laplace transformations as above, if $m > n$ we don't get a solution of (2) after n steps since (3) with $PQ_n = (\partial_y + a)P(\partial_x + b_n)$ would not have a solution with P of the order 0. If $m \le n$ then successively following Laplace transformations we arrive to (6) in which (7) is obtained from equality $PQ_m = (\partial_y + a)P(\partial_x + b_m)$ (3)) and taking into account that $K_m = 0$. $\square$

Remark 2 *Due to (2) existence of a Laplace divisor is equivalent to reducibility (in a special form) of* Q *in the ring* $(F[\partial_x])^{-1} F[\partial_x, \partial_y]$ *(see above).*

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