REIDEMEISTER NUMBERS OF SATURATED WEAKLY BRANCH GROUPS

ALEXANDER FEL'SHTYN, YURIY LEONOV, AND EVGENIJ TROITSKY

ABSTRACT. We prove for a wide class of saturated weakly branch group (including the (first) Grigorchuk group and the Gupta-Sidki group) that the Reidemeister number of any automorphism is infinite.

CONTENTS

1. INTRODUCTION

Let $\phi : G \to G$ be an automorphism of a group G. A class of equivalence $x \sim gx\phi(g^{-1})$ is called the *Reidemeister class* or *ϕ-conjugacy class* or *twisted conjugacy class of ϕ*. The number $R(\phi)$ of Reidemeister classes is called the *Reidemeister number.* The interest in twisted conjugacy relations has its origins, in particular, in the Nielsen-Reidemeister fixed point theory (see, e.g. [18, 4]), in Selberg theory (see, eg. [22, 1]), and Algebraic Geometry (see, e.g. [16]). The main current problem of the field is to obtain a twisted analogue of the celebrated Burnside-Frobenius theorem [6, 4, 9, 10, 27, 8, 7]. For this purpose it is important to describe the class of groups G, such that $R(\phi) = \infty$ for any automorphism $\phi : G \to G$. First attempts to localize this class of groups go up to [6]. After that it was proved that the following groups belong to this class: non-elementary Gromov hyperbolic groups [11, 21], Baumslag-Solitar groups $BS(m,n) = \langle a, b | ba^m b^{-1} = a^n \rangle$ except for $BS(1,1)$ [5], generalized Baumslag-Solitar groups, that is, finitely generated groups

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which act on a tree with all edge and vertex stabilizers infinite cyclic [20], lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ iff $2|n$ or $3|n$ [12], the solvable generalization Γ of $BS(1,n)$ given by the short exact sequence

$$1 \to \mathbb{Z}[\frac{1}{n}] \to \Gamma \to \mathbb{Z}^k \to 1$$

as well as any group quasi-isometric to Γ [26], groups which are quasi-isometric to $BS(1,n)$ [25] (while this property is not a quasi-isometry invariant), the chameleon R. Thompson group [2].

In paper [26] a terminology for this property was suggested. Namely, a group G has *property* R_∞ if any its automorphism ϕ has $R(\phi) = \infty$.

For the immediate consequences of R_∞ property for the topological fixed point theory e.g. see [25].

In the present paper we prove that a wide class of weakly branch groups including the Grigorchuk group and the Gupta-Sidki group, has R_∞ property.

The results of the present paper demonstrate that the further study of Reidemeister theory for this class of groups has to go along the lines specific for the infinite case. On the other hand these results make smaller the class of groups, for which the twisted Burnside-Frobenius conjecture [6, 9, 10, 27, 8, 7] has to be verified.

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2. Preliminaries on weakly branch groups

Let $\mathcal{T}$ be a (spherically symmetric) rooted tree. A group G acting faithfully on a rooted tree is said to be a *weakly branch group* if for every vertex v of $\mathcal{T}$ there exists an element of G which acts nontrivially on the subtree $\mathcal{T}_v$ with the root vertex v and trivially off it.

The group G is said to be *saturated* if for every positive integer n there exists a characteristic subgroup $H_n \subset G$ acting trivially on the n-th level of $\mathcal{T}$ and level transitive on any subtree $\mathcal{T}_v$ with v in the n-th level.

Theorem 2.1 ([19])**.** *Let G be a saturated weakly branch group. Then the automorphism group* $\operatorname{Aut} G$ *coincides with the normalizer of G in the full automorphism group* $\operatorname{Iso} \mathcal{T}$ *of the rooted tree; i.e., every automorphism of the group G is induced by conjugation from the normalizer and the centralizer of G in* $\operatorname{Iso} \mathcal{T}$ *is trivial.*

All groups in what follows will be supposed to be *saturated weakly branch.*

Theorem 2.2 ([19, Theorem 7.1])**.** *Let G be a level-transitive isometry group of a rooted tree $\mathcal{T}$ with a stabilizer sequence $(G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots)$. An automorphism $\phi \in \operatorname{Aut} G$ is induced by an element of the full automorphism group* $\operatorname{Iso} \mathcal{T}$ *of the rooted tree if and only if for every $n \geq 0$ there exists $a_n \in G$ such that $a_n G_i a_n^{-1} = \phi(G_i)$ for every $i \leq n$.*

3. Reidemeister classes and inner automorphisms

Let us denote by $\tau_g : G \to G$ the automorphism $\tau_g(\widetilde{g}) = g\widetilde{g}\,g^{-1}$ for $g \in G$. Its restriction on a normal subgroup we will denote by τ_g as well. We will need the following statements.

Lemma 3.1. $\{g\}_\phi k = \{g\,k\}_{\tau_{k^{-1}} \circ \phi}$.

Proof. Let $g' = f\,g\,\phi(f^{-1})$ be ϕ-conjugate to g. Then

$$g'\,k = f\,g\,\phi(f^{-1})\,k = f\,g\,k\,k^{-1}\,\phi(f^{-1})\,k = f\,(g\,k)\,(\tau_{k^{-1}} \circ \phi)(f^{-1}).$$

Conversely, if g' is $\tau_{k^{-1}} \circ \phi$-conjugate to g, then

$$g'\,k^{-1} = f\,g\,(\tau_{k^{-1}} \circ \phi)(f^{-1})k^{-1} = f\,g\,k^{-1}\,\phi(f^{-1}).$$

Hence a shift maps ϕ-conjugacy classes onto classes related to another automorphism. $\square$

Corollary 3.2. $R(\phi) = R(\tau_g \circ \phi)$.

4. Binary trees

Consider the subset $K_n \subset G$ formed of elements of the stabilizer $\mathrm{St}_n\,G$ such that on the $n+1$-th level they are switching each pair of (neighboring) vertexes.

Lemma 4.1. *Under our assumptions, K_n is not empty.*

Proof. Let us encode the action of an element of G, which consists only of switches or trivial actions on neighboring vertices as a sequence of -1 and 1. For $n = 1$ the statement is evident. Let us argue by induction and suppose that the statement is true for the levels up to $n-1$. By the supposition on G to be saturated, among the elements of $\mathrm{St}_n(G)$ there are some elements, such that for any pair of neighboring vertices one of these elements has -1 on the corresponding places. We have two possibilities: in any pair of neighbors both entries are equal to each other, or there is a pair with $+1$ and -1. In the second case let us conjugate our element by the desired element at the level $n-1$. The result of its action is the following: it transposes each pair. Indeed, since the automorphism group of $\mathbb{Z}_2$ is $\mathbb{Z}_2$, the conjugation can only permutate. For example, the conjugation sends

$$((1,-1),(1,1),(-1,-1),(1,-1),\dots)$$

to

$$((-1,1),(1,1),(-1,-1),(-1,1),\dots).$$

Their product will have $(-1,-1)$ on the place under consideration and pairs of the same elements (i.e., $(-1,-1)$ or $(1,1)$) on the remaining ones. So, we have reduced the second possibility to the first one. We go further taking the conjugation by the element, which was obtained at the level $n-2$. After an analogous multiplication we obtain a (nontrivial) element with quadruples of neighbors formed by the same elements. And so on. The end step of the induction (may be the first one) is the desired element at the level n. $\square$

Lemma 4.2. *Let an automorphism t (after the elimination of action on upper levels) have at level $m+1$ the number of switches, which is distinct from 2^{m-1}. Then the Reidemeister class of an element from K_m does not intersect $\mathrm{St}_{m+1}\,G$.*

Proof. Suppose, $g \in K_m$, $h \in G$. Let us consider

$$hg\phi(h^{-1}) = hgth^{-1}t^{-1} = (hgh^{-1})(hth^{-1}t^{-1}). \tag{1}$$

By the same argument with $\mathbb{Z}_2$ as in proof of Lemma 4.1, $hgh^{-1} \in K_m$. If $hth^{-1}t^{-1}$ is non-identical on some of levels $1,\dots,m$, then the product (1) is non-identical as well, since $hgh^{-1} \in K_m$, and we are done. Otherwise, let us remark, that after elimination of the action on the previous levels the number of switches on the level $m+1$ in $hgth^{-1}$ is 2^m-(the number of switches of t), which is distinct from the number of switches of t (or

t^{-1}) provided that it is not equal to 2^{m-1}. Hence the total number of switches in (1) is non-zero. So it is non-trivial on the level $m+1$. $\square$

Theorem 4.3. *Let G be a saturated weakly branch group acting on a binary tree $\mathcal{T}$. Suppose, $\phi : G \to G$ is an automorphism, such that for any $k \in \mathbb{N}$ there exists an inner automorphism of G such that its composition ϕ' with ϕ satisfies the condition of Lemma 4.2 at some collection of levels of number k. Then $R(\phi) = \infty$.*

Proof. Let us take an arbitrary $k \in \mathbb{N}$ and show that the number of Reidemeister classes is not less than k. For this purpose, take an appropriate inner automorphism in accordance with the supposition. Then the Reidemeister numbers of ϕ and ϕ' are the same (cf. Lemma 3.2). So it is sufficient to prove that $R(\phi') \geq k$. For the notation brevity, suppose that the levels with mentioned parity properties are $1, \dots, k$. By Lemma 4.2, for elements $g_i \in K_i \subset \mathrm{St}_i\, G$ one has $\{g_i\}_{\phi'} \cap \mathrm{St}_{i+1}\, G = \varnothing$, $i = 1, \dots, k$. Hence the classes $\{g_i\}_{\phi'}$, $i = 1, \dots, k$, are distinct. $\square$

Remark 4.4. It is clear, that the following condition can serve as an alternative for the supposition of the theorem: there exists a $g_i \in K_i$ with an odd number of switches.

5. Grigorchuk group

Now we want to prove that the Grigorchuk group G ([15], see also [3, 13, 14]) satisfies the conditions of Theorem 4.3.

Consider the following presentation of G (cf. [14]). It has the generators a, b, c, and element d, where $d = bc$, a is defined at Fig. 1, where -1 is a switch, b and c are defined inductively by

$$b = (a, c), \qquad c = (a, d), \qquad d = (1, b),$$

where brackets mean the action on the corresponding sub-trees.

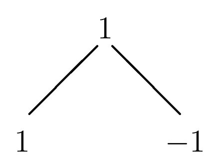

Figure 1

Then b and c are as at Fig. 2 (we partially omit 1's).

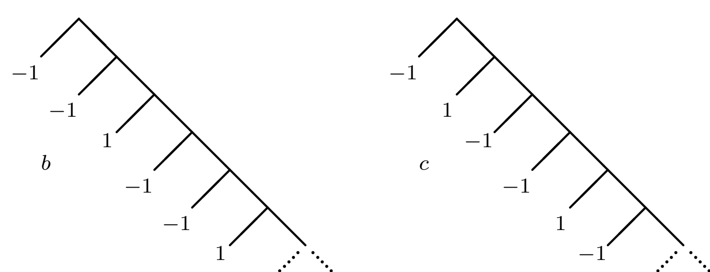

Figure 2

In particular

$$a^2 = b^2 = c^2 = d^2 = 1, \qquad a^{-1}ca = (d,a), \qquad a^{-1}da = (b,1). \tag{2}$$

By [14], any automorphism of Grigorchuk group, up to taking a product by an inner one, is a finite product of commuting involutions of the form (at some level):

$$(1,(ad)^2,1,(ad)^2,1,(ad)^2,1,(ad)^2,\dots).$$

Hence, by (2) on the next level we will have:

$$(1,1,b,b,1,1,b,b,1,1,b,b,\dots).$$

Keeping in mind the form of b (see Fig. 2), we conclude that the number of switches at low levels is bounded (by the number of b's in the above formula). Hence, some uniform estimation holds for their finite product (i.e., our automorphism). Thus, starting from some level, the number of switches is less than the half of places, and we can apply Theorem 4.3.

6. Some generalizations: strongly saturated groups

We return to the case of a general spherical tree $\mathcal{T}$.

Definition 6.1. Let us denote by $l(m)$ the number of vertexes at the level m.

We will make the following supposition about t.

Assumption 6.2. *There exist a constant $s \in (0,1)$ such that for any j the isometry t has the number of fixed vertexes at the level j not less than $s \cdot l(j)$.*

We will need the following definition.

Definition 6.3. A saturated group G is called *strongly saturated* if for any $i \in \mathbb{N}$ there exists an element $g_i \in \mathrm{St}_i(G)$ such that it has no fixed points on the level $i+1$.

As it is proved above, any saturated group on a binary tree is strongly saturated. We will see (Remark 8.2) that the Gupta-Sidki group is strongly saturated as well.

Lemma 6.4. *Suppose, an automorphism ϕ of a strongly saturated group G is defined by $t \in \operatorname{Iso}\mathcal{T}$ and Assumption 6.2 holds. Then there exists an element $\widehat{g} \in \mathrm{St}_m G$ such that its Reidemeister class $\{\widehat{g}\}_\phi$ does not intersect $\mathrm{St}_{m+r}(G)$, where*

$$\left(\frac{1}{2}\right)^r < s.$$

Proof. An element of $\{\widehat{g}\}_\phi$ has the form $h\widehat{g}th^{-1}t^{-1} =: \widehat{g}_h$. Note that for any $\widehat{g} \in \mathrm{St}_m G$ this element is not in $\mathrm{St}_m G$ and hence not in $\mathrm{St}_{m+l(m)}(G)$, if $hth^{-1}t^{-1} \notin \mathrm{St}_m G$. Indeed,

$$\widehat{g}_h = (h\widehat{g}h^{-1})(hth^{-1}t^{-1}),$$

where the first factor belongs $\mathrm{St}_m G$. Thus we need to check for the desired $\widehat{g}$ only that

$$\widehat{g}_h = (h(\widehat{g}t)h^{-1})t^{-1} \notin \mathrm{St}_{m+r}(G), \qquad \text{for } h \text{ such that } hth^{-1}t^{-1} \in \mathrm{St}_m G, \tag{3}$$

i.e., h and t commute on the level m. In this case one has

$$t(hv_m) = htv_m = hv_m,$$

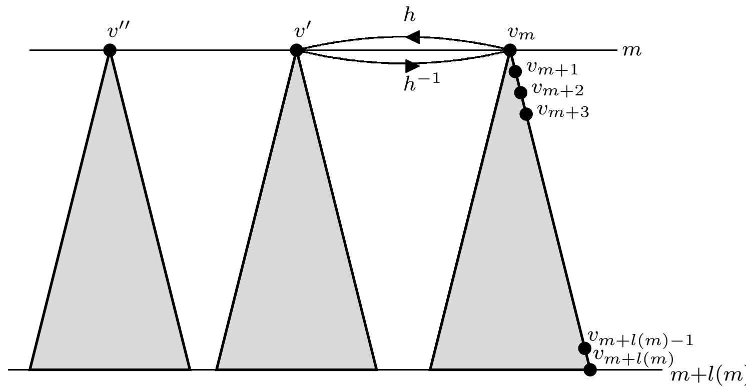

FIGURE 3

i.e. h maps v_m to some other fixed point of t (cf. Fig. 3, where thick points mean some fixed points of t).

We will construct the desired $\widehat{g}$ as a product

$$\widehat{g} = g_{m+l(m)}^{\varepsilon(r)} \cdots g_{m+1}^{\varepsilon(1)} g_m^{\varepsilon(0)}, \qquad \text{where } \varepsilon(i) = 0 \text{ or } 1,\ x^0 := e,$$

where g_i are from Definition 6.3. We will chose these $\varepsilon(i)$ in such a way that $\widehat{g}t$ will have the number of fixed points in the set B_r of the vertexes of the level $m+r$ is less than $s \cdot l(m+r) = s \cdot \#B_r$. This $\widehat{g}$ will be the desired one. Indeed, the map t on the $m+r$ level has at least $s \cdot l(m+r)$ fixed points, and to obtain the identity in the composition (3) one should suppose that $\widehat{g}t$ has the same number of fixed points at the level $m+r$.

Now we pass to the determination of $\varepsilon(i)$ (cf. Fig. 4). If the number of fixed points of t at the set B_1 of all vertexes of $\mathcal{T}$ at the level $m+1$ is less than $1/2$ of $\#B_1$, then we take $\varepsilon(0) = 0$. Otherwise, $\varepsilon(0) = 1$. Then $g_m^{\varepsilon(0)} t$ has the number of fixed points in B_1 less or equal than $\frac{1}{2}\#B_1$. Define B_2 in the same way as B_1, and let $C_2 \subset B_2$ be the set of that points, which do not belong to subtrees with roots at level $m+1$, which are not fixed by $g_m^{\varepsilon(0)} t$. Thus, $\#C_2 \le \frac{1}{2}\#B_2$. The points of $B_2 \setminus C_2$ are not-fixed by $g_m^{\varepsilon(0)} t$. Now we consider the fixed points of $g_m^{\varepsilon(0)} t$ in C_2. If the number of them is less than $\frac{1}{2}\#C_2$, then we take $\varepsilon(1) = 0$. Otherwise, $\varepsilon(1) = 1$. Take the composition $g_{m+1}^{\varepsilon(1)} g_m^{\varepsilon(0)} t$. Since $g_{m+1} \in \mathrm{St}_{m+1} G$, the vertexes which come from the vertexes at the level $m+1$, which were not fixed by $g_m^{\varepsilon(0)} t$, are still not fixed by $g_{m+1}^{\varepsilon(1)} g_m^{\varepsilon(0)} t$. Hence, if $C_3 \subset B_3$ is defined as the complement to the set formed by the vertexes of that subtrees, whose roots were moved by $g_m^{\varepsilon(0)} t$ at the level $m+1$ and by $g_{m+1}^{\varepsilon(1)} g_m^{\varepsilon(0)} t$ at the level $m+2$, then

(1) $\#C_3 \le \left(1 - \frac{1}{2} - \frac{1}{4}\right) \#B_3$;
(2) $g_{m+1}^{\varepsilon(1)} g_m^{\varepsilon(0)} t$ has no fixed points on $B_3 \setminus C_3$.

Now we count the number of fixed points of $g_{m+1}^{\varepsilon(1)} g_m^{\varepsilon(0)} t$ on C_3, and so one. Since at each step at least a half of points from C_i comes to "the world of non-fixed points", while the points which are in the subtrees, whose roots were "joined to the world of non-fixed points" at the previous steps, can not "leave this world", because $g_i \in \mathrm{St}_{i-1}(G)$, we will obtain the desired result in (no more than) r steps. Indeed, the number of fixed points of

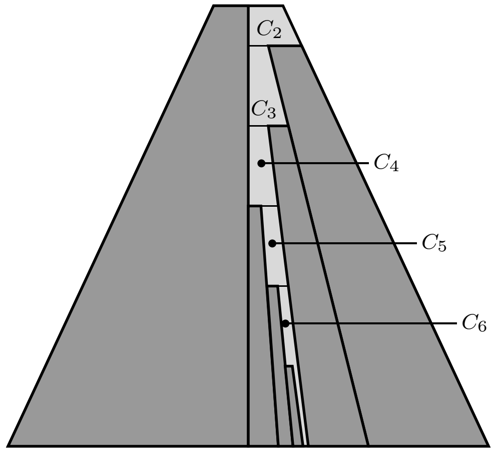

FIGURE 4

$\widehat{g}t$ on the level $m+r$ is less than

$$\#C_r \leq \left(\frac{1}{2}\right)^r \#B_r = \left(\frac{1}{2}\right)^r l(m+r),$$

while, by the supposition, the number of fixed points of t on the level $m+r$ is more than

$$s \cdot l(m+r) > \left(\frac{1}{2}\right)^r l(m+r).$$

□

Theorem 6.5. *Suppose, G is a weakly branch group on a spherical tree $\mathcal{T}$ and ϕ its automorphism induced by $t \in \operatorname{Iso}\mathcal{T}$ restricted to satisfy Assumption* 6.2*. Then $R(\phi) = \infty$.*

Proof. Let us prove that $R(\phi) \geq n$ for any $n \in \mathbb{N}$.

Now we apply Lemma 6.4 for the purpose to produce elements $\widehat{g}_0, \ldots, \widehat{g}_{n-1}$, such that

(1) $\widehat{g}_i \in \mathrm{St}_{ir}(G)$, $i = 0, \ldots, n-1$;
(2) $\{\widehat{g}_i\}_{\phi'} \cap \mathrm{St}_{(i+1)r}(G) = \varnothing$, $i = 0, \ldots, n-1$.

Hence, $R(\phi') \geq n$. □

Remark 6.6. In fact, we need much weaker assumptions, than 6.2. For example, one can suppose the existence of a large number of fixed vertexes, say at each k-th level., etc. This is the case of the Grigorchuk group, as it is evident from Section 5.

Also, it is possible to suppose that Assumption 6.2 holds for gt for some g, etc.

7. Further results: locally normal groups

We will introduce the following definition, which is related to the property of G to be residually finite (cf. [19, p. 171]).

Definition 7.1. Let G be a saturated group on $\mathcal{T}$. It is called *locally normal*, if

(1) the (transitive) subgroup $H(v)$ of $\Sigma_{k(v)}$, which represents the action of $\mathrm{St}_m(G)$ on the branches coming from a vertex v (supposing that the branching index of v is $k(v)$ and v is at the level m), is normal in $\Sigma_{k(v)}$ for any vertex $v \in \mathcal{T}$;

(2) for any automorphism $\phi : G \to G$ which is defined by an isometry t of $\mathcal{T}$ fixing v, the corresponding element of $\Sigma_{k(v)}$ belongs to $H(v)$.

The following statement is evident.

Lemma 7.2. *If $\mathcal{T}$ is a binary, then G is always locally normal.*

Lemma 7.3. *Let $k \in \mathbb{N}$, $k > 1$, $k \neq 4$. Suppose, $H \subset \Sigma_k$ is a normal transitive subgroup of the symmetric group of permutations on k-set. Let $g \in H$ be a non-trivial element. Then the normal subgroup $N_H(g)$ of H generated by g is transitive.*

Proof. The group H is Σ_k or A_k (the alternating group), while the last one is simple for these k. Hence, $N_H(g)$ is Σ_k or A_k. It is transitive. $\square$

Lemma 7.4. *Suppose, ϕ is induced by $t \in \operatorname{Iso}\mathcal{T}$ and $(v_0, v_1, \dots)$ is an end of $\mathcal{T}$. Then for any $n \in \mathbb{N}$ there exists an element $\alpha_n \in G$ such that $\alpha_n t(v_i) = v_i$, $i = 1, \dots, n$.*

Proof. We construct α_n inductively as a composition $\alpha_n = \beta_{n-1} \dots \beta_0$, $\beta_i \in \mathrm{St}_i(G)$. Since $\mathrm{St}_0(G) = G$ acts transitively on the level 1, one can find $\beta_0 \in \mathrm{St}_0(G) = G$ such that $\beta_0 t(v_1) = v_1$. Since $\mathrm{St}_1(G)$ acts transitively on the level 2, one can find $\beta_1 \in \mathrm{St}_1(G)$ such that $\beta_1\beta_0 t(v_2) = v_2$. Moreover, since $\beta_1 \in \mathrm{St}_1(G)$, $\beta_1\beta_0 t(v_1) = \beta_1 v_1 = v_1$. And so on. $\square$

Definition 7.5. Let G be a weakly branch group. Consider the level m. Define the *weak branch index of the level m* as

$$\mathrm{WBI}(m) = \max_v \mathrm{WBI}(\mathcal{T}_v),$$

where v runs over vertexes of the level m, while $\mathrm{WBI}(\mathcal{T}_v)$ is equal to the minimal level in $\mathcal{T}_v$ of non-trivial action of $G[v]$.

Lemma 7.6. *Suppose, a saturated weakly branch locally normal group G acts on a (spherically symmetric) tree $\mathcal{T}$ with no vertex of branching index 4. Let $t \in \operatorname{Iso}\mathcal{T}$ induce an automorphism $\phi : G \to G$. Then for any m there exists an element $\widehat{g} \in \mathrm{St}_m(G)$ such that $\{\widehat{g}\}_\phi \cap \mathrm{St}_{m+\mathrm{WBI}(m)}(G) = \varnothing$, provided that t has a fixed vertex at the level m.*

Proof. Let v_0 be this fixed vertex. In the locally normal case we can make $g't$ act on the first step successors of v_0 without fixed points for some $g' \in \mathrm{St}_m(G)$ (hence, the unique fixed point on $\mathcal{T}(v_0)$ is v_0). If $hg'th^{-1}t^{-1} = g'_h$ is still in the stabilizer of the next level, then we continue as follows.

By the weakly branch condition one can find a non-trivial element $g'' \in G[v]$. Let $m' \leq \mathrm{WBI}(m)$ be its first non-trivial level. Let v be a vertex of the level $m-1$, such that g'' moves its first step successors. Consider the permutation group H of these successors obtained by the (transitive) action of $\mathrm{St}_m(G)$. This group is normal (by the local normality) and contains the representing element of g''. The action of the normalizer of g'' in $\mathrm{St}_m(G)$ is a subgroup of $G[v]$ and its representation on the first step successors of v is transitive by Lemma 7.3. Hence, there is an element $\widetilde{g} \in G[v]$ such that $\widetilde{g}g't$ has at least one fixed point at the level $m + m'$ while t does not have. Hence they can not be conjugate by any h and $(\widetilde{g}g')_h \notin \mathrm{St}_{m+\mathrm{WBI}(m)}(G)$ for any h. So we are done. $\square$

Theorem 7.7. *Suppose, a saturated weakly branch locally normal group G acts on a (spherically symmetric) tree $\mathcal{T}$ with no vertex of branching index 4. Then $R(\phi) = \infty$ for any automorphism of G.*

Proof. Let us take an arbitrary $n \in \mathbb{N}$. We will prove that $R(\phi) \geq n$. By Lemma 7.4 find an element $g \in G$, such that gt has a fixed vertex at each level $1, \dots, w(n)$, where

$$w(n) = \mathrm{WBI}(1) + \mathrm{WBI}(\mathrm{WBI}(1)) + \cdots + \underbrace{\mathrm{WBI}(\mathrm{WBI}(\mathrm{WBI}(\cdots(\mathrm{WBI}}_{n \text{ times}}(1))\cdots)))$$

By Corollary 3.2 $R(\phi) = R(\phi')$, where ϕ' is induced by gt.
We apply inductively n times Lemma 7.6 to prove that $R(\phi') \geq n$. $\square$

If $\mathcal{T}$ is a binary tree, then Lemma 7.2 shows that G is locally normal. Thus, by Theorem 2.1 from Theorem 7.7 we obtain the following statement.

Theorem 7.8. *Let G be a saturated weakly branch group on a binary tree $\mathcal{T}$. Then G has the R_∞-property.*

8. Ternary trees and the Gupta-Sidki group

The Gupta-Sidki group G [17, 24, 23] acts on the ternary tree $\mathcal{T}$ (see Fig. 5) with

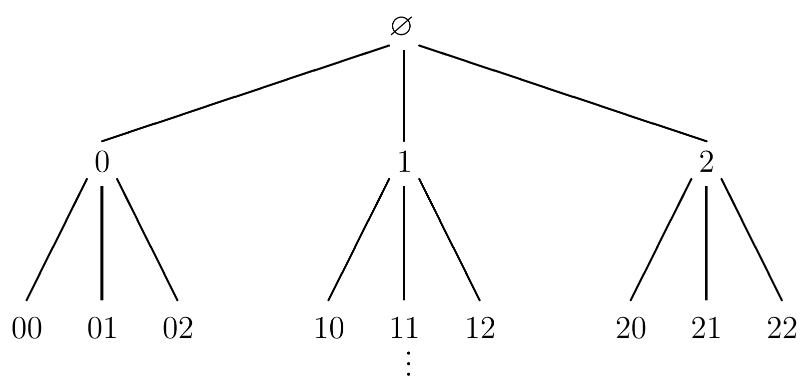

Figure 5

generators x and γ:

$$x : 0 \to 1 \to 2 \to 0, \quad 00 \to 10 \to 20 \to 00, \dots \quad 0s \to 1s \to 2s \to 0s,$$

where s is any finite sequence on $0, 1, 2$ and γ is presented on Fig. 6.

Consider $g = x^{-1}\gamma^{-1}x\gamma$. One checks up directly that $g \in \mathrm{St}_1(G)$ and g acts without fixed points on the level 2. Now let $(g, g, g) \in \mathrm{Iso}\,\mathcal{T}$. It evidently stabilizes the level 2 and acts without fixed points on the level 3. Moreover, $(g, g, g) \in G$. This is proved in [23, Theorem 1], but also can be seen immediately from the facts that $(\gamma, \gamma, \gamma) \in G$ and $(x, x, x) \in \mathrm{Iso}\,\mathcal{T}$ is in the normalizer of G. Hence, $(g, g, g) \in \mathrm{St}_2(G)$. By [23, Theorem 1] (note that there is a misprint in the formulation of that theorem: $i+1$ should be replaced by $i-1$ three times) $g_i := (g, g, \dots, g) \in \mathrm{St}_i(G)$ for the corresponding i. In particular, the subgroup of Σ_3 mentioned in the first item of Definition 7.1 contains $A_3 \cong \mathbb{Z}_3$. On the other hand, G is constructed by actions 1, x, and x^{-1}, hence, it is not larger than A_3. Thus the first item of Definition 7.1 holds.

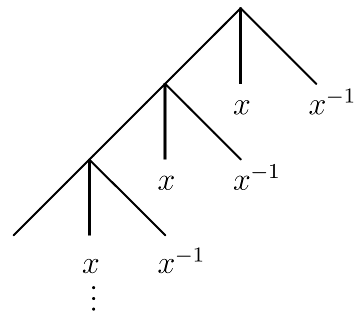

FIGURE 6

Let us remind the description of automorphisms of G [23] (Theorem 3 and pp. 39–41): $\mathrm{Aut}(G) = (G \rtimes X) \rtimes V$, where X is an elementary abelian 3-group of infinite rank with basis $x^{(i)}$ ($i \in \mathbb{N}$):

$$x^{(1)} = (x, x, x), \quad x^{(i+1)} = (x^i, x^i, x^i) \quad \text{for } i \geq 1,$$

and $V \cong \mathbb{Z}_4$ with nontrivial elements τ_1, τ_2, τ_3, which map generators of G in the following way

$$\begin{aligned} \tau_1(\gamma) &= \gamma, & \tau_1(x) &= x^{-1}, \\ \tau_2(\gamma) &= \gamma^{-1}, & \tau_2(x) &= x, \\ \tau_3(\gamma) &= \gamma^{-1}, & \tau_3(x) &= x^{-1}, \end{aligned}$$

and act on the base of X by the formulas ($\tau_3 = \tau_2 \circ \tau_1$):

$$\tau_1(x^i) = (x^i)^{-1}, \quad \tau_2(x^{2i-1}) = (x^{2i-1})^{-1}, \quad \tau_2(x^{2i}) = (x^{2i}),$$

for $i \geq 1$. Thus the action of t, mentioned in the second item of Definition 7.1, can be only defined by 1, x or x^{-1}, i.e., belongs to A_3. So, G is locally normal.

Also, this group is saturated weakly branch [19, Prop. 8.6], [23]. Hence applying Theorem 7.7 we obtain

Theorem 8.1. *For any automorphism ϕ of the Gupta-Sidki group one has $R(\phi) = \infty$.*

Remark 8.2. The argument above concerning the first property of Definition 7.1 shows that the Gupta-Sidki group is strongly saturated (Definition 6.3).

In fact, the unique proper transitive subgroup of Σ_3 is A_3. Hence, one has the following statement.

Theorem 8.3. *Any saturated group on a ternary tree enjoys the first property of Definition* 7.1*.*

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